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ON SINGULAR GEODESICS IN A SUB-RIEMANNIAN METRIC

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ABSTRACT. This paper is an investigation of approximation of a singular geodesic by nonsingular geodesics. It is proved that under certain assumptions a singular geodesic can be approximated by sequences of two types. On a sequence of the first type the conjugate points do not get close. On a sequence of the second type the distance between the conjugate points tends to zero.

CONTENTS

In the case of a submetric, conjugate variables can take values different from zero on the zero level of the Hamiltonian. Connected with this is the possibility of singular geodesics, that is, geodesics generated by an extremal on which the Hamiltonian vanishes. An important characteristic of a singular geodesic is independence of the submetric, a property nonsingular geodesics do not enjoy. This is the reason for the steady interest in singular geodesics and in a clarification of their minimization possibilities.

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*The prominent Russian mathematician Alekseĭ Alekseevich Milyutin died suddenly on 20 April 2001. He made substantial contributions to the theory of Banach spaces and the theory of extrema, especially to the theory of optimal control.

In recent times significant progress has been made in the author's view in the theory of singular geodesics, due to the attraction of the theory of quadratic conditions for singular extremals in optimal control developed by the author and especially A. V. Dmitruk. Namely, Dmitruk proved that the quadratic conditions for rigidity obtained somewhat earlier by the author [1] are also quadratic conditions for a singular geodesic [2]. Thus, the deep connection between the theory of rigidity and the theory of singular geodesics was explained. The most striking result of Dmitruk was that each quadratically rigid (Milyutin) trajectory is a singular geodesic that realizes a strong minimum in any strictly convex submetric, in particular, in any sub-Riemannian metric. Nonsingular geodesics never realize a minimum independent of the submetric. Thus, a new and extremely strong minimum condition has been found for singular geodesics.

In the present paper we investigate locally quadratic rigid geodesics from the point of view of the influence on sequences of nonsingular geodesics approximating them. Our main result complements and, as it were, balances the above result of Dmitruk.

CHAPTER I. APPROXIMATION OF A SINGULAR EXTREMAL BY NONSINGULAR EXTREMALS

§ 1. General facts

Suppose that in a domain Ω in R^n we are given a distribution $\Gamma(x)$, $x \in \Omega$, of dimension $m < n$ (that is, $\Gamma(x)$ is a subspace of R^n of dimension m), and assume that on Ω we are given twice continuously differentiable vector fields $\mathbf{r}_0(x), \dots, \mathbf{r}_{m-1}(x)$ with values in R^n such that $\Gamma(x)$ is the linear span of the fields $\mathbf{r}_0, \dots, \mathbf{r}_{m-1}$. In other words,

$$\Gamma(x) = \operatorname{Lin}(\mathbf{r}_0(x), \dots, \mathbf{r}_{m-1}(x)) \quad \forall x \in \Omega.$$

Thus, for each $x \in \Omega$ the vectors $\mathbf{r}_0, \dots, \mathbf{r}_{m-1}$ form a basis of $\Gamma(x)$. We denote by $\widehat{\mathbf{b}}$ the basis $\mathbf{r}_0(x), \dots, \mathbf{r}_{m-1}(x)$, and by $\widehat{\mathbf{B}}$ the set of bases.

We define a metric on $\Gamma(x)$ as follows. Let $Q(x)$ be a positive-definite matrix on R^n that is twice continuously differentiable on Ω. For $x \in \Omega$ and $\xi \in \Gamma(x)$ we let

$$n(x, \xi) = \sqrt{Q(x)\xi\xi}\,.$$

This determines a metric on $\Gamma(x)$.

Let us consider the differential inclusion

$$\dot{x} \in \Gamma(x). \tag{1}$$

By a trajectory of the inclusion (1) we mean an absolutely continuous function $x(t)$ defined on some interval $[t_0, t_1]$ and satisfying (1) almost everywhere on it. In what follows, the function $x(t)$ will often be denoted by $\widehat{x}$. This applies also to other functions. Suppose that we are given a trajectory of the inclusion (1): $\widehat{x} = x(t)|[t_0, t_1]$. Let

$$l(\widehat{x}) = \int_{t_0}^{t_1} n(x, \dot{x})\, dt.$$

We consider the problem of the shortest path:

$$l(x) \to \min; \quad x(t_0) = x_*,\ x(t_1) = x_{**}.$$

The function $\widehat{x}$ is a trajectory of (1). Let us rewrite this problem as follows. A basis $\widehat{\mathbf{b}} = \mathbf{r}_0(x), \dots, \mathbf{r}_{m-1}(x)$ is said to be orthonormal if

$$Q(x)\mathbf{r}_i(x)\mathbf{r}_i(x) = 1, \quad i = 0, \dots, m-1,$$
$$Q(x)\mathbf{r}_i(x)\mathbf{r}_k(x) = 0, \quad i, k = 0, \dots, m-1,\ i \neq k.$$

The set of orthonormal bases is denoted by $\widehat{\mathbf{B}}_0$. It is clear that $\widehat{\mathbf{B}}_0 \neq \emptyset$. Let $\widehat{\mathbf{b}} \in \widehat{\mathbf{B}}_0$. We consider a control system

$$\dot{x} = \sum_{i=0}^{m-1} u_i \mathbf{r}_i(x). \tag{2_1}$$

Here x is the phase variable and $u = (u_0, \dots, u_{m-1})$ is the control. Clearly, the control system (2_1) is equivalent to the inclusion (1).

For a solution of the system (2_1) we have

$$n(x, \dot{x}) = |u| = \sqrt{\sum_{i=0}^{m-1} u_i^2}.$$

We supplement the control system (2_1) by the requirement

$$u^2 = 1. \tag{2_2}$$

Then the problem of a shortest path can be formulated as a time-optimal problem in the system (2_1), (2_2):

$$t_1 - t_0 \to \min; \quad x(t_0) = x_*,\ x(t_1) = x_{**}. \tag{2_3}$$

We say that the trajectory $(x(t), u(t)|[t_0, t_1])$ of the system (2_1), (2_2) is a geodesic if it satisfies the maximum principle in the problem (2_3). It is important to note that, as shown by the author in [1], this definition does not depend on the choice of basis in the set $\widehat{\mathbf{B}}_0$.

For a trajectory $(x(t), u(t)|[t_0, t_1])$ of the system (2_1), (2_2) we write out the conditions of the maximum principle.

The Pontryagin function has the form

$$H(\psi, x, u, t) = \sum_{i=0}^{m-1} u_i \psi \mathbf{r}_i(x);$$

correspondingly, the Hamiltonian has the form

$$\mathcal{H}(\psi, x, t) = \max_{u|u^2=1} H(\psi, x, u, t) = \sqrt{\sum_{i=0}^{m-1} (\psi \mathbf{r}_i)^2}.$$

The conditions of the maximum principle are the following: there exists a function

$$\psi(t) \not\equiv 0|[t_0, t_1]$$

such that

$$H(\psi(t), x(t), u(t), t) = \mathcal{H}(\psi(t), x(t), t)|[t_0, t_1], \tag{3_1}$$
$$-\dot{\psi} = H_x^{'}(\psi(t), x(t), u(t), t), \tag{3_2}$$
$$H(\psi(t), x(t), u(t), t) = \text{const}. \tag{3_3}$$

A triple of functions

$$\gamma = (\psi(t), x(t), u(t) | [t_0, t_1])$$

satisfying the conditions (3_1), (3_2), (3_3) is called an extremal of the control system (2_1), (2_2).

It follows from the conditions (3_1), (3_2) that the extremals break up into two classes: the extremals on which $\mathcal{H} \neq 0$, and then $\mathcal{H} > 0$, and the extremals on which

$$\mathcal{H} = 0. \tag{4}$$

We call the extremals of the first class nonsingular extremals, and those of the second class singular extremals.

In connection with this definition it is important to make the following clarification. We call the function $\psi(t)$ the conjugate component of the extremal, and the function $x(t)$ its phase component. Then [1] under a change of the basis in the set $\widehat{\mathbf{B}}_0$ the conjugate and phase components remain unchanged, and hence the value of the Hamiltonian also does not change. Thus, the concept of a nonsingular or singular extremal does not depend on the basis $\widehat{\mathbf{b}} \in \widehat{\mathbf{B}}_0$.

Proposition 1. *The set of phase and conjugate components of nonsingular extremals coincides with the set of solutions of the Hamiltonian system*

$$\begin{aligned} \dot{x} &= \mathcal{H}_\psi'(\psi, x), \\ -\dot{\psi} &= \mathcal{H}_x'(\psi, x). \end{aligned} \tag{5}$$

Proof. Before starting the proof we remark that, since $\mathcal{H}$ does not depend on t, we omit this argument. We do the same with the Pontryagin function H. Let us now proceed to the proof.

Let $\gamma = (\psi(t), x(t), u(t) | [t_0, t_1])$ be a nonsingular extremal, and let $t_* \in [t_0, t_1]$ be a point at which the functions $\widehat{\psi}$ and $\widehat{x}$ are differentiable and satisfy the equalities

$$\begin{aligned} \dot{x}(t_*) &= \sum_{i=0}^{m-1} u_i \mathbf{r}_i(x(t_*)), \\ -\dot{\psi}(t_*) &= H_x'(\psi(t_*), x(t_*), u(t_*)), \\ H(\psi(t_*), x(t_*), u(t_*)) &= \mathcal{H}(\psi(t_*), x(t_*)). \end{aligned}$$

It follows from (2_1) and (3_1), (3_2) that the set of points t_* satisfying these conditions is a set of full measure on the interval $[t_0, t_1]$.

In a neighborhood of the point ψ_*, x_* $(\psi_* = \psi(t_*), x_* = x(t_*))$ we consider the function $H(\psi, x, u(t_*))$. This function satisfies the inequality $H(\psi, x, u(t_*)) \leq \mathcal{H}(\psi, x)$ and the equality $H(\psi_*, x_*, u(t_*)) = \mathcal{H}(\psi_*, x_*)$. By definition, $\mathcal{H}$ is a differentiable function at each point at which it is nonzero. Since γ is a nonsingular extremal, it follows that $\mathcal{H}(\psi_*, x_*) > 0$, and hence $\mathcal{H}$ is differentiable at the point (ψ_*, x_*). Consequently,

$$\begin{aligned} H_\psi'(\psi_*, x_*, u(t_*)) &= \mathcal{H}_\psi'(\psi_*, x_*), \\ H_x'(\psi_*, x_*, u(t_*)) &= \mathcal{H}_x'(\psi_*, x_*). \end{aligned}$$

Hence,

$$\dot{x}(t_*) = \mathcal{H}_\psi'(\psi_*, x_*), \quad -\dot{\psi}(t_*) = \mathcal{H}_x'(\psi_*, x_*).$$

These equalities hold almost everywhere on $[t_0, t_1]$. But the right-hand side is a continuous function, and consequently the pair $\psi(t), x(t)$ is a solution of the Hamiltonian system (5).

Suppose that $\psi(t), x(t)$ satisfy the system (5). Since $\mathcal{H}$ does not depend on t, $\mathcal{H}$ is constant on a solution. Let $\mathcal{H} > 0$. Then, setting

$$u_i(t) = \frac{\psi(t)\mathbf{r}_i(x(t))}{\mathcal{H}},$$

it is easy to see that $\gamma = (\psi(t), x(t), u(t)|[t_0, t_1])$ is an extremal of the system (2_1), (2_2). Proposition 1 is proved. □

When referring to an extremal, we shall say that it belongs to some level of $\mathcal{H}$. Complete information about nonsingular extremals can be obtained from the solutions of the system (5) for a single level of $\mathcal{H}$. This follows from the fact that $\mathcal{H}$ is a positive-homogeneous function of ψ of degree 1.

Let us proceed to the investigation of singular extremals. We cannot use the system (5), because the Hamiltonian $\mathcal{H}(\psi, x)$ is not differentiable at any point of the zero level of $\mathcal{H}$. Therefore, we have to rely only on the conditions (3_1), (3_2), (3_3) of the maximum principle.

For a trajectory $(x(t), u(t)|[t_0, t_1])$ of the control system (2_1), (2_2) we let $\Psi_0(\widehat{x})$ be the set of functions $\psi(t)$ such that $(\psi(t), x(t), u(t)|[t_0, t_1])$ is an extremal, with $\psi(t)$ normalized in some way. We have already noted that the pairs $\psi(t), x(t)$ corresponding to extremals do not depend on the choice of basis, and hence not on the control $u(t)$. Therefore, we write $\Psi_0(\widehat{x})$ and omit the dependence on the control.

An element

$$\psi(t)|[t_0, t_1] \in \Psi_0(\widehat{x})$$

is said to be nonsingular if the corresponding extremal is nonsingular, and singular if the corresponding extremal is singular. Taking the terminology further, we say that a geodesic $(x(t), u(t)|[t_0, t_1])$ is nonsingular if all the elements of $\Psi_0(\widehat{x})$ are nonsingular. We say that a geodesic is singular if all the elements of $\Psi_0(\widehat{x})$ are singular. Finally, we say that a geodesic has mixed type if $\Psi_0(\widehat{x})$ contains both nonsingular and singular elements. There are two quite distinct cases from the point of view of approximation of a singular extremal by nonsingular extremals.

Before formulating these cases, we give one more explanation of the terminological order. Since a geodesic and its set Ψ_0 do not depend on the basis $\widehat{\mathbf{b}}$, in referring to a geodesic we shall have in view the phase component $\widehat{x} = x(t)|[t_0, t_1]$ of the trajectory.

Let us return to our topic. First, a singular extremal can turn out to be such that its phase component is a geodesic of mixed type. The second case consists in the phase component of a singular extremal being a singular geodesic.

In the first case the approximation of the singular extremal by nonsingular extremals can be implemented in an elementary way.

Indeed, let $\gamma^0 = (\psi^0(t), x^0(t), u^0(t)|[t_0, t_1])$ be a singular extremal, and let $\gamma^1 = (\psi^1(t), x^0(t), u^0(t)|[t_0, t_1])$ be a nonsingular extremal. Then

$$\gamma^\varepsilon = (\varepsilon\psi^1(t) + \psi^0, x^0(t), u^0(t)|[t_0, t_1])$$

is easily seen to be a nonsingular extremal for any $\varepsilon > 0$, and $\gamma^\varepsilon \to \gamma^0$ as $\varepsilon \to 0$ in the sense that $\lim\limits_{\varepsilon\to 0} \|\psi^\varepsilon(t) - \psi^0(t)\|_\infty = 0$, $\lim\limits_{\varepsilon\to 0} \|x^\varepsilon(t) - x^0(t)\|_\infty = 0$, and $\lim\limits_{\varepsilon\to 0} \|u^\varepsilon(t) - u^0(t)\|_\infty = 0$, where $\psi^\varepsilon(t) = \varepsilon\psi^1(t) + \psi^0(t)$, $x^\varepsilon(t) = x^0(t)$, and $u^\varepsilon(t) = u^0(t)$. Here $\|\cdot\|_\infty$ is the norm in the space L_∞ on the interval $[t_0, t_1]$.

Approximation of a singular extremal by nonsingular extremals is incomparably more complicated in the second case. In addition we have to make certain assumptions about the extremal γ^0, though the assumptions are natural in the author's opinion. First of all we assume that the phase component of the extremal $\gamma^0 = (x^0(t)|[t_0, t_1])$ has on $[t_0, t_1]$

a twice continuously differentiable derivative $\dot{x}(t)$ and does not have self-intersections. Further, we assume that at each point t of $[t_0, t_1]$

$$\psi^0(t)[\mathbf{r}_i, \mathbf{r}_k](x^0(t)) = 0, \quad i, k = 0, 1, \ldots, m-1. \tag{6}$$

Here $[\rho_1, \rho_2](x) = \rho_1'(x)\rho_2(x) - \rho_2'(x)\rho_1(x)$. This condition is connected with necessary conditions that the geodesic realize a shortest distance. Namely, if $\Psi_0(\widehat{x})$ does not contain such extremals, then the phase component $\widehat{x}$ never realizes a shortest distance.

Therefore, it is natural to be interested in the approximation of such extremals. Dmitruk, who proved the necessity of the existence of such extremals in the problem of a shortest distance [2], called them Goh extremals. Accordingly, we consider not just any extremal, but a Goh extremal.

However, this is not yet all. The following assumption requires some preliminary definitions. We say that a basis $\widehat{\mathbf{b}} \in \mathbf{B}_0$ is an associated basis with respect to the phase component $x^0(t)|[t_0, t_1]$ if $\mathbf{r}_0(x^0(t)) = \dot{x}^0(t)$ $\forall t \in [t_0, t_1]$. The set of such bases will be denoted by $\widehat{\mathbf{B}}_0(\widehat{x}^0)$. It is not hard to see that $\widehat{\mathbf{B}}_0(\widehat{x}^0) \neq \emptyset$. In an associated basis $u_0^0(t) = 1$ and $u_i^0(t) = 0|[t_0, t_1]$, $i = 1, \ldots, m-1$.

Let $\widehat{\mathbf{b}} \in \widehat{\mathbf{B}}_0(\widehat{x}^0)$. We denote by $D^0(t)$ the $(m-1) \times (m-1)$ matrix with elements $d_{ik}^0(t) = \psi^0(t)\big[[\mathbf{r}_i, \mathbf{r}_0]\mathbf{r}_k\big](x^0(t))$, $i, k = 0, 1, \ldots, m-1$. It is easy to see that the Goh condition (6) implies that the matrix $D^0(t)$ is symmetric for any $t \in [t_0, t_1]$.

As proved by Dmitruk, the condition that $\Psi_0(\widehat{x}^0)$ must contain Goh elements such that the matrix $D^0(t)$ is nonnegative definite for any $t \in [t_0, t_1]$ is a necessary condition on a geodesic in the problem of a shortest path. This class of extremals is called the Legendre class of extremals. However, it is not Legendre extremals that appear in the sufficient conditions, but extremals which we call plus-Legendre extremals. These are extremals such that the matrix $D^0(t)$ is positive definite for any $t \in [t_0, t_1]$. Therefore, we shall consider extremals of two types: plus-Legendre and minus-Legendre extremals. The conjugate component of the latter differs only in sign from the plus-Legendre elements of the set $\Psi_0(\widehat{x}^0)$. Such extremals will be called definite extremals.

Accordingly, we consider singular geodesics and choose in Ψ_0 the Goh definite elements. It is such extremals that will be approximated by nonsingular extremals. We note that the concept of a Goh extremal does not depend on the basis $\widehat{\mathbf{b}} \in \mathbf{B}_0$, and the concept of a definite extremal does not depend on the choice of $\widehat{\mathbf{b}} \in \widehat{\mathbf{B}}_0(\widehat{x}^0)$. This was proved by the author in [1].

§ 2. The extended system

In this section we let $w = (\psi, x)$. For a definite extremal $\gamma^0 = (w^0(t), u^0(t)|\Delta^0)$, where $\Delta^0 = \left[t_0^0, t_1^0\right]$, we choose an associated orthonormal basis. Then, obviously, $u_0^0(t) = 1$ and $u_i^0(t) = 0|\Delta^0$. Let $z_i(w) = \psi\mathbf{r}_i(x)$, $i = 0, \ldots, m-1$. Then $\mathcal{H}(w) = \sqrt{\sum_{i=0}^{m-1} z_i(w)}$. Also, let $Z_i(w) = \psi[\mathbf{r}_i, \mathbf{r}_0](x)$, $i = 1, \ldots, m-1$. Obviously, $z_i(w^0(t)) = 0$ for $i = 0, \ldots, m-1$, and $Z_i(w^0(t)) = 0$ for $i = 1, \ldots, m-1$.

Let $\rho(x)$ be an arbitrary field. Then from the extremality equations it follows that

$$\frac{d}{dt}\psi\rho(x) = \sum_{i=0}^{m-1} u_k \psi[\rho, \mathbf{r}_k](x). \tag{1}$$

This formula will be used repeatedly.

Proposition 1. *Let t_* be an arbitrary point of Δ^0, and let $w_* = w^0(t_*)$. Then the gradients $z_{iw}'(w_*)$, $i = 0, \ldots, m-1$, and $Z_{iw}'(w_*)$, $i = 1, \ldots, m-1$, are linearly independent.*

Proof. Let $\overline{\mu} = \big(\overline{\mu}_0, \dots, \overline{\mu}_{m-1}\big)$, $\overline{v} = \big(\overline{v}_1, \dots \overline{v}_{m-1}\big)$, and $\overline{w} = \big(\overline{\psi}, \overline{x}\big)$. We show that the system of equations

$$\begin{aligned} \overset{\circ}{z}{}'_{iw}(w_*)\overline{w} &= \overline{\mu}_i, \quad i = 0, \dots, m-1, \\ \overset{\circ}{z}{}'_{iw}(w_*)\overline{w} &= \overline{v}_i, \quad i = 1, \dots, m-1, \end{aligned} \tag{2}$$

is solvable for any $\overline{\mu}$ and $\overline{v}$.

We introduce the vector field

$$\rho(x) = \alpha_0(x - x_*, \mathbf{r}_0(x_*))\psi_* + \sum_{i=1}^{m-1} \alpha_i \mathbf{r}_i(x) + \sum_{i=1}^{m-1} \beta_i [\mathbf{r}_i, \mathbf{r}_0](x).$$

Here $\alpha_0, \dots, \alpha_{m-1}$ and $\beta_1, \dots, \beta_{m-1}$ are undetermined coefficients which we find by solving (2). We look for $\overline{w}$ in the form

$$\overline{\psi} = (\psi_* \, \rho'(x_*)), \quad \overline{x} = -\rho(x_*). \tag{3}$$

Let $\overline{w}$ be substituted in (2). We have

$$\begin{aligned} &\overset{\circ}{z}{}'_{iw}(w_*)\overline{w} = \overline{\psi}\mathbf{r}_i(x_*) + \overline{\psi}_* \mathbf{r}_i'(x_*)\overline{x} = \psi_*[\rho, \mathbf{r}_i](x_*), \quad i = 0, \dots, m-1, \\ &\overset{\circ}{z}{}'_{iw}(w_*)\overline{w} = \psi_* \big[\rho, [\mathbf{r}_i, \mathbf{r}_0]\big](x_*), \quad i = 1, \dots, m-1. \end{aligned}$$

Then (2) passes into the system

$$\begin{aligned} \psi_*[\rho, \mathbf{r}_i](x_*) &= \overline{\mu}_i, \quad i = 0, \dots, m-1, \\ \psi_* \big[\rho, [\mathbf{r}_i, \mathbf{r}_0]\big](x_*) &= \overline{v}_i, \quad i = 1, \dots, m-1. \end{aligned} \tag{4}$$

Let us consider the zeroth equation of the first group. The equations (4) break up into three blocks:

$$\psi_*[\rho, \mathbf{r}_0](x_*) = \overline{\mu}_0, \tag{4_1}$$

$$\psi_*[\rho, \mathbf{r}_i](x_*) = \overline{\mu}_i, \quad i = 1, \dots, m-1, \tag{4_2}$$

$$\psi_* \big[\rho, [\mathbf{r}_i, \mathbf{r}_0]\big](x_*) = \overline{v}_i, \quad i = 1, \dots, m-1. \tag{4_3}$$

The left-hand side of the first block does not contain the variables $\alpha_1, \dots, \alpha_{m-1}$ nor $\beta_1, \dots, \beta_{m-1}$. Indeed, the coefficient of $\alpha_i\big|_{i=1,\dots,m-1}$ is equal to $\psi_*[\mathbf{r}_i, \mathbf{r}_0](x_*)$, and by (6) of §1 this is zero. The coefficient of $\beta_i\big|_{i=1,\dots,m-1}$ is equal to $\psi_* \big[[\mathbf{r}_i, \mathbf{r}_0], \mathbf{r}_0\big](x_*)$. By (1),

$$\psi_* \big[[\mathbf{r}_i, \mathbf{r}_0], \mathbf{r}_0\big](x_*) = \frac{d}{dt}\psi^0(t)[\mathbf{r}_i, \mathbf{r}_0](x^0(t))\big|_{t=t_*} = 0.$$

Finally, the coefficient of α_0 is equal to $\mathbf{r}_0^2(x_*)\psi_*^2 > 0$.

We proceed to the equations of the second block. This system does not contain the variables $\alpha_1, \dots, \alpha_{m-1}$. Indeed, the elements of the matrix of coefficients of $\alpha_1, \dots, \alpha_{m-1}$ are equal to $\psi_*[\mathbf{r}_i, \mathbf{r}_k](x_*)$ $(i, k = 1, \dots, m-1)$, which are equal to zero in view of (6.1). The matrix of coefficients of $\beta_i\big|_{i=1,\dots,m-1}$ has the form $\psi_* \big[[\mathbf{r}_i, \mathbf{r}_0], \mathbf{r}_k\big](x_*)$ $(i, k = 1, \dots, m-1)$. By assumption, the extremal γ^0 is definite, and therefore this matrix (it coincides with $D^0(t_*)$) is nonsingular.

Finally, we consider the equations of the third block. The matrix of coefficients of $\alpha_1, \dots, \alpha_{m-1}$ in the system (4_3) has the form $\psi_* \big[\mathbf{r}_i, [\mathbf{r}_k, \mathbf{r}_0]\big](x_*)$ $(i, k = 1, \dots, m-1)$. It is easily seen to coincide with $D^0(t_*)$, and hence is nonsingular. From this consideration of all three blocks in the system (4) it follows that the system is uniquely solvable for any $\overline{\mu}_0, \dots, \overline{\mu}_{m-1}$ and $\overline{v}_1, \dots \overline{v}_{m-1}$. Consequently, the system (2) is always solvable. Proposition 1 is proved. □

We consider the Hamiltonian system and its solutions belonging to the domain of positivity of the Hamiltonian $\mathcal{H}(w)$. Setting

$$U_i(w) = \frac{z_i(w)}{\mathcal{H}(w)}, \quad i = 0, \dots, m-1,$$

we write the system in the form

$$\begin{aligned} \dot{x} &= \sum_{i=0}^{m-1} U_i(w)\mathbf{r}_i(x), \\ -\dot{\psi} &= \sum_{i=0}^{m-1} U_i(w)(\psi \mathbf{r}_i'(x)). \end{aligned} \tag{5}$$

Setting $\overset{\circ}{z}(w) = \big(\overset{\circ}{z}_1(w), \dots, \overset{\circ}{z}_{m-1}(w)\big)$, we write the system (5) in the form

$$\dot{w} = \frac{\overset{\circ}{z}_0(w)}{\mathcal{H}(w)} a(w) + A(w)\frac{\overset{\circ}{z}(w)}{\mathcal{H}(w)}, \tag{6}$$

where $a(w) = \big(-\psi \mathbf{r}_0'(x), \mathbf{r}_0(x)\big)$, and $A(w)$ is a $2n \times (m-1)$ matrix. The equation (6) will be needed in what follows.

We determine how $\overset{\circ}{z}(w)$ varies along a solution of the system (5). For $i = 1, \dots, m-1$ we have, by (1),

$$\frac{d}{dt}\overset{\circ}{z}_i = \sum_{k=0}^{m-1} U_k(w)\psi[\mathbf{r}_i, \mathbf{r}_k](x), \tag{7}$$

$$\begin{aligned} \frac{d^2}{dt^2}\overset{\circ}{z}_i = &\sum_{k,s=0}^{m-1} \frac{1}{\mathcal{H}(w)} U_s(w)\psi[\mathbf{r}_k, \mathbf{r}_s](x)\psi[\mathbf{r}_i, \mathbf{r}_k](x) \\ &+ \sum_{k,s=0}^{m-1} U_k(w)U_s(w)\psi\big[[\mathbf{r}_i, \mathbf{r}_k], \mathbf{r}_s\big](x). \end{aligned} \tag{8}$$

We need to transform the formula (8). To avoid unnecessary awkwardness, we introduce the notation

$$\begin{aligned} [\mathbf{r}_i, \mathbf{r}_k](x) &= [i, k](x), \quad i, k = 0, \dots, m-1, \\ \big[[\mathbf{r}_i, \mathbf{r}_k], \mathbf{r}_s\big](x) &= [i, k; s](x), \quad i, k, s = 0, \dots, m-1. \end{aligned}$$

We have

$$\begin{aligned} \frac{d^2\overset{\circ}{z}_i}{dt^2} = &\frac{\overset{\circ}{z}_0}{\mathcal{H}^2} \sum_{k=0}^{m-1} \psi[k, 0](x)\psi[i, k](x) + \Big(\frac{\overset{\circ}{z}_0}{\mathcal{H}}\Big)^2 \psi[i, 0; 0](x) \\ &+ \sum_{\substack{k=0 \\ s=1}}^{m-1} \frac{\overset{\circ}{z}_s}{\mathcal{H}^2}\psi[k, s](x)\psi[i, k](x) + \frac{\overset{\circ}{z}_0}{\mathcal{H}^2} \sum_{k=1}^{m-1} \overset{\circ}{z}_k \psi[i, k; 0](x) \\ &+ \sum_{k,s=1}^{m-1} \frac{\overset{\circ}{z}_k}{\mathcal{H}}\frac{\overset{\circ}{z}_s}{\mathcal{H}}\psi[i, k; s](x) + \frac{\overset{\circ}{z}_0}{\mathcal{H}} \sum_{s=1}^{m-1} \frac{\overset{\circ}{z}_s}{\mathcal{H}}\psi[i, 0; s](x). \end{aligned} \tag{9}$$

The following notation will be used:

$$
\begin{aligned}
&b_i^1(w) = \sum_{k=0}^{m-1} \psi[k,0](x)\psi[i,k](x), && b^1(w) = \big(b_1^1(w),\dots,b_{m-1}^1\big),\\
&b_i^2(w) = \psi[i,0;0](x), && b^2(w) = \big(b_1^2(w),\dots,b_{m-1}^2\big);\\
&B_{is}^1(w) = \sum_{k=0}^{m-1} \psi[k,s](x)\psi[i,k](x), && B^1(w) = \big\{B_{is}^1(w)\big\},\ i,s=1,\dots,m-1,\\
&B_{ik}^2(w) = \psi[i,k;0](x), && B^2(w) = \big\{B_{ik}^2(w)\big\},\ i,k=1,\dots,m-1,\\
&B_{iks}^3(w) = \psi[i,k;s](x), && B^3(w) = \big\{B_{iks}^3(w)\big\},\ i,k=1,\dots,m-1;\\
&D_{is}(w) = \psi[i,0;s](x), && D(w) = \big\{D_{is}(w)\big\},\ i,k=1,\dots,m-1.
\end{aligned}
\tag{10}
$$

According to (9), $\dfrac{d^2 z_i}{dt^2}$ can be represented in the form

$$
\frac{d^2\mathring{z}_i}{dt^2} = \frac{\mathring{z}_0}{\mathcal{H}^2} b^1(w) + \left(\frac{\mathring{z}_0}{\mathcal{H}}\right)^2 b^2(w) + \frac{1}{\mathcal{H}^2} B^1(w) z + \frac{\mathring{z}_0}{\mathcal{H}^2} B^2(w) z + B^3(w)\frac{\mathring{z}}{\mathcal{H}}\cdot\frac{\mathring{z}}{\mathcal{H}} + \frac{\mathring{z}_0}{\mathcal{H}} D(w)\frac{\mathring{z}}{\mathcal{H}}.
$$

We can now define the extended system. Let $z = (z_1,\dots,z_{m-1})$. The phase space of the extended system is the space of triples $\overline{\mathcal{H}}, w, z$. The domain of the extended system consists of the triples for which $\overline{\mathcal{H}} > 0$ and $\overline{\mathcal{H}}^2 - z^2 > 0$. Let $\breve{z}_0(\overline{\mathcal{H}}, z) = \sqrt{\overline{\mathcal{H}}^2 - z^2}$. The extended system has the form

$$
\begin{aligned}
&\dot{\overline{\mathcal{H}}} = 0,\\
&\dot{w} = \frac{\breve{z}_0}{\overline{\mathcal{H}}} a(w) + A(w)\frac{z}{\overline{\mathcal{H}}},\\
&\frac{d^2 z}{dt^2} = \frac{\breve{z}_0}{\overline{\mathcal{H}}^2} b^1(w) + \left(\frac{\breve{z}_0}{\overline{\mathcal{H}}}\right)^2 b^2(w) + \frac{1}{\overline{\mathcal{H}}^2} B^1(w) D + \frac{\breve{z}_0}{\overline{\mathcal{H}}^2} B^2(w) z\\
&\qquad + B^3(w)\frac{z}{\overline{\mathcal{H}}}\cdot\frac{z}{\overline{\mathcal{H}}} + \frac{\breve{z}_0}{\overline{\mathcal{H}}} D(w)\frac{z}{\overline{\mathcal{H}}}.
\end{aligned}
\tag{11}
$$

The right-hand side of the last equation will be denoted by $F(\overline{\mathcal{H}}, w, z)$. There is a connection between the solutions of the Hamiltonian system (6) and the solutions of the extended system.

Suppose that $w(t)|\Delta$ is a solution of (6) such that $\mathring{z}_0(w(t)) > 0|\Delta$. Then $\overline{\mathcal{H}} = H(w(t)), w(t), z(t) = \mathring{z}(w(t))$ is a solution of the system (11) on Δ. The transition from solutions of (11) to solutions of (6) is given by the following proposition.

Proposition 2. *Let* $(\overline{\mathcal{H}}, w(t), z(t)|[t_0,t_1])$ *be a solution of the system* (11) *such that*:

1) $\overline{\mathcal{H}} = H(w(t_0))$;
2) $\mathring{z}_0(w(t_0)) = \breve{z}_0(\overline{\mathcal{H}}, z(t_0))$;
3) $\mathring{z}(w(t_0)) = z(t_0)$;
4) $\left.\dfrac{d\mathring{z}}{dt}\right|_{w(t_0)} = \left.\dfrac{dz}{dt}\right|_{t_0}$.

Then $(w(t)|[t_0,t_1])$ *is a solution of* (6) *such that*

$$
\mathcal{H}(w(t)) = \overline{\mathcal{H}}, \quad z_0(w(t)) = \breve{z}_0(\overline{\mathcal{H}}, z(t)), \quad z(w(t)) = z(t).
$$

Proof. Let $\widetilde{w}(t)$ be a solution of (6) such that $\widetilde{w}(t_0) = w(t_0)$. Then on some interval $[t_0,t']$, $t' > t_0$, the solution $\widetilde{w}(t)$ is defined and the condition $\mathring{z}_0(\widetilde{w}(t)) > 0|[t_0,t']$ is satisfied. Since $\widetilde{w}(t_0) = w(t_0)$, it follows that $\mathcal{H}(w(t)) = \overline{\mathcal{H}}|[t_0, t']$, and hence $\mathring{z}(\widetilde{w}(t)) = \breve{z}_0(\overline{\mathcal{H}}, z(\widetilde{w}(t)))$. From this it follows that $(\overline{\mathcal{H}}, \widetilde{w}(t), \mathring{z}(\widetilde{w}(t))|[t_0, t'])$ is a solution of the system (11). By assumption, $z(t_0) = \mathring{z}(\widetilde{w}(t_0))$ and $\dot{z}(t_0) = \frac{d}{dt}\mathring{z}|_{\widetilde{w}(t_0)}$. From this, by the uniqueness theorem for the extended system, it follows that $\mathcal{H}(\widetilde{w}(t)) = \overline{\mathcal{H}}$, $\widetilde{w}(t) = w(t)$, and $\mathring{z}(\widetilde{w}(t)) = z(t)|[t_0, t']$, and moreover, $\mathring{z}_0(\widetilde{w}(t)) = \breve{z}_0(\overline{\mathcal{H}}, z(t))|[t_0, t']$. The statement of the proposition follows from what has been proved according to the induction principle.□

Below we use the extended system to construct a sequence of nonsingular extremals approximating a given definite extremal.

§ 3. Estimate of solutions of a Cauchy problem

We consider a finite-dimensional space of elements ρ of dimension $d(\rho)$, that is, $\rho = (\rho_1, \dots, \rho_{d(\rho)})$. Let $V(t)$ be a function defined on the interval $[0, T]$, where $V(t)$ is a symmetric $d(\rho) \times d(\rho)$ matrix satisfying the following conditions on $[0, T]$:

a) there is a constant $\lambda_0 > 0$ such that

$$\lambda_0 \rho^2 \le V(t)\rho\rho|[0, T] \quad \forall \rho \in R^{d(\rho)}; \tag{1a}$$

b) there is a constant $c_0 > 0$ such that

$$|\dot{V}(t)\rho\rho| \le \frac{c_0}{T}|\rho|^2|[0, T] \quad \forall \rho \in R^{d(\rho)}. \tag{1b}$$

On $[0, T]$ we consider the equation

$$\ddot{\rho} = -V(t)\rho + f(t), \tag{2}$$

with the initial conditions

$$\rho(0) = \rho_0, \quad \dot{\rho}(0) = \dot{\rho}_0. \tag{3}$$

Let us estimate the solution in terms of $T, \lambda_0, c_0, \|f\|_\infty$, where $\|f\|_\infty$ is understood to be $\operatorname{ess\,max}_{t\in[0,T]} |f(t)|$. Multiplying (2) by $2\dot{\rho}$, we get

$$\frac{d}{dt}\big(V(t)\rho\rho + \dot{\rho}^2\big) = 2f(t)\dot{\rho} + \dot{V}(t)\rho\rho.$$

Setting $p = \sqrt{V\rho\rho + \dot{\rho}^2}$, we find from this and (1) that

$$\frac{d}{dt}p^2 \le 2\|f\|_\infty p + \frac{c_0}{\lambda_0}\frac{1}{T}p^2.$$

Consequently,

$$\frac{dp}{dt} \le \|f\|_\infty + \frac{c_0}{2\lambda_0}\frac{1}{T}p.$$

This implies the estimate

$$\|p\|_\infty \le p_0 \exp\{c_0/2\lambda_0\} + T\|f\|_\infty \frac{\exp\{c_0/2\lambda_0\} - 1}{c_0/2\lambda_0}, \tag{4}$$

where $p_0 = p(0)$.

Let us estimate $\left\|\int_0^t V(t)\rho(t)\,dt\right\|_\infty$. Integrating (2) with respect to t, we get

$$\int_0^t V(t)\rho(t)\,dt = \int_0^t f(t)\,dt + \dot{\rho}_0 - \dot{\rho}(t).$$

Consequently,

$$\left\| \int_0^t V\rho\,dt \right\|_\infty \le T\|f\|_\infty + |p_0| + \|p\|_\infty.$$

¿From this and the estimate (4) we get

$$\left\| \int_0^t V\rho\,dt \right\|_\infty \le p_0\left(\exp\{c_0/2\lambda_0\}+1\right) + T\|f\|_\infty \left(\frac{\exp\{c_0/2\lambda_0\}-1}{c_0/2\lambda_0} + 1 \right). \tag{5}$$

Let

$$C(\lambda_0, c_0) = \max\left\{ \exp\{c_0/2\lambda_0\}+1;\ \frac{\exp\{c_0/2\lambda_0\}-1}{c_0/2\lambda_0} + 1 \right\}.$$

Then the estimates (4) and (5) can be coarsened to the following:

$$\begin{gathered} \|p\|_\infty \le c(\lambda_0, c_0)(p_0 + T\|f\|_\infty), \\ \left\| \int_0^t V\rho\,dt \right\|_\infty \le c(\lambda_0, c_0)(p_0 + T\|f\|_\infty). \end{gathered} \tag{6}$$

We remark that the estimates (6) do not depend on the dimension $d(\rho)$ of the space of elements ρ.

§ 4. Estimates of solutions of a boundary-value problem

As in the preceding section, in this section we consider the space $R^{d(\rho)}$ of elements ρ together with a function $V(t)$ defined on $[0, 1]$, where $V(t)$ is a $d(\rho)\times d(\rho)$ matrix assumed to satisfy the following conditions:

a) there is a $\lambda_0 > 0$ such that

$$\lambda_0\rho^2 \le V(t)\rho\rho|[0,T] \quad \forall \rho \in R^{d(\rho)}; \tag{1a}$$

b) there is a $c_0 > 0$ such that

$$|V(t)\rho| \le c_0|\rho||[0,T] \quad \forall \rho \in R^{d(\rho)}. \tag{1b}$$

On $[0, T]$ we consider the equation

$$\ddot\rho = f + V(t)\rho \tag{2}$$

and the boundary conditions

$$\rho(0) = \rho(T) = 0. \tag{3}$$

We shall get an estimate of the solution of this problem in terms of $\lambda_0, c_0, \|f\|_\infty$.

The solution of the boundary-value problem (3), if it exists, is unique. Indeed, otherwise for $f(t) = 0|[0, T]$ there would be a nonzero solution $\widetilde\rho(t)$ of (3). Multiplying the equation (2) by $\widetilde\rho$, we get

$$\frac{d^2}{dt^2}\left(\frac{1}{2}\widetilde\rho^2\right) = \dot{\widetilde\rho}^2 + V\widetilde\rho\widetilde\rho.$$

It is clear that $\frac{d}{dt}\widetilde\rho^2\big|_{t=0} = 0$. Since the first part is not identically equal to zero and is not negative, $\widetilde\rho^2$ is a strictly convex function on $[0, T]$, and hence $\widetilde\rho(T) \ne 0$, a contradiction. This proves the uniqueness.

Let us prove existence. To this end we prove the existence of a Green's function, that is, a solution of problem (2), (3) for $f(t) = f_*\delta(t - t_*)$, where f_* is an arbitrary element of $R^{d(\rho)}$, and t_* is an arbitrary point of the interval $(0,T)$. Let f_* and t_* be fixed, and denote the Green's function by $\overline{\rho}(t,t_*,f_*)$. Obviously, it is determined by the following conditions.

1) $\overline{\rho}(t,\cdot)$ is continuous on $[0,T]$.
2) $\overline{\rho}(t,\cdot)$ is a solution of the homogeneous equation with initial condition $\overline{\rho}(0,\cdot)=0$ on the interval $[0,t_*]$, and $\overline{\rho}(t,\cdot)$ is a solution of the homogeneous equation with the condition $\overline{\rho}(T,\cdot)=0$ on the interval $[t_*,T]$.
3)

$$\dot{\overline{\rho}}\big|_{\mathrm{right}}(t_*,\cdot)-\dot{\overline{\rho}}\big|_{\mathrm{left}}(t_*,\cdot)=f_*. \tag{4}$$

Let $A_{\mathrm{left}}\colon R^{d(\rho)}\to R^{d(\rho)}$, $\dot\rho_0(0)\to\rho(t_*)$, where $\ddot\rho=V(t)\rho$ and $\rho(0)=0$. Similarly, let $A_{\mathrm{right}}\colon R^{d(\rho)}\to R^{d(\rho)}$, $\dot\rho(T)\to\rho(t_*)$, where $\rho(t)$ satisfies the homogeneous equation on $[t_*,T]$ and the condition $\rho(T)=0$. It follows from the uniqueness just proved that the operators A_{left} and A_{right} are invertible.

Let $A\colon R^{d(\rho)}\to R^{d(\rho)}$, $\rho_*\to\dot\rho_{\mathrm{right}}(t_*)-\dot\rho_{\mathrm{left}}(t_*)$, where $\rho_{\mathrm{right}}(t)$ satisfies on $[t_*,T]$ the homogeneous equation and the initial conditions $\rho_{\mathrm{right}}(T)=0$ and $\dot\rho_{\mathrm{right}}(T)=A^{-1}_{\mathrm{right}}\rho_*$, and where $\rho_{\mathrm{left}}(t)$ satisfies on $[0,t_*]$ the homogeneous equation and the initial conditions $\rho_{\mathrm{left}}(0)=0$ and $\dot\rho_{\mathrm{left}}(0)=A^{-1}_{\mathrm{left}}\rho_*$. It follows from the proven uniqueness that A is an invertible operator. Choosing $\rho_*=A^{-1}f_*$, we get

$$\overline{\rho}(t,t_*,f_*)=\begin{cases}\rho_{\mathrm{left}}(t), & t\in[0,t_*],\\ \rho_{\mathrm{right}}(t), & t\in[t_*,T].\end{cases} \tag{5}$$

The conditions (4) are all easily verified, and the existence of a Green's function is proved.

Let us estimate $|\overline{\rho}(t,t_*,f_*)|$. According to the formula (5), we should estimate $|\rho_{\mathrm{left}}(t)|$ on $[0,t_*]$ and $|\rho_{\mathrm{right}}(t)|$ on $[t_*,T]$. It has already been noted that $\rho^2_{\mathrm{left}}(t)$ satisfies on $[0,t_*]$ the equation

$$\frac{d^2}{dt^2}\left(\frac12\rho^2_{\mathrm{left}}\right)=\dot{\rho}^2_{\mathrm{left}}(t)+V(t)\rho_{\mathrm{left}}(t)\rho_{\frac12}(t)$$

and the conditions $\rho^2_{\mathrm{left}}(0)=0$ and $\dot\rho^2_{\mathrm{left}}(0)=0$. According to the condition (1a), $\dfrac{d^2}{dt^2}\rho^2_{\mathrm{left}}\ge 2\lambda_0\rho^2_{\mathrm{left}}$. From this it follows that $\rho^2_{\mathrm{left}}(t)\le\rho^2_*\exp\bigl\{\sqrt{2\lambda_0}\,(t-t_*)\bigr\}|[0,t_*]$. Similarly, $\rho^2_{\mathrm{right}}(t)\le\rho^2_*\exp\bigl\{\sqrt{2\lambda_0}\,(t_*-t)\bigr\}|[t_*,T]$. Consequently, by (5),

$$|\overline{\rho}(t,t_*,f_*)|^2\le\rho^2_*\exp\bigl\{-\sqrt{2\lambda_0}\,|t-t_*|\bigr\}|[0,T]. \tag{6}$$

¿From this we first estimate $|\rho_*|$ in terms of $|f_*|$. Indeed, by (6) and (5),

$$2\rho_*\dot\rho_{\mathrm{left}}(t_*)\ge\rho^2_*\sqrt{2\lambda_0}\,,$$
$$2\rho_*\dot\rho_{\mathrm{right}}(t_*)\le-\rho^2_*\sqrt{2\lambda_0}\,.$$

Subtracting, we get $2\rho_*f_*\le-2\rho^2_*\sqrt{2\lambda_0}$. Consequently, $|f_*|\ge|\rho_*|\sqrt{2\lambda_0}$. This tells us that $|\rho_*|\le\dfrac{1}{\sqrt{2\lambda_0}}\,|f_*|$. Using the given estimate, we have

$$|\overline{\rho}(t,t_*,f_*)|\le\frac{1}{\sqrt{2\lambda_0}}\,|f_*|\exp\left\{-\sqrt{\frac{\lambda_0}{2}}\,|t-t_*|\right\}. \tag{7}$$

Let $\rho(t)$ be the solution of problem (2), (3) for given $f(t)$. Then $\rho(t) = \int\limits_0^T \overline{\rho}(t, t_*, f(t_*))\, dt_*$. From this and (7),

$$|\rho(t)| \leq \int\limits_0^T \frac{1}{\sqrt{2\lambda_0}}\, |f_*| \exp\left\{ -\sqrt{\frac{\lambda_0}{2}}\, |t - t_*| \right\} dt_*$$

$$\leq \frac{1}{\sqrt{2\lambda_0}}\, \|f\|_\infty \int\limits_{-\infty}^{\infty} \exp\left\{ -\sqrt{\frac{\lambda_0}{2}}\, |\tau| \right\} d\tau = \frac{2}{\lambda_0}\|f\|_\infty.$$

Thus,

$$\|\rho\|_\infty \leq \frac{2}{\lambda_0}\|f\|_\infty. \tag{8}$$

We now get an estimate for $\|\dot\rho\|_\infty$. For our purposes it is sufficient to get an estimate in the case $T \geq 2$. The estimate, like (8), does not depend on T, but for small T it is established somewhat differently, and we shall not consider it here. Let $T \geq 2$ and $t_* \in [0, T]$. Then either $t_* + 1 \in [0, T]$ or $t_* - 1 \in [0, T]$. Suppose for definiteness that $t_* + 1 \in [0, T]$. Integrating (2) with respect to t on the interval $[t_*, t_* + 1]$, we get

$$\rho(t_* + 1) - \rho(t_*) = \dot\rho(t_*) + \int\limits_{t_*}^{t_*+1} dt \int\limits_{t_*}^{t} \big(f(t') + V(t')\rho(t')\big)\, dt'.$$

¿From this it follows easily that

$$|\dot\rho(t_*)| \leq 2\|\rho\|_\infty + \frac{1}{2}(\|f\|_\infty + c_0\|\rho\|_\infty).$$

Then, using (8), we have

$$|\dot\rho(t_*)| \leq \left(\frac{4}{\lambda_0} + \frac{1}{2} + \frac{c_0}{\lambda_0}\right)\|f\|_\infty.$$

Therefore,

$$\|\dot\rho\|_\infty \leq \left(\frac{4}{\lambda_0} + \frac{1}{2} + \frac{c_0}{\lambda_0}\right)\|f\|_\infty. \tag{9}$$

Setting $c_1(\lambda_0, c_0) = \left(\frac{4}{\lambda_0} + \frac{1}{2} + \frac{c_0}{\lambda_0}\right)$, we finally get

$$\begin{aligned} \|\rho\|_\infty &\leq c_1(\lambda_0, c_0)\|f\|_\infty, \\ \|\dot\rho\|_\infty &\leq c_1(\lambda_0, c_0)\|f\|_\infty. \end{aligned} \tag{10}$$

We see that the estimate (10) also does not depend on the dimension $d(\rho)$ of the space of elements ρ.

§ 5. A theorem on approximation of a plus-Legendre extremal

Let $\gamma^0 = (w^0(t), u^0(t)|\Delta^0)$ be a plus-Legendre extremal. We prove the following theorem.

Theorem 1. *There exists a $\delta_0 \le |\Delta^0|$ such that for any interval $\Delta \subset \Delta^0$ with $|\Delta| = \delta_0$ there is a sequence $\{\gamma^s = (w^s(t), u^s(t)|\Delta)\}_{s=1,2,\dots}$ of nonsingular extremals such that*

$$\begin{gathered}\mathcal{H}(w^s(t)) = \overline{\mathcal{H}}^s \to 0 \quad \text{as } s \to \infty,\\ \left\|w^s(t) - w^0(t)\right\|_\infty \le O(\overline{\mathcal{H}}^s),\\ \left\|u^s(t) - u^0(t)\right\|_\infty \le O\big(\sqrt{\overline{\mathcal{H}}^s}\big).\end{gathered}$$

Proof. We break up the proof into several parts. The treatment of a plus-Legendre extremal is somewhat more complicated than the treatment of a minus-Legendre extremal. Therefore, certain parts of the proof will be common to both.

Part 1. We make some necessary deductions from Proposition 1 in § 2.

Let $\eta = (\eta_0, \eta_1, \dots, \eta_{m-1}, \eta_m, \dots \eta_{2m-2})$. It is convenient for us to define $|\eta| = \max |\eta_i| |i = 0, \dots, 2m-2$. It follows from Proposition 1 in § 2 that for all sufficiently small η and for any $t \in \Delta^0$ there exists a smooth function $w_*(\eta, t)$ such that

$$\begin{aligned} w_*(0,t) &= w^0(t)|\Delta^0,\\ \overset{\circ}{z}_0(w_*(\eta,t)) &= \eta_0,\\ \overset{\circ}{z}_i(w_*(\eta,t)) &= \eta_i, \quad i = 1,\dots,m-1,\\ \overset{\circ}{z}_i(w_*(\eta,t)) &= \eta_{m-1+i}, \quad i = 1,\dots,m-1. \end{aligned} \tag{1}$$

By (1), there is a $\overline{\mu} > 0$ such that

$$\left|w_*(\eta,t) - w^0(t)\right| \le \overline{\mu}\,|\eta| \tag{2}$$

for any $t \in \Delta^0$ and all sufficiently small η.

The formulas (1) and the estimate (2) will also be used in the treatment of a minus-Legendre extremal.

Part 2. Let us return to the function $F(\overline{\mathcal{H}}, w, z)$ defined in § 2. For convenience we give an expression for it:

$$\begin{aligned} F(\overline{\mathcal{H}}, w, z) = \frac{\breve{z}_0}{\overline{\mathcal{H}}^2} b^1(w) + \left(\frac{\breve{z}_0}{\overline{\overline{\mathcal{H}}}}\right)^2 b^2(w) + \frac{1}{\overline{\mathcal{H}}^2} B^1(w) z + \frac{\breve{z}_0}{\overline{\mathcal{H}}^2} B^2(w) z&\\ + B^3(w) \frac{z}{\overline{\overline{\mathcal{H}}}} \cdot \frac{z}{\overline{\overline{\mathcal{H}}}} + \frac{\breve{z}_0}{\overline{\overline{\mathcal{H}}}} D(w) \frac{z}{\overline{\overline{\mathcal{H}}}}.& \end{aligned} \tag{3}$$

Let $w(t) = w^0(t) + \theta(t)|\Delta^0$. We estimate the functions depending on w in the expression for F when $\|\theta\|_\infty$ is small, using the formulas (10.2).* Then by (6.1), $b^1(w^0(t)) = 0$ and $(b^1)'_w(w^0(t)) = 0$. According to (1.2) and (6.1), $b^2(w^0(t)) = 0$. Further, according to (6.1), $B^1(w^0(t)) = 0$ and $(B^1)'_w(w^0(t)) = 0$. By (6.1) and (1.2), $B^2(w^0(t)) = 0$. These

*(10.2) means (10) in §2.

equalities imply the estimates

$$
\begin{aligned}
\left\|b^1(w^0(t)+\theta(t))\right\|_\infty &\le \overline{b}^1 \,\|\theta\|_\infty^2,\\
\left\|b^2(w^0(t)+\theta(t))\right\|_\infty &\le \overline{b}^2 \,\|\theta\|_\infty,\\
\left\|B^1(w^0(t)+\theta(t))\right\|_\infty &\le \overline{B}^1 \,\|\theta\|_\infty^2,\\
\left\|B^2(w^0(t)+\theta(t))\right\|_\infty &\le \overline{B}^2 \,\|\theta\|_\infty,\\
\left\|B^3(w^0(t)+\theta(t))\right\|_\infty &\le \overline{B}^3,\\
\left\|D(w^0(t)+\theta(t))-D(w^0(t))\right\|_\infty &\le \overline{D}\,\|\theta\|_\infty,\\
\left\|D(w^0(t)+\theta(t))\right\|_\infty &\le \overline{D}^0,
\end{aligned}
\tag{4}
$$

where $\overline{b}^1, \overline{b}^2, \overline{B}^1, \overline{B}^2, \overline{B}^3, \overline{D}, \overline{D}^0$ are positive constants. The inequalities (4) will also play a role in the treatment of a minus-Legendre extremal.

Part 3. Let $\Theta = (\theta_\psi, \theta_x) \in R^{2n}$, $\xi = (\xi_1, \dots, \xi_{m-1}) \in R^{m-1}$, and $h = (\theta, \xi)$. We take an interval $\Delta = [t_0, t_1] \subset \Delta^0$ and denote by $\widehat{h}$ the function $h(t)|\Delta$. Let $L(\Delta)$ be the space $L_\infty^{(2n+m-1)}(\Delta)$ of elements $\widehat{h}$. Let $P[\Delta, \overline{\mathcal{H}}](\widehat{h}) = \widehat{z}$, where $\widehat{z} \in L_\infty^{m-1}(\Delta)$, and

$$
\ddot{z} = F\big(\overline{\mathcal{H}}, w^0(t)+\theta(t), \xi(t)\big) - D(w^0(t))\frac{\xi(t)}{\overline{\mathcal{H}}} + D(w^0(t))\frac{z}{\overline{\overline{\mathcal{H}}}}, \quad z(t_0) = \dot{z}(t_0) = 0.
$$

As is clear from (3), the domain of $P[\Delta, \overline{\mathcal{H}}]$ is given by the condition $\overline{\mathcal{H}}^2 - \xi^2(t) \ge 0|\Delta$. On this set $P[\Delta, \overline{\mathcal{H}}]$ is a continuous operator.

Let $\widehat{z} \in C_1^{m-1}(\Delta)$. We set $G[\Delta, \overline{\mathcal{H}}](\widehat{z}) = \delta\widehat{w}$, where $\delta\widehat{w} \in L_\infty^{2n}(\Delta)$ satisfies on Δ the equation

$$
\delta\dot{w} = \frac{\breve{z}_0(\overline{\mathcal{H}}, z(t))}{\overline{\mathcal{H}}}\, a(w^0(t)+\delta w(t)) - a(w^0(t)) + A(w^0(t)+\delta w(t))\frac{z(t)}{\overline{\mathcal{H}}}
\tag{5}
$$

and the initial condition

$$
\delta w(t_0) = w_*(\eta, t_0) - w^0(t_0),
$$

where

$$
\begin{aligned}
\eta_0 &= \breve{z}_0(\overline{\mathcal{H}}, z(t_0)),\\
\eta_i &= z_i(t_0), \quad i = 1, \dots, m-1,\\
\eta_{m-1+i} &= \dot{z}_i(t_0), \quad i = 1, \dots, m-1.
\end{aligned}
$$

The operator $G[\Delta, \overline{\mathcal{H}}]$ is also continuous on its domain.

Let $J[\Delta, \overline{\mathcal{H}}](\widehat{h}) = (\delta\widehat{w}, \widehat{z})$, where

$$
\begin{aligned}
\dot{\widehat{w}} &= G[\Delta, \overline{\mathcal{H}}](\widehat{z}),\\
\widehat{z} &= P[\Delta, \overline{\mathcal{H}}](\widehat{h}).
\end{aligned}
\tag{6}
$$

We fix $\mu > 0$ and $v > 0$ and denote

$$
\left\{ \widehat{h} \;\middle|\; \|\widehat{\theta}\|_\infty \le \mu\overline{\mathcal{H}},\ \|\widehat{\xi}\|_\infty \le v\overline{\mathcal{H}}\,\sqrt{\overline{\overline{\mathcal{H}}}} \right\}
$$

by $L(\Delta, \overline{\mathcal{H}}, \mu, v) \subset L(\Delta)$. In what follows we investigate $J[\Delta, \overline{\mathcal{H}}]$ on $L(\Delta, \overline{\mathcal{H}}, \mu, v)$ as $\overline{\mathcal{H}} \to 0$.

Part 4. For an arbitrary function $f(t)|\Delta$, $\widehat{f} \in L_\infty^{m-1}(\Delta)$, we investigate the solution $\widehat{z} = z(t)|\Delta$ of the Cauchy problem

$$\ddot{z} = f(t) + D(w^0(t))\frac{z}{\overline{\mathcal{H}}}, \quad z(0) = \dot{z}(0) = 0. \tag{7}$$

We introduce a change of variable, setting $t = t_0 + \tau\sqrt{\overline{\mathcal{H}}}$. Then obviously, $\tau \in [0, T]$, where $T = \Delta/\sqrt{\overline{\mathcal{H}}}$. Setting $\widetilde{z}(\tau) = z(t(\tau))$ and $V(\tau) = -D\big(w^0(t(\tau))\big)$, we get

$$\frac{d^2\widetilde{z}}{d\tau^2}\widetilde{z} = f(t(\tau))\overline{\mathcal{H}} - V(\tau)\widetilde{z}, \quad \widetilde{z}(0) = 0, \quad \frac{d\widetilde{z}}{d\tau}\bigg|_{\tau=0} = 0. \tag{8}$$

By definition, $D(w^0(t)) = -D^0(t)$, and for the plus-Legendre extremal $D^0(t)$ is a positive-definite matrix for any $t \in \Delta^0$. Consequently, $V(\tau)$ is a positive-definite matrix for any $\tau \in [0, T]$, and we can use the results obtained in § 3 for an estimate of $\widetilde{z}(\tau)$.

To get uniform estimates with respect to $\overline{\mathcal{H}}$, we verify conditions (1a) and (1b) in § 3. Obviously, $V(\tau)\overline{z} \cdot \overline{z} \ge \overline{\lambda}^0\overline{z}^2$ for any τ and $\overline{z}$, where $\overline{\lambda}^0$ is the smallest eigenvalue of the matrix $D^0(t)$ on the interval Δ^0. Thus, (1a) is satisfied.

Let $\overline{\lambda}^1$ be the eigenvalue of $\dot{D}^0(t)$ on Δ^0 that is maximal in modulus. Then obviously,

$$\left|\frac{dV}{d\tau}\overline{z}\overline{z}\right| \le \overline{\lambda}^1\sqrt{\overline{\mathcal{H}}}\overline{z}^2 \quad \forall \tau, \overline{z}.$$

Setting $\overline{c}^0 = \overline{\lambda}^1|\Delta^0|$, we get from this inequality that

$$\left|\frac{dV}{d\tau}\overline{z}\overline{z}\right| \le \frac{\overline{c}^0}{T}\overline{z}\overline{z} \quad \forall \tau, \overline{z}, \overline{\mathcal{H}}.$$

The condition (1b) is thus also satisfied. Then, by (6) in §3,

$$\left\|\sqrt{V(\tau)\widetilde{z}\widetilde{z} + \left(\frac{d\widetilde{z}}{d\tau}\right)^2}\right\|_{\infty,\tau} \le C\big(\overline{\lambda}^0, \overline{c}^0\big)T\overline{\mathcal{H}}\,\|f\|_{\infty,\tau},$$

$$\left\|\int_0^\tau V(\tau)\widetilde{z}\,d\tau\right\|_{\infty,\tau} \le C\big(\overline{\lambda}^0, \overline{c}^0\big)T\overline{\mathcal{H}}\,\|f\|_{\infty,\tau}.$$

Passing from the variable τ back to the variable t and taking into account that $\|\cdot\|_{\infty,\tau} = \|\cdot\|_{\infty,t}$, we get

$$\left\|\sqrt{D^0(t)zz + \overline{\mathcal{H}}\dot{z}^2}\right\|_\infty \le C\big(\overline{\lambda}^0, \overline{c}^0\big)|\Delta|\sqrt{\overline{\mathcal{H}}}\,\|f\|_\infty,$$

$$\left\|\int_{t_0}^t D^0(t)\frac{z}{\overline{\mathcal{H}}}\,dt\right\|_\infty \le C\big(\overline{\lambda}^0, \overline{c}^0\big)|\Delta|\,\|f\|_\infty.$$

It will be convenient to use estimates that follow clearly from the given estimates. Setting $C_0 = \max\Big\{1/\sqrt{\overline{\lambda}^0}, 1\Big\}C\big(\overline{\lambda}^0, \overline{c}^0\big)$, we get from the estimates above that

$$\begin{gathered} \|z\|_\infty \le C_0|\Delta|\sqrt{\overline{\mathcal{H}}}\,\|f\|_\infty, \\ \|\dot{z}\|_\infty \le C_0|\Delta|\,\|f\|_\infty, \\ \left\|\int_{t_0}^t D^0(t)\frac{z}{\overline{\mathcal{H}}}\,dt\right\|_\infty \le C_0|\Delta|\,\|f\|_\infty. \end{gathered} \tag{9}$$

We underscore that C_0 does not depend on $\overline{\mathcal{H}}, \Delta$.

Part 5. For $\widehat{h} \in L(\Delta)$, $\widehat{h} = (\theta(t), \xi(t))|\Delta$, we set

$$f[\overline{\mathcal{H}}, \widehat{h}](t) = F\big(\overline{\mathcal{H}}, w^0(t) + \theta(t), \xi(t)\big) - D(w^0(t))\frac{\xi(t)}{\overline{\mathcal{H}}}.$$

Let $\mu, v > 0$ be fixed. Then for sufficiently small $\overline{\mathcal{H}}$ we have the estimate

$$\big\|f[\overline{\mathcal{H}}, \widehat{h}](t)\big\|_\infty \le \big(\overline{b}^1\mu^2 + \overline{b}^2\mu + \overline{b}^3 v^2\big)\overline{\mathcal{H}} + \big(\overline{B}^1\mu^2 v + \overline{B}^2\mu v + \overline{D}\mu v + \overline{D}^0 v^3\big)\overline{\mathcal{H}}\sqrt{\overline{\mathcal{H}}}$$

for any $\widehat{h} \in L(\Delta, \overline{\mathcal{H}}, \mu, v)$ in view of (3) and (4).

Let $\overline{b} = \max\big\{\overline{b}^1, \overline{b}^2, \overline{B}^3\big\}$. Then for small $\overline{\mathcal{H}}$

$$\big\|f[\overline{\mathcal{H}}, \widehat{h}](t)\big\|_\infty \le \overline{b}(\mu^2 + \mu + v^2)\overline{\mathcal{H}}. \tag{10}$$

Note that $\overline{b}$ does not depend on $\Delta, \overline{\mathcal{H}}, \mu, v$.

It follows from (10) and (9) that for small $\overline{\mathcal{H}}$ and any $\widehat{h} \in L(\Delta, \overline{\mathcal{H}}, \mu, v)$ the estimates

$$\begin{aligned} \|z\|_\infty &\le C_0\overline{b}|\Delta|(\mu^2 + \mu + v^2)\overline{\mathcal{H}}\sqrt{\overline{\mathcal{H}}}, \\ \|\dot{z}\|_\infty &\le C_0\overline{b}|\Delta|\,(\mu^2 + \mu + v^2)\overline{\mathcal{H}}, \\ \left\|\int_{t_0}^{t} D^0(t)\frac{z}{\overline{\mathcal{H}}}\,dt\right\|_\infty &\le C_0\overline{b}|\Delta|\,(\mu^2 + \mu + v^2)\overline{\mathcal{H}} \end{aligned} \tag{11}$$

are valid for $\widehat{z} = P[\Delta, \overline{\mathcal{H}}](\widehat{h})$.

Part 6. Suppose that $\widehat{z}$ satisfies (11) and the initial condition $z(t_0) = \dot{z}(t_0) = 0$. Let us estimate $\delta\widehat{w} = G[\Delta, \overline{\mathcal{H}}](\widehat{z})$. We rewrite equation (5) in the form

$$\begin{aligned} \delta\dot{w} = \frac{\breve{z}_0}{\overline{\overline{\mathcal{H}}}}\big(a(w^0(t) + \delta w) - a(w^0(t))\big) + \left(\frac{\breve{z}_0}{\overline{\overline{\mathcal{H}}}} - 1\right)a(w^0(t)) \\ + A(w^0(t))\,\frac{z(t)}{\overline{\mathcal{H}}} + \big(A(w^0(t) + \delta w) - A(w^0(t))\big)\,\frac{z(t)}{\overline{\mathcal{H}}}. \end{aligned} \tag{12}$$

For small δw we have the estimates

$$\begin{aligned} \big|a(w^0(t))\big| &\le \overline{a}^0, \\ \big|a(w^0(t) + \delta w) - a(w^0(t))\big| &\le \overline{a}^1|\delta w|, \\ \big|A(w^0(t) + \delta w) - A(w^0(t))\big| &\le \overline{A}|\delta w| \quad \forall t \in \Delta^0 \end{aligned} \tag{13}$$

for some constants $\overline{a}^0 > 0$, $\overline{a}^1 > 0$, $\overline{A} > 0$. Let $N(t) = A(w^0(t))(D^0)^{-1}(t)$. There exist constants $\overline{v}^0 > 0$ and $\overline{v}^1 > 0$ such that $|N(t)| \le \overline{v}^0$ and $|\dot{N}(t)| \le \overline{v}^1$ for any $t \in \Delta^0$. Let

$$\delta w(t) = \xi(t) + N(t)\int_{t_0}^{t} D^0(t)\frac{z}{\overline{\mathcal{H}}}\,dt.$$

Substituting in (12) and using the fact that $A(w^0(t))\frac{z}{\overline{\mathcal{H}}} = N(t)D^0(t)\frac{z}{\overline{\mathcal{H}}}$, we get

$$(14)\quad \dot{\xi} = \dot{N}(t)\int_{t_0}^{t} D^0(t)\frac{z}{\overline{\mathcal{H}}}\,dt + \left(\frac{\breve{z}_0}{\overline{\mathcal{H}}} - 1\right)a(w^0(t)) + \frac{\breve{z}_0}{\overline{\mathcal{H}}}\left[a\left(w^0(t) + \xi(t) + N(t)\int_{t_0}^{t} D^0(t)\frac{z}{\overline{\mathcal{H}}}\,dt\right) - a(w^0(t))\right] + \left[A\left(w^0(t) + \xi(t) + N(t)\int_{t_0}^{t} D^0(t)\frac{z}{\overline{\mathcal{H}}}\,dt\right) - A(w^0(t))\right]\frac{z}{\overline{\mathcal{H}}}.$$

Let $q = C_0\overline{b}|\Delta|(\mu^2 + \mu + v^2)$. Then, obviously,

$$|\dot{\xi}| \le \overline{\nu}^1 q\overline{\mathcal{H}} + \overline{a}^0 q^2\overline{\mathcal{H}} + \overline{a}^1|\xi| + \overline{a}^1\overline{\nu}^0 q\overline{\mathcal{H}} + \overline{A}^1 q\overline{\mathcal{H}}|\xi|\overline{A}^1 q^2\overline{\mathcal{H}}\sqrt{\overline{\mathcal{H}}} \quad \forall t \in \Delta.$$

Let $\overline{a} = \overline{\nu}^1 q + \overline{a}^0 q^2 + \overline{a}^1\overline{\nu}^0 q$. Then for small $\overline{\mathcal{H}}$ we have

$$(15)\qquad |\dot{\xi}| \le \overline{a}(|\xi| + \overline{\mathcal{H}}),$$

where $\overline{a}$ does not depend on $\overline{\mathcal{H}}$.

We have $\xi(t_0) = \delta w(t_0) = w_*(\eta, t_0) - w^0(t_0)$, where $\eta_0 = \overline{\mathcal{H}}$, $\eta_1 = \eta_2 = \cdots = \eta_{2m-2} = 0$. Consequently,

$$(16)\qquad |\xi(t_0)| \le \overline{\mu}\overline{\mathcal{H}}.$$

It follows from (15) and (16) that

$$(17)\qquad \|\xi\|_\infty \le \overline{\mathcal{H}}\overline{\mu}\exp\{\overline{a}\Delta\} + \overline{\mathcal{H}}\big(\exp\{\overline{a}\Delta\} - 1\big).$$

¿From (17) we easily get

$$(18)\qquad \|\delta w\|_\infty \le \|\xi\|_\infty + \overline{\nu}^0 q\overline{\mathcal{H}} \le \overline{\mathcal{H}}\big(\overline{\mu}\exp\{\overline{a}\Delta\} + \exp\{\overline{a}\Delta\} - 1 + \overline{\nu}^0 q\big).$$

Part 7.

Proposition 1. *There exist $\delta > 0, \mu > 0, v > 0$ such that for sufficiently small $\overline{\mathcal{H}}$ any operator $J[\Delta, \overline{\mathcal{H}}]\,|\,|\Delta| = \delta$ has a fixed point on $L(\Delta, \overline{\mathcal{H}}, \mu, v)$.*

Proof. First of all we prove the existence of $\delta > 0, \mu > 0, v > 0$ such that for sufficiently small $\overline{\mathcal{H}}$ and any Δ ($\Delta \subset \Delta^0, |\Delta| = \delta$) the operator $J[\Delta, \overline{\mathcal{H}}]$ maps $L(\Delta, \overline{\mathcal{H}}, \mu, v)$ into itself.

It follows from (11) and (18) that for this the following two inequalities suffice:

$$(19)\qquad \begin{gathered} C_0\overline{b}\,\delta(\mu^2 + \mu + v^2) \le v, \\ \overline{\mu}\exp\{\overline{a}\Delta\} + \exp\{\overline{a}\Delta\} - 1 + \overline{\nu}^0 q \le \mu, \end{gathered}$$

where $q = C_0\overline{b}\,\delta\,(\mu^2 + \mu + v^2)$. From this it is clear that v can be given arbitrarily, while μ must satisfy the inequality $\overline{\mu} < \mu$. In this case it suffices to take a small δ, and the inequalities (19) will be satisfied.

Suppose that δ, μ, v satisfy (19). Then for sufficiently small $\overline{\mathcal{H}}$ and for any Δ ($|\Delta| = \delta$) the set $L(\Delta, \overline{\mathcal{H}}, \mu, v)$ is in the domain of the operator $J[\Delta, \overline{\mathcal{H}}]$. Indeed, let $\overline{\mathcal{H}}$ be small enough that $\overline{\mathcal{H}}^2 > v^2\overline{\mathcal{H}}^3$. Then $\overline{\mathcal{H}}^2 > \xi^2(t)|\Delta$ for any $\widehat{h} \in L(\Delta, \overline{\mathcal{H}}, \mu, v)$, $|\Delta| = \delta$. Thus, $\widehat{h}$ belongs to the domain of the operator $P[\Delta, \overline{\mathcal{H}}]$. Since the inequalities (19) are satisfied, it follows that $\|z\|^2 \le v\overline{\mathcal{H}}^3$ for $\widehat{z} = P[\Delta, \overline{\mathcal{H}}](\widehat{h})$. Consequently, $\overline{\mathcal{H}}^2 > z^2(t)|\Delta$, and hence $\widehat{z}$ belongs to the domain of the operator $G[\Delta, \overline{\mathcal{H}}]$. This implies that $\widehat{h}$ belongs to the domain of the operator $J[\Delta,\overline{\mathcal{H}}]$. But $\widehat{h}$ is an arbitrary element of $L(\Delta,\overline{\mathcal{H}},\mu,v)$. This proves that $J[\Delta,\overline{\mathcal{H}}]$ is defined on $L(\Delta,\overline{\mathcal{H}},\mu,v)$.

It follows from the inequalities (11), (17), and (18) that the operator $J[\Delta,\overline{\mathcal{H}}]$ maps $L(\Delta,\overline{\mathcal{H}},\mu,v)$ into a compact subset, namely, into the subset of the elements $\delta\widehat{w},\widehat{z}$ for which $\|\delta\dot{w}\|_\infty$, $\|\dot{z}\|_\infty$ are uniformly bounded by the product of $\overline{\mathcal{H}}$ and a constant depending on μ,v and the coefficients $\overline{b},\overline{a},\overline{v}^0$. Let $\widetilde{L}(\Delta,\overline{\mathcal{H}},\mu,v)\subset L(\Delta,\overline{\mathcal{H}},\mu,v)$, $\|\delta\dot{w}\|\le 1$, and $\|\dot{z}\|\le 1$. Obviously, $\widetilde{L}(\Delta,\overline{\mathcal{H}},\mu,v)$ is a compact convex set, and for small $\overline{\mathcal{H}}$ it is mapped into itself by the operator $J[\Delta,\overline{\mathcal{H}}]$ for any $\Delta||\Delta|=\delta$. The statement of the proposition follows from this. $\square$

Part 8. Here we conclude the proof of the theorem.

Let δ,μ,v be such that for sufficiently small $\overline{\mathcal{H}}$ and any $\Delta||\Delta|=\delta$ the operator $J[\Delta,\overline{\mathcal{H}}]$ has a fixed point belonging to $L(\Delta,\overline{\mathcal{H}},\mu,v)$. We choose a closed interval $\Delta\subset\Delta^0$ with $|\Delta|=\delta$. Let $\delta\widehat{w}^{\overline{\mathcal{H}}}$, $\widehat{z}^{\overline{\mathcal{H}}}$ be a fixed point of $J[\Delta,\overline{\mathcal{H}}]$ belonging to $L(\Delta,\overline{\mathcal{H}},\mu,v)$. We set $w^{\overline{\mathcal{H}}}(t)=w^0(t)+\delta w^{\overline{\mathcal{H}}}(t)|\Delta$. Then it is not hard to see that $\widehat{w}^{\overline{\mathcal{H}}}$, $\widehat{z}^{\overline{\mathcal{H}}}$ form a solution of the extended system (2.11). We show that for any small $\overline{\mathcal{H}}$ the solution $\widehat{w}^{\overline{\mathcal{H}}}$, $\widehat{z}^{\overline{\mathcal{H}}}$ satisfies the conditions of Proposition 2 in § 2. By the definition of the operator $J[\Delta,\overline{\mathcal{H}}]$, we have $\delta\widehat{w}^{\overline{\mathcal{H}}}(t_0)=w_*(\eta,t_0)-w^0(t_0)$, where $\eta=(\overline{\mathcal{H}},0,\dots,0)$. Consequently, $w^{\overline{\mathcal{H}}}(t_0)=w_*(\eta,t_0)$. According to the definition of $w_*(\eta,t)$ (part 1), this implies that

$$\begin{aligned}&\mathring{z}_0(w^{\overline{\mathcal{H}}}(t_0))=\overline{\mathcal{H}},\\&\mathring{z}_i(w^{\overline{\mathcal{H}}}(t_0))=0,\quad i=1,\dots,m-1,\\&\mathring{z}_i(w^{\overline{\mathcal{H}}}(t_0))=0,\quad i=1,\dots,m-1.\end{aligned}\tag{20}$$

Consequently,

$$\begin{aligned}&\mathcal{H}(w^{\overline{\mathcal{H}}}(t_0))=\overline{\mathcal{H}},\\&\mathring{z}_0(w^{\overline{\mathcal{H}}}(t_0))=\breve{z}_0(\overline{\mathcal{H}},z^{\overline{\mathcal{H}}}(t_0)),\\&\mathring{z}(w^{\overline{\mathcal{H}}}(t_0))=z^{\overline{\mathcal{H}}}(t_0).\end{aligned}\tag{21}$$

Thus, the conditions 1), 2), and 3) of Proposition 2 in § 2 are satisfied. By (7) in §2,

$$\left.\frac{d\mathring{z}_i}{dt}\right|_{w^{\overline{\mathcal{H}}}(t_0)}=\frac{\mathring{z}_0(w^{\overline{\mathcal{H}}}(t_0))}{\mathcal{H}(w^{\overline{\mathcal{H}}}(t_0))}\,\mathring{z}_i(w^{\overline{\mathcal{H}}}(t_0))+\sum_{k=1}^{m-1}\frac{\mathring{z}_k(w^{\overline{\mathcal{H}}}(t_0))}{\mathcal{H}(w^{\overline{\mathcal{H}}}(t_0))}\psi^{\overline{\mathcal{H}}}(t_0)[\mathbf{r}_i,\mathbf{r}_k](x^{\overline{\mathcal{H}}}(t_0)).$$

Then it follows from (21) that

$$\left.\frac{d\mathring{z}}{dt}\right|_{w^{\overline{\mathcal{H}}}(t_0)}=0=\dot{z}^{\overline{\mathcal{H}}}(t_0).$$

Thus, condition 4) of Proposition 2 in § 2 is also satisfied. Accordingly, all the conditions of that proposition are satisfied.

In view of this proposition, $\dot{w}^{\overline{\mathcal{H}}}$ is a solution of the Hamiltonian system (6) of §2 on Δ, and hence is a nonsingular extremal, and the following conditions hold:

$$\begin{aligned}&\mathcal{H}(w^{\overline{\mathcal{H}}}(t))=\overline{\mathcal{H}},\\&\mathring{z}_0(w^{\overline{\mathcal{H}}}(t))=\breve{z}_0(\overline{\mathcal{H}},z^{\overline{\mathcal{H}}}(t)),\\&\mathring{z}(w^{\overline{\mathcal{H}}}(t))=z^{\overline{\mathcal{H}}}(t)\quad\forall t\in\Delta.\end{aligned}\tag{22}$$

Since $(\delta\widehat{w}^{\overline{\mathcal{H}}}, \widehat{z}^{\overline{\mathcal{H}}}) \in L(\Delta, \overline{\mathcal{H}}, \mu, v)$, it follows that

$$\left\|w^{\overline{\mathcal{H}}}(t) - w^0(t)\right\|_\infty \le \mu\overline{\mathcal{H}}. \tag{23}$$

We then have

$$u_0^{\overline{\mathcal{H}}}(t) = \frac{\mathring{z}_0(w^{\overline{\mathcal{H}}}(t))}{\mathcal{H}(w^{\overline{\mathcal{H}}}(t))} = \frac{\breve{z}_0(\overline{\mathcal{H}}, z^{\overline{\mathcal{H}}}(t))}{\overline{\mathcal{H}}}.$$

Consequently,

$$\left\|u_0^{\overline{\mathcal{H}}}(t) - u_0^0(t)\right\|_\infty = \left\|\frac{\breve{z}_0}{\overline{\mathcal{H}}} - 1\right\|_\infty \le v^2\,\overline{\mathcal{H}}. \tag{24}$$

Further,

$$u_i^{\overline{\mathcal{H}}}(t) = \frac{\mathring{z}_i(w^{\overline{\mathcal{H}}}(t))}{\mathcal{H}(w^{\overline{\mathcal{H}}}(t))} = \frac{z_i^{\overline{\mathcal{H}}}(t)}{\overline{\mathcal{H}}}, \quad i = 1, \dots, m-1.$$

Consequently,

$$\left\|u_i^{\overline{\mathcal{H}}}(t) - u_i^0(t)\right\|_\infty = \left\|\frac{z_i^{\overline{\mathcal{H}}}(t)}{\overline{\mathcal{H}}}\right\|_\infty \le v\,\sqrt{\overline{\mathcal{H}}}, \quad i = 1, \dots, m-1. \tag{25}$$

The assertion of the theorem follows from (23), (24), and (25), and the proof is complete.□

§ 6. A theorem on approximation of a minus-Legendre extremal

Let $\gamma^0 = (w^0(t), u^0(t)|\Delta^0)$ be a minus-Legendre extremal.

Theorem 1. *There is a $\delta > 0$ such that for any closed interval $\Delta \subset \Delta^0||\Delta| = \delta$ there exists a sequence $\{\gamma^s = (w^s(t), u^s(t) \mid \Delta)\}_{s=1,2,\dots}$ of nonsingular extremals such that as $s \to \infty$*

$$\mathcal{H}\big(w^s(t)\big) = \mathcal{H}^s \to 0,$$
$$\|w^s(t) - w^0(t)\|_\infty \le O(\mathcal{H}^s),$$
$$\|u^s(t) - u^0(t)\|_\infty \le O(\mathcal{H}^s).$$

Proof. As in the case of the preceding theorem, we break up the proof into several parts. Some of the results are taken from the preceding section.

Part 1. For the elements of $L(\Delta)$, where $\Delta \subset \Delta^0$ is an arbitrary closed interval, we let $P^1[\Delta, \overline{\mathcal{H}}](\widehat{h}) = \widehat{z}$, where

$$\widehat{z} \in L_\infty^{m-1}(\Delta),$$
$$\ddot{z} = F\big(\overline{\mathcal{H}}, w^0(t) + \theta(t), \xi(t)\big) - D(w^0(t))\,\frac{\xi(t)}{\overline{\mathcal{H}}} + D(w^0(t))\,\frac{z(t)}{\overline{\mathcal{H}}},$$
$$z(t_0) = z(t_1) = 0,\ \Delta = [t_0, t_1].$$

The operator $P^1[\Delta, \overline{\mathcal{H}}]$ is defined and continuous on the set $\{\widehat{h}|\overline{\mathcal{H}}^2 - \xi^2(t) \ge 0|\Delta\}$.

Let $J^1[\Delta, \overline{\mathcal{H}}]\colon L(\Delta) \to L(\Delta)$, where

$$J^1[\Delta, \overline{\mathcal{H}}](\widehat{h}) = (\delta\widehat{w}; \widehat{z}), \quad \widehat{z} = P^1[\Delta, \overline{\mathcal{H}}](\widehat{h}), \quad \delta\widehat{w} = G[\Delta, \overline{\mathcal{H}}](\widehat{z}). \tag{1}$$

The operator $G[\Delta, \overline{\mathcal{H}}]$ was defined in Part 3 of § 5.

Let $\mu > 0, \sigma > 0$. We define $L^1(\Delta, \overline{\mathcal{H}}, \mu, v) \subset L(\Delta)$ by

$$L^1 = \{\widehat{h}|\|\theta\|_\infty \le \mu\overline{\mathcal{H}};\ \|\xi\|_\infty \le v\overline{\mathcal{H}}^2\}.$$

Part 2. For an arbitrary function $\widehat{f} \in L^{m-1}_{\infty}(\Delta)$ we investigate the solution $\widehat{z} = z(t)|\Delta$ of the boundary value problem

$$\ddot{z} = f(t) + D(w^0(t))\frac{z(t)}{\overline{\mathcal{H}}}, \quad z(t_0) = z(t_1) = 0. \tag{2}$$

Let $t = t_0 + \tau\sqrt{\overline{\mathcal{H}}}$. Then $\tau \in [0, T(\overline{\mathcal{H}})]$, where $T(\overline{\mathcal{H}}) = \frac{|\Delta|}{\sqrt{\overline{\mathcal{H}}}}$. We next set $\widetilde{z}(\tau) = z(t(\tau))$ and $V(\tau) = D\big(w^0(t(\tau))\big)$. Then from (2) we get

$$\frac{d^2\widetilde{z}}{d\tau^2} = f(t(\tau))\overline{\mathcal{H}} + V(\tau)\widetilde{z}, \quad \widetilde{z}(0) = \widetilde{z}(T(\overline{\mathcal{H}})) = 0. \tag{3}$$

By definition, $D(w^0(t)) = -D^0(t)|\Delta^0$. For a minus-Legendre extremal the matrix $D^0(t)$ is negative definite on Δ^0, so the matrix $V(\tau)$ is positive definite on $[0, T(\overline{\mathcal{H}})]$. Therefore, it is natural for us to turn to the results in § 4. Strictly speaking, the matrix $V(\tau)$ depends on $\overline{\mathcal{H}}$, and hence to get estimates of the solution of (3) that are uniform with respect to $\overline{\mathcal{H}}$ we should verify the conditions (1) of §4. Let $\overline{\lambda}_0$ be the smallest eigenvalue of $D(w^0(t))$ on the interval Δ^0. Then it is obvious that for any $\overline{\mathcal{H}}$

$$\overline{\lambda}_0\overline{z}^2 \le V(\tau)\overline{z}\overline{z} \quad \forall\, \overline{z},\ \tau \in [0, T(\overline{\mathcal{H}})].$$

Thus, condition (1a) holds.

Let $\overline{c}^0$ be the maximal value of the norm of the matrix $D(w^0(t))$ on Δ^0. Then, obviously,

$$|V(\tau)\overline{z}| \le \overline{c}^0\,|\overline{z}| \quad \forall\, \overline{z},\ \tau.$$

Thus, condition (1b) also holds. We can now use the results in § 4. According to (10) of §4, we have

$$\|\widetilde{z}\|_{\infty,\tau} \le c_1(\overline{\lambda}^0, \overline{c}^0)\|f\|_{\infty,\tau}\,\overline{\mathcal{H}},$$
$$\|\dot{\widetilde{z}}\|_{\infty,\tau} \le c_1(\overline{\lambda}^0, \overline{c}^0)\|f\|_{\infty,\tau}\,\overline{\mathcal{H}}.$$

Going back to the variable t, we get from these estimates that

$$\begin{aligned} \|z\|_\infty &\le c_1(\overline{\lambda}^0, \overline{c}^0)\|f\|_\infty\,\overline{\mathcal{H}}, \\ \|\dot{z}\|_\infty &\le c_1(\overline{\lambda}^0, \overline{c}^0)\|f\|_\infty\,\sqrt{\overline{\mathcal{H}}}. \end{aligned} \tag{4}$$

Here the constant c_1 is independent of Δ, $\widehat{f}$, and $\overline{\mathcal{H}}$.

Part 3. As in § 4, let

$$f[\overline{\mathcal{H}}, \widehat{h}](t) = F\big(\overline{\mathcal{H}}, w^0(t) + \theta(t), \xi(t)\big) - D(w^0(t))\,\frac{\xi(t)}{\overline{\mathcal{H}}}.$$

Using (3) and (4) of §5, we estimate $\big\|f[\overline{\mathcal{H}}, \widehat{h}](t)\big\|_\infty$ on $L^1(\Delta, \overline{\mathcal{H}}, \mu, v)$ for fixed μ, v and small $\overline{\mathcal{H}}$. We have

$$\big\|f[\overline{\mathcal{H}}, \widehat{h}](t)\big\|_\infty \le (\overline{b}^1\mu^2 + \overline{b}^2\mu)\overline{\mathcal{H}} + (\overline{B}^1\mu^2 v + \overline{B}^2\mu v + \overline{B}^3 v^2 + \overline{D}^1\mu v)\overline{\mathcal{H}}^2 + \overline{D}^0 v^3\overline{\mathcal{H}}^3.$$

Let $\overline{b} = \max\{\overline{b}^1, \overline{b}^2\} + 1$. Then for sufficiently small $\overline{\mathcal{H}}$ and for all $\widehat{h} \in L^1(\Delta, \overline{\mathcal{H}}, \mu, v)$ we have the estimate

$$\big\|f[\overline{\mathcal{H}}, \widehat{h}](t)\big\|_\infty \le \overline{b}\,(\mu^2 + \mu)\,\overline{\mathcal{H}}. \tag{5}$$

¿From this and (4) it follows that, for $\widehat{z} \in P^1[\Delta, \overline{\mathcal{H}}](L^1(\Delta, \overline{\mathcal{H}}, \mu, v))$,

$$\begin{aligned} \|z\|_\infty &\le c_1\,(\overline{\lambda}^0, \overline{c}^0)\,\overline{b}\,(\mu^2 + \mu)\overline{\mathcal{H}}^2, \\ \|\dot{z}\|_\infty &\le c_1(\overline{\lambda}^0, \overline{c}^0)\,\overline{b}\,(\mu^2 + \mu)\overline{\mathcal{H}}\,\sqrt{\overline{\mathcal{H}}}. \end{aligned} \tag{6}$$

Part 4. Suppose that $\widehat{z}$ satisfies the conditions (6) and belongs to $C_1^{m-1}(\Delta)$. Let us estimate $\delta\widehat{w} = G[\Delta, \overline{\mathcal{H}}](\widehat{z})$. We write (5) of §5 in the form

$$\delta\dot{w} = \frac{\breve{z}_0}{\overline{\overline{\mathcal{H}}}}\left(a(w^0(t)+\delta w(t)) - a(w^0(t))\right) + \left(\frac{\breve{z}_0}{\overline{\overline{\mathcal{H}}}} - 1\right) a(w^0(t)) + A(w^0(t)+\delta w(t))\frac{z(t)}{\overline{\overline{\mathcal{H}}}}. \tag{7}$$

¿From (6) and (7), by (13) of §5, we get the estimate

$$|\delta\dot{w}| \le \overline{a}^1|\delta w| + \overline{a}^0 q^2 \overline{\mathcal{H}}^2 + \overline{A}^0 q\overline{\mathcal{H}},$$

where $q = c_1 \overline{b}\,(\mu^2+\mu)$. Setting $\overline{a} \ = \overline{a}^1 + \overline{A}^0 q$, we get that for small $\overline{\mathcal{H}}$

$$|\delta\dot{w}| \le \overline{a}(|\delta w| + \overline{\mathcal{H}}). \tag{8}$$

Let us estimate the initial data. We have $\eta_0 = \overline{\mathcal{H}}$; $\eta_i = 0\,(i=1,\dots,m-1)$; $\eta_i \le q\overline{\mathcal{H}}\sqrt{\overline{\mathcal{H}}}$ $(i=m,\dots,2m-2)$. Consequently, $|\delta\dot{w}(t_0)| \le \overline{\mu}\overline{\mathcal{H}}$ for small $\overline{\mathcal{H}}$. Then from (8) it follows that

$$\|\delta w\|_\infty \le \bigl(\overline{\mu}\exp\{\overline{a}\Delta\} + \exp\{\overline{a}\ \Delta\} - 1\bigr)\overline{\mathcal{H}}. \tag{9}$$

We have obtained an estimate in which $\overline{\mu}$ and $\overline{a}$ are independent of $\widehat{h}$ and $\overline{\mathcal{H}}$.

Thus, the operator $J^1[\Delta, \overline{\mathcal{H}}]$ carries the set $L^1(\Delta, \overline{\mathcal{H}}, \mu, v)$ into the set of pairs of functions $\delta\widehat{w}$, $\widehat{z}$ for which the estimates (9) and (4) hold.

Part 5. In this part we prove the following result.

Proposition 1. *There exist $\delta > 0, \mu > 0, v > 0$ such that the operator $J^1[\Delta, \overline{\mathcal{H}}]$ maps the set $L^1(\Delta, \overline{\mathcal{H}}, \mu, v)$ into itself for any $\Delta \subset \Delta^0||\Delta| = \delta$ and for small $\overline{\mathcal{H}}$.*

Proof. Obviously, this property is ensured by the conditions

$$\begin{aligned} c_1\,(\overline{\lambda}^0, \overline{c}^0)\,\overline{b}\,(\mu^2+\mu) &\le v, \\ \overline{\mu}\exp\{\overline{a}\delta\} + \exp\{\overline{a}\delta\} - 1 &\le \mu. \end{aligned} \tag{10}$$

But the left-hand side of (10) does not contain v. Consequently, it suffices to ensure the second inequality. If $\mu > \overline{\mu}$, then this can always be done by choosing δ sufficiently small. This proves the proposition. □

As a corollary of this proposition we get that for small $\overline{\mathcal{H}}$ and any $\Delta \subset \Delta^0||\Delta| = \delta$ the operator $J^1[\Delta, \overline{\mathcal{H}}]$ has a fixed point belonging to $L^1(\Delta, \overline{\mathcal{H}}, \mu, v)$. We choose some closed interval $\Delta \subset \Delta^0||\Delta| = \delta$ and denote the fixed point by $\delta w^{\overline{\mathcal{H}}}(t)$, $z^{\overline{\mathcal{H}}}(t)|\Delta$. Setting $w^{\overline{\mathcal{H}}}(t) = w^0(t) + \delta w^{\overline{\mathcal{H}}}(t)|\Delta$, we get that $w^{\overline{\mathcal{H}}}(t)$, $z^{\overline{\mathcal{H}}}(t)$ is a solution of the extended system. The verification of the conditions of Proposition 2 in § 2 is similar to the proof of the preceding theorem.

The verification of condition 4) looks a little different. According to (7) in §2,

$$\frac{d}{dt}\mathring{z}_i(w^{\overline{\mathcal{H}}}(t))\,\big|_{t=t_0} = \frac{\mathring{z}_0(w^{\overline{\mathcal{H}}}(t_0))}{\mathcal{H}(w^{\overline{\mathcal{H}}}(t_0))}\mathring{z}_i(w^{\overline{\mathcal{H}}}(t_0)) = \dot{z}_i^{\overline{\mathcal{H}}}(t_0).$$

Thus, condition 4) holds.

The rest of the proof is a literal repetition of the proof of Theorem 1 in § 5. This completes the proof of the theorem. □

We direct our attention to the invariance of our theorems with respect to the sub-Riemannian metric. The singular extremals remain singular extremals in any submetric up to a parametrization. The nonsingular extremals do not enjoy this property. They are determined only as a result of a choice of submetric. However, despite the fact that the set of nonsingular extremals changes in dependence on the choice of submetric, each definite extremal will always be a limit point of this set. In general, the quadratic properties of a singular extremal will always be insensitive to the submetric. We go into this circumstance in greater detail in the next chapter.

CHAPTER II. NEIGHBORHOODS OF A GEODESIC

The results of this chapter are closely connected both with the preceding chapter and with the theory of quadratic conditions developed by the author, A. V. Dmitruk, and N. P. Osmolovskiĭ for optimal control problems.

§ 7. Sufficient and necessary extremum conditions for the geodesic generated by a nonsingular extremal

Let $\gamma = \psi(t), x(t), u(t)|\Delta$ be a nonsingular extremal, and denote its trajectory component by $\sigma = x(t), u(t)|\Delta$.

Below we present some consequences of Osmolovskiĭ's results in [3] in connection with geodesics.

The cone K_Δ of critical variations has the following form in this case. Let $\widehat{\overline{x}} = \overline{x}(t)|\Delta$ be an absolutely continuous function on Δ, and let $\widehat{\overline{u}} = \overline{u}(t)|\Delta$ be a measurable function on Δ that is square integrable. Then $(\widehat{\overline{x}}, \widehat{\overline{u}}) \in K_\Delta$ is equivalent to the conditions

$$\dot{\overline{x}} = \sum_{i=0}^{m-1} u_i(t)\mathbf{r}_i'(x(t))\overline{x} + \sum_{i=0}^{m-1} \overline{u}_i \mathbf{r}_i(x(t)), \quad \overline{x}(t_0) = \overline{x}(t_1) = 0, \tag{1}$$

where $\Delta = [t_0, t_1]$. The quadratic form ω is defined in this case as

$$\omega(\gamma; \widehat{\overline{x}}, \widehat{\overline{u}}) = \int \Bigg(-\frac{1}{2} \sum_{i=0}^{m-1} u_i(t)\big(\psi(t), \mathbf{r}_i(x(t))\big)'' \overline{x}\overline{x} - \sum_{i=0}^{m-1} \overline{u}_i \big(\psi(t), \mathbf{r}_i(x(t))\big)' \overline{x} + \frac{1}{2} \mathcal{H}(\psi(t), x(t)) \overline{u}\overline{u} \Bigg)\, dt. \tag{2}$$

Here the prime denotes differentiation with respect to x. According to Osmolovskiĭ, the condition below implies a strong minimum:

There exists a constant $c > 0$ such that

$$\omega(\gamma; \widehat{\overline{x}}, \widehat{\overline{u}}) \geq c \int_{t_0}^{t_1} \overline{u}^2\, dt \quad \forall\, (\widehat{\overline{x}}, \widehat{\overline{u}}) \in K_\Delta. \tag{3}$$

We present a necessary condition for the case when $\Psi_0(\sigma)$ consists of a single element. It is a necessary condition for a weak minimum and is at the same time very close to the sufficient condition (3). It has the form

$$\omega(\gamma; \widehat{\overline{x}}, \widehat{\overline{u}}) \geq 0 \quad \forall\, \big(\widehat{\overline{x}}, \widehat{\overline{u}}\big) \in K_\Delta. \tag{4}$$

Take a closed interval $\Delta' \subset \Delta$, and let $\gamma|_{\Delta'}$ be the restriction of γ to Δ'. Denote by $g_{\Delta'}$ the operator imbedding the space of functions defined on Δ' in the space of functions defined on Δ by setting a function equal to zero outside Δ'. Then it follows from (1) and (2) that

$$\begin{gathered} g_{\Delta'} K_{\Delta'} \subset K_\Delta, \\ \omega(\gamma|_{\Delta'}; \widehat{\overline{x}}, \widehat{\overline{u}}) = \omega\big(\gamma; g_{\Delta'}\widehat{\overline{x}}, g_{\Delta'}\widehat{\overline{u}}\big). \end{gathered} \tag{5}$$

For a nonsingular extremal, $\mathcal{H}(\psi(t), x(t)) = \text{const} > 0$. This implies that $\gamma|_{\Delta'}$ satisfies (3) for all sufficiently small Δ'. Thus, each geodesic generated by a nonsingular extremal is locally optimal; that is, it realizes a shortest path on each sufficiently short closed interval of it.

Returning to the necessary condition, we prove the following result.

Proposition 1. *Let $\sigma = x(t)|\Delta$ be a geodesic. Then for $\Psi_0(\sigma)$ to consists of a single element, it is necessary and sufficient that σ be a nonsingular geodesic.*

Proof. For simplicity we assume that the normalization in the definition of $\Psi_0(\sigma)$ is symmetric. In this case if $\widehat{\psi}$ is a singular element of $\Psi_0(\sigma)$, then $-\widehat{\psi}$ is also an element of $\Psi_0(\sigma)$. Consequently, if $\Psi_0(\sigma)$ consists of a single element, then the geodesic σ is nonsingular. Let σ be a nonsingular geodesic. We show that $\Psi_0(\sigma)$ consists of a single element. Assume the contrary. Then there exist $\widehat{\psi}_1, \widehat{\psi}_2 \in \Psi_0(\sigma)$ with $\widehat{\psi}_1 \neq \widehat{\psi}_2$. Let $\psi(t) = \dfrac{\psi_1(t)}{\mathcal{H}^1} - \dfrac{\psi_2(t)}{\mathcal{H}^2}$, where $\mathcal{H}^1 = \mathcal{H}(\psi_1(t), x(t))$ and $\mathcal{H}^2 = \mathcal{H}(\psi_2(t), x(t))$. Since σ is a nonsingular geodesic, it follows that $\mathcal{H}^1, \mathcal{H}^2 > 0$. Obviously, $\psi(t)$ satisfies the conjugate equation, because the latter is linear with respect to ψ, and $\mathcal{H}^1, \mathcal{H}^2 = \text{const}$. Let us verify the maximum conditions. According to these conditions,

$$u_i(t) = \frac{\psi_1(t)\mathbf{r}_i(x(t))}{\mathcal{H}^1} = \frac{\psi_2(t)\mathbf{r}_i(x(t))}{\mathcal{H}^2}.$$

¿From this, $\psi(t)\mathbf{r}_i(x(t)) = 0$, $i = 0, \dots, m-1$, $\forall\, t \in \Delta$.

Consequently, the maximum conditions hold for $\widehat{\psi}$, and $\mathcal{H}(\psi(t), x(t)) = 0$.

Thus, $\psi(t)$ is the conjugate component in the maximum principle, and is singular. This implies that $\Psi_0(\sigma)$ contains a singular element, and hence the geodesic is not nonsingular. We have come to a contradiction and have thereby proved that $\Psi_0(\sigma)$ consists of a single element if σ is a nonsingular geodesic. Proposition 1 is proved. □

In this section we have considered quadratic conditions for a nonsingular extremal. In the next section we present the theory of conjugate points for the form ω.

§ 8. Theory of conjugate points for nonsingular geodesics

Let $(\gamma = \psi(t), x(t), u(t)|\Delta)$ be a nonsingular extremal, $\Delta = [t_0, t_1]$, and let $\Delta' = [t_0, t']$. We define

$$I(t') = \inf \omega(\gamma|_{\Delta'}; \widehat{\overline{x}}, \widehat{\overline{u}}),$$

where the infimum is taken over all $\widehat{\overline{x}}$, $\widehat{\overline{u}} \in K_{\Delta'}$ with $\int_{t_0}^{t'} \overline{u}^2\, dt = 1$. According to (5) of §7, $I(t')$ is a nonincreasing function of t'. It is obvious that $I(t') \to \frac{1}{2}\mathcal{H}$ as $t' \to t_0$. Let

$$\text{(1)} \qquad \Gamma(t') = \left\{ \overline{x} \,\middle|\, \overline{x} = \overline{x}(t');\ \frac{d\overline{x}}{dt} = \sum_{i=0}^{m-1} u_i(t)\mathbf{r}_i'(x(t))\overline{x}(t) - \sum_{i=0}^{m-1} \overline{u}_i \mathbf{r}_i(x(t));\ \overline{x}(t_0) = 0 \right\}.$$

Then $d(\Gamma(t'))$ is clearly a left-continuous nondecreasing function. This implies left continuity also for $I(t')$

Let us prove this. We fix $t' = t_*$, let $\overline{x}_*, \overline{u}_*|[t_0, t_*] \in K_{[t_0,t_*]}$, and choose a basis $\rho_1, \dots, \rho_{k_*}$ in $\Gamma(t_*)$, where $k_* = d(\Gamma(t_*))$. Suppose that $\overline{x}^k(t), \overline{u}^k(t)|[t_0, t_*]$, $k = 1, \dots, k_*$, satisfy the equation (1) and the conditions $\overline{x}^k(t_*) = \rho_k$ at the right-hand endpoint. Then $\overline{x}^k(t') = \rho_k'$, $k = 1, \dots, k_*$, is a basis in $\Gamma(t')$ for $t' < t_*$ and sufficiently close to t_*. This follows from the continuity of $\overline{x}^k(t)$ and the monotonicity of $d(\Gamma(t'))$. It is clear that

$\rho'_k \to \rho_k$ as $t' \to t_* - 0$. It follows from this argument that for $t' \to t_* - 0$ there exist $\varepsilon_k(t')$ $(k = 1, \dots, k_*)$ such that

$$\left(\overline{x}_* + \sum_{k=1}^{k_*} \varepsilon(t')\overline{x}^k(t);\ \overline{u}_* + \sum_{k=1}^{k_*} \varepsilon(t')\overline{u}^k(t)\right) \in K_{\Delta'},$$

and from the continuity of the basis with respect to t' at the point t_* it follows that $\varepsilon_k(t') \to 0$ $(k = 1, \dots, k_*)$ as $t' \to t_* - 0$. From this it is easy to get our assertion about the left continuity of $I(t')$.

The assertion proved is in no way based on the strengthened Legendre condition $\mathcal{H} > 0$. The important proposition below makes essential use of it.

Proposition 1. *Let $I(t_*) \le 0$. Then*:

a) *There exists an element of $K_{[t_0,t_*]}$ on which the value $I(t_*)$ is realized.*

b) *The point t_* is a point of continuity of the function $I(t')$.*

Proof. We shall define some sequences. Suppose that $t^s \to t_* + 0$, $\varepsilon^s > 0$, $\varepsilon^s \to 0$ as $s \to 0$, and $\{\widehat{\overline{x}}^s, \widehat{\overline{u}}^s\}_{s=1,2,\dots} \in K_{\Delta^s}$, where $\Delta^s = [t_0, t^s]$ and

$$\int_{t_0}^{t^s} |\overline{u}^s(t)|^2\,dt = 1, \qquad \omega(\gamma|_{\Delta^s}; \widehat{\overline{x}}^s, \widehat{\overline{u}}^s) \le I(t^s) + \varepsilon^s. \tag{2}$$

It can be assumed without loss of generality that the sequence $\{\widehat{\overline{u}}^s\}$ converges weakly in L_2 to some function $\widehat{\overline{u}}^0 = \overline{u}^0(t)|\Delta_*$, where $\Delta_* = [t_0, t_*]$. Then $\{\widehat{\overline{x}}^s\}$ converges uniformly to $\widehat{\overline{x}}^0 = \overline{x}^0(t)|_{\Delta_*}$ and $(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0) \in K_{\Delta_*}$. But the dependence of $\omega(\gamma|_{\Delta^s}; \widehat{\overline{x}}^s, \widehat{\overline{u}}^s) - \frac{1}{2}\int \mathcal{H}\overline{u}^2\,dt$ on $\widehat{\overline{u}}^s$ is weakly continuous in L_2. Consequently,

$$\lim_s \omega(\gamma|_{\Delta^s}; \widehat{\overline{x}}^s, \widehat{\overline{u}}^s) = \omega(\gamma|_{\Delta_*}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0) - \frac{1}{2}\int_{t_0}^{t_*} \mathcal{H}(\overline{u}^0(t))^2\,dt + \frac{1}{2}\mathcal{H}.$$

On the other hand, it is clear from (2) that

$$I(t_*) \ge \omega(\gamma|_{\Delta_*}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0) - \frac{1}{2}\int_{t_0}^{t_*} (\overline{u}^0(t))^2\,dt + \mathcal{H}. \tag{3}$$

This implies that if $-\frac{1}{2}\int_{t_0}^{t_*}(\overline{u}^0(t))^2\,dt + \mathcal{H} > 0$, then $\omega(\gamma|_{\Delta_*}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0) < I(t_*)$ and $\int_{t_0}^{t_*}(\overline{u}^0(t))^2\,dt < 1$. But then $I(t_*) < \omega(\gamma|_{\Delta_*}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0)$. We have arrived at a contradiction. Consequently, $\int_{t_0}^{t_*}(\overline{u}^0(t))^2\,dt = 1$, and hence the sequence $\{\widehat{\overline{u}}^s\}$ converges to $\overline{u}^0(t)|\Delta_*$ in the norm of L_2. In this case

$$\lim_s \omega(\gamma|_{\Delta^s}; \widehat{\overline{x}}^s, \widehat{\overline{u}}^s) = \omega(\gamma|_{\Delta_*}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0), \qquad \int_{t_0}^{t_*} (\overline{u}^0(t))^2\,dt = 1.$$

By the monotonicity of $I(t')$ we thus have that $I(t_*) = \omega(\gamma|_{\Delta_*}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0)$. This completes the proof of part a). Part b) follows from the proven property of left continuity of the function $I(t')$ and from the right continuity proved in part a). Proposition 1 is proved.□

Proposition 1 is the basis for the method of using the Euler–Jacobi equation. The property in the proposition has come to be called “passing through zero”.

We can now see the function $I(t')$ in greater detail. Its zero set is a closed interval $[t_0', t_1']$, which is called the conjugate interval or table. If the conjugate interval lies outside $\Delta = [t_0, t_1]$, then condition (3) of §7 holds for the form. If the conjugate interval lies interior to Δ, then condition (4) of §7 does not hold for the form. In the remaining cases (4) holds, and there exist elements of K_Δ on which the form ω is zero.

It is of interest to find conditions under which the conjugate interval degenerates into a point. With this in mind we prove the following proposition.

We call a geodesic absolutely nonsingular if every restriction of it is also a nonsingular geodesic.

Proposition 2. *If a geodesic is absolutely nonsingular, then the conjugate interval always degenerates into a point.*

Proof. Let the conjugate interval be $\Delta_c = [t_0', t_1']$. According to Proposition 1, there exists on $[t_0, t_0']$ an element $\big(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0\big) \in K_{[t_0, t_0']}$ such that

$$\omega(\gamma|_{[t_0,t_0']}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0) = 0, \quad \big(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0\big) \neq 0. \tag{4}$$

Consider $\overline{x}_*(t), \overline{u}_*(t)|_{[t_0,t_1']}$, where $\overline{x}_*(t), \overline{u}_*(t)$ coincide with $\overline{x}(t)^0, \overline{u}(t)^0$ on $[t_0, t_0']$ and are equal to zero outside this interval. Obviously,

$$\big(\widehat{\overline{x}}_*(t), \widehat{\overline{u}}_*(t)\big) \in K_{[t_0,t_1']}, \quad \omega(\gamma|_{[t_0,t_1']}; \widehat{\overline{x}}_*, \widehat{\overline{u}}_*) = \omega(\gamma|_{[t_0,t_0']}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0) = 0.$$

By the definition of the conjugate interval, $\omega(\gamma|_{[t_0,t_1']}; \widehat{\overline{x}}, \widehat{\overline{u}}) \geq 0$ on $K_{[t_0,t_1']}$. Thus, $\widehat{\overline{x}}_*, \widehat{\overline{u}}_*$ form a solution of the variational problem

$$\omega(\gamma|_{[t_0,t_1']}; \widehat{\overline{x}}, \widehat{\overline{u}}) \to \min, \quad \big(\widehat{\overline{x}}, \widehat{\overline{u}}\big) \in K_{[t_0,t_1']}. \tag{5}$$

The pair $\big(\widehat{\overline{x}}_*, \widehat{\overline{u}}_*\big)$ must satisfy the maximum principle of the problem (5). The Pontryagin function is denoted by $\overline{H}$. Then

$$\overline{H} = -\overline{\alpha}\bigg(-\frac{1}{2}\sum_{i=0}^{m-1} u_i(t)\psi \mathbf{r}_i''(x(t))\overline{x}\overline{x} - \sum_{i=0}^{m-1} \overline{u}_i \psi \mathbf{r}_i'(x(t))\overline{x} + \frac{1}{2}\int \mathcal{H}\overline{u}^2\,dt\bigg) + \overline{\psi}(t)\bigg(\sum_{i=0}^{m-1} u_i \mathbf{r}_i'(x(t))\overline{x} + \sum_{i=0}^{m-1} \overline{u}_i \mathbf{r}_i(x(t))\bigg). \tag{6}$$

Two cases are possible. First, $\overline{\alpha} \neq 0$; second, $\overline{\alpha} = 0$. In the second case the conjugate equation, the maximum condition, and the nontriviality condition have the following forms:

$$\begin{gathered} -\dot{\overline{\psi}} = \sum_{i=0}^{m-1} u_i(t)\big(\overline{\psi}\mathbf{r}_i(x(t))\big)', \\ \overline{\psi}\mathbf{r}_i(x(t)) = 0, \quad i = 0, \dots, m-1, \\ \overline{\psi}(t) \neq 0 \quad \forall\, t \in [t_0, t_1']. \end{gathered} \tag{7}$$

The conditions (7) show that $\overline{\psi}(t)$ is the singular conjugate component of the maximum principle for the geodesic $x(t), u(t)|[t_0, t_1']$.

Let us consider the first case. In differential form, the maximum condition is

$$\overline{\alpha}\psi(t)\mathbf{r}_i'(x(t))\overline{x}_*(t) - \overline{\alpha}\mathcal{H}\overline{u}_*(t) + \overline{\psi}(t)\mathbf{r}_i(x(t)) = 0, \quad i = 0, \dots, m-1, \quad t \in [t_0, t_1'].$$

This implies that we can eliminate $\overline{u}_*(t)$ from the conjugate equation, replacing it by a linear form depending on $\overline{\psi}(t), \overline{x}_*(t)$. Then the conjugate system together with the linearized equations takes the form of a system of equations that is linear with respect to

$\overline{\psi}(t), \overline{x}_*(t)$. This implies that $\overline{\psi}(t_0') \neq 0$, because $\overline{x}_*(t_0') = 0$ by our choice, and therefore in the opposite case $(\overline{\psi}(t), \overline{x}_*(t)) \equiv (0,0)|[t_0, t_1']$, which contradicts the nontriviality of the pair $(\overline{x}_*(t), \overline{u}_*(t))$. On the conjugate interval the conjugate equation has the form

$$-\dot{\overline{\psi}} = \sum_{i=0}^{m-1} u_i(t)\big(\overline{\psi}\mathbf{r}_i(x(t))\big)',$$

and the maximum condition is

$$\overline{\psi}(t)\mathbf{r}_i(x(t)) = 0, \quad i = 0, \dots, m-1.$$

¿From this and the condition $\overline{\psi}(t_0') \neq 0$ it follows that $\overline{\psi}(t)$ is a singular conjugate component of the maximum principle for the geodesic $x(t), u(t)|_{[t_0, t_1']}$.

Thus, if there is a conjugate interval different from a point, then the geodesic is not absolutely nonsingular. This proves the proposition. □

§ 9. Neighborhoods of a nonsingular geodesic

Let

$$(\sigma^0 = x^0(t), u^0(t)|\Delta)$$

be a geodesic, and let

$$\{\sigma^s = x^s(t), u^s(t)|\Delta\}_{s=1,2,\dots}$$

be a sequence of geodesics converging to σ^0. By this we mean that

$$\|x^s(t) - x^0(t)\|_\infty \to 0 \quad \text{as } s \to \infty.$$

Let us investigate this situation. First of all, we note that

$$\int_{t_0}^{t_1} |u^s(t) - u^0(t)|\, dt \to 0. \tag{1}$$

Indeed, it follows from the uniform convergence of the phase components that the controls converge weakly in L_∞:

$$u^s(t) \xrightarrow{w.\, L_\infty} u^0(t)$$

as $s \to \infty$. (More precisely, this is weak$-*$ convergence in L_∞ with respect to L_1.) Consequently,

$$u^0(t)u^s(t) \xrightarrow{w.\, L_\infty} {u^0}^2(t). \tag{2}$$

But ${u^0}^2(t) = {u^s}^2(t) = 1$, and hence $u^0(t)u^s(t) \leq 1|\Delta$. Then it follows from (2) that

$$u^0(t)u^s(t) \xrightarrow{L_1} 1.$$

This gives us (1) immediately.

We next consider the sequence $\{\Psi_0(\sigma^s)\}_{s=1,2,\dots}$. Denote by $\underset{s}{\overline{\ell t}}\, \Psi_0(\sigma^s)$ the set of functions $\widehat{\psi}$ for which there is a sequence $\{\widehat{\psi}^s\}_s$, $\widehat{\psi}^s \in \Psi_0(\sigma^s)$ $(s = 1, 2, \dots)$, converging in the norm of L_∞ to $\widehat{\psi}$.

Proposition 1.

$$\Psi_0(\sigma^0) \supseteq \overline{\ell t}_s\, \Psi_0(\sigma^s).$$

Proof. Let $\widehat{\psi}^0 \in \overline{\ell t}_s\, \Psi_0(\sigma^s)$ and suppose that $\widehat{\psi}^s \xrightarrow{L_\infty} \widehat{\psi}^0$, $\widehat{\psi}^s \in \Psi_0(\sigma^s)$. According to the conjugate equation,

$$-\psi^s(t) + \psi^s(t_0) = \int_{t_0}^{t} H'_x(\psi^s(t), x^s(t), u^s(t))\, dt.$$

The left-hand side tends to $(-\widehat{\psi}^0(t) + \widehat{\psi}^0(t_0))$. Since H'_x is a continuous function of its arguments and $\{\widehat{u}^s\}$ is a bounded sequence, it follows from (1) that the integrand tends to $H'_x(\psi^0(t), x^0(t), u^0(t))$ in the norm of L_1. Therefore $\widehat{\psi}^0$ satisfies the conjugate equation of the geodesic σ^0. Further, since $\|\psi^0(t) - \psi^s(t)\|_\infty \to 0$, $\|x^0(t) - x^s(t)\|_\infty \to 0$, and $u(t)$ belongs to the compact subset $|u| = 1$ of R^m, it follows that $\mathcal{H}(\psi^s(t), x^s(t)) \to \mathcal{H}(\psi^0(t), x^0(t))$, uniformly with respect to t. On the other hand,

$$H(\psi^s(t), x^s(t), u^s(t)) \xrightarrow{L_1} H(\psi^0(t), x^0(t), u^0(t)).$$

By the maximum condition for $\widehat{\psi}^s$, σ^s, this implies that

$$H(\psi^0(t), x^0(t), u^0(t)) = \mathcal{H}(\psi^0(t), x^0(t))|\Delta.$$

Thus, $\widehat{\psi}^0$ also satisfies the maximum condition for the geodesic σ^0. Finally, it is clear that the normalization condition holds for $\widehat{\psi}^0$, and hence $\widehat{\psi}^0 \in \Psi_0(\sigma^0)$. Proposition 1 is proved. □

We remark that any sequence of elements of the sets $\Psi_0(\sigma^s)$, $s = 1, 2, \dots$, is compact in the sense of uniform convergence. ¿From this it follows that if some subsequence of the sequence $\{\sigma^s\}_s$ consists of geodesics either of mixed type or of singular type, then σ^0 is also a geodesic of either mixed or singular type.

This has the following consequence.

Proposition 2. *Let $\sigma^0 = x^0(t), u^0(t)|\Delta$ be a nonsingular geodesic. Then it has a neighborhood containing only nonsingular geodesics.*

Proof. This follows from Proposition 1, the remark after it about singular elements, and Proposition 1 in § 7. □

We can now refine the formula (1). If σ^0 is a nonsingular geodesic, then the σ^s are also nonsingular geodesics beginning with some index. The extremals corresponding to them are unique and satisfy the Hamiltonian system. This gives us

$$\|u^0(t) - u^s(t)\|_\infty \xrightarrow{s\to\infty} 0.$$

Finally, let us consider the case when σ^0 and σ^s $(s = 1, 2, \dots)$ are absolutely nonsingular geodesics. Let t^0_* and t^s_* $(s = 1, 2, \dots)$ be the points conjugate to t_0 for these geodesics, respectively. We have the following result.

Theorem 1. $t^s_* \to t^0_*$.

Proof. Without loss of generality it can be assumed that $t^\infty_* = \lim_s t^s_*$ exists. Assume first that $t^\infty_* < t^0_*$, and let $t^\infty_* < \overline{t} < t^0_*$. Since $\overline{t}$ is beyond the limits of the conjugate points of the sequence $\{\sigma^s\}$, there is a sequence $\overline{x}^s(t), \overline{u}^s(t)|[t_0, \overline{t}]$ such that:

a) $(\overline{x}^s(t), \overline{u}^s(t)) \in K^s_{[t_0,\overline{t}]}$, where K^s is the cone of critical variations of the geodesic σ^s;

b) $\int\limits_{t_0}^{\overline{t}} (\overline{u}^s(t))^2\, dt = 1$;

c) $\omega^s(\widehat{\overline{x}}^s, \widehat{\overline{u}}^s) < 0$, where $\omega^s = \omega(\gamma^s, \cdot)$, γ^s is the extremal (unique!) corresponding to the nonsingular geodesic σ^s.

Without any loss of generality it will be assumed that the sequence $\{\widehat{\overline{u}}^s\}$ converges weakly to $\widehat{\overline{u}}^0 = \overline{u}^0(t)|[t_0, \overline{t}]$ in L_2. Then the sequence $\{\widehat{\overline{x}}^s\}$ converges uniformly to $\widehat{\overline{x}}^0$, where $(\overline{x}^0(t), \overline{u}^0(t)) \in K^0_{[t_0,\overline{t}]}$. It is not hard to see that

$$\lim_s \omega^s(\widehat{\overline{x}}^s, \widehat{\overline{u}}^s) = \int_{t_0}^{\overline{t}} \Big(-\frac{1}{2} \sum_{i=0}^{m-1} u_i(t) \big(\psi^0(t), \mathbf{r}_i(x^0(t))\big)'' \overline{x}^0 \overline{x}^0 - \sum_{i=0}^{m-1} \overline{u}_i^0 \big(\psi^0(t), \mathbf{r}_i(x^0(t))\big)' \overline{x}^0 \Big)\, dt + \frac{1}{2}\mathcal{H}^0.$$

The first part of this equality is not positive, and at the same time $\mathcal{H}^0 > 0$. Consequently, $(\widehat{\overline{x}}^0, \widehat{\overline{u}^0}) \neq 0$. But $\int_{t_0}^{\overline{t}} (\overline{u}^0(t))^2\, dt \le 1$, and hence $w^0(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0) \le 0$. However, this is impossible if $\overline{t} < t^0_*$. But then $t^\infty_* \ge t^0_*$. Assume that $t^\infty_* > t^0_*$, and let $t^0_* < \overline{t} < t^\infty_*$. Since $\overline{t}$ lies beyond the conjugate point of the geodesic σ^0, it follows that there is an $(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0) \neq 0$ with $(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0) \in K^0_{[t_0,\overline{t}]}$ such that $w^0(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0) < 0$.

We note that $\Gamma^s(\overline{t}) = R^n$, $s = 0, 1, \dots$, since the geodesics are absolutely nonsingular. Here $\Gamma^s(\overline{t})$ is the subspace of values of $\overline{x}(\overline{t})$ that was defined in the preceding section for the geodesic σ^s on the interval $[t_0, \overline{t}\,]$.

Let $\overline{\xi}^{i,0}$, $\overline{\eta}^{i,0}(t)|[t_0, \overline{t}\,]$, $i = 1, \dots, m$, satisfy on $[t_0, \overline{t}\,]$ the equation

$$\dot{\overline{\xi}} = \sum_0^{m-1} u_i^0(t) r_i'(x^0(t)) \overline{\xi} + \sum_0^{m-1} \overline{\eta}_i r_i(x^0(t)) \tag{3}$$

and the initial condition $\overline{\xi}(t_0) = 0$, with $\overline{\xi}^{i,0}(\overline{t})$ forming a basis in R^n.

Let $\overline{\xi}^{i,s}(t)$, $s = 1, 2, \dots$, be solutions of the equation

$$\dot{\overline{\xi}} = \sum_0^{m-1} u_i^s(t) r_i'(x^s(t)) \overline{\xi} + \sum_0^{m-1} \overline{\eta}^{i,0} r_i(x^s(t)) \tag{4}$$

with initial condition $\overline{\xi}(t_0) = 0$. Since $\widehat{x}^s$, $\widehat{u}^s$, $s = 1, 2, \dots$, converges uniformly on $[t_0, \overline{t}\,]$ to $\widehat{x}^0$, $\widehat{u}^0$, it follows that $\overline{\xi}^{i,s}(\overline{t})$ converges to $\overline{\xi}^{i,0}(\overline{t})$ as $s \to \infty$. Thus, $\overline{\xi}^{i,s}(\overline{t})$ is also a basis in R^n, and it is close to $\overline{\xi}^{i,0}(\overline{t})$, $i = 1, \dots, n$, for large values of s.

Let $\overline{x}^{s,0}(t)|[t_0, \overline{t}\,]$ be a solution of the equation (4) for $s = 1, 2, \dots$, with $\overline{u}^0(t)$ substituted for $\overline{\eta}$. Then obviously $\|\overline{x}^{s,0}(t) - \overline{x}^0(t)\|_\infty \to 0$. We determine $\varepsilon^{i,s}$ from the conditions

$$\overline{x}^{s,0}(\overline{t}) + \sum_{i=1}^n \varepsilon^{i,s} \overline{\xi}^{i,s}(\overline{t}) = 0.$$

Then, since $\overline{x}^{s,0}(\overline{t}) \to 0$ and $\overline{\xi}^{i,s}(\overline{t}) \to \overline{\xi}^{i,0}(\overline{t})$, it follows that $\varepsilon^{i,s} \to 0$, $i = 1, \dots, n$.

Let

$$\overline{x}^s(t) = \overline{x}^{s,0}(t) + \sum_{i=1}^n \varepsilon^{i,s} \overline{\xi}^{i,s}(t),$$

$$\overline{u}^s(t) = \overline{u}^0(t) + \sum_{i=1}^n \varepsilon^{i,s} \overline{\eta}^{i,0}(t).$$

Then $\overline{x}^s(t)$, $\overline{u}^s(t)|[t_0, \overline{t}\,] \in K_{[t_0,\overline{t}\,]}$. On the other hand, $\widehat{\overline{x}}^s, \widehat{\overline{u}}^s$ converge uniformly to $\widehat{\overline{x}}^0, \widehat{\overline{u}}^0$. Consequently, $w^s\big(\widehat{\overline{x}}^s, \widehat{\overline{u}}^0\big) \to w^0\big(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0\big) < 0$. However, this inequality contradicts the condition $\overline{t} < t_*^\infty$. Thus, it remains only to assume that $t_*^0 = t_*^\infty$, and the theorem is proved. □

Thus, nonsingular geodesics or absolutely nonsingular geodesics can be neighbors only with similar geodesics.

Quite a different picture presents itself when we consider singular geodesics or geodesics of mixed type.

§ 10. Neighborhoods of a geodesic generated by a definite extremal

It is known [1] that the definiteness of an extremal can be established in an associated basis, but does not depend on the choice of the latter. In this section we assume that the geodesic σ^0 is written in an associated orthonormal basis.

Accordingly, $\sigma^0 = x^0(t)$, $u^0(t)|\Delta$ is a geodesic generated by a definite extremal. This means that $\Psi_0(\sigma^0)$ contains two elements $\widehat{\psi}_+^0$ and $\widehat{\psi}_-^0$ such that $\widehat{\psi}_+^0 = -\widehat{\psi}_-^0$, $\gamma^+ = (\psi_+^0(t), x^0(t), u^0(t)|\Delta)$ is a plus-Legendre extremal, and $\gamma^- = (\psi_-^0(t), x^0(t), u^0(t)|\Delta)$ is a minus-Legendre extremal. In other respects σ^0 can be either a singular extremal or of mixed type.

We use the extremals γ^+ or γ^- to formulate a sufficient condition, following from results of Dmitruk [2], for a strong minimum for the geodesic σ^0. Formally it is very similar to condition (3) of §7. It is represented in an associated basis. We should recall that the extremals γ^+ and γ^- belong to the zeroth level of the Hamiltonian $\mathcal{H}$.

In this case the cone $K_{oc}^0(\Delta)$ of critical variations is somewhat broader than the cone defined in § 7. Moreover, the difference concerns only a local condition. In § 7 it had the form $\overline{u}_0(t) = 0|\Delta$ in the case of an associated basis. It now has the form $\overline{u}_0(t) = \mathrm{const}|\Delta$. In § 7 the comparison function on K_Δ would have the form $\int_{t_0}^{t_1} \sum_{i=1}^{m-1} \overline{u}_i{}^2(t)\,dt$. Now it is considerably easier:

$$\sum_{i=1}^{m-1} \int_{t_0}^{t_1} \overline{y}_i{}^2(t)\,dt,$$

where $\dot{\overline{y}}_i = \overline{u}_i$ and $\overline{y}_i(t_0) = 0$. Then a sufficient condition for a strong minimum can be written in the form

$$\omega\big(\gamma^-, \widehat{\overline{x}}, \widehat{\overline{u}}\big) \geq c \int_{t_0}^{t_1} \sum_{i=1}^{m-1} \overline{y}_i{}^2(t)\,dt | K_{oc}^0(\Delta), \tag{1}$$

where c is a positive constant.

We remark that $\omega(\gamma^-, \cdot)$ differs from $\omega(\gamma^+, \cdot)$ only in sign. In contrast to condition (3) of §7, the condition (1) is not transmitted to the nearest geodesics. We shall see this later. Sequences of nonsingular extremals approximating definite extremals according to the theorems of Chapter I have different characteristics.

First of all we consider approximation of a minus-Legendre extremal by nonsingular extremals. Let $\gamma^0 = (\psi^0(t), x^0(t), u^0(t)|\Delta^0)$ be a minus-Legendre extremal, let $\Delta^0 = [t_0^0, t_1^0]$, and let $\{\gamma^s = (\psi^s(t), x^s(t), u^s(t)|\Delta^0)\}$ be a sequence of nonsingular extremals approximating γ^0 in the sense of Theorem 1 in § 6. This means the following: as $s \to \infty$,

$$
\begin{aligned}
&\mathcal{H}^s = \mathcal{H}(\psi^s(t), x^s(t)) \to +0|\Delta^0, \\
&\|\psi^s(t) - \psi^0(t)\|_\infty \le O(\mathcal{H}^s), \\
&\|x^s(t) - x^0(t)\|_\infty \le O(\mathcal{H}^s), \\
&\|u^s(t) - u^0(t)\|_\infty \le O(\mathcal{H}^s).
\end{aligned} \tag{2}
$$

It is this convergence that we established in Theorem 1 of § 6. True, the approximation (2) was established not on the whole interval Δ^0, but on any closed subinterval of it with given length. However, this is not so important, because any part of a definite extremal is a definite extremal. It is important only that convergence of the type (2) always holds.

Theorem 1. *Let $\gamma^0 = (\psi^0(t), x^0(t), u^0(t)|\Delta^0)$ be a minus-Legendre extremal, and let $\{\gamma^s = (\psi^s(t), x^s(t), u^s(t)|\Delta^0)\}_{s=1,2,\dots}$ be a sequence of nonsingular extremals approximating γ^0 in the sense of* (2). *Then there exists a $\delta > 0$ such that for all sufficiently large values of s the following conditions hold*: *for any $\Delta \subset \Delta^0$ with $|\Delta| \le \delta$ the extremal $\gamma^s|_\Delta$ satisfies the sufficient condition* (3) *of* §7.

Proof. Assume that the theorem is false. Then there exist a subsequence $\{s'\} \to +\infty$ and a sequence $\{\Delta_{s'}\}$, where $\Delta_{s'} \subset \Delta^0$ and $|\Delta_{s'}| \to 0$, such that (3) of §7 does not hold for the extremal $\gamma^{s'}|_{\Delta^{s'}}$. It can be assumed without loss of generality that the sequence of closed intervals $\{\Delta_{s'}\}$ accumulates at a point t_* as $s' \to \infty$. According to Proposition 1 of §8, the failure of condition (3) of §7 for the nonsingular extremal $\gamma^{s'}|_{\Delta^{s'}}$ means that there exists a pair $\big(\widehat{\overline{x}}^{s'}(t), \widehat{\overline{u}}^{s'}(t)\big) \in K^{s'}_{\Delta_{s'}}$ such that

$$
\int\limits_{\Delta_{s'}} \left|\overline{u}^{s'}(t)\right|^2 dt = 1, \quad \omega(\gamma^{s'}|_{\Delta_{s'}}; \widehat{\overline{x}}^{s'}, \widehat{\overline{u}}^{s'}) \le 0. \tag{3}
$$

We consider a sequence of pairs $\left\{x^{s'}, u^{s'}\right\}_{s'=1,2,\dots}$ satisfying (3). Let $\gamma^1 = (-\psi^0(t), x^0(t), u^0(t)|\Delta^0)$. Obviously, γ^1 is a plus-Legendre extremal. It follows from results of Dmitruk in [4] that condition (1) holds for γ^1 on any sufficiently small closed interval.

Suppose that $\Delta_* \subset \Delta^0$ is a closed interval on which (1) holds, with the point t_* interior to Δ_* if t_* lies interior to Δ^0 and an endpoint of Δ_* if it is an endpoint of Δ^0. Then it can be assumed without loss of generality that $\Delta_{s'} \subset \Delta_* \ \forall s'$. We agree to regard the functions $\widehat{\overline{x}}^{s'}$ and $\widehat{\overline{u}}^{s'}$ as being extended to $\Delta_* \setminus \Delta_{s'}$ by zero. Thus, $\big(\widehat{\overline{x}}^{s'}(t), \widehat{\overline{u}}^{s'}(t)\big) \in K^{s'}_{\Delta_*} \ \forall s'$, and $\Delta_{s'}$ is the support of the pair. This concludes our necessary technical preparation for the proof, and we now begin it.

We consider an equation, more precisely a sequence of equations, on the interval Δ_*:

$$
\dot{\overline{x}} = \sum_{i=0}^{m-1} \overline{u}_i^0(t)\mathbf{r}_i'(x^0(t))\overline{x} + \sum_{i=0}^{m-1} \overline{u}_i^{s'}(t)\mathbf{r}_i(x^0(t)), \quad \overline{x}(t_{\text{left}}) = 0, \tag{4}
$$

where t_{left} is the left-hand endpoint of the interval Δ_*. Let us estimate $|\overline{x}(t_{\text{right}})|$ on the solutions of (4), where t_{right} is the right-hand endpoint of Δ_*. As $s' \to \infty$ we have

$$
\left\|\overline{x}^{s'}(t)\right\|_\infty \le c_0 \left\| \int\limits_{t_{\text{left}}}^{t} \overline{u}^{s'}(t)\, dt \right\|_\infty \le c_0 \sqrt{|\Delta_{s'}|}, \tag{5}
$$

where c_0 does not depend on s'. Subtracting from $\overline{x}(t)$ the solution $\overline{x}^{s'}$ of (4), we get

$$\frac{d}{dt}\big(\overline{x}(t)-\overline{x}^{s'}(t)\big)=\sum_{i=0}^{m-1}\overline{u}_i^0(t)\mathbf{r}_i'\big(x^0(t)\big)\overline{x}-\sum_{i=0}^{m-1}\overline{u}_i^{s'}(t)\mathbf{r}_i'\big(x^{s'}(t)\big)\overline{x}^{s'}$$
$$+\sum_{i=0}^{m-1}\overline{u}_i^{s'}(t)\Big(\mathbf{r}_i\big(x^0(t)\big)-\mathbf{r}_i\big(x^{s'}(t)\big)\Big).$$

Using the condition of the theorem, we get from this

$$\left|\frac{d}{dt}\big(\overline{x}(t)-\overline{x}^{s'}(t)\big)\right|\le O\big(\mathcal{H}^{s'}\big)\big|\overline{x}-\overline{x}^{s'}\big|+O\big(\mathcal{H}^{s'}\big)c_0\sqrt{|\Delta_{s'}|}+\big|\overline{u}^{s'}\big|O\big(\mathcal{H}^{s'}\big),$$
$$\overline{x}(t_{\text{left}})-\overline{x}^{s'}(t_{\text{left}})=0. \tag{6}$$

This implies that there is a constant $c_1>0$ independent of s' such that as $s'\to\infty$ we have the estimate

$$\big\|\overline{x}(t)-\overline{x}^{s'}(t)\big\|_\infty\le c_1\mathcal{H}^{s'}\sqrt{|\Delta_{s'}|}. \tag{7}$$

But $\overline{x}^{s'}(t_{\text{right}})=0$. Consequently,

$$|\overline{x}(t_{\text{right}})|\le c_1\mathcal{H}^{s'}\sqrt{|\Delta_{s'}|}. \tag{8}$$

¿From (4) and (8) we finally get the existence of a constant $c_2>0$ independent of s' such that for $s'\to\infty$ there exist $\delta\overline{x}^{s'}(t),\delta\overline{u}^{s'}(t)$ with

$$\big\|\delta\overline{x}^{s'}(t)\big\|_\infty,\ \big\|\delta\overline{u}^{s'}(t)\big\|_\infty\le c_2\mathcal{H}^{s'}\sqrt{|\Delta_{s'}|},$$
$$\big(\overline{x}^{s'}(t)+\delta\overline{x}^{s'}(t),\ \overline{u}^{s'}(t)+\delta\overline{u}^{s'}(t)\big)\in K^0_{\Delta_*}. \tag{9}$$

Let us consider $\omega(\gamma^{s'}|_{\Delta_*};\widehat{\overline{x}}^{s'},\widehat{\overline{u}}^{s'})$. Setting $u^{s'}(t)=u^0(t)+\delta u^{s'}(t)$, $\psi^{s'}(t)=\psi^0(t)+\delta\psi^{s'}(t)$, $x^{s'}(t)=x^0(t)+\delta x^{s'}(t)$, $\overline{x}^{s'}(t)=\big(\overline{x}^{s'}(t)+\delta\overline{x}^{s'}(t)\big)-\delta\overline{x}^{s'}(t)$, and $\overline{u}^{s'}(t)=\big(\overline{u}^{s'}(t)+\delta\overline{u}^{s'}(t)\big)-\delta\overline{u}^{s'}(t)$, we see from (9) that

$$\omega(\gamma^{s'}|_{\Delta_*};\widehat{\overline{x}}^{s'},\widehat{\overline{u}}^{s'})=\omega(\gamma^0|_{\Delta_*};\widehat{\overline{x}}^{s'}+\delta\widehat{\overline{x}}^{s'},\widehat{\overline{u}}^{s'}+\delta\widehat{\overline{u}}^{s'})+o\big(\mathcal{H}^{s'}\big)+\mathcal{H}^{s'}. \tag{10}$$

But by the choice of Δ_*, the first term is positive, and hence for $s'\to\infty$ the left-hand side of (10) is also positive. We have arrived at a contradiction to (3), and the theorem is proved. □

A geodesic generated by a definite extremal is generated both by a minus- and by a plus-Legendre extremal. It is clear from the theorem that as one of its aspects the geodesic shows a positive influence on the approximating geodesics. However, there is also another aspect. We shall see that it shows a sharply negative influence.

Let $\gamma^0=(\psi^0(t),x^0(t),u^0(t)|\Delta^0)$ be a plus-Legendre extremal, and let

$$\{\gamma^s=(\psi^s(t),x^s(t),u^s(t)|\Delta^0)\}_{s=1,2,\ldots}$$

be a sequence of nonsingular extremals approximating γ^0, so that

$$\|\psi^s(t)-\psi^0(t)\|_\infty\to 0,$$
$$\|x^s(t)-x^0(t)\|_\infty\to 0,$$
$$\|u^s(t)-u^0(t)\|_\infty\to 0. \tag{11}$$

Theorem 2. *For any $\delta > 0$ there is an s_0 such that*

$$\inf_{K^s_\Delta} \omega(\gamma^s|_\Delta; \cdot, \cdot) = -\infty$$

for any s and Δ with $s \geq s_0$, $\Delta \subset \Delta^0$, and $|\Delta| \geq \delta$.

Proof. Obviously, the theorem will be proved if we prove that for any $\Delta \subset \Delta^0$

$$\lim_s \inf_{K^s_\Delta} \omega(\gamma^s|_\Delta; \cdot, \cdot) = -\infty. \tag{12}$$

Assume the contrary. Then there exist a $\Delta^1 \subset \Delta^0$ and a subsequence $\{s'\}$ of the sequence $\{s\}$ such that

$$\lim_{s'} \inf_{K^{s'}_{\Delta^1}} \omega(\gamma^{s'}|_{\Delta^1}; \cdot, \cdot) = 0. \tag{13}$$

According to the result of Dmitruk mentioned above, for all sufficiently small $\Delta \subset \Delta^0$ we have the inequality

$$\omega(\gamma^0|_{\Delta^1}; \widehat{\overline{x}}, \widehat{\overline{u}}) < 0 \quad \forall \left(\widehat{\overline{x}}, \widehat{\overline{u}}\right) \in K^0_\Delta \setminus \{0\}. \tag{14}$$

Suppose that $\overline{\Delta} \subset \Delta^1$ satisfies (14). We give some preliminaries for the proof.

For $\Delta \subset \Delta^0$ let V^s_Δ, $s = 0, 1, \dots$, be the space of functions $\overline{u}(t)$ that are measurable and square integrable on Δ and that satisfy the condition

$$u^s(t)\overline{u}(t) = 0|\Delta.$$

Thus, V^s_Δ is defined. We consider $\overline{\Delta}$, and for $t \in \overline{\Delta}$ we define a linear transformation $\varphi^s(t)\colon R^m \to R^m$. If $u^s(t) = u^0(t)$, then $\varphi^s(t) = 1$. Suppose that $u^s(t) \neq u^0(t)$. Then in the plane of $u^s(t), u^0(t)$ the transformation $\varphi^s(t)$ is the rotation carrying $u^0(t)$ into $u^s(t)$ through an angle less than π. Since $|u^s(t)| = 1$, $s = 0, 1, \dots$, and $\|u^s(t) - u^0(t)\|_\infty \to 0$, this definition is correct. In the subspace orthogonal to the plane of $u^s(t), u^0(t)$, the operator $\varphi^s(t)$ is the identity. Next, in the space $L^m_2(\overline{\Delta})$ we define the operator $\Phi^s\colon L^m_2(\overline{\Delta}) \to L^m_2(\overline{\Delta})$ by setting $\Phi^s\left(\widehat{\overline{u}}\right) = \varphi^s(t)\overline{u}(t)$. Obviously, Φ^s is an invertible linear operator preserving the norm and carrying $V^0_{\overline{\Delta}}$ into $V^s_{\overline{\Delta}}$, and

$$\|\Phi^s(t) - \Phi^0(t)\|_\infty \to 0. \tag{15}$$

Let $\widetilde{V}^0$ be an $(n+1)$-dimensional subspace of V^0, and let $\widetilde{V}^s = \Phi^s\widetilde{V}^0$. Then it is clear that $\widetilde{V}^s$ is also an $(n+1)$-dimensional subspace of V^s.

For $s = 1, 2, \dots$ we consider the following equation on the interval $\overline{\Delta} = [\overline{t}_0, \overline{t}_1]$:

$$\dot{\overline{x}} = \sum_{i=0}^{m-1} \overline{u}^s_i(t)\mathbf{r}'_i(x^s(t))\overline{x} + \sum_{i=0}^{m-1} \overline{u}_i(t)\mathbf{r}_i(x^s(t)), \quad \overline{x}(\overline{t}_0) = 0.$$

We choose $\overline{u}(t)$ from the conditions

$$\overline{u}(t) \in \widetilde{V}^s, \quad \int_{\overline{\Delta}} \overline{u}^2(t)\, dt = 1, \quad \overline{x}(\overline{t}_1) = 0. \tag{16}$$

Since the dimension of $\widetilde{V}^s$ is $n+1$, such a choice is possible. Denote the chosen pair of functions by $\overline{x}^s(t)$, $\overline{u}^s(t)|\overline{\Delta}$. Obviously, $\left(\widehat{\overline{x}}^s, \widehat{\overline{u}}^s\right) \in K^s_{\overline{\Delta}}$. This concludes our preparation, and we return to the subsequence $\{s'\}$.

Since the nonnegativity property of the form ω on the cone of critical variations carries over to any closed subinterval, it follows from (13) that

$$\omega(\gamma^{s'}|_{\overline{\Delta}}; \widehat{\overline{x}}^{s'}, \widehat{\overline{u}}^{s'}) \geq 0. \tag{17}$$

On the other hand, it follows from (15) that the sequence $\{\widehat{\overline{u}}^{s'}\}$ is strongly compact. Therefore, it can be assumed without loss of generality that the sequence $\widehat{\overline{u}}^{s'}$ converges in the L_2^m norm to some function $\widehat{\overline{u}}^0$. From this we get

$$\int_{\overline{\Delta}} \overline{u}^0(t)^2\, dt = 1, \quad \overline{x}^{s'}(t) \xrightarrow{C} \overline{x}^0(t), \quad \left(\widehat{\overline{x}}^0, \widehat{\overline{u}}^0\right) \in K_{\overline{\Delta}}^0.$$

By (14),

$$\omega(\gamma^0|_{\overline{\Delta}}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0) < 0.$$

On the other hand,

$$\omega(\gamma^{s'}|_{\overline{\Delta}}; \widehat{\overline{x}}^{s'}, \widehat{\overline{u}}^{s'}) \to \omega(\gamma^0|_{\overline{\Delta}}; \widehat{\overline{x}}^0, \widehat{\overline{u}}^0).$$

Consequently, for $s' \to \infty$

$$\omega(\gamma^{s'}|_{\overline{\Delta}}; \widehat{\overline{x}}^{s'}, \widehat{\overline{u}}^{s'}) < 0,$$

which contradicts the condition (17). This proves the theorem. □

We assume that the sequence of extremals γ^s in Theorems 1 and 2 is such that the geodesics generated by them are absolutely nonsingular. Then it follows from Theorems 1 and 2 that on the approximating sequence the conjugate points do not get close if we are talking about a minus-Legendre extremal, and do get close if we are talking about a plus-Legendre extremal.

Thus, a neighborhood of a geodesic generated by a definite extremal has a clearly expressed nonhomogeneous structure. According to Theorems 1 and 2, this nonhomogeneity always accompanies a geodesic generated by a definite extremal, regardless of the closed interval on which it realizes a strong minimum.

§ 11. An example

In the four-dimensional space of elements $x = (x_0, x_1, x_2, x_3)$ we define a two-dimensional distribution $\Gamma(x)$. To do this we set $\mathbf{e}_i = x$ with $x_i = 1$ and $x_k = 0$ for $k \neq i$, where $i, k = 0, 1, 2, 3$. A basis $\mathbf{r}_0(x), \mathbf{r}_1(x)$ of the distribution $\Gamma(x)$ is defined by setting $\mathbf{r}_0 = \mathbf{e}_0$ and $\mathbf{r}_1 = x_0\mathbf{e}_1 + x_1\mathbf{e}_2 + \mathbf{e}_3$. We have

$$\begin{gathered} [\mathbf{r}_0, \mathbf{r}_1](x) = -\mathbf{e}_1, \\ \big[[\mathbf{r}_0, \mathbf{r}_1], \mathbf{r}_0\big](x) = 0, \\ \big[[\mathbf{r}_0, \mathbf{r}_1], \mathbf{r}_1\big](x) = \mathbf{e}_2. \end{gathered} \tag{1}$$

It follows from (1) that $\Gamma(x)$ is a bracket-generating distribution.

The given basis determines the control system

$$\dot{x} = u_0\mathbf{r}_0 + u_1\mathbf{r}_1(x).$$

A submetric is given by the condition $u_0^2 + u_1^2 = 1$. Thus, we have the control system

$$\begin{gathered} \dot{x}_0 = u_0, \quad \dot{x}_1 = u_1 x_0, \quad \dot{x}_2 = u_1 x_1, \quad \dot{x}_3 = u_1; \\ u_0^2 + u_1^2 = 1. \end{gathered} \tag{2}$$

Below we give a description of all extremals of this control system. The Pontryagin function for the system (2) has the form

$$H = u_0\psi_0 + u_1(x_0\psi_1 + x_1\psi_2 + \psi_3).$$

Setting $z = x_0\psi_1 + x_1\psi_2 + \psi_3 = (\psi, r_1(x))$, we represent the Hamiltonian in the form

$$\mathcal{H} = \sqrt{\psi_0^2 + z^2}.$$

The conjugate system has the form

$$\dot\psi_0 = -u_1\psi_1, \quad \dot\psi_1 = -u_1\psi_2, \quad \dot\psi_2 = 0, \quad \dot\psi_3 = 0. \tag{3}$$

Since H does not depend on t, $H = \mathcal{H} = \text{const}$ on an extremal. Let us find the singular extremals, that is, the extremals on which $\mathcal{H} = 0$. Thus, $\psi_0 = 0$, $z = 0$ on a singular extremal. Differentiating these equalities with respect to t, we get $u_1\psi_1 = 0$ and $u_0\psi_1 = 0$. Since $u_0^2 + u_1^2 = 1$, $\psi_1 = 0$ on a singular extremal. Differentiating this equality with respect to t, we get by (3) that $u_1\psi_2 = 0$. Since $\psi_2 = \text{const}$ on an extremal, it follows that either $\psi_2 = 0$ at each point of the extremal, or $\psi_2 \neq 0$ at each point of the extremal. If we assume the first possibility, then from $z = 0$ and $\psi_0 = \psi_1 = \psi_2 = 0$ it will follow that $\psi_3 = 0$, and thus the extremal turns out to be trivial. But we are considering nontrivial extremals. Consequently, $\psi_2 \neq 0$ at each point of the extremal, and hence $u_1 = 0$ at each point of the extremal. Thus, $u_0 = \pm 1$ and $u_1 = 0$ on a singular extremal. From this it follows that there are two families of geodesics: $u_0 = 1, u_1 = 0$ and $u_0 = -1, u_1 = 0$. The geodesics of the two families differ only in the direction of motion. On each geodesic of the first family x_0 is increasing, and on each geodesic of the second family x_0 is decreasing. For simplicity we confine ourselves to the first family. The equations (2) take the form

$$\dot x_0 = 1, \quad \dot x_1 = 0, \quad \dot x_2 = 0, \quad \dot x_3 = 0. \tag{4}$$

The equations (3) take the form $\dot\psi = 0$, and

$$\psi_0 = \psi_1 = 0, \quad \psi_2 \neq 0, \quad x_1\psi_2 + \psi_3 = 0. \tag{5}$$

We normalize the conjugate component of a singular extremal by the condition $|\psi_2| = 1$. Then it is easily established from (4) and (5) that through each point x there passes a unique geodesic generated by a singular extremal. Any solution of (4), and only it, is such a geodesic.

Let us proceed to nonsingular extremals, that is, extremals on which $\mathcal{H} \neq 0$ and hence $\mathcal{H} > 0$. According to the maximum condition, $u_0 = \dfrac{\psi_0}{\mathcal{H}}$ and $u_1 = \dfrac{z}{\mathcal{H}}$. The Hamiltonian system can be written in the form of the equations (2) and (3), with the assumption that u_0 and u_1 are connected with the conjugate and phase variables by the given equalities. Thus, for a nonsingular extremal we get

$$\dot x_0 = u_0, \quad \dot x_1 = u_1 x_0, \quad \dot x_2 = u_1 x_1, \quad \dot x_3 = u_1, \tag{6}$$

where $u_0 = \dfrac{\psi_0}{\mathcal{H}}$ and $u_1 = \dfrac{z}{\mathcal{H}}$, and also

$$\dot\psi_0 = -u_1\psi_1, \quad \dot\psi_1 = -u_1\psi_2, \quad \dot\psi_2 = \dot\psi_3 = 0. \tag{7}$$

It follows from (6) and (7) that $\dot z = u_0\psi_1$. Thus, from the eight equations in (6), (7) we can single out a block of equations, adding the equation in z to (7). We have

$$\dot\psi_0 = -u_1\psi_1, \quad \dot\psi_1 = -u_1\psi_2, \quad \dot z = u_0\psi_1, \quad \dot\psi_2 = \dot\psi_3 = 0. \tag{8}$$

The system (8) has four first integrals:

(9) $$\mathcal{H}=\sqrt{\psi_0^2+x^2},\quad c_0=\frac{1}{2}\psi_1^2-\psi_0\psi_2,\quad \psi_2,\quad \psi_3.$$

When the values of the integrals are specified, one parameter remains free, and we can take it to be the time shift, since the variable t does not appear explicitly in (8).

The functions $x_0, x_1, x_2, x_3, \psi_0, \psi_1, \psi_2, z$ are easily verified to be independent. This implies that if the functions $u_0(t)$ and $u_1(t)$ are generated by some solution of (8), then by substituting them in (2) we get a system of differential equations whose solutions are each geodesics.

We describe the solutions of the system (8). Three cases are possible:

I) $|c_0|>\mathcal{H}|\psi_2|$; II) $|c_0|=\mathcal{H}|\psi_2|$; III) $|c_0|<\mathcal{H}|\psi_2|$.

Let us now consider them.

I) Since $|\psi_0\psi_2|\le\mathcal{H}|\psi_2|$, it follows that $\operatorname{sgn}(\psi_0\psi_2+c_0)=\operatorname{sgn} c_0\neq 0$. From this, by (9) $c_0>0$ and $\psi_1\neq 0$. For brevity we write $s(\cdot)$ instead of $\operatorname{sgn}(\cdot)$ in what follows. Setting $d\tau=|\psi_1|dt$, we get from (8) that

(10) $$\frac{d\psi_0}{d\tau}=-s(\psi_1)\frac{z}{\mathcal{H}},\quad \frac{dz}{d\tau}=s(\psi_1)\frac{\psi_0}{\mathcal{H}}.$$

The pair

(11) $$\psi_0=\mathcal{H}\cos\frac{\tau}{\mathcal{H}},\quad z=s(\psi_1)\mathcal{H}\sin\frac{\tau}{\mathcal{H}}$$

is a solution of this system. Then

$$\psi_1=s(\psi_1)\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\Big(s(\psi_2)\cos\frac{\tau}{\mathcal{H}}+1\Big)+\delta c_0},$$

where $\delta c_0=c_0-\mathcal{H}|\psi_2|>0$. Let

(12) $$t-t_0=\int_0^{\tau}\frac{d\tau}{\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\Big(s(\psi_2)\cos\frac{\tau}{\mathcal{H}}+1\Big)+\delta c_0}}.$$

Replacing τ in (11) by its expression in terms of t according to (12), we get the general solution of (8) for the given integrals (9).

It follows from (11) and (12) that on any solution $u_0(t)$ and $u_1(t)$ are periodic functions with period

(13) $$\widetilde{\tau}=\int_0^{2\pi\mathcal{H}}\frac{d\tau}{\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\Big(\cos\frac{\tau}{\mathcal{H}}+1\Big)+\delta c_0}}.$$

The sign of ψ_1 determines the direction of motion of the point (ψ_0, z) along the circle $\psi_0^2+z^2=\mathcal{H}$.

II) Three subcases are possible in case II, namely, II_1: $c_0<0$, II_2: $c_0=0$, and II_3: $c_0>0$. Let us consider them.

II_1. In this case $\psi_2\neq 0$. Let $\widetilde{\psi}_0=s(\psi_0)\mathcal{H}$. Then

$$\psi_0\psi_2+c_0<0\quad\text{if}\quad \psi_0\neq\widetilde{\psi}_0,$$
$$\psi_0\psi_2+c_0=0\quad\text{if}\quad \psi_0=\widetilde{\psi}_0.$$

This implies that $\psi_0(t) = \widetilde{\psi}_0$, $z(t) = 0$, and $\psi_1(t) = 0$. Therefore, $u_0(t) = \pm 1$ and $u_1(t) = 0$. The corresponding geodesic has mixed type.

II$_2$. Since $\mathcal{H} > 0$, it follows that $\psi_2 = 0$, and $\psi_1 = 0$ by (9). From this, by (8),

$$\psi_0 = \text{const}, \ z = \text{const}. \tag{14}$$

Then $u_0(t) = \text{const}$ and $u_1(t) = \text{const}$. The values of u_0 and u_1 are in no way limited except for the condition that $u_0^2 + u_1^2 = 1$. If $u_1 \neq 0$, then the geodesic is absolutely nonsingular, but if $u_1 = 0$, then we have the geodesic of mixed type already known to us.

II$_3$. In this case $\psi_2 \neq 0$. Let $\widetilde{\psi}_0 = -s(\psi_2)\mathcal{H}$. Then

$$\begin{aligned} \psi_0\psi_2 + c_0 > 0 \quad \text{if} \quad \psi_0 \neq \widetilde{\psi}_0, \\ \psi_0\psi_2 + c_0 = 0 \quad \text{if} \quad \psi_0 = \widetilde{\psi}_0. \end{aligned}$$

If $\psi_0 = \widetilde{\psi}_0$, then $\psi_1 = 0$, and we get the solution $\psi_0 = \widetilde{\psi}_0 = \text{const}$, $z = 0$, $\psi_1 = 0$. The corresponding geodesic has mixed type, because $u_0(t) = \pm 1$ and $u_1(t) = 0$.

We find a solution on which $\psi_0 \neq \widetilde{\psi}_0$. According to (9), $\psi_1 \neq 0$ on this solution. Setting $d\tau = |\psi_1|\,dt$, we go to the system (10), and we choose the following solution of it:

$$\psi_0 = s(\psi_2)\mathcal{H}\cos\frac{\tau}{\mathcal{H}}, \quad z = s(\psi_1)s(\psi_2)\mathcal{H}\sin\frac{\tau}{\mathcal{H}}, \quad |\tau| < \pi\mathcal{H}. \tag{15}$$

By (9),

$$\psi_1 = s(\psi_1)\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\left(\cos\frac{\tau}{\mathcal{H}} + 1\right)}.$$

¿From this we get

$$t = t_0 + \int_0^\tau \frac{d\tau}{\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\left(\cos\frac{\tau}{\mathcal{H}} + 1\right)}}. \tag{16}$$

It is not hard to see that $t \xrightarrow{\tau \to \pi\mathcal{H}} +\infty$ and $t \xrightarrow{\tau \to -\pi\mathcal{H}} -\infty$. The formulas (15) and (16) give the general solution of the system (8) in this case.

Since $u_0(t), u_1(t) \neq \text{const}$, it follows that the corresponding geodesic is absolutely nonsingular. The control $u_0(t), u_1(t)$ is obviously not periodic, since $|u_0(t)| \to 1$ and $u_1(t) \to 0$ as $|t| \to 0$. As we shall see below, this is the only family on which the control is not a periodic function.

This concludes the treatment of Case II.

III) In this case it is clear that $\psi_2 \neq 0$. We determine $\widetilde{\psi}_0$ from the equation $\psi_0\psi_2 + c_0 = 0$. Since $|c_0| < \mathcal{H}|\psi_2|$, it follows that $|\widetilde{\psi}_0| < \mathcal{H}$. Consequently, the circle $\psi_0^2 + z^2 = \kappa^2$ is divided into two arcs, namely, $\sigma_1\colon (\psi_0, z)|\psi_0 \leq \widetilde{\psi}_0$ and $\sigma_2\colon (\psi_0, z)|\psi_0 \geq \widetilde{\psi}_0$. The two arcs are symmetric with respect to the axis of ψ_0. They have the common endpoints $\widetilde{\psi}_0, \widetilde{z}_0$ and $\widetilde{\psi}_0, -\widetilde{z}_0$, where $\widetilde{z}_0 > 0$. We choose the arc containing the point $\psi_0 = s(\psi_2)\mathcal{H}$, $z = 0$, denote it by $\widetilde{\sigma}$, and denote the complementary arc by $\widetilde{\sigma}'$. Then

$$\begin{aligned} \psi_0, z \in \widetilde{\sigma} \Longrightarrow \psi_0\psi_2 + c_0 > 0, \\ \psi_0, z \in \widetilde{\sigma}' \Longrightarrow \psi_0\psi_2 + c_0 < 0. \end{aligned}$$

To be more precise, these inequalities hold on open arcs. They imply that the solution ψ_0, z must belong only to $\widetilde{\sigma}$, and $\psi_1 \neq 0$ on the open arc. We find the solution on the open arc. Introducing the substitution $d\tau = |\psi_1|\,dt$, we pass to the system (10). Let θ

be the angle between the axis of ψ_0 and the vector $\big(\widetilde{\psi}_0, \widetilde{z}_0\big)$. The system (10) has the following solution on the interval $|\tau| < \theta\mathcal{H}$:

$$\psi_0 = s(\psi_2)\mathcal{H}\cos\frac{\tau}{\mathcal{H}}, \quad z = s(\psi_1)s(\psi_2)\mathcal{H}\sin\frac{\tau}{\mathcal{H}}. \tag{17}$$

Since $c_0 = -\widetilde{\psi}_0\psi_2 = -\mathcal{H}|\psi_2|\cos\theta$, we have

$$\psi_1 = s(\psi_1)\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\left(\cos\frac{\tau}{\mathcal{H}} - cos\theta\right)}.$$

Then

$$t = t_0 + \int\limits_0^\tau \frac{d\tau}{\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\left(\cos\frac{\tau}{\mathcal{H}} - cos\theta\right)}}. \tag{18}$$

The integral (18) converges as $|\tau| \to \theta\mathcal{H}$. Thus, the value $\psi_1 = 0$ is attained in a finite time. Since $\widetilde{z} \neq 0$, it follows from (8) that ψ_1 changes sign at the point $\widetilde{\psi}_0$. The motion along the arc $\widetilde{\sigma}$ is repeated, but in the opposite direction.

The given description admits the following formalization. Let $\tau_\nu = 2\nu + 1$, $\nu = 0, \pm 1, \dots$, and denote by Δ_ν the interval $[\tau_{\nu-1}, \tau_\nu]$. We define the function $\varphi(\tau)$ by the conditions $\varphi(0) = 0$ and $\dot{\varphi}(\tau) = (-1)^\nu|\Delta_\nu$. Finally, we denote by s_1 the quantity $s(\psi_1)$ in (17). Then the solution admits the expression

$$\psi_0 = s(\psi_2)\mathcal{H}\cos\theta\,\varphi\left(\frac{\tau}{\theta\mathcal{H}}\right), \quad z = s_1 s(\psi_2)\mathcal{H}\sin\theta\,\varphi\left(\frac{\tau}{\theta\mathcal{H}}\right). \tag{19}$$

According to (9),

$$\psi_1 = s_1\frac{d\varphi}{d\tau}\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\left(\cos\theta\,\varphi\left(\frac{\tau}{\theta\mathcal{H}}\right) - cos\theta\right)}. \tag{20}$$

Let

$$t = t_0 + \int\limits_0^\tau \frac{d\tau}{\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\left(\cos\theta\,\varphi\left(\frac{\tau}{\theta\mathcal{H}}\right) - cos\theta\right)}}. \tag{21}$$

The formulas (19), (20), and (21) give the general solution of the system (8) in this case. This concludes the treatment of Case III.

And with this we have completed our investigation of extremals of the control system (2).

We can now write out families of systems of ordinary differential equations for determining geodesics. To this end it suffices to write out the families of controlling functions, for we have already remarked above that if $u_0(t)$, $u_1(t)$ are obtained from a solution of (8), then any solution of the system

$$\dot{x}_0 = u_0(t), \quad \dot{x}_1 = u_1(t)x_0, \quad \dot{x}_2 = u_1(t)x_1, \quad \dot{x}_3 = u_1(t) \tag{22}$$

is a geodesic. If $u_1(t) \not\equiv 0$, then the geodesic is always absolutely nonsingular, but if $u_1(t) \equiv 0$, then it is of mixed type. To describe the controlling families, we use the above analysis of extremals—more precisely, of the solutions of the system (8).

Let us consider type I. Setting $v = \dfrac{\tau}{\mathcal{H}}$, we get from (11)

$$u_0(v) = \cos v, \quad u_1(v) = s(\psi_1)\sin v. \tag{23}$$

Setting $t_0 = 0$ in (12), we get

$$t = \int_0^v \frac{\mathcal{H}\,dv}{\sqrt{2}\,\sqrt{\mathcal{H}|\psi_2|\,(\cos v + 1) + \delta c_0}}, \quad \mathcal{H},\ \delta c_0 > 0.$$

Let

$$\overline{\alpha}_0 = \frac{\dfrac{|\psi_2|}{\mathcal{H}}}{\left(\dfrac{|\psi_2|}{\mathcal{H}} + \dfrac{\delta c_0}{\mathcal{H}^2}\right)}, \quad \overline{\alpha}_1 = \frac{\dfrac{\delta c_0}{\mathcal{H}^2}}{\left(\dfrac{|\psi_2|}{\mathcal{H}} + \dfrac{\delta c_0}{\mathcal{H}^2}\right)}, \quad \zeta = \sqrt{\frac{|\psi_2|}{\mathcal{H}} + \frac{\delta c_0}{\mathcal{H}^2}} \cdot t.$$

Then

$$\zeta = \int_0^v \frac{dv}{\sqrt{2}\,\sqrt{\overline{\alpha}_0(\cos v + 1) + \overline{\alpha}_1}}, \quad \overline{\alpha}_1 > 0. \tag{24}$$

The formula (24) determines a family $v(\zeta, \overline{\alpha})$, where $\overline{\alpha} = (\overline{\alpha}_0, \overline{\alpha}_1)$. Substituting in (23), we have

$$u_0(\zeta) = \cos v(\zeta, \overline{\alpha}), \quad u_1(v) = s(\psi_1)\sin v(\zeta, \overline{\alpha}). \tag{25}$$

Let $\sqrt{\dfrac{|\psi_2|}{\mathcal{H}} + \dfrac{\delta c_0}{\mathcal{H}^2}} = \mu$. Obviously, μ can take any positive value. Thus, we have obtained a family of controls

$$u_0(t) = \cos v(\mu t, \overline{\alpha}), \quad u_1(t) = s(\psi_1)\sin v(\mu t, \overline{\alpha}). \tag{26}$$

In this family, which we call the first family, there are two continuous parameters μ and $\overline{\alpha}$ and one discrete parameter $s(\psi_1)$. Substituting the functions of this family into (22), we get equations whose solutions are all absolutely nonsingular geodesics. The functions of the first family are periodic, and the vector $(u_0(t), u_1(t))$ describes a circle without changing direction.

Let us proceed to extremals of type II. Extremals of type II_2 are characterized by the condition that $\psi_0 = \text{const}$ and $z = \text{const}$ on them. Consequently, corresponding to this case is the family $u_0(t) = \text{const}, u_1(t) = \text{const}$. We call this the second family. If $u_1(t) \neq 0$, then the corresponding geodesic is absolutely nonsingular. The conjugate components corresponding to it are characterized by the conditions $\psi_1 = \psi_2 = 0$ and $\psi_3 \neq 0$. If $u_1(t) = 0$, then the geodesic is of mixed type, and the set of conjugate components corresponding to it contains extremals: $\psi_0 \neq 0, \psi_1 = \psi_2 = \psi_3 = 0$ and $\psi_0 = \psi_1 = 0, \psi_2 \neq 0$. The quantity ψ_3 is determined from the condition $z = 0$.

The case II_3 supplies us with one more family of controlling functions. Let us turn to the formulas (15) and (16). Setting $v = \dfrac{\tau}{\mathcal{H}}$, we get from (15) that

$$u_0(v) = s(\psi_2)\cos v, \quad u_1(v) = s(\psi_1)s(\psi_2)\sin v, \quad |v| < \pi. \tag{27}$$

Setting $t_0 = 0$ in (16), we have

$$t = \int_0^v \frac{\mathcal{H}\,dv}{\sqrt{2}\,\sqrt{\mathcal{H}|\psi_2|\,(\cos v + 1)}}, \quad \psi_2 \neq 0.$$

Setting $\zeta = t\sqrt{\dfrac{|\psi_2|}{\mathcal{H}}}$, we have

$$\zeta = \int_0^v \frac{dv}{\sqrt{2}\sqrt{\cos v + 1}}, \quad |v| < \pi. \tag{28}$$

Taking the inverse of (28), we obtain $v(\zeta)$. Then in view of (27)

$$u_0(\zeta) = s(\psi_2)\cos v(\zeta), \quad u_1(\zeta) = s(\psi_1)s(\psi_2)\sin v(\zeta). \tag{29}$$

Setting $\sqrt{\dfrac{|\psi_2|}{\mathcal{H}}} = \mu$, we get the third family:

$$u_0(\zeta) = s(\psi_2)\cos v(\mu t), \quad u_1(t) = s(\psi_1)s(\psi_2)\sin v(\mu t). \tag{30}$$

As $t \to +\infty$ we have $v \to \pi$, and as $t \to -\infty$ we have $v \to -\pi$. Consequently, as $|t| \to \infty$ we have $|u_0(t)| \to 1$ and $u_1(t) \to 0$. All the geodesics of this family are absolutely nonsingular. The family depends on one continuous parameter and two discrete parameters. The vector $\big(u_0(t), u_1(t)\big)$ traverses the whole circle except for the point $\psi_0 = -s(\psi_2)\mathcal{H}$, $z = 0$ in infinite time, without changing the direction of motion.

Finally, we describe the fourth and last family, based on extremals of type III.

Setting $v = \dfrac{\tau}{\mathcal{H}}$ as before, we get from (19) that

$$u_0(v) = s(\psi_2)\cos\theta\,\varphi\left(\frac{v}{\theta}\right), \quad u_1(v) = s_1 s(\psi_2)\sin\theta\,\varphi\left(\frac{v}{\theta}\right), \quad 0 < \theta < \pi. \tag{31}$$

Setting $t_0 = 0$ in (21), we obtain

$$t = \int_0^v \frac{\mathcal{H}\,dv}{\sqrt{2}\sqrt{\mathcal{H}|\psi_2|\big(\cos\theta\,\varphi(\frac{v}{\theta}) - \cos\theta\big)}}, \quad |v| < \pi. \tag{32}$$

Setting $\zeta = t\sqrt{\dfrac{|\psi_2|}{\mathcal{H}}}$, we obtain

$$\zeta = \int_0^v \frac{dv}{\sqrt{2}\sqrt{\cos\theta\,\varphi(\frac{v}{\theta}) - \cos\theta}}, \quad |v| < \pi. \tag{33}$$

Taking the inverse of the dependence (33), we get $v = v(\zeta,\theta)$. Consequently,

$$u_0(\zeta) = s(\psi_2)\cos\theta\,\varphi\left(\frac{v(\zeta,\theta)}{\theta}\right), \quad u_1(\zeta) = s_1 s(\psi_2)\sin\theta\,\varphi\left(\frac{v(\zeta,\theta)}{\theta}\right). \tag{34}$$

Let $\mu = \sqrt{\dfrac{|\psi_2|}{\mathcal{H}}}$. We obtain the following family of functions $u_0(t)$, $u_1(t)$:

$$u_0(t) = s(\psi_2)\cos\theta\,\varphi\left(\frac{v(\mu t,\theta)}{\theta}\right), \quad u_1(t) = s_1 s(\psi_2)\sin\theta\,\varphi\left(\frac{v(\mu t,\theta)}{\theta}\right) \tag{35}$$

The family (35) depends on two continuous parameters μ and θ and two discrete parameters s_1 and $s(\psi_2)$. The geodesics of this family are absolutely nonsingular. A distinguishing feature of the functions in this family is that the vector $(u_0(t), u_1(t))$ changes direction of motion along the circle periodically.

Accordingly, we have indicated four families of controlling functions that supply all the geodesics of the given control system. It is important to note that these families do not intersect; that is, there are no closed intervals on which they have common controls.

Let $u^0(t) = \big(u_0^0(t) = 1,\, u_1^0(t) = 0\big)|\Delta$. Thus, $u^0(t)$ is a control determining a geodesic of mixed type on Δ. The function $u^0(t)|\Delta$ can be uniformly approximated by functions of any family.

Our goal is to determine the properties of geodesics approximating a geodesic of mixed type on the closed interval Δ, or, which is the same, on some closed interval of the same length, since a geodesic is not changed by a time shift.

To this end we write the quadratic conditions. According to § 7, the cone of critical variations is given by the equations

$$\dot{\overline{x}}_0 = \overline{u}_0, \quad \dot{\overline{x}}_1 = u_1\overline{x}_0 + \overline{u}_1 x_0, \quad \dot{\overline{x}}_2 = u_1\overline{x}_1 + \overline{u}_1 x_1, \quad \dot{\overline{x}}_3 = \overline{u}_1 \tag{36}$$

and by the boundary conditions $\overline{x}(t_0) = \overline{x}(t_1) = 0$, where $\Delta = [t_0, t_1]$. Moreover, the condition $u_0(t)\overline{u}_0(t) + u_1(t)\overline{u}_1(t) = 0$ must hold. By § 7, the form ω has the appearance

$$\omega = \int_{t_0}^{t_1} \left(-\overline{u}_1 z'\overline{x} + \frac{\mathcal{H}}{2}\big(\overline{u}_0^2 + \overline{u}_1^2\big) \right) dt, \tag{37}$$

which can be simplified. We have

$$\frac{d}{dt} z'\overline{x} = \psi_1\overline{u}_0 + \psi_2 x_0 \overline{u}_1.$$

Integrating the first term in (37) by parts, we get

$$\omega = \int_{t_0}^{t_1} \left(\psi_1\overline{x}_3\overline{u}_0 - \frac{1}{2}\psi_2 u_0 \overline{x}_3^2 + \frac{\mathcal{H}}{2}\big(\overline{u}_0^2 + \overline{u}_1^2\big) \right) dt. \tag{38}$$

Assume that u_0 is nonzero everywhere on Δ. Then

$$\overline{u}_0 = -\frac{u_1}{u_0}\overline{u}_1, \quad \frac{\mathcal{H}}{2}\big(\overline{u}_0^2 + \overline{u}_1^2\big) = \frac{\mathcal{H}}{2u_0^2}\,\overline{u}_1^2.$$

Substituting in (38) and using (9), we get

$$\omega = \int_{t_0}^{t_1} \left(\frac{1}{2\mathcal{H}}\,\frac{\psi_0\psi_2 + 2c_0}{u_0^2}\overline{x}_3^2 + \frac{\mathcal{H}}{2u_0^2}\dot{\overline{x}}_3^2 \right) dt. \tag{39}$$

We can now carry out the necessary analysis.

Suppose that the function $u^0(t)|\Delta$ is uniformly approximated by a sequence of functions in the first family. Then $u_0^2 \sim 1$ and $\operatorname{sgn}(\psi_0\psi_2 + 2c_0) = 1$. Consequently, an approximating geodesic realizes a strong minimum on Δ.

Suppose that $u^0(t)|\Delta$ is uniformly approximated by functions of the second family. Then since $u_0^2 \sim 1$, the formula (39) is applicable. The conjugate variables of the second family are such that $\psi_2 = c_0 = 0$. Consequently,

$$\omega = \int_{t_0}^{t_1} \frac{\mathcal{H}}{2u_0^2}\dot{\overline{x}}_3^2\, dt.$$

Thus, an approximating geodesic realizes a strong minimum also in this case.

We consider approximation by functions of the third family. According to II_3, $\mathcal{H}|\psi_2| = c_0 > 0$, and therefore $\operatorname{sgn}(\psi_0\psi_2 + 2c_0) > 0$. Thus, it follows from (39) that an approximating geodesic realizes a strong minimum.

It remains to consider the fourth family. Since $u_0^2 \sim 1$, we use the formula (39). By (35), $\theta \to 0$ and $s(\psi_2) = 1$ on an approximating sequence. Consequently, $c_0 = -\mathcal{H}\cos\theta$. Then the form ω can be written as

$$\omega = \int_{t_0}^{t_1} \left(\frac{1}{2}\psi_2 \frac{u_0 - 2\cos\theta}{u_0^2} \overline{x}_3^2 + \frac{\mathcal{H}}{2u_0^2} \dot{\overline{x}}_3^2 \right) dt,$$

and so

$$\frac{2}{\psi_2}\omega = \int_{t_0}^{t_1} \left(\frac{u_0 - 2\cos\theta}{u_0^2} \overline{x}_3^2 + \frac{1}{\mu^2 u_0^2} \dot{\overline{x}}_3^2 \right) dt. \tag{40}$$

If the sequence of functions $u_0(t)$, $u_1(t)$ tends to $u_0^0(t) = 1$, $u_1^0(t) = 0|\Delta$, then the form $\dfrac{2}{\psi_2}\omega$ tends to the form

$$\omega' = \int_{t_0}^{t_1} \left(-\overline{x}_3^2 + \frac{1}{\mu^2} \dot{\overline{x}}_3^2 \right) dt. \tag{41}$$

¿From this it is clear that as $\mu \to \infty$ the negative index of the form ω' on any fixed closed interval tends to infinity. Consequently, ω changes sign on the cone of critical variations. Thus, the conjugate points become close as $\mu \to \infty$ and $\theta \to 0$, so that the distance between them tends to zero.

We remark that the fourth family along with the first is basic with regard to the number of parameters.

It remains for us to make some comments about geodesics determined by the control $u_0^0(t) = 1$, $u_1^0(t) = 0$. It is quite clear that these geodesics realize an absolute minimum in system (2) of §1, because the (x_0, x_3)-components cannot be joined by a shorter path in the framework of the given system. The same property is enjoyed by all the geodesics in the second family. However, the geodesics of mixed type occupy a special place even among the geodesics of the second family. They do not simply realize an absolute minimum in the given submetric; they realize a strong minimum in any strictly convex submetric. Indeed, the basis $\mathbf{r}_0, \mathbf{r}_1$ is an associated basis for these geodesics. Definite extremals are characterized by the conditions $\psi_0 = 0$, $\psi_1 = 0$, $\psi_2 \neq 0$, $z = 0$. The form ω then looks like

$$\omega = \int_{t_0}^{t_1} (-\overline{u}_1 z' \overline{x})\, dt = \int_{t_0}^{t_1} (-\overline{u}_1 \psi_2 \overline{x}_1)\, dt = \int_{t_0}^{t_1} \psi_2 (u_1 \overline{x}_0 + \overline{u}_1 x_0) \overline{x}_3\, dt.$$

Since $u_1 = 0$, it follows that

$$\omega = \int_{t_0}^{t_1} \psi_2 x_0 \overline{x}_3 \overline{u}_1\, dt = \int_{t_0}^{t_1} \psi_2 \frac{1}{2} \overline{x}_3^2\, dt.$$

Setting $\psi_2 = -1$, we see that the sufficient condition in § 10 is satisfied. Our assertion then follows from a theorem of Dmitruk.

§ 12. On absolutely nonsingular geodesics

In this section we consider in first approximation the question of the existence among the nonsingular extremals of a dense set of absolutely nonsingular extremals, that is, extremals corresponding to absolutely nonsingular geodesics.

Suppose that the submetric problem has been reduced to a control system

$$\dot{x} = \sum_{i=0}^{m-1} u_i \mathbf{r}_i(x), \quad \sum_{i=0}^{m-1} u_i^2 = u^2 = 1. \tag{1}$$

Here the $\mathbf{r}_i(x)$ are smooth vector fields that are linearly independent at each point of some domain in the space $R^{d(x)}$. The Hamiltonian $\mathcal{H}$ of the system (1) has the form

$$\mathcal{H}(w) = \sqrt{\sum_{i=0}^{m-1} (\psi, \mathbf{r}_i(x))^2}, \tag{2}$$

where $w = (\psi, x)$. If w is such that $\mathcal{H}(w) > 0$, then the control u can be expressed in terms of w, and we get

$$u_i(w) = \frac{1}{\mathcal{H}(w)} (\psi, \mathbf{r}_i(x)), \quad i = 0, \dots, m-1. \tag{3}$$

Let $W^+ = \{w | \mathcal{H}(w) > 0\}$. The nonsingular extremals are those that belong to W^+.

To each nonsingular extremal there corresponds either an absolutely nonsingular geodesic or a geodesic of mixed type on some time intervals. We present a condition sufficient for the existence of a dense set of absolutely nonsingular extremals in the set of all nonsingular extremals.

Let

$$\begin{aligned} \widehat{u}(w) &= \mathcal{H}(w) u(w), \\ \widehat{u}^1(w) &= \mathcal{H}(w) \frac{d}{dt} \widehat{u}(w), \\ &\dots\dots\dots\dots\dots\dots \\ \widehat{u}^k(w) &= \mathcal{H}(w) \frac{d}{dt} \widehat{u}^{k-1}(w), \\ &\dots\dots\dots\dots\dots\dots \end{aligned} \tag{4}$$

We define two sets in W^+:

$$\begin{aligned} Q &= \left\{ w \middle| d\left(\frac{\partial}{\partial \psi} \widehat{u}(w), \dots, \frac{\partial}{\partial \psi} \widehat{u}^k(w), \dots \right) = d(\psi) \right\}, \\ F &= \left\{ w \middle| d\left(\frac{\partial}{\partial \psi} \widehat{u}(w), \dots, \frac{\partial}{\partial \psi} \widehat{u}^k(w), \dots \right) < d(\psi) \right\}. \end{aligned} \tag{5}$$

It is obvious that Q is an open subset of W^+ and that $W^+ = Q \bigcup F$. Consequently, F is a closed subset of W^+. We have the following result.

Theorem 1. *If the set Q is dense in W^+, then W^+ contains a dense set of points through which absolutely nonsingular extremals pass.*

Proof. First we prove the following lemma.

Let X be a topological space of second category, that is, a space in which the complement of any set of first category is dense. Let Y be a topological space with a countable base, and let $Z = X \times Y$ be the topological direct product of X and Y.

Suppose that $f\colon Z \to X$ is a continuous open mapping, and let M be a closed nowhere dense subset of X. Let $N \subset X$ be the set of those x for which there is an open subset V of Y such that $f(x, V) \subset M$. Denote by N' the complement of N in X.

Lemma 1. *The set N' is dense in X.*

Proof. Let V_s, $s = 1, 2, \dots$, be a countable base in Y, and let $N_s = \{x | f(x, V_s) \subset M\}$. Obviously, N_s can be represented in the form

$$N_s = \bigcap_{y \in V_s} \operatorname{pr}_x((X \cdot y) \cap f^{-1}(M)). \tag{6}$$

¿From this it is clear first of all that N_s is a closed set for any s. Next, it is clear that N_s does not contain interior points, because otherwise there would be an open subset of Z carried by f into M. Since f is an open mapping, M would have to contain interior points, contrary to a condition of the lemma. Thus, N_s is a closed nowhere dense subset of X. But $N = \bigcup_s N_s$, and hence N' is dense in X, because X is a space of second category. The lemma is proved. □

We apply the lemma to the product $W^+ \times R$, the set F, and the mapping $f \colon W^+ \times R \to W^+$, $f(w, T) = w(T)$, where $w(t)$ is a solution of the Hamiltonian system with initial data $w(0) = w$. Since Q is a dense open subset of W^+ by a condition of the theorem, F is a nowhere dense closed subset of W^+. Then by the lemma there is a dense subset W^0 of W^+ such that on each extremal passing through some point of W^0 the set of times for which the extremal belongs to F is nowhere dense on the time axis. It remains for us to show that each extremal with this property is absolutely nonsingular. To this end it suffices to prove that each nonsingular extremal generated by a geodesic of mixed type belongs entirely to F. Thus, let $(x(t), u(t))|\Delta$ be a geodesic of mixed type, let $(\psi(t), x(t), u(t))$ be a nonsingular extremal for it, and let $(\overline{\psi}(t), x(t), u(t))|\Delta$ be a singular extremal for it.

We introduce some notation. Let $D = \dfrac{1}{\mathcal{H}(\psi(t), x(t))} \dfrac{d}{dt}$. We note that the conjugate components of the extremals corresponding to the given geodesic satisfy one and the same conjugate equation:

$$\dot{\psi} = -\sum_{i=0}^{m-1} u_i(t) \big(\psi \mathbf{r}_i(x(t))\big)'_x. \tag{8}$$

In this case

$$u_i(t) = \frac{1}{\mathcal{H}(\psi(t), x(t))} \big(\psi(t) \mathbf{r}_i(x(t))\big). \tag{9}$$

For any solution $\eta(t)$ of the conjugate equation we have

$$D \cdot \big(\eta(t), \mathbf{r}_i(x(t))\big) = \sum_{k=0}^{m-1} \eta(t) [\mathbf{r}_i, \mathbf{r}_k](x(t)) \widehat{u}_k(\psi(t), x(t)) = \ell_i^1 \big(x(t), \widehat{u}(\psi(t), x(t))\big) \eta(t),$$

where $\ell_i^1(\cdot)$ is a linear functional of η. Using induction, we get that

$$\begin{gathered} D^s \cdot \big(\eta(t), \mathbf{r}_i(x(t))\big) = \ell_i^s \big(x(t), \widehat{u}(\psi(t), x(t)), \dots, \widehat{u}^{s-1}(\psi(t), x(t))\big) \eta(t), \\ s = 1, 2, \dots, \end{gathered} \tag{10}$$

where $\ell_i^s(\cdot)$ is a linear functional of η for any $t \in \Delta$. We now have the obvious equality

$$\big(\overline{\psi} \mathbf{r}_i(x(t))\big) = \overline{\psi}(t) \frac{\partial}{\partial \psi} \widehat{u}_i(\psi(t), x(t)).$$

But the left-hand side of this is zero, because the extremal $(\overline{\psi}(t), x(t), u(t))|\Delta$ is singular. Consequently, the right-hand side is also zero on Δ. Thus,

$$\overline{\psi}(t)\frac{\partial}{\partial\psi}\widehat{u}_i(\psi(t), x(t)) = 0. \tag{11}$$

Further,

$$D \cdot \big(\overline{\psi}(t), \mathbf{r}_i(x(t))\big) = \ell_i^1\big(x(t), \widehat{u}(\psi(t), x(t))\big)\overline{\psi}(t).$$

But $D \cdot \widehat{u}(\psi(t), x(t)) = \widehat{u}^1(\psi(t), x(t))$. Since $D \cdot \big(\overline{\psi}(t), \mathbf{r}_i(x(t))\big) = 0$, (11) then gives us

$$\overline{\psi}(t)\frac{\partial}{\partial\psi}\widehat{u}_i^1(\psi(t), x(t)) = 0. \tag{12}$$

Next, arguing by induction, we find from (10) that

$$\overline{\psi}(t)\frac{\partial}{\partial\psi}\widehat{u}_i^s(\psi(t), x(t)) = 0|\Delta, \quad s = 1, 2, \dots. \tag{13}$$

¿From this it follows that on Δ the extremal $(\psi(t)x(t), u(t))$ belongs to the set F. But then the nonsingular extremal belonging to F on at most a nowhere dense set of times is absolutely nonsingular. The theorem is proved. □

Using the above theorem, we prove that in the case of a distribution of maximal dimension (1 less than the dimension of the manifold: $m = d(x) - 1$) there always exists a dense set of absolutely nonsingular extremals whenever the distribution is bracket generating on a dense subset of points in $R^{d(x)}$.

Accordingly, we prove the following theorem.

Theorem 2. *Let $\Gamma(x)$ be a distribution of maximal dimension that is bracket generating on some dense set of values x in its domain. Then W^+ contains a dense set such that each extremal passing through some point of this set is absolutely nonsingular.*

Proof. We prove that the condition of Theorem 2 implies the condition of Theorem 1. This will prove Theorem 2. To do this we prove that if the condition of Theorem 1 does not hold, then neither does the condition of Theorem 2. This is the scheme of proof of Theorem 2.

Accordingly, assume that the condition of Theorem 1 fails. Then the set F contains an open set $\widehat{Q}$. Let $x_* \in \mathrm{pr}_x\widehat{Q}$. We denote by $\widehat{Q}|_x$ the set of ψ such that $\big(\psi, x\big) \in \widehat{Q}$. Obviously, if $x \in \mathrm{pr}_x\widehat{Q}$, then $\widehat{Q}|_x$ is open in $R^{d(x)}$. Thus, the set $\widehat{Q}|_{x_*}$ is open in $R^{d(x)}$. Let $\psi \in \widehat{Q}|_{x_*}$. We have

$$\widehat{u}_i(\psi, x_*) = \psi(t)\mathbf{r}_i(x_*),$$
$$\widehat{u}_i^1(\psi, x_*) = \sum_{k=0}^{m-1} \psi[\mathbf{r}_i, \mathbf{r}_k](x_*)\widehat{u}_k(\psi, x_*).$$

¿From this we get

$$\frac{\partial}{\partial\psi}\widehat{u}_i(\psi, x_*) \in \Gamma(x_*). \tag{14}$$

Consequently,

$$\frac{\partial}{\partial\psi}\widehat{u}_i^1(\psi, x_*) \in \sum_{k=0}^{m-1}[\mathbf{r}_i, \mathbf{r}_k](x_*)\widehat{u}_k(\psi, x_*) + \Gamma(x_*), \quad i = 0, \dots, m-1. \tag{15}$$

But the dimension of $\Gamma(x_*)$ is equal to $m = d(x) - 1$, and the mapping $\psi \to \widehat{u}(\psi, x_*)$ is linear and open. Therefore, it follows from the condition

$$\frac{\partial}{\partial\psi}\widehat{u}_i^1(\psi, x_*) \in \Gamma(x_*) \quad \forall\, \psi \in \widehat{Q}|_{x_*}$$

and (15) that

$$[\mathbf{r}_i, \mathbf{r}_k](x_*) \in \Gamma(x_*), \quad i, k = 0, \ldots, m-1. \tag{16}$$

Consequently, $[\Gamma]^2\,(x_*) \subseteq \Gamma(x_*)$. But $\mathrm{pr}_x\widehat{Q}$ is an open subset of $R^{d(x)}$. Thus, the inclusion $[\Gamma]^2\,(x) \subset \Gamma(x)$ holds on an open subset of $R^{d(x)}$. But then $\Gamma(x)$ is not a bracket-generating distribution on this subset. The condition of Theorem 2 thereby fails. The proof of Theorem 2 is complete. □

The bracket-generating requirement is an invariable assumption in the theory of submetrics. Thus, we have established that if a distribution in the theory of submetrics has maximal dimension, then there always exists a dense set of absolutely nonsingular extremals.

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