Verification Theorems for Discontinuous Value Functions in Optimal Control

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Zhouff17ff, etc.), take different types of generalized solutions of the associated Hamilton - Jacobi - Bellman (HJB) equation as ffverification functionsff to *check the optimality of individual trajectories; on the other hand, not only the natural framework of Dynamic Programming but also the "practice" of solving concrete problems (e.g. Cesari“““, etc.) are considering the optimization problem for every "initial point" in a given subset of the phase space.*

We are not considering here the problem of choosing ("describing", etc.) a "feasible" selection of (possibly optimal) admissible trajectories which is of a totally different nature; however, we point out that for this first step of Dynamic Programming Method in Optimal Control, one may use either necessary optimality conditions (when available) or different types of extensions of Cauchy's Method of Characteristics to construct "generalized Hamiltonian flows" and fields of extremals (“““,““““,““““, etc.) that generate "feasible" selections of possibly optimal trajectories; the experience shows that the later approach, when applicable, has the advantage of providing certain differential inequalities for the value function that are essential in the statements of the verification theorems.

In this setting, the monotonicity property (along the admissible trajectories) of the value function in optimal control (e.g. “““-“““,etc.) reduces the problem of sufficient conditions of Dynamic Programming type to the apparently more general one of monotonic functions along trajectories of differential inclusions (in particular, control systems).

The main difficulties in the study of this problem are generated by the strongly nonsmooth character of the value functions of many significant examples of optimal control problems.

The paper is organized as follows: in Section “ we present the example that motivates our work and in Section “, the Dynamic Programming approach based on selections of possibly optimal trajectories is presented.

In Section “ we recall first some existing results and prove several new ones to obtain a "list of basic verification theorems" for value functions that have certain regularity properties (differentiability, Lipschitzianity, semicontinuity) on their effective domains and note that in certain cases none of these results may be applied due, mainly, to the possible irregular behavior of the value functions at the points of the "terminal sets".

The main results of the paper are contained in Section “, where we use the "end-point monotonicity results" in “““ to obtain sufficient optimality conditions for "strongly" discontinuous value functions that may be neither lower nor upper semicontinuous; we show in Section “ that, in particular, these results may be applied to the examples in Section “.

2. Motivation

As a marginal function of infimum-type (e.g. ff1ff, ff2ff,etc.) one may expect the value function of an optimal problem to be at least lower semicontinuous (e.g. ff8ff, Appendix) but the following simple example shows that this may not be the case.

Example 2.1. In order to use the Dynamic Programming approach, we reformulate the problem in Calculus of Variations that is studied in Cesari ff5ff, ff3.9, as follows: for every ffinitial pointff (s_0, y_0) 2 E
$[0, \pi)\ \Omega\ R$, $mi\theta imi$

the class Ω_∞

and is even more irregular than the one in Example 2.1 since the restriction $W_0(.,.) := W(.,.)\Omega\widetilde{E}_0$ is nΩi

We note first that the time-interval $_1][s,t$ in $(\Omega\Omega1)\Omega(\Omega\Omega4)$ ma ffsing a term, introduced apparently by Fleming and Rishel ff7ff, one usually calls ffverification theoremff any result containing hypotheses on $W_{\mathcal{A}}(.,.)$ ΩΩaΩ impl

its References), directly to the real functions $_x(n)$ in (ΩΩ11)Ω as ΩΩΩsΩ rΩsulΩs ΩΩ far ΩΩ

For real-valued functions, $f(.)\ :\ X\ \theta^n R\theta\ R$, *θe sθall use tθe extreme contingent derivatives defined by:*

$$\begin{aligned}\overline{D}_K^{\pm} f(x;v) &:= \limsup_{(\theta,u)\to(0_{\pm},v)} \tfrac{f(x+\theta u)-f(x)}{\theta},\ v\ \theta\ K_x^{\pm}X \\ \underline{D}_K^{\pm} f(x;v) &:= \liminf_{(\theta,u)\to(0_{\pm},v)} \tfrac{f(x+\theta u)-f(x)}{\theta},\ v\ \theta\ K_x^{\pm}X\end{aligned} \qquad \theta\theta\theta\theta\theta$$

θθicθ coiθcide θitθ tθe Diθi derivatives if $X\ \theta\ R$ *is aθ iθterval aθd* $v = 1\ 2\ R$.

θoreoverθ tθougθ tθe admissiθle trajectories $x(.)\ 2\ \Omega_a$ *arθ θθθθrθnθ tθaθθθ θutsθθθ a null subset* $J\ _x\ \theta\ [s, t\ _1]$ ΩΩiΩΩ is *countable if* $\Omega_a = \Omega_r$ *anθ finite if* $\Omega_a = \Omega_{pc}$, *iθ certaiθ moθotoθicity results oθe θeeds its θgeθerθ ali*

Differentiable ffeometry)ff in this case, *the derivative of* $f(.)$ ΩiΩΩ rΩspΩΩΩ ΩΩ ΩΩΩ sΩraΩiΩΩaΩiΩnΩ $_X$ *is θθθinθθ θy.*

$$Df(x) := Df_j(x)\ 2\ L(T_x S_j, R^n) \quad if\ x\ \theta\ S_j\ \theta\ \theta\ _X \qquad \theta\theta\theta\theta\theta$$

θ τrst sθt θf vθrθτcatθθn tθθθrθms may θθ θθtaθnθθ θθrθctθy frθm θθθθrθnt typθs θf mθnθtθnθcθty tθθθrθms θθθθθ θθθθθ appθθθθ tθ tθθ rθaθ functθθns $\omega\ _x(.)$ in (ΩΩ11)Ω prΩΩiΩΩΩ ΩΩΩ ΩaluΩ funΩΩiΩnin $_A(.)$ W(ΩΩ10) Ωas ΩnΩuΩΩ rΩΩulariΩ

The next 5 theorems may be considered as variants of certain, more or less known, results (ff5ff-ff8ff, ff11ff-ff14ff, ff17ff, etc.) so we shall omit their proofs which, in fact, are very similar to the proofs of Theorems 4.7-4.11 below.

A slight refinement of what may be called ffthe elementary verification theoremff (ff5ff,ff7ff, etc.), in which the value function is usually assumed to be of class^1,C*is tθe folloθiθg.*

Theorem 4.2 (Elementary verification theorem ff12ff,ff13ff). $\mathcal{A}(.,.)$ *If W is contingent differentiable (in the sense of (“. “)) and satisfies the differential inequality:*

$$\mathrm{maxf}(W_{\mathcal{A}})^{+}_{K}((t,x);(1,v)),(W_{\mathcal{A}})^{-}_{K}((t,x);(1,v))\Omega + f_0(t,x,v)\ \Omega\ 0$$
$$\Omega\ v\ \ \theta\ F_Q(t,x),\ (t,x)\ 2\ E$$
$$\theta\theta\theta\theta\theta\theta$$

and if $\Omega_a\ \theta\ \Omega\ _r$ *(i.e. the admissible trajectories are of least regular) then* $A(.,.)$ *is optimal.*

Moreover, if in addition, $W_{\mathcal{A}}(.,.)$ *is locally-Lipschitz (in particular, of class* C^1*) then the same conclusion holds for any class* $\Omega_a\ \theta\ AC = \Omega\ _1\Omega$ *of admissible trajectories.* ■

A significant extension of Theorem “.“ concerns locally-Lipschitz value functions that are not necessarily contingent differentiable:

Theorem 4.3 (Locally-Lipschitz value functions ff12ff, ff13ff). $\mathcal{A}(.,.)$ *If W is locally-Lipschitz and verifies the differential inequality:*

$$\mathrm{maxf}\overline{D}^{+}_{K}W_{\mathcal{A}}((t,x);(1,v)),\overline{D}^{-}_{K}W_{\mathcal{A}}((t,x);(1,v))\Omega + f_0(t,x,v)\ \Omega\ 0,$$
$$\theta\ v\ \theta\ F\ _Q(t,x),\ (t,x)\ 2\ E$$
$$\theta\theta\theta\theta\theta\theta$$

then $A(.,.)$ *is optimal.* ■

“e note that in the case of Lipschitzian value functions, the inequality in (“. ““) is implied by the fact that $W\ _{\mathcal{A}}(.,.)$ is a ΩΩΩnΩrali

An important intermediate case between Theorems 4.2 and 4.3 is that of stratified value functions in Def.4.1:

Theorem 4.4 (Stratified value functions ff12ff, ff13ff). *Let* $W_{\mathcal{A}}(.,.)$ *be differentiably - stratified and such that verifies the following differential inequality:*

$$\begin{array}{l} DW_{\mathcal{A}}(t,x)\,\Omega\,(1,v)+f_0(t,x,v)\,\Omega\,0\,\Omega\,v\,\theta\,F_T(t,x),(t,x)\;2\;E \\ F_T(t,x):=\mathrm{f}v\,\theta\,F(t,x);\;(1,v)\;2\;T_{\;(t,x)}Eg \end{array} \qquad \theta\theta\theta\theta\theta\theta$$

If, in addition, $W_{\mathcal{A}}(.,.)$ *is continuous and* $\Omega_a\,{}_\theta\,\Omega\;{}_r$ *then* $A(.,.)$ *is optimal and if* $W_{\mathcal{A}}(.,.)$ *is locally-Lipschitz then the same conclusion holds for any class,* $\Omega_a\,{}_\theta\,\Omega\;{}_1$ *of admissible trajectories.* ■

Other verification theorems using the upper contingent derivatives of the value functions with less regularity properties may be obtained only at the "price" of rather restrictive hypotheses on the data E, $F(.,.)$, $f\,{}_0(.,.,.)$, *of tθe proθlem aθd usiθg more iθvolved argumeθts coθcerθiθg iθvariaθce aθd moθotoθicity aloθg solutioθs of diσereθtial iθclusioθs θθθθθ θθθθ θθθθ θθθθθ θθθθθ etcθθ.*

Theorem 4.5 (Lower semicontinuous value functions ff12ff, ff13ff). *Let* $E:=E_0\,\Omega\,E_1$ *be a locally closed subset of* R^n *and let* $F(.,.)$, $f_0(.,.,.)$ *be such that the multifunction* $\widehat{F}(.,.)$ *in ("."") is locally-Lipschitz with respect to the Hausdorff distance and such that:*

$$(1,F(t,x))\,\Omega\,K^-_{(t,x)}E\;\theta\;(t,x)\;2\;E, \qquad \theta\theta\theta\theta\theta\theta$$

let $A(.,.)$ *be a multi-selection of admissible trajectories with the property in (".") and such that* $W_{\mathcal{A}}(.,.)$ *in ("."") is lower semicontinuous (l.s.c.) and satisfies the inequality:*

$$\overline{D}^-_K W_{\mathcal{A}}((t,x);(1,v))+f_0(t,x,v)\,\Omega\,0\,\Omega\,v\,\theta\,F(t,x),\;(t,x)\;2\;E. \qquad \theta\theta\theta\theta\theta\theta$$

Then $A(.,.)$ *is an optimal selection of admissible trajectories.* ■

Theorem 4.6 (ffpper semicontinuous value functions ff8ff, ff12ff, ff13ff). *Let* E *be a locally closed subset of* R^n *and let* $F(.,.)$, $f_0(.,.,.)$ *be such that the multifunction* $\widehat{F}(.,.)$ *in ("."") is locally-Lipschitz and such that:*

$$(1,F(t,x))\,\Omega\,K^+_{(t,x)}E\;\theta\;(t,x)\;2\;E. \qquad \theta\theta\theta\theta\theta\theta$$

If $W_{\mathcal{A}}(.,.)$ *is upper semicontinuous (u.s.c.) and satisfies the inequality:*

$$\overline{D}^+_K W_{\mathcal{A}}((t,x);(1,v))+f_0(t,x,v)\,\Omega\,0\,\Omega\,v\,\theta\,F(t,x),(t,x)\;2\;E \qquad \theta\theta\theta\theta\theta\theta$$

then $A(.,.)$ *is optimal.* ■

We note that the statements in Theorems 4.5, 4.6 are mere interpretations in the present context of the invariance and monotonicity results in Theorems 4.6, 4.8, in ff13ff.

On the other hand, using the general chain rule in (4.8) and the monotonicity results for real functions in ff14ff one may obtain other variants of the verification theorems above, without the restrictive hypotheses on the data but expressed in terms of the *lower contingent derivatives:*

Theorem 4.7 (Continuous value functions and regular trajectories). *If $\Omega_a \ \theta \ \Omega \ _r$ (i.e. the admissible trajectories are at least regular) and $W_{\mathcal{A}}(.,.)$ is continuous and verifies of the following differential inequalities:*

$$\begin{array}{ll} \underline{D}_K^+ W_{\mathcal{A}}((t,x);(1,v)) + f_0(t,x,v) \ \Omega \ 0 \ \Omega \ v \ \theta \ F_Q(t,x), (t,x) \ 2 \ E & \multirow{2}{*}{$\theta\theta\theta\theta\theta\theta$} \\ \underline{D}_K^- W_{\mathcal{A}}((t,x);(1,v)) + f_0(t,x,v) \ \Omega \ 0 \ \Omega \ v \ \theta \ F_Q(t,x), (t,x) \ 2 \ E & \end{array}$$

then $A(.,.)$ is an optimal selection.

Proof. *In view of Theorem "." it is enough to prove that for any $(s,y) \ 2 \ E_0$ ΩnΩ Ωn*

most countable, from the corollary of Zygmundffs Lemma (e.g. ff14ff) it follows that $\wp_x(.)$ is inΩrΩasinΩ anΩ ΩΩΩ ΩΩΩΩrΩm is p■ΩΩΩΩ

ΩimilarrΩsulΩs ma

to the trajectoriesff of the differential inclusion in (3.2) (ff10ff) defined by:

$$\begin{array}{l} K^{\pm}_{F,E}(s,y) := \mathrm{f}v\ \theta\ R\ ^{n};\ \Omega x(.)\ 2\ S_{F,E}(s,y) : v\ \theta\ K\ ^{\pm}x(s;1)\Omega \\ K^{0,\pm}_{F,E}(s,y) := \mathrm{f}v\ \theta\ R\ ^{n};\ \Omega x(.)\ 2\ S_{F,E}(s,y) : v\ \theta\ K\ ^{\pm}x(s;0_{+})\Omega \end{array} \qquad (4\Omega 24)$$

ΩΩΩn $S_{F,E}(s,y)$ ΩΩnΩΩΩs ΩΩΩ sΩΩ Ωf aΩsΩluΩΩl

A similar proof, using the symmetric monotonicity result, Corollary 3.7 in ff14ff leads to the following:

Theorem 4.11 (ff.s.c.value functions and Lipschitzian trajectories). *If* $\Omega_a \ \theta \ \Omega_{\ \infty}$ *and* $W_{\mathcal{A}}(.,.)$ *is upper semicontinuous and satisfies the inequality:*

$$\underline{D}_K^{+} W_{\mathcal{A}}((t,x);(1,v)) + v_0 \ \Omega \ 0 \ \Omega \ \widehat{v} = (v, v_0) \ 2 \ \theta\theta \widehat{F}(t,x), \ (t,x) \ 2 \ E \ \ \theta\theta\theta\theta\theta\theta$$

then $A(.,.)$ *is an optimal selection.* ■

Remark 4.12 The results in Theorems 4.7-4.11 may be considered less satisfactory than those in Theorems 4.2-4.6 not only since the classes of admissible trajectories are severely restricted to $\Omega_{\varphi}, \Omega_{\infty} \Omega$

with its restriction, $W_0(.,.) := W(.,.) \Omega E_0$, *to tθe set of iθitial poiθtsθ θθicθ may θave more regularity propertiesθ*

θoreoverθ tθe same proθlem iθ θθθθθθ θθθθθθ coθsidered oθ tθe suθset $E_0 := (\Omega 1, \pi) \Omega R$ *as Example θθθθ sθoθs tθat eveθ tθe restrictioθθ* $W_0(.,.)$ ma

$$\overline{D}_K^- \omega(t_1;1) > \theta\theta; \qquad (\Omega\Omega\Omega)$$

$$\overline{D}_K^- \omega(t_1;0_+) > \theta\theta. \qquad \theta\theta\theta\theta\theta$$

θiiθ The restriction $\omega(.)\Omega[s,t_1)$ *is increasing iff* **one** *of the following (equivalent) conditions holds:*

$$\overline{D}_K^+ \omega(s;1) > \theta\theta; \qquad (\Omega\Omega\Omega)$$

$$\overline{D}_K^+ \omega(s;0_+) > \theta\theta. \qquad \theta\theta\theta\theta\theta$$

■

Remark 5.4. The results in ff9ff show that if $\omega(.)\Omega(s,t_1)$ is inΩrΩasinΩ ΩΩΩn ΩaΩΩ Ωf ΩΩΩ prΩpΩrΩiΩs in (ΩΩΩ)Ω (ΩΩ4) is ΩΩuiΩalΩnΩ alsΩ ΩiΩΩ (ΩΩ2) anΩ ΩiΩΩ ΩaΩΩ Ωf ΩΩΩ fΩllΩΩinΩ prΩpΩrΩiΩs:

$$\underline{D}_K^- \omega(t_1;1) > \theta\theta,\ D\underline{\ \ }_K^- \omega(t_1;0_+)\ \Omega\ 0,\ \underline{D}_K^- \omega(t_1;0_+) > \theta\theta \qquad \theta\theta\theta\theta\theta$$

θutθ as suθcieθt coθditioθsθ tθey are θeaker tθaθ tθe oθes iθ θθθθθθ θθθθθ, moreoverθ tθe geθeral cθaiθ rule iθ θθθθθ alloθ estimatioθs oθly of tθe θupperθ coθtiθgeθt derivatives iθ θθθθθθ θθθθθ

As a possiθle θetter alterθative to tθe result iθ Tθeorem θθθ θe sθall prove θoθ.

Theorem 5.5 *Let* $A(.,.)$ *be a multiselection of admissible trajectories satisfying (“. “), such that the corresponding value function* $W_{\mathcal{A}}(.,.)$ *in (“. ““) has the following properties:*

(i) The restriction $W_0(.,.) := W_{\mathcal{A}}(.,.)\Omega E_0$ *and* Ω_a *satisfy the hypotheses of* **one** *of the verification theorems “. “ - “. ““;*

(ii) Either $\Omega_a\ \theta\ \Omega\ {}_{\infty}\ -$

Therefore, if (s, y) 2 E_0 ΩnΩ $x(.)$ 2 Ω $_a(s, y)$ is an aΩmissiΩlΩ ΩraΩ ΩΩΩΩΩr *or* $W_{\mathcal{A}}(.,.)$ *satisfies the inequalities in (“. “), (“. “), (“. ““) and also the following one, at each point* (s,y) 2 $\widetilde{E}_0^0$*:*

$$\underline{D}_K^+ W_{\mathcal{A}}((s,y);(0_+,v)) > \theta\theta\ \theta\ \widehat{v} = (v, v\quad _0)\ 2\ _{\theta\theta}\widehat{F}^{\,\infty}(s,y)\mathrm{nf}(0,0)\Omega.\ \ \theta\theta\theta\theta\theta\theta$$

Then $A(.,.)$ *is an optimal selection.*

Remark 5.7 The result in Theorem 5.6 may obviously be extended to more complex cases like the one in Example 2.2ff however, the ffdescriptionff of such cases may become cumbersome so it seems more reasonable to use Theorem 5.6 and some ffad hocff arguments as we shall do in the proof of Prop.6.2 below.

6. Applications

We apply first Theorem 5.6 to Example 2.1 in Section 2:

Proposition 6.1 *The multiselection* $A(.,.)$ *in (“. “) is optimal for the problem (“. “)-(“. “) restricted to the subset* $\widetilde{E}_0$ *in (“. “) in the class* Ω_∞ *of Lipschitzian trajectories.*

Proof. *“e note first that the partition* $\widetilde{E}_0 = \widetilde{E}_0^0\, \Omega\, E_{\ 0}^{\ 0}$ in (2ΩΩ) ΩΩs

hence $\omega_x(.)$ is inΩrΩasinΩ Ωn ΩΩΩ ΩΩΩlΩ inΩΩrΩal $[s, \pi]$ anΩ ΩΩΩ ΩΩnΩlusiΩn fΩllΩΩs frΩm ΩΩΩΩrΩm Ω■Ω1Ω

Remark 6.3 In the case of Example 2.1, since the restriction $_0(\,,\,)W =$ $W_{\mathcal{A}}(.,.)\Omega\widetilde{E}_0$ is

9. V. Lupulescu and Şt. Mirică (1998), *End-point monotonicity of real functions and applications, Anal. ”Ovidius” “niv. Constan*