On analysis and design of oscillatory control systems

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Abstract

This paper presents analysis and design results for control systems subject to oscillatory inputs, i.e., inputs of large amplitude and high frequency. The key analysis results are a series expansion characterizing the averaged system and various Lie-algebraic conditions that guarantee the series can be summed. Various example systems provide insight into the results. With regards to design, we recover and extend a variety of point stabilization and trajectory tracking results using oscillatory controls. We present novel developments on stabilization of systems with positive trace and on tracking for second order systems.

Keywords

oscillatory control, averaging, geometric methods, point stabilization, trajectory tracking

I. Introduction

The paper investigates the behavior of finite dimensional analytic systems subject to oscillatory controls. We present averaging analysis and control design results for systems described by a differential equation of the form

$$\dot{x} = f(t,x) + \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}, t, x\right) ,$$

where the vector field g is periodic in its first argument, ϵ is a small positive parameter, and both vector fields f and g are analytic in x. Our objective is to provide a rigorous and general framework that allows to obtain (1) a coordinate-free expression of the averaged system, and a series expansion representation for it; (2) control design tools for point stabilization and trajectory planning in underactuated systems.

Motivation

The study of oscillations in nonlinear differential equations is a classic and widespread research topic. Related research areas include nonlinear dynamical systems [1], nonlinear and geometric control [2], [3], analysis of animal locomotion [4], design of robotic locomotion and manipulation devices [5], analysis of switching circuit models and power conversion circuits [6], control of quantum dynamics [7] and chemical reactions [8], analysis, design, and control of biomineralization and crystallization processes [9], [10], and so forth.

Furthermore, averaging analysis seems well suited to tackle novel applications in the field of micro-electro mechanical systems and vibrational control is being investigated within the context of active control of fluids and separation control. Examples include [11] on the scale dependence in oscillatory control of mechanical systems, [12] on unsteady flow and separation control using oscillatory blowing.

From a control theoretical viewpoint, we study oscillatory controls for the purpose of stabilization and tracking problems. For classes of nonlinear underactuated systems, it is interesting to investigate what control objectives can be obtained via the use of high frequency, high amplitude inputs. Since modern textbooks [13], [14], [15], [16] do not present a comprehensive approach on perturbation methods in control theory, we endeavor to develop novel tools and shed further light onto these problems.

Literature review

This work has connections with classic averaging theory (see [17] for a standard treatment), as well as with numerous ongoing research efforts. First of all, our analysis complements the study of differential equations subject to periodic high frequency, high amplitude forcing terms; see [18], [19], [20]. In these works, the coupling effect between the input vector fields plays a key role: typically, Lie brackets between them appear, and in the averaging approximation the trajectories of the original system converge to those of the averaged system. Here we shall focus our attention on systems where the interaction takes place between the drift and the input vector fields.

A second set of related results deals with the analysis of high frequency vibrations in mechanical systems [21], [22], [23], [24], [25], [26], and more generally averaging analysis in locomotion, rectification and other physical phenomenon where non-commuting vector fields play a role; see [27], [28], [29].

Finally, there are three related areas within the context of control design. One of these is the design of time-varying stabilizing laws for driftless systems, (sometimes referred to as nonholonomic), see for instance [30], [31], [32], [33], [34]. A second area deals with the design of oscillatory controls for point stabilization in general nonlinear and mechanical control systems; see [35], [36], [37], [2], and a third area is devoted to the design of oscillatory controls for trajectory planning in driftless systems [38], [39], and for constructive controllability and approximate inversion [40], [41].

Statement of contributions

The first contribution of this paper is a general averaging analysis in a coordinate-free differential geometric setting. The first step in the averaging analysis is the computation of the averaged system based on a generalized variation of constants formula and on the notion of pull-back vector field. Next, we provide a more explicit representation of the averaged system as an infinite sum of Lie brackets of the input vector fields with the drift and iterated integrals of the open-loop controls. Finally, we present various examples systems including bilinear, Hamiltonian, and second order systems, and collect sufficient conditions that guarantee the series expansion for the averaged system is summable.

We perform this step for analytic control systems with two-time scales, and for the single and multiple input cases. In the case of multiple input systems, the key assumption is that input vector fields commute. The technical approach relies on a collection of tools that were originally developed in [42], [43] to analyse time-varying vector fields within the context of the so-called "chronological calculus."

After completing this general analysis, we present various design tools and results for vibrational control. With regards to point stabilization, we provide sufficient conditions for the existence of an equilibrium point for the averaged system, we prove that the order of linearizing and averaging is non-influential, and we design oscillatory controls to stabilize the averaged systems. In particular, we recover the known result on stabilization of systems with negative linearization trace, and prove a novel result on stabilization of systems with positive trace (via nonlinear feedback). With regards to trajectory tracking, we exploit our analysis results on nonlinear systems with two time scales to steer the averaged system along arbitrary paths. We focus on second order underactuated systems and illustrate how to use oscillatory signals to track an arbitrary smooth trajectory.

Organization

We introduce some preliminary concepts in Section II. Section III presents the main averaging analysis, and Section IV treats various example systems. Section V contains assumptions that simplify the averaging analysis and open the way to design. Section VI and Section VII discuss respectively stabilization and tracking via oscillatory controls. Finally, we present our conclusions in Section VIII.

II. Preliminaries and notation

This section contains some basic definitions and results on iterated integrals of scalar functions and on differential geometry.

A. Iterated integrals and their averages

Let $\mathbb{N}$ be the set of strictly positive integers and $\mathbb{R}_+ = [0, +\infty)$. Given a bounded function $u : \mathbb{R}_+ \to \mathbb{R}$, define its *iterated integrals* $\{U_k : \mathbb{R}_+ \to \mathbb{R}, k \in \mathbb{N}\}$ as

$$U_k(t) = \int_0^t u(t_k) \int_0^{t_k} u(t_{k-1}) \cdots \int_0^{t_2} u(t_1)\, dt_1 \dots dt_k\,.$$

Lemma II.1: Let u be a bounded function. Its iterated integrals satisfy

$$U_k(t) = \frac{1}{k!} \left(\int_0^t u(\tau) d\tau \right)^k , \tag{1}$$

and are T-periodic functions if and only if u is T-periodic and zero-mean, i.e., $\int_0^T u(\tau)d\tau = 0$.

We postpone the proof to Lemma II.3 below. Given a T-periodic function $V(t)$, let us denote

$$\overline{V} = \frac{1}{T} \int_0^T V(t) dt\,.$$

If u is T-periodic and zero-mean, the sequence of its *averages* $\{\overline{U}_k \in \mathbb{R}, k \in \mathbb{N}\}$ is defined by means of its iterated integrals as

$$\overline{U}_k = \frac{1}{T} \int_0^T U_k(t) dt = \frac{1}{k! T} \int_0^T \left(\int_0^t u(\tau) d\tau \right)^k dt.$$

Let $||u||_{\mathcal{L}_\infty}$ be the supremum of the absolute value $u(t)$ for all $t \in \mathbb{R}_+$. Note that, since u is T-periodic,

$$||u||_{\mathcal{L}_\infty} = \sup_{t \in [0,T]} |u(t)|\,.$$

Lemma II.2: Let u be a bounded, T-periodic, and zero-mean function. Its iterated integrals and averages are bounded according to

$$\|U_k\|_{\mathcal{L}_\infty} \leq \frac{(T\,\|u\|_{\mathcal{L}_\infty})^k}{k!}, \qquad |\overline{U}_k| \leq \frac{(T\,\|u\|_{\mathcal{L}_\infty})^k}{(k+1)!}.$$

We postpone the proof to Lemma II.4 below. Finally, we consider the example functions

$$v(t) = a\sin\omega t\ , \quad u(t) = a\cos\omega t\ , \quad \omega \in \mathbb{N},$$

and easily compute the $(2\pi/\omega)$-periodic signals

$$V_k(t) = \frac{1}{k!}\left(\frac{2a}{\omega}\sin^2\left(\frac{\omega t}{2}\right)\right)^k, \quad U_k(t) = \frac{1}{k!}\left(\frac{a}{\omega}\sin\omega t\right)^k.$$

From the identity $4^m\int_0^{2\pi}(\sin t)^{2m}dt = 2\pi\begin{pmatrix}2m\\ m\end{pmatrix}$ in [44], the averages become

$$\overline{V}_k = \frac{1}{k!}\begin{pmatrix}2k\\ k\end{pmatrix}\left(\frac{a}{2\omega}\right)^k\ , \quad \overline{U}_k = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{1}{k!}\begin{pmatrix}k\\ k/2\end{pmatrix}\left(\dfrac{a}{2\omega}\right)^k & \text{if } k \text{ is even}\end{cases} \tag{2}$$

B. Iterated integrals of multiple functions

Let $\mathcal{I}$ be the set of all nontrivial multiindices $I = \{i_1, \ldots, i_k\}$, where the integers $i_1, \ldots, i_k$ take values in $\{1, \ldots, m\}$. Given m bounded functions $u_i : \mathbb{R}_+ \to \mathbb{R}$, define their *iterated integrals* $\{U_I : \mathbb{R}_+ \to \mathbb{R}, I \in \mathcal{I}\}$ by

$$U_{\{i_1,\ldots,i_k\}}(t) = \int_0^t u_{i_k}(t_k)\int_0^{t_k} u_{i_{k-1}}(t_{k-1})\cdots\int_0^{t_2} u_{i_1}(t_1)dt_1\ldots dt_k\,.$$

Note that $U_2(t)$ and $U_{\{2\}}(t)$ denote different functions.

Let $C_{k_1,\ldots,k_m}$ be the collection of all possible ways of taking m classes of $k_1 + \cdots + k_m$ different objects with k_i objects in the ith class. The cardinality of $C_{k_1,\ldots,k_m}$ is the multinomial coefficient

$$\begin{pmatrix}k_1 + \cdots + k_m\\ k_1, \ldots, k_m\end{pmatrix} = \frac{(k_1 + \cdots + k_m)!}{k_1!\ldots k_m!}.$$

To each element $\alpha \in C_{k_1,\ldots,k_m}$, we associate a multiindex $I(\alpha)$ of length $k_1 + \cdots + k_m$ as follows: as $i \in \{1, \ldots, m\}$, place the index i in the k_i places corresponding to the ith class of α.

Given m bounded functions $u_i : \mathbb{R}_+ \to \mathbb{R}$, define their *multinomial iterated integrals* $\{\mathcal{U}_{k_1,\dots,k_m} : \mathbb{R}_+ \to \mathbb{R}, k_1, \dots, k_m \in \mathbb{N}\}$ according to

$$\mathcal{U}_{k_1,\dots,k_m}(t) = \sum_{\alpha \in C_{k_1,\dots,k_m}} U_{I(\alpha)}(t)\,. \tag{3}$$

Furthermore, let $\mathcal{U}_{0,\dots,0}(t) \equiv 1$.

Lemma II.3: Let $u_1, \dots, u_m$ be bounded functions. Their multinomial iterated integrals satisfy

$$\mathcal{U}_{k_1,\dots,k_m}(t) = \frac{1}{k_1! \dots k_m!} \left(\int_0^t u_1(\tau) d\tau\right)^{k_1} \dots \left(\int_0^t u_m(\tau) d\tau\right)^{k_m}\,. \tag{4}$$

The functions $\mathcal{U}_{k_1,\dots,k_m}$ are T-periodic if and only if $u_1, \dots, u_m$ are T-periodic and zero-mean.

Proof: We prove the result by induction on $k = k_1 + \dots + k_m$. For $k = 1$, we have that $k_i = \delta_{ij}$, for some $j \in \{1, \dots, m\}$. Then, $\mathcal{U}_{k_1,\dots,k_m}(t) = \int_0^t u_j(\tau) d\tau = U_{\{j\}}(t)$.

Assume the claim is true for $k-1$ and let us prove it for k. From (4), one can compute

$$\begin{aligned}\frac{d}{dt}\left(\mathcal{U}_{k_1,\dots,k_m}(t)\right) &= \frac{1}{k_1! \dots k_m!}\left(k_1 u_1 U_{\{1\}}^{k_1-1} \dots U_{\{m\}}^{k_m} + \dots + k_m u_m U_{\{1\}}^{k_1} \dots U_{\{m\}}^{k_m-1}\right)\\ &= u_1(t)\mathcal{U}_{k_1-1,\dots,k_m}(t) + \dots + u_m(t)\mathcal{U}_{k_1,\dots,k_m-1}(t)\,,\end{aligned}$$

where $\mathcal{U}_{l_1,\dots,l_m}(t) \equiv 0$ if any of the l_i is negative. Integrating with respect to time we have that the expression in (4) satisfies

$$\mathcal{U}_{k_1,\dots,k_m}(t) = \int_0^t \Big(u_1(s)\mathcal{U}_{k_1-1,\dots,k_m}(s) + \dots + u_m(s)\mathcal{U}_{k_1,\dots,k_m-1}(s)\Big) ds\,.$$

The claim now follows by noting that this equation is equivalent to the definition of multinomial iterated integral (3).

Next, we prove the second statement. All functions $\mathcal{U}_{k_1,\dots,k_m}$ are T-periodic if and only if all functions $\int_0^t u_i(\tau)d\tau$ are T-periodic. Since

$$\int_0^{T+t} u_i(\tau) d\tau = \int_0^T u_i(\tau) d\tau + \int_T^{T+t} u_i(\tau) d\tau,$$

the functions $\int_0^t u_i(\tau)d\tau$ are T-periodic if u_i are zero-mean and T-periodic. Furthermore, the u_i are T-periodic if their time integrals are T-periodic, and they are zero-mean if $\int_0^T u_i(\tau)d\tau = \int_0^0 u_i(\tau)d\tau = 0$. ■

Note that for the single-input case, $m = 1$, $u_1 = u$, we have that $\mathcal{U}_k(t) = \sum_{\alpha \in C_k} U_{I(\alpha)}(t) = U_{\{1,\dots,1\}}(t) = U_k(t)$ and we recover the result in Lemma II.1

$$\mathcal{U}_k(t) = \frac{1}{k!} U^k_{\{1\}}(t)\,.$$

Lemma II.4: Let $u_1, \dots, u_m$ be bounded, T-periodic and zero-mean functions. Then

$$\begin{aligned} \|\mathcal{U}_{k_1,\dots,k_m}\|_{\mathcal{L}\infty} &\le \frac{T^{k_1+\cdots+k_m}}{k_1!\dots k_m!} \|u_1\|^{k_1}_{\mathcal{L}\infty} \dots \|u_m\|^{k_m}_{\mathcal{L}\infty}\,, \\ |\overline{\mathcal{U}}_{k_1,\dots,k_m}| &\le \frac{T^{k_1+\cdots+k_m}}{k_1!\dots k_m!(1+\sum_{j=1}^m k_j)} \|u_1\|^{k_1}_{\mathcal{L}\infty} \dots \|u_m\|^{k_m}_{\mathcal{L}\infty}\,. \end{aligned}$$

Proof: Recall that, under the given assumptions, $\mathcal{U}_{k_1,\dots,k_m}(t)$ are T-periodic. For $0 \le t \le T$,

$$\begin{aligned} |\mathcal{U}_{k_1,\dots,k_m}(t)| &\le \frac{1}{k_1!\dots k_m!} \left(\int_0^t |u_1(\tau)| d\tau\right)^{k_1} \dots \left(\int_0^t |u_m(\tau)| d\tau\right)^{k_m} \\ &\le \frac{1}{k_1!\dots k_m!} t^{k_1} \|u_1\|^{k_1}_{\mathcal{L}\infty} \dots t^{k_m} \|u_1\|^{k_m}_{\mathcal{L}\infty}\,, \end{aligned}$$

which gives the first bound. The second one is proven via the chain of inequalities

$$|\overline{\mathcal{U}}_{k_1,\dots,k_m}| \le \frac{1}{T}\int_0^T |\mathcal{U}_{k_1,\dots,k_m}(t)| dt \le \frac{\|u_1\|^{k_1}_{\mathcal{L}\infty} \dots \|u_1\|^{k_m}_{\mathcal{L}\infty}}{k_1!\dots k_m!} \frac{1}{T}\int_0^T t^{k_1+\cdots+k_m} dt\,.$$

■

As an example, consider the functions $u_i(t) = a_i \cos\omega t$, $\omega \in \mathbb{N}$. Then,

$$\mathcal{U}_{k_1,\dots,k_m}(t) = \frac{a_1^{k_1}\dots a_m^{k_m}}{k_1!\dots k_m!} \left(\frac{1}{\omega}\sin\omega t\right)^{k_1+\cdots+k_m}\,,$$

and their averages are

$$\overline{\mathcal{U}}_{k_1,\dots,k_m} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{a_1^{k_1}\dots a_m^{k_m}}{k_1!\dots k_m!} \left(\dfrac{1}{2\omega}\right)^k \begin{pmatrix} k \\ k/2 \end{pmatrix} & \text{if } k \text{ is even} \end{cases} \tag{5}$$

with $k = \sum_{j=1}^m k_j$.

C. Elements of differential geometry and complex analysis

We refer to [42], [43], [45] for comprehensive references on these topics. Let $x, x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, and let the parameter ϵ vary in the range $(0, \epsilon_0]$ with $\epsilon_0 \ll 1$. Let $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to$

$\mathbb{R}^n$ be smooth time-varying vector fields. Define their Lie bracket according to

$$[g, f] = \frac{\partial f(t,x)}{\partial x} g(t,x) - \frac{\partial g(t,x)}{\partial x} f(t,x)\,.$$

In what follows, we will use the notation $\mathrm{ad}_g^0 f = f$, $\mathrm{ad}_g f = [g, f]$ and $\mathrm{ad}_g^k f = \mathrm{ad}_g^{k-1}[g, f]$. Given a diffeomorphism ϕ and a vector field f, the *pull-back of g along ϕ*, denoted by $\phi^* f$, is the vector field

$$(\phi^* f)(x) = \left(\frac{\partial \phi^{-1}}{\partial x} \circ f \circ \phi\right)(x).$$

A useful diffeomorphism is the flow map $x(t) = \Phi^g_{0,T}(x_0)$ describing the solution at time T to the initial value problem $\dot{x} = g(t,x), \quad x(0) = x_0$. Let $T > 0$ be small enough so that the flow map $\Phi^g_{0,T}$ is a local diffeomorphism in a neighborhood of x_0.

Lemma II.5 (Variation of constants formula) Consider
the initial value problem $\dot{x} = f(t,x) + g(t,x)$, $x(0) = x_0$. The final value $x(T) = \Phi^{f+g}_{0,T}(x_0)$ can be written as

$$\begin{aligned} x(T) &= \Phi^g_{0,T}(z(T)), \\ \dot{z}(t) &= \left(\left(\Phi^g_{0,t}\right)^* f\right)(z), \quad z(0) = x_0. \end{aligned}$$

Given a positive scalar σ, define the complex σ-neighborhood of x_0 in $\mathbb{C}^n$ as $B_\sigma(x_0) = \{z \in \mathbb{C}^n \,:\, ||z - x_0|| < \sigma\}$. Let f be a real analytic function on $\mathbb{R}^n$ that admits a bounded analytic continuation over $B_\sigma(x_0)$. The norm of f is defined as

$$||f||_\sigma = \max_{z \in B_\sigma(x_0)} |f(z)|,$$

where f denotes both the function over $\mathbb{R}^n$ and its analytic continuation. Given a time-varying vector field $(q,t) \mapsto Y(q,t) = Y_t(q)$, let Y_t^i be its ith component with respect to the usual basis on $\mathbb{R}^n$. Assuming $t \in [0,T]$, and assuming that every component function Y_t^i is analytic over $B_\sigma(x_0)$, we define the norm of Y as

$$||Y||_{\sigma,T} = \max_{t\in[0,T]} \max_{i \in \{1,\dots,n\}} ||Y_t^i||_\sigma.$$

In what follows, we shall simplify notation by neglecting the subscript T in the norm of a time-varying vector field.

III. Coordinate-free averaging under oscillatory controls

We study averaging under oscillatory controls, borrowing ideas from the standard treatment on averaging in [17], and introducing some differential geometric aspects from [42], [26], [46].

Let $x : [0,T] \to \mathbb{R}^n$ be the solution to the initial value problem

$$\frac{dx}{dt} = f(t,x) + \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}, t, x\right), \qquad x(0) = x_0 \,. \tag{6}$$

Enlarge the state space by considering $x' = (t,x)$, denote by $\tau = t/\epsilon$ the fast time scale, and rewrite equation (6) as

$$\frac{dx'}{d\tau} = \epsilon f'(x') + g'(\tau, x') \,, \quad x'_0 = (0, x_0) \,, \tag{7}$$

where the vector fields f' and g' are defined according to

$$f'(x') = (1, f(t,x)) \,, \qquad g'(\tau, x') = (0, g(\tau, t, x)) \,.$$

In the extended space, τ is the independent variable and (t,x) are dependent variables. We shall write the flow of g' as

$$\Phi_{0,\tau}^{g'(\tau,x')}(t,x) = \left(t, \Phi_{0,\tau}^{g(\tau,t,x)}(x)\right) \,,$$

and define the *pull-back vector field* F' as

$$F'(\tau, x') = \left(\left(\Phi_{0,\tau}^{g'(\tau,x')}\right)^* f'\right)(x') \,. \tag{8}$$

Note that F' is of the form

$$F'(\tau, x') = (1, F(\tau, x')) \,. \tag{9}$$

Lemma III.1: Assume that the vector fields in $\{g_\tau, \ \tau \in [0,T]\}$ commute and that they are T-periodic and zero mean in their first argument, i.e., $g(\tau + T, t, x) = g(\tau, t, x)$, and $\int_0^T g(\tau, t, x) d\tau = 0$. Then, the flow $\Phi_{0,\tau}^{g'(\tau,x')}$ and the vector field F' are T-periodic.

Proof: The assumptions on the family of vector fields $\{g_\tau, \ \tau \in [0,T]\}$ are automatically verified by the family $\{g'_\tau, \ \tau \in [0,T]\}$. Let $x'(t)$ be the solution to the initial value problem

$$\dot{x}'(\tau) = g'(\tau, x') \,, \quad x'(0) = x'_0 \,. \tag{10}$$

Note that $x'(\tau) = \Phi_{0,\tau}^{g'(\tau,x')}(x'_0)$. Let $X'(\tau) = x'(\tau + T)$. Then

$$\dot{X}'(\tau) = g'(\tau + T, X') = g'(\tau, X')\,, \quad X'(0) = x'(T)\,.$$

Consequently, $X'(\tau) = x'(\tau)$ iff $x'(T) = x'(0)$. To prove the latter statement we introduce the Volterra series [42], [14]. The flow of (10) is formally represented by the expansion

$$\begin{aligned} x'(\tau) &\equiv \mathrm{Id}(x'_0) + \sum_{k=1}^{+\infty} \int_0^\tau ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k (g'(s_k, x'_0) \circ \cdots \circ g'(s_1, x'_0)) \\ &= x'_0 + \int_0^\tau g'(s, x'_0) ds + \int_0^\tau \int_0^{s_1} (g'(s_2, x'_0) \circ g'(s_1, x'_0)) ds_2 ds_1 + \ldots \end{aligned} \tag{11}$$

where the vector fields g' are interpreted as derivations of $C^\infty(\mathbb{R}^{n+1})$. Now, using integration by parts and the fact that the vector fields commute, we have

$$\begin{aligned} &\int_0^\tau \left(\int_0^{s_1} g'(s_2, x'_0) ds_2 \right) \circ g'(s_1, x'_0) ds_1 = \\ &\qquad \left(\int_0^\tau g'(s, x'_0) ds \right)^2 - \int_0^\tau g'(s_1, x'_0) \circ \left(\int_0^{s_1} g'(s_2, x'_0) ds_2 \right) ds_1\,, \end{aligned}$$

and hence

$$\int_0^\tau \int_0^{s_1} (g'(s_2, x'_0) \circ g'(s_1, x'_0)) ds_2 ds_1 = \frac{1}{2} \left(\int_0^\tau g'(s, x'_0) ds \right)^2.$$

By induction, one can show that

$$\int_0^\tau ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{k-1}} ds_k (g'(s_k, x'_0) \circ \cdots \circ g'(s_1, x'_0)) = \frac{1}{k!} \left(\int_0^\tau g'(s, x'_0) ds \right)^k. \tag{12}$$

Since by hypothesis $g'(\tau, x')$ is zero-mean, we conclude from (11) that $x'(T) = x'(0)$. ■

Given the result in the lemma, we define the averaged vector field $\overline{F'}$ as

$$\overline{F'}(x') = \frac{1}{T} \int_0^T F'(\tau, x') d\tau\,.$$

It is straightforward to see that $\overline{F'}(x') = \left(1, \overline{F}(x')\right)$. Finally, let $y, z : [0, T] \to \mathbb{R}^n$ be solutions to the initial value problems

$$\frac{dz}{dt} = F\left(\frac{t}{\epsilon}, t, z\right), \quad z(0) = x_0, \tag{13}$$

$$\frac{dy}{dt} = \overline{F}(t, y), \quad y(0) = x_0. \tag{14}$$

Theorem III.2 (Coordinate-free averaging) Assume that the vector fields in $\{g_\tau,\ \tau \in [0,T]\}$ commute and that they are T-periodic and zero mean in their first argument. For $t \in \mathbb{R}_+$, we have

$$x(t) = \Phi^{g(\tau,t,x)}_{0,t/\epsilon}(z(t))\,,$$

and, as $\epsilon \to 0$ on the time scale 1, we have

$$z(t) - y(t) = O(\epsilon)\,.$$

Additionally, assume both f and g do not depend explicitly on the slow time scale t, i.e. $f = f(x)$ and $g = g(t/\epsilon, x)$. If the origin is a hyperbolically stable critical point for $\overline{F} = \overline{F}(x)$, then $z(t) - y(t) = O(\epsilon)$ as $\epsilon \to 0$ for all $t \in \mathbb{R}_+$ and the differential equation (13) possesses a unique periodic orbit which is hyperbolically stable and belongs to an $O(\epsilon)$ neighborhood of the origin.

Proof: Applying the variation of constants formula to (7), we get

$$\begin{aligned} \frac{dx'}{d\tau} &= g'(\tau, x')\,, \qquad x'(0) = z'(\tau)\,, \\ \frac{dz'}{d\tau} &= \epsilon F'(\tau, z')\,, \qquad z'(0) = x'_0\,. \end{aligned} \tag{15}$$

Averaging this last system in z', we obtain

$$\frac{dy'}{d\tau} = \epsilon \overline{F'}(y')\,, \qquad y'(0) = x'_0\,.$$

By the theorem of first order averaging [17], we know that $z'(\tau) - y'(\tau) = O(\epsilon)$ over the time scale $\tau = 1/\epsilon$. Now, if we write $y' = (v, y)$, we get from the previous equation that $v = t$ and, changing the time scale back to $t = \epsilon\tau$,

$$\frac{dy}{dt} = \overline{F}(t, y)\,, \qquad y(0) = x_0\,,$$

which is the definition of equation (14). Putting $z' = (u, z)$, we also deduce that $u = t$ and $z(t) - y(t) = O(\epsilon)$ over the time scale 1. In addition, we recover equation (13)

$$\frac{dz}{dt} = F\left(\frac{t}{\epsilon}, t, z\right)\,, \qquad z(0) = x_0\,.$$

Finally, we get from (15) that

$$x(t) = \Phi^{g(\tau,t,x)}_{0,t/\epsilon}(z(t))\,.$$

In case $f = f(x)$ and $g = g(t/\epsilon, x)$, if the origin is a hyperbolically stable critical point for $\overline{F}$, then the second result follows from the theorem of first order averaging [17]. ■

Next, we refine the approximation result as follows. Let $\lfloor s \rfloor$ be the greatest integer less than or equal to $s \in \mathbb{R}$, and let $(t/\epsilon \mod T)$ denote the time instant $t/\epsilon - \lfloor t/(\epsilon T) \rfloor T$.

Proposition III.3: Under the hypothesis of Theorem III.2, if $T = O(1)$, then $x(t) = \Phi^{g(\tau,t,x)}_{0,(t/\epsilon \mod T)}(y(t)) + O(\epsilon)$, on the time scale 1.

Proof: Since g is T-periodic and zero-mean, we have that

$$\int_0^{t/\epsilon} g(s,t,x)ds = \int_0^{(t/\epsilon \mod T)} g(s,t,x)ds + \int_{(t/\epsilon \mod T)}^{t/\epsilon} g(s,t,x)ds = \int_0^{(t/\epsilon \mod T)} g(s,t,x)ds,$$

and, using the Volterra series expansion for the flow $\Phi^{g(s,t,x)}_{0,t/\epsilon}$ (see [42], [14]) and equation (12), we have

$$\begin{aligned}\Phi^{g(\tau,t,x)}_{0,t/\epsilon}(z(t)) &= z(t) + \sum_{k=1}^{+\infty} \frac{1}{k!}\left(\int_0^{t/\epsilon} g(\tau, z(t))ds\right)^k \\ &= z(t) + \sum_{k=1}^{+\infty} \frac{1}{k!}\left(\int_0^{t/\epsilon \mod T} g(\tau, z(t))ds\right)^k = \Phi^{g(\tau,t,x)}_{0,(t/\epsilon \mod T)}(z(t))\,.\end{aligned}$$

Recall that $z(t) = y(t) + O(\epsilon)$ on the time scale 1. Since $(t/\epsilon \mod T) = O(1)$ and since the flow along g depends continuously on its initial condition we have

$$x(t) = \Phi^{g(\tau,t,x)}_{0,t/\epsilon}(z(t)) = \Phi^{g(\tau,t,x)}_{0,(t/\epsilon \mod T)}(y(t) + O(\epsilon)) = \Phi^{g(\tau,t,x)}_{0,(t/\epsilon \mod T)}(y(t)) + O(\epsilon).$$

■

A. Series expansions for the averaged system

In this section, we develop series expansions for the averaged system for particular forms of the vector field g.

We start by studying the single input case. Assume that the input vector field in equation (6) is of the form $g\,(t/\epsilon, t, x) = u(t/\epsilon, t) g(x)$, where u is a bounded function. In other words, equation (6) reads

$$\frac{dx}{dt} = f(t,x) + \frac{1}{\epsilon} u\left(\frac{t}{\epsilon}, t\right) g(x), \qquad x(0) = x_0. \tag{16}$$

Accordingly, we shall consider the iterated integrals $U_k(\tau, t)$ and their averages $\overline{U}_k(t)$ with respect to the first variable of the input $u(\tau, t)$.

Theorem III.4 (Single input system) Let $(\tau, t) \mapsto u(\tau, t)$ be a bounded function, T-periodic and zero-mean in τ, continuously differentiable in t. The vector field F defined in equation (9) satisfies the formal expansion

$$F(\tau, t, x) = f(t, x) + \sum_{k=1}^{+\infty} U_k(\tau, t)\, \mathrm{ad}_g^k\, f(t, x) - \frac{\partial U_1}{\partial t}(\tau, t) g(x) \tag{17}$$

and its average $\overline{F}$ satisfies

$$\overline{F}(t, x) = f(t, x) + \sum_{k=1}^{+\infty} \overline{U}_k(t)\, \mathrm{ad}_g^k\, f(t, x) - \frac{d\overline{U}_1}{dt}(t) g(x). \tag{18}$$

Proof: We compute F' as in equation (8), where we let $f' = f'(x')$ be τ-invariant and $g' = g'(\tau, x')$ be τ-varying. The following statement is proved in [45, Theorem 4.2.31]

$$\frac{d}{d\tau}\left(\left(\Phi_{0,\tau}^{g'}\right)^* f'\right)(\tau, x') = \left(\Phi_{0,\tau}^{g'}\right)^* [g'(\tau, x'), f'(x')].$$

At fixed $x' \in \mathbb{R}^{n+1}$, we integrate the previous equation from time 0 to τ to obtain

$$\left(\left(\Phi_{0,\tau}^{g'}\right)^* f'\right)(\tau, x') = f'(x') + \int_0^\tau (\Phi_{0,s}^{g'})^* [g'(s, x'), f'(x')] ds.$$

Iteratively applying the previous equality, we get

$$\left(\left(\Phi_{0,\tau}^{g'}\right)^* f'\right)(\tau, x') = f'(x') + \sum_{k=1}^{+\infty} \int_0^\tau \dots \int_0^{s_{k-1}} \left(\mathrm{ad}_{g'(s_k, x')} \dots \mathrm{ad}_{g'(s_1, x')}\, f'(x')\right) ds_k \dots ds_1$$

Now, it can be proven by induction that

$$\begin{aligned}
\mathrm{ad}_{g'(s_1, t, x)}\, f' &= \left(0, u(s_1, t)\, \mathrm{ad}_{g(x)}\, f - \frac{\partial u}{\partial t}(s_1, t) g(x)\right), \\
\mathrm{ad}_{g'(s_k, x')} \dots \mathrm{ad}_{g'(s_1, x')}\, f' &= \left(0, u(s_k, t) \dots u(s_1, t)\, \mathrm{ad}_{g(x)}^k\, f\right).
\end{aligned}$$

with $k \geq 2$. Finally,

$$F(\tau, t, x) = f(t, x) + \sum_{k=1}^{+\infty} U_k(\tau, t)\, \mathrm{ad}_g^k\, f(t, x) - \frac{\partial U_1}{\partial t}(\tau, t) g(x).$$

■

Next, we consider the multiple input system

$$\frac{dx}{dt} = f(x) + \frac{1}{\epsilon}\left(u_1\left(\frac{t}{\epsilon}, t\right) g_1(x) + \cdots + u_m\left(\frac{t}{\epsilon}, t\right) g_m(x)\right).$$

The following result will be useful to obtain a series expansion for the averaged system. It is readily proved by induction.

Lemma III.5: Let $g_1, \ldots, g_m$ be commuting vector fields. Take a multiindex $\{i_1, \ldots, i_k\} \in \mathcal{I}$ and let $\sigma \in \Sigma_k$ be a permutation of k symbols. Then

$$\mathrm{ad}_{g_{i_{\sigma(1)}}} \ldots \mathrm{ad}_{g_{i_{\sigma(k)}}} f = \mathrm{ad}_{g_{i_1}} \ldots \mathrm{ad}_{g_{i_k}} f\,.$$

We now present the main analysis result in this section.

Theorem III.6 (Multiple input system) Let $(\tau, t) \mapsto u_1(\tau, t), \ldots, u_m(\tau, t)$ be bounded functions, T-periodic and zero-mean in τ, continuously differentiable in t. Let $g_1, \ldots, g_m$ be commuting vector fields. The pull-back vector field F defined in equation (9) satisfies the formal expansions

$$\begin{aligned} F(\tau, t, x) &= f(t, x) + \sum_{k=1}^{+\infty} \sum_{\{i_1 \ldots i_k\} \in \mathcal{I}} U_{\{i_1, \ldots, i_k\}}(\tau, t)\, \mathrm{ad}_{g_{i_1}} \ldots \mathrm{ad}_{g_{i_k}} f(t, x) - \sum_{i=1}^{m} \frac{\partial U_{\{i\}}}{\partial t}(\tau, t) g_i(x) \\ &= \sum_{k_1, \ldots, k_m = 0}^{+\infty} \mathcal{U}_{k_1, \ldots, k_m}(\tau, t)\, \mathrm{ad}_{g_1}^{k_1} \ldots \mathrm{ad}_{g_m}^{k_m} f - \sum_{i=1}^{m} \frac{\partial U_{\{i\}}}{\partial t}(\tau, t) g_i(x), \end{aligned} \tag{19}$$

and its average $\overline{F}$ satisfies

$$\begin{aligned} \overline{F}(t, x) &= f(t, x) + \sum_{k=1}^{+\infty} \sum_{\{i_1 \ldots i_k\} \in \mathcal{I}} \overline{U}_{\{i_1, \ldots, i_k\}}(t)\, \mathrm{ad}_{g_{i_1}} \ldots \mathrm{ad}_{g_{i_k}} f(t, x) - \sum_{i=1}^{m} \frac{d\overline{U}_{\{i\}}}{dt}(t) g_i(x) \\ &= \sum_{k_1, \ldots, k_m = 0}^{+\infty} \overline{\mathcal{U}}_{k_1, \ldots, k_m}(t)\, \mathrm{ad}_{g_1}^{k_1} \ldots \mathrm{ad}_{g_m}^{k_m} f - \sum_{i=1}^{m} \frac{d\overline{U}_{\{i\}}}{dt}(t) g_i(x). \end{aligned} \tag{20}$$

Proof: The first series expansion for F and $\overline{F}$ are deduced in the same way as in the proof of Theorem III.4. As for the second ones, note that using Lemma III.5, we have for each k,

$$\sum_{\{i_1, \ldots, i_k\} \in \mathcal{I}} U_{\{i_1, \ldots, i_k\}}(\tau, t)\, \mathrm{ad}_{g_{i_1}} \ldots \mathrm{ad}_{g_{i_k}} f(t, x) = \sum_{\substack{k_1, \ldots, k_m \geq 0 \\ k_1 + \cdots + k_m = k}} \mathcal{U}_{k_1, \ldots, k_m}(\tau, t)\, \mathrm{ad}_{g_1}^{k_1} \ldots \mathrm{ad}_{g_m}^{k_m} f(t, x)\,.$$

■

Note that in the single-input case, $m = 1$, both series in (19) coincide with the series in (17). Finally, we characterize the convergence properties of the series. For

$(\tau,t) \mapsto u(\tau,t)$, let $\|u\|_{\mathcal{L}\infty}$ denote the supremum of the absolute value of $u(\tau,t)$ for $\tau, t \in \mathbb{R}_+$.

Lemma III.7: Let f and $g_1,\dots,g_m$, be analytic vector fields admitting bounded analytical continuations over $B_\sigma(x_0)$. Assume that the input vector fields commute. Let $(\tau,t) \mapsto u_1(\tau,t),\dots,u_m(\tau,t)$ be bounded functions, T-periodic and zero-mean in τ, continuously differentiable in t. Let $0<\sigma'<\sigma$ and assume further that

$$T\sum_{j=1}^{m} \|u_j\|_{\mathcal{L}\infty}\|g_j\|_\sigma < \frac{\sigma-\sigma'}{4n}\,. \tag{21}$$

Then, the series expansions (19) and (20) converge absolutely and uniformly for $x \in B_{\sigma'}(x_0)$ and t $\in \mathbb{R}_+$.

Proof: From Lemma II.4 and Proposition 3.1 in [42], we have that

$$\begin{aligned}\left\|\mathcal{U}_{k_1,\dots,k_m}(\tau,t)\,\mathrm{ad}_{g_1}^{k_1}\dots\mathrm{ad}_{g_m}^{k_m} f\right\|_{\sigma'} &\le \left(\frac{4nT}{\sigma-\sigma'}\right)^{k}\|u_1\|_{\mathcal{L}\infty}^{k_1}\dots\|u_m\|_{\mathcal{L}\infty}^{k_m}\|g_1\|_\sigma^{k_1}\dots\|g_m\|_\sigma^{k_m}\|f\|_\sigma\\ &\le \left(\frac{4nT}{\sigma-\sigma'}\right)^{k}\left(\sum_{j=1}^{m}\|u_j\|_{\mathcal{L}\infty}\|g_j\|_\sigma\right)^{k}\|f\|_\sigma\,,\end{aligned}$$

where $k=k_1+\dots+k_m$. As a consequence, equation (21) implies that the series in F is convergent. This also implies the convergence of the series expansions of $\overline{F}$. ■

IV. Example systems

In this section we investigate classes of differential equations for which the series expansions in Section III-A assume a particular structure. By doing so, we recover and extend a variety of earlier results on bilinear, polynomial and Hamiltonian systems.

Before proceeding, we summarize the averaging procedure from Theorem III.2 and Proposition III.3 as follows,

$$x(t) = \Phi^{g(t,x)}_{0,(t/\epsilon \bmod T)}(y(t)) + O(\epsilon)\,,\ \dot{y}=\overline{F}(y),\ y(0)=x_0\,.$$

Accordingly, for each example we shall compute $\Phi^{g(t,x)}_{0,(t/\epsilon \bmod T)}$ and $\overline{F}$. For simplicity, we will mainly treat single input systems with a single time scale, i.e. $g(\tau,t,x)=u(\tau)g(x)$.

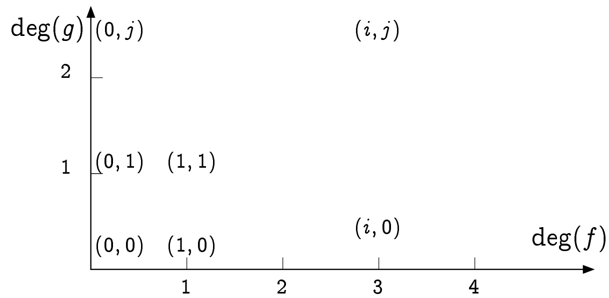

Fig. 1. Table of degrees of the drift vector field f and the input vector field g for homogeneus systems. The (i,j)th position refers to the case when $f \in \mathcal{H}_i$ and $g \in \mathcal{H}_j$.

A. Homogeneus systems

Here we focus our attention on homogeneus systems. Let f be a vector field on $\mathbb{R}^n$. We say that f is homogeneus of degree i if each of its components with respect to the usual basis of $\mathbb{R}^n$ is a homogeneus function of degree i. The set of homogeneus vector fields of degree i is denoted by $\mathcal{H}_i$. We have, for instance, that $\mathcal{H}_0$ is the set of constant vector fields and $\mathcal{H}_1$ is the set of linear vector fields. By convention, $\mathcal{H}_i = \{0\}$, for $i \leq -1$. If $f \in \mathcal{H}_i$ and $g \in \mathcal{H}_j$, then $[f,g] \in \mathcal{H}_{i+j-1}$.

It is straightforward to obtain the relevant quantities from Theorems III.2 and III.4 for the lower triangular cases $(0,0)$, $(0,1)$ and $(1,0)$ in Figure 1. Indeed, we have that

Case (0,0): $\dot{x} = a + \frac{1}{\epsilon} u\left(\frac{t}{\epsilon}\right) b\,, \quad x(0) = x_0\,,$

$$\Phi^{u(t)b}_{0,(t/\epsilon \mod T)}(x_0) = b \int_0^{(t/\epsilon \mod T)} u(\tau) d\tau + x_0,\ \overline{F} = a\,.$$

Case (0,1): $\dot{x} = a + \frac{1}{\epsilon} u\left(\frac{t}{\epsilon}\right) Bx\,, \quad x(0) = x_0\,,$

$$\Phi^{u(t)Bx}_{0,(t/\epsilon \mod T)}(x_0) = \mathrm{e}^{B \int_0^{(t/\epsilon \mod T)} u(\tau) d\tau} x_0\,,\ \overline{F} = a + \sum_{k=1}^{+\infty} \overline{U}_k (-B)^k a\,.$$

Case (1,0): $\dot{x} = Ax + \frac{1}{\epsilon} u\left(\frac{t}{\epsilon}\right) b\,, \quad x(0) = x_0\,,$

$$\Phi^{u(t)b}_{0,(t/\epsilon \mod T)}(x_0) = b \int_0^{(t/\epsilon \mod T)} u(\tau) d\tau + x_0\,,\ \overline{F} = Ax + \overline{U}_1 Ab\,.$$

In the following sections, we consider the cases of bilinear and polynomial systems.

B. Bilinear systems

We refer the reader to [14, Section 2.4] for a treatment on bilinear systems. Consider the system

$$\dot{x} = Ax + \frac{1}{\epsilon} u\left(\frac{t}{\epsilon}\right) Bx, \quad x(0) = x_0. \tag{22}$$

Note that this system corresponds to the case $(1,1)$ in Figure 1. Lie brackets between linear vector fields are expressed in terms of matrix commutators

$$\operatorname{ad}_{Bx} Ax = (\operatorname{ad}_B A)x, \quad \text{where} \quad \operatorname{ad}_B A = AB - BA.$$

One can compute,

$$\Phi^{gu(t)}_{0,(t/\epsilon \bmod T)}(x_0) = \mathrm{e}^{B\int_0^{(t/\epsilon \bmod T)} u(\tau)d\tau} x_0\,, \quad \overline{F}(x) = \left(A + \sum_{k=1}^{+\infty} \overline{U}_k \operatorname{ad}_B^k A\right)x.$$

C. Polynomial systems

Consider the system

$$\dot{x} = f(x) + \frac{1}{\epsilon} u\left(\frac{t}{\epsilon}\right) g, \quad x(0) = x_0, \tag{23}$$

where the components of the vector field f are polynomials in x of degree at most M, and where the vector field $g(x) = g$ is constant. This system is the combination of a finite number of $(i,0)$ cases in Figure 1. This structure leads to the following simplifications. The degree of $\operatorname{ad}_g^k f$ is $M-k$, and only the first M Lie brackets are non-vanishing. Accordingly, we have that

$$\Phi^{gu(t)}_{0,(t/\epsilon \bmod T)}(x_0) = x_0 + \left(\int_0^{(t/\epsilon \bmod T)} u(\tau)d\tau\right) g\,,\ \overline{F}(x) = f(x) + \sum_{k=1}^{M} \overline{U}_k \frac{\partial^k f}{\partial x^k}(\underbrace{g,\ldots,g}_{k\,\text{times}})(x)\,.$$

Note that $\overline{F}$ is a finite sum of polynomial vector fields.

D. Second order systems

We next focus on control systems described by second order differential equations. This setting is representative of interesting examples and very instructive. Consider the second order system on $\mathbb{R}^n$

$$\ddot{x} = \frac{1}{\epsilon} u\left(\frac{t}{\epsilon}\right) g(x)\,. \tag{24}$$

To write the equation in the standard (first order) form (16), define the vector fields on $\mathbb{R}^{2n}$

$$f(x,\dot{x}) = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}, \quad g(x,\dot{x})^{\text{lift}} = \begin{bmatrix} 0 \\ g(x) \end{bmatrix},$$

and compute the relevant Lie brackets as

$$\mathrm{ad}_{g^{\mathrm{lift}}} f = \begin{bmatrix} g \\ -\frac{\partial g}{\partial x}\dot{x} \end{bmatrix}, \ \mathrm{ad}^2_{g^{\mathrm{lift}}} f = -\langle g : g\rangle^{\mathrm{lift}}, \ \mathrm{ad}^k_{g^{\mathrm{lift}}} f = 0\,,$$

for $k > 2$ and where we define the operation of *symmetric product* between vector fields g_a, g_b on $\mathbb{R}^n$ as

$$\langle g_a : g_b\rangle = \frac{\partial g_a}{\partial x} g_b + \frac{\partial g_b}{\partial x} g_a\,.$$

The relevant quantities from Theorem III.2 and Theorem III.4 are

$$\Phi^{g(x,\dot{x})^{\mathrm{lift}} u(t)}_{0,t} \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} + \begin{bmatrix} 0 \\ \left(\int_0^t u(\tau)d\tau\right) g(x_0) \end{bmatrix},$$

$$\overline{F} = f + \overline{U}_1\,\mathrm{ad}_{g^{\mathrm{lift}}} f + \overline{U}_2\,\mathrm{ad}^2_{g^{\mathrm{lift}}} f = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} + \overline{U}_1 \begin{bmatrix} g \\ -\frac{\partial g}{\partial x}\dot{x} \end{bmatrix} - \overline{U}_2 \begin{bmatrix} 0 \\ \langle g : g\rangle \end{bmatrix},$$

so that, using the variables (y_1, y_2) for the averaged system, we write

$$\begin{aligned} \dot{y}_1 &= y_2 + \overline{U}_1 g(y_1) \\ \dot{y}_2 &= -\overline{U}_1 \frac{\partial g}{\partial x}(y_1) y_2 - \overline{U}_2 \langle g : g\rangle(y_1) \end{aligned}$$

with initial conditions $(y_1(0), y_2(0)) = (x(0), \dot{x}(0))$. It is instructive to compute the second time derivative of y_1, and write the averaged system again as an equation of second order. Some straightforward simplifications lead to

$$\ddot{y}_1 = \left(\frac{1}{2}\overline{U}_1^2 - \overline{U}_2\right) \langle g : g\rangle(y_1), \tag{25}$$

with initial conditions $(y_1(0), \dot{y}_1(0)) = (x(0), \dot{x}(0) + \overline{U}_1 g(x(0)))$. In summary, we have

$$\begin{aligned} x(t) &= y_1(t) + O(\epsilon) \\ \dot{x}(t) &= \dot{y}_1(t) + g(y_1(t)) \left(\int_0^{(t/\epsilon \mod T)} u(s)ds - \overline{U}_1\right) + O(\epsilon). \end{aligned}$$

Remark IV.1: Analogues to the result in equation (25) and their physical meaning have been long studied; e.g., see [21], [22], [24], [25], [26]. In particular, if g is a potential field, $g = \partial V/\partial x$, then one can compute $\langle g : g\rangle = \partial W/\partial x$, where $W = (\partial V/\partial x)^2$ is the classical Kapitsa's potential [21], [24] (also called the averaged potential). It is easy to see that every isolated critical point of V is a minimum of W. Using Hölder inequality, we obtain

$$\frac{1}{2}\overline{U}_1^2 - \overline{U}_2 = \frac{1}{2T}\left(\frac{1}{T}\left(\int_0^T U_1(s)ds\right)^2 - \int_0^T U_1^2(s)ds\right)$$
$$< \frac{1}{2T}\left(\frac{1}{T}\left(\left(\int_0^T U_1^2(s)ds\right)^{1/2}\left(\int_0^T 1ds\right)^{1/2}\right)^2 - \int_0^T U_1^2(s)ds\right) = 0\,,$$

and we conclude that every isolated equilibrium point of the original system is a Lyapunov-stable equilibrium point for the averaged system [25].

E. Hamiltonian control systems

Second order systems as in equation (24) are examples of Lagrangian control systems, and the analysis presented above can be generalized to a coordinate-free setting on manifolds; e.g., the result in equation (25) agrees with the results in [26]. We present here a coordinate-free based treatment for Hamiltonian control systems as described for example in [47], [13]. Consider the control system

$$\dot{x} = X_H + \sum_{i=1}^{m} \frac{1}{\epsilon} u_i\left(\frac{t}{\epsilon}\right) X_{H_i}\,,\ x(0) = x_0\,,$$

where $x = (q,p) \in T^*\mathbb{R}^n$, H, $H_i \in C^\infty(T^*\mathbb{R}^n)$, and X_H, X_{H_i} denote the corresponding Hamiltonian vector fields with respect to the canonical symplectic form $\Omega_{\mathbb{R}^n}$ on $T^*\mathbb{R}^n$. Let $\{\cdot,\cdot\}$ denote the Poisson bracket associated with $\Omega_{\mathbb{R}^n}$. Assume that the vector fields $X_{H_1}, \ldots, X_{H_m}$ commute, or equivalently, that the Poisson bracket $\{H_i, H_j\}$ is constant for any $i, j \in \{1, \ldots, m\}$.

The pull-back vector field F in Theorem III.4 is again Hamiltonian with respect to

$$H^* = H + \sum_{k=1}^{+\infty} (-1)^k \sum_{\{i_1,\ldots,i_k\}\in I} U_{\{i_1,\ldots,i_k\}} \{H_{i_1}, \{H_{i_2}, \ldots, \{H_{i_k}, H\}\} \ldots\}\,.$$

In particular, let $\varphi_1, \ldots, \varphi_m$ be functions defined on $\mathbb{R}^n$ and consider their natural lift, $\varphi_i^{\mathrm{lift}} = \varphi_i \circ \pi_{\mathbb{R}^n}$, where $\pi_{\mathbb{R}^n} : T^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the canonical projection. Then, it is straightforward to verify that $\{\varphi_i^{\mathrm{lift}}, \varphi_j^{\mathrm{lift}}\} = 0$, and we can consider a control system with input Hamiltonian functions $H_i = \varphi_i^{\mathrm{lift}}$. If the Hamiltonian H has a polynomial dependence on the momentum variables p, say of order l, then one can see that the series for H^* is finite,

$$H^* = H + \sum_{k=1}^{l} (-1)^k \sum_{\{i_1,\dots,i_k\}\in I} U_{\{i_1,\dots,i_k\}} \{\varphi_{i_1}^{\mathrm{lift}}, \{\varphi_{i_2}^{\mathrm{lift}}, \dots, \{\varphi_{i_k}^{\mathrm{lift}}, H\}\} \dots \} \, .$$

This is the case, for instance, of the so-called simple mechanical systems, where the Hamiltonian corresponds to kinetic plus potential energy, $H = \frac{1}{2} p M^{-1}(q) p + V(q)$, with M the mass matrix. Indeed, one gets $\{\varphi_i^{\mathrm{lift}}, \{\varphi_j^{\mathrm{lift}}, \{\varphi_k^{\mathrm{lift}}, H\}\}\} = 0$.

V. On summable series

In this section we present sufficient conditions under which the series for the averaged system in equation (20) can be summed. The sufficient conditions are assumptions on the Lie algebraic structure of $\{f, g_1, \dots, g_m\}$. In what follows, let i, j, k take values in $\{1, \dots, m\}$.

We start by extending a result described in [23].

Proposition V.1: Consider the bilinear control system

$$\dot{x} = Ax + \frac{1}{\epsilon} \sum_i u_i \left(\frac{t}{\epsilon}, t \right) B_i x \, ,$$

and assume $B_i B_j = 0$ for all i, j. Consider also the initial value problem

$$\dot{y} = \left(A + \sum_i \overline{U}_{\{i\}}(t) \, \mathrm{ad}_{B_i} A + \sum_{i,j} \overline{U}_{\{i,j\}}(t) \, \mathrm{ad}_{B_i} \mathrm{ad}_{B_j} A - \sum_i \left(\frac{d}{dt} \overline{U}_{\{i\}}(t) \right) B_i \right) y \, ,$$

with initial condition $y(0) = x(0)$. Then,

$$x(t) = \mathrm{e}^{\sum_i B_i \int_0^{(t/\epsilon \mod T)} u_i(\tau, t) d\tau} y(t) + O(\epsilon) \, .$$

The result follows from equation (20) by noting that $\mathrm{ad}_{B_i} \mathrm{ad}_{B_j} \mathrm{ad}_{B_k} A = 0$ for all i, j, k. Next, we present a formal statement for second order systems.

Proposition V.2 (Second order systems) Consider the control system

$$\ddot{x} + f_1(x)\dot{x} + f_0(x) = \frac{1}{\epsilon} \sum_i u_i \left(\frac{t}{\epsilon}, t \right) \, g_i(x) \, ,$$

and the initial value problem

$$\ddot{y} + f_1(y)\dot{y} + f_0(y) = \frac{1}{2} \sum_{i,j} \left(\overline{U}_{\{i\}}(t) \overline{U}_{\{j\}}(t) - \overline{U}_{\{i,j\}}(t) - \overline{U}_{\{j,i\}}(t) \right) \langle g_i : g_j \rangle (y)$$

with initial conditions $y(0) = x(0)$, $\dot{y}(0) = \dot{x}(0) + \sum_i \overline{U}_{\{i\}}(0) g_i(x(0))$. Then, we have

$$\begin{aligned} x(t) &= y(t) + O(\epsilon) \\ \dot{x}(t) &= \dot{y}(t) + \sum_i g_i(y(t)) \left(\int_0^{(t/\epsilon \mod T)} u_i(\tau, t) d\tau - \overline{U}_{\{i\}}(t) \right) + O(\epsilon) \, . \end{aligned}$$

Next, we investigate a summing method based on recursive Lie bracket relationships and generating functions [48]. Let $\langle g \,|f \rangle$ be the smallest g-invariant distribution containing f. Let the distribution $\langle g \,|f \rangle$ be finite-dimensional. Note that any pair of linear vector fields satisfies this assumption because of the Cayley-Hamilton theorem.

Lemma V.3: Assume $\mathrm{ad}_g^p f = \lambda \, \mathrm{ad}_g^q f$ for some integers $p > q \geq 0$, with $\lambda(x) \in \ker g$. Then

$$\overline{F}(x) = \sum_{k=0}^{q-1} \overline{U}_k \, \mathrm{ad}_g^k f + \left(\sum_{k=0}^{+\infty} \overline{U}_{q+(p-q)k} \lambda^k \right) \mathrm{ad}_g^q f + \cdots + \left(\sum_{k=0}^{+\infty} \overline{U}_{(p-1)+(p-q)k} \lambda^k \right) \mathrm{ad}_g^{p-1} f \, .$$

Let ${}_m H_n$ be the generalized hypergeometric function [44]. If $u(t) = a \sin \omega t$, we compute the generating function as

$$\sum_{k=0}^{+\infty} \overline{U}_{mk+n} \lambda^k = \frac{1}{\pi n!} \beta(\frac{1}{2}, \frac{1}{2} + n) \left(\frac{2a}{\omega} \right)^n {}_{(m+1)}H_{(2m)} \left(\begin{bmatrix} 1, \frac{1}{2m} + \frac{n}{m}, \frac{3}{2m} + \frac{n}{m}, \ldots, \frac{2m-1}{2m} + \frac{n}{m} \\ \frac{1+n}{m}, \frac{1+n}{m}, \frac{2+n}{m}, \frac{2+n}{m}, \ldots, \frac{m+n}{m}, \frac{m+n}{m} \end{bmatrix}, \lambda \left(\frac{2a}{\omega m} \right)^m \right) .$$

If $u(t) = a \cos \omega t$, we compute the generating function for m even as

$$\begin{aligned} &\sum_{k=0}^{+\infty} \overline{U}_{mk+n} \lambda^k = 0 \, , \\ &\sum_{k=0}^{+\infty} \overline{U}_{mk+n} \lambda^k = \frac{1}{\Gamma\left(1 + \frac{n}{2}\right)^2} \left(\frac{a}{2\omega} \right)^n {}_{(1)}H_{(m)} \left(\begin{bmatrix} 1 \\ \frac{2+n}{m}, \frac{2+n}{m}, \frac{4+n}{m}, \frac{4+n}{m}, \ldots, \frac{m+n}{m}, \frac{m+n}{m} \end{bmatrix}, \lambda \left(\frac{a}{m\omega} \right)^m \right) , \end{aligned}$$

for n odd and even respectively; and for m odd we compute

$$\begin{aligned} &\sum_{k=0}^{+\infty} \overline{U}_{mk+n} \lambda^k = \frac{1}{\Gamma\left(1 + \frac{m+n}{2}\right)^2} \lambda \left(\frac{a}{2\omega} \right)^{m+n} {}_{(1)}H_{(2m)} \left(\begin{bmatrix} 1 \\ \frac{2+m+n}{2m}, \frac{2+m+n}{2m}, \ldots, \frac{3m+n}{2m}, \frac{3m+n}{2m} \end{bmatrix}, \lambda^2 \left(\frac{a}{2m\omega} \right)^{2m} \right) , \\ &\sum_{k=0}^{+\infty} \overline{U}_{mk+n} \lambda^k = \frac{1}{\Gamma\left(1 + \frac{n}{2}\right)^2} \left(\frac{a}{2\omega} \right)^n {}_{(1)}H_{(2m)} \left(\begin{bmatrix} 1 \\ \frac{2+n}{2m}, \frac{2+n}{2m}, \frac{4+n}{2m}, \frac{4+n}{2m}, \ldots, \frac{2m+n}{2m}, \frac{2m+n}{2m} \end{bmatrix}, \lambda^2 \left(\frac{a}{2m\omega} \right)^{2m} \right) , \end{aligned}$$

for n odd and even respectively.

The simplest example of Lemma V.3 is $(p, q) = (1, 0)$. Accordingly,

$$\overline{F}_{\sin}(x) = f(x) \left(\mathrm{e}^{\left(\frac{a\lambda}{\omega}\right)} I_0 \left(\frac{a\lambda}{\omega} \right) \right) , \; \overline{F}_{\cos}(x) = f(x) \, I_0 \left(\frac{a\lambda}{\omega} \right) ,$$

where I_0 denote the modified Bessel function of the first kind. Another interesting example which we will use later is $(p,q)=(4,2)$,

$$\begin{aligned}\overline{F}_{\sin}(x) &= f(x) + \frac{a}{\omega}\operatorname{ad}_g f + \frac{1}{\lambda}\operatorname{ad}_g^2 f\left(-1 + I_0\left(\frac{a\sqrt{\lambda}}{\omega}\right)\cosh\left(\frac{a\sqrt{\lambda}}{\omega}\right)\right)\\ &\quad + \frac{1}{\lambda\sqrt{\lambda}}\operatorname{ad}_g^3 f\left(-\frac{a}{\omega}\sqrt{\lambda} + I_0\left(\frac{a\sqrt{\lambda}}{\omega}\right)\sinh\left(\frac{a\sqrt{\lambda}}{\omega}\right)\right),\\ \overline{F}_{\cos}(x) &= f(x) + \frac{1}{\lambda}\operatorname{ad}_g^2 f\left(-1 + I_0\left(\frac{a\sqrt{\lambda}}{\omega}\right)\right).\end{aligned}$$

VI. On stabilization via oscillatory controls

In this section we discuss the problem of stabilization of the nonlinear system $\dot{x} = f(x)$ by means of highly oscillatory controls of the form $(1/\epsilon)u\,(t/\epsilon)\,g(x)$. The starting point is the result in Theorem III.2 about the existence of hyperbolically stable periodic orbits for $\dot{x} = f(x)$ provided the origin is a hyperbolically stable equilibrium for $\overline{F}$. In some cases, we shall prove asymptotic stability for the original equilibrium point (this is what was termed as t-stabilizability in [36]) and in some others we shall prove that the equilibrium bifurcates to an asymptotically stable periodic orbit contained in an $O(\epsilon)$-neighbor (v-stabilizability in [36]).

We start by studying whether the origin is an equilibrium point for the averaged system.

Lemma VI.1 (Equilibrium points) The origin is an equilibrium point of the averaged system, $\overline{F}(0)=0$, if either of the following conditions are satisfied:
(i) $f(0) = g_1(0) = \cdots = g_m(0) = 0$,
(ii) $f(0)=0$ and f is an odd function, g_j is an even function for all $1 \le j \le m$, there exists i such that $g_i(0) \neq 0$, and the odd multinomial iterated averages of the inputs vanish, i.e., $\overline{\mathcal{U}}_{k_1,\dots,k_m} = 0$ whenever $k_1 + \cdots + k_m$ is odd.

Proof: In case *(i)*, one can prove recursively that

$$\operatorname{ad}_{g_1}^{k_1}\dots\operatorname{ad}_{g_m}^{k_m} f(0) = 0\,, \quad k_1,\dots,k_m \ge 0\,,$$

and hence $\overline{F}(0)=0$. To see *(ii)*, consider the $(k_1,\dots,k_m)$-th term $\overline{\mathcal{U}}_{k_1,\dots,k_m}\operatorname{ad}_{g_1}^{k_1}\dots\operatorname{ad}_{g_m}^{k_m} f$ in the expansion of $\overline{F}$. If f is odd and g_j is even for all j, the vector field $\operatorname{ad}_{g_1}^{k_1}\dots\operatorname{ad}_{g_m}^{k_m} f$

is odd whenever $k_1 + \cdots + k_m$ is even. Accordingly, each term $\overline{\mathcal{U}}_{k_1,\ldots,k_m} \operatorname{ad}_{g_1}^{k_1} \ldots \operatorname{ad}_{g_m}^{k_m} f$ is either odd or it vanishes. Therefore, $\overline{F}$ is an odd function, and $\overline{F}(0) = 0$. ■

Next, we study the linearization of the averaged system.

Proposition VI.2: Assume $f(0) = g_1(0) = \cdots = g_m(0) = 0$. At the origin, the linearization of the averaged system equals the average of the linearized system.

Proof: We prove it for the single input setting. Let $f = \sum_{i=1}^{+\infty} f^{[i]}$, $g = \sum_{i=1}^{+\infty} g^{[i]}$ be the Taylor expansions around $x = 0$ of f and g. Accordingly, $f^{[i]}, g^{[i]} \in \mathcal{H}_i$ and

$$\operatorname{ad}_g^k f = \sum_{\substack{j=1 \\ i_1,\ldots,i_k=1}}^{+\infty} \operatorname{ad}_{g^{[i_1]}} \ldots \operatorname{ad}_{g^{[i_k]}} f^{[j]} = \operatorname{ad}_{g^{[1]}}^k f^{[1]} + h \,,$$

where h is an infinite sum of homogeneus polynomials of degree ≥ 2. Consequently,

$$\frac{\partial}{\partial x} \left(\operatorname{ad}_g^k f \right)(0) = \operatorname{ad}_{\frac{\partial g}{\partial x}(0)}^k \frac{\partial f}{\partial x}(0) \,,$$

where one adjoint operator is a Lie bracket and the other a matrix commutator. This implies that the linearization of the averaged system is equal to

$$\frac{\partial \overline{F}}{\partial x}(0) = \frac{\partial f}{\partial x}(0) + \sum_{k=1}^{+\infty} \overline{U}_k \operatorname{ad}_{\frac{\partial g}{\partial x}(0)}^k \frac{\partial f}{\partial x}(0) \,,$$

which is the average of the linearized system (see the previous section of bilinear systems). ■

Note that the setting of bilinear systems (cf. Section IV-B) is very important as it represents the linearization of the average of any nonlinear system with $f(0) = g_1(0) = \cdots = g_m(0) = 0$.

Corollary VI.3: Let $f(0) = g_1(0) = \cdots = g_m(0) = 0$. If the trace of the linearization of the drift vector field f is positive, then the averaged system is unstable for any oscillatory control law.

Proof: Since $\operatorname{tr}(\operatorname{ad}_C D) = 0$ for any matrix C, D, we have that

$$\operatorname{tr}\left(\frac{\partial \overline{F}}{\partial x}(0)\right) = \operatorname{tr}\left(\sum_{k_1,\ldots,k_m \geq 0}^{+\infty} \overline{\mathcal{U}}_{k_1,\ldots,k_m} \operatorname{ad}_{\frac{\partial g_1}{\partial x}(0)}^{k_1} \ldots \operatorname{ad}_{\frac{\partial g_m}{\partial x}(0)}^{k_m} \frac{\partial f}{\partial x}(0)\right) = \operatorname{tr}\left(\frac{\partial f}{\partial x}(0)\right) > 0 \,,$$

and therefore the averaged system is unstable. ■

The corollary is a generalization in two directions of the result in [36] about the stabilizability of the system (22) by linear multiplicative vibrations. First, we do not require

$\frac{\partial f}{\partial x}(0)$ to be nonderogatory. Recall that a matrix is nonderogatory if its eigenvalues have geometric multiplicity equal to one [49]. Second, we consider general nonlinear systems and vibrations. Next, we present a classical result on stabilization by means of oscillatory controls.

Proposition VI.4: Consider the nonlinear system $\dot{x} = f(x)$, with $f(0) = 0$. If $A = \partial f/\partial x(0)$ is nonderogatory and $\operatorname{tr} A < 0$, then there exist commuting linear vector fields $\{g_1, \ldots, g_{n-1}\}$ and oscillatory controls $\{u_1, \ldots, u_{n-1}\}$ such that the equilibrium $x = 0$ is asymptotically stable for

$$\dot{x} = f(x) + \frac{1}{\epsilon} \sum_{i=1}^{n-1} u_i \left(\frac{t}{\epsilon}\right) g_i(x) . \tag{26}$$

Proof: The proof goes along the lines of [35], [36]. Assume A is in companion form (otherwise, we first perform a change of coordinates). This means that

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & \vdots & & \\ 0 & \ldots & \ldots & 0 & 1 \\ a_n & a_{n-1} & \ldots & a_2 & a_1 \end{pmatrix} = \mathrm{Comp}(a_n, \ldots, a_2, a_1) .$$

Let $i \in \{1, \ldots, n-1\}$ and consider the linear vector fields

$$g_i(x) = E_{ni} x ,$$

where E_{ni} is the matrix which has a 1 in the row n and column i and zero otherwise. Since $E_{ni} E_{nj} = 0$, for all $i, j \in \{1, \ldots, n-1\}$, the input vector fields commute and the result in Proposition V.1 applies. Furthermore, the only non-vanishing second order commutators are

$$\begin{aligned} \mathrm{ad}^2_{E_{n(n-1)}} A &= -2E_{n(n-1)} x , \\ \mathrm{ad}_{E_{nj}} \mathrm{ad}_{E_{n(n-1)}} A &= -E_{nj} x, \end{aligned} \tag{27}$$

for $j < n-1$ and therefore the linearized averaged system is again in companion form. Taking the input functions $u_i = r_i \cos(\omega t)$, the first averages $\overline{U}_{\{i\}}$ vanish, and from

Proposition V.1 we have that $\overline{A} = \frac{\partial \overline{F}}{\partial x}(0)$ equals

$$\begin{aligned} A - 2\,\overline{U}_{\{n-1,n-1\}} E_{n(n-1)} - \sum_{i=1}^{n-2} (\overline{U}_{\{i,n-1\}} + \overline{U}_{\{n-1,i\}}) E_{ni} \\ = \mathrm{Comp}\left(a_n - \frac{r_1 r_{n-1}}{2\omega^2},\, \ldots,\, a_2 - \frac{r_{n-1}^2}{2\omega^2},\, a_1 \right) , \end{aligned}$$

where in the last equality we have used the result in Lemma II.3 and the equality (5). By assumption $\mathrm{tr}(A) = a_1 < 0$. Let λ_i denote the ith eigenvalue of A, define

$$\bar{\lambda}_i = \frac{a_1}{n} + j\mathrm{Im}\lambda_i \,,$$

and the Hurwitz polynomial $(x - \bar{\lambda}_1)\ldots(x - \bar{\lambda}_1) = x^n + \bar{a}_1 x^{n-1} + \cdots + \bar{a}_{n-1}x + \bar{a}_n$. Note that $\bar{a}_1 = a_1$ and $\bar{a}_2 \leq a_2$. Now, it is clear that there exists an appropriate selection of the amplitudes r_i that makes $\overline{A} = \mathrm{Comp}\,(\bar{a}_n,\, \ldots,\, \bar{a}_2,\, a_1)$. Therefore $y = 0$ is asymptotically stable for the averaged system. From Theorem III.2, we have that equation $\dot{z}(t) = F(t/\epsilon, z)$ possesses an asymptotically stable periodic orbit $z_p(t)$. Resorting to the theorem of first order averaging [17], one can indeed deduce that $z_p(t) \equiv 0$, and finally $x = \Phi_{0,t/\epsilon}^{g(t,x)}(0) = 0$ is an asymptotically stable equilibrium point for equation (26). ∎

An interesting observation is that systems with positive trace may be stabilized by means of vibrations g with $g(0) \neq 0$: the example in [50] is a linear system $\dot{x} = Ax$, with $\mathrm{tr}(A) > 0$ and a control input $g = g^{[0]} + g^{[2]}$, $g^{[0]} \in \mathcal{H}_0$, $g^{[2]} \in \mathcal{H}_2$. Here, we give the following result.

Proposition VI.5: Consider the linear system $\dot{x} = Ax$, and let A be a nonderogatory matrix with $\mathrm{tr}\,A > 0$. Then, there exist a nonlinear vector field g_{nl}, commuting vector fields $\{g_1, \ldots, g_{n-1}\}$ and controls $\{u_1, \ldots, u_{n-1}\}$ such that the equilibrium $x = 0$ of the linear system becomes an asymptotically stable periodic orbit contained in an $O(\epsilon)$ neighborhood of the origin for the equation

$$\dot{x} = Ax + g_{\mathrm{nl}}(x) + \frac{1}{\epsilon} \sum_{i=1}^{n-1} u_i \left(\frac{t}{\epsilon}\right) g_i(x) \,.$$

Proof: Assume A is in companion form. Let

$$g_{\mathrm{nl}}(x) = f^{[3]}(x) = (0, \ldots, 0, bx_1^3 + cx_1^2 x_n) \in \mathcal{H}_3.$$

For $i \in \{2, \dots, n-1\}$, define the commuting vector fields

$$g_1(x) = (1, 0, \dots, 0) \in \mathcal{H}_0 \,, \quad g_i(x) = E_{ni} x \in \mathcal{H}_1 \,,$$

and the controls $u_i = r_i \cos(\omega t)$. The averaged system is

$$\overline{F}(x) = \sum_{k_1, \dots, k_m = 0}^{+\infty} \overline{\mathcal{U}}_{k_1, \dots, k_m} \operatorname{ad}_{g_1}^{k_1} \dots \operatorname{ad}_{g_m}^{k_m} (Ax + f^{[3]})$$

We first verify that the origin is an equilibrium point for the averaged system. Because of homogeneity arguments, and using Lemma III.5, the 0th order term in $\overline{F}$ is

$$\begin{aligned} \overline{F}(0) = \sum_{k_2, \dots, k_m = 0}^{+\infty} \Big(& \overline{\mathcal{U}}_{1, k_2, \dots, k_m} \operatorname{ad}_{g_2}^{k_2} \dots \operatorname{ad}_{g_m}^{k_m} (\operatorname{ad}_{g_1} Ax) \\ & + \overline{\mathcal{U}}_{3, k_2, \dots, k_m} \operatorname{ad}_{g_2}^{k_2} \dots \operatorname{ad}_{g_m}^{k_m} \left(\operatorname{ad}_{g_1}^3 f^{[3]} \right) \Big)_{x=0} \,. \end{aligned}$$

Since $\operatorname{ad}_{g_i} \operatorname{ad}_{g_1} Ax = 0$ and $\operatorname{ad}_{g_i} \operatorname{ad}_{g_1}^3 f^{[3]}(x) = 0$ for all $i, j \in \{2, \dots, n-1\}$,

$$\overline{F}(0) = \overline{\mathcal{U}}_{1,0,\dots,0} \operatorname{ad}_{g_1} Ax + \overline{\mathcal{U}}_{3,0,\dots,0} \operatorname{ad}_{g_1} f^{[3]} = 0 \,,$$

where in the last equality we have used equation (5). With regards to the linearization of $\overline{F}$, we have that

$$\begin{aligned} \overline{A} = \frac{\partial \overline{F}}{\partial x}(0) = \sum_{k_2, \dots, k_m = 0}^{+\infty} \Big(& \overline{\mathcal{U}}_{0, k_2, \dots, k_m} \operatorname{ad}_{g_2}^{k_2} \dots \operatorname{ad}_{g_m}^{k_m} Ax \\ & + \overline{\mathcal{U}}_{2, k_2, \dots, k_m} \operatorname{ad}_{g_2}^{k_2} \dots \operatorname{ad}_{g_m}^{k_m} \left(\operatorname{ad}_{g_1} f^{[3]} \right) \Big)_{x=0} \,. \end{aligned}$$

The first term is computed as in the previous proposition via equation (27). As for the second one, note that for $i, j \in \{2, \dots, n-1\}$,

$$\begin{aligned} & \operatorname{ad}_{g_1}^2 f^{[3]} = (0, \dots, 6bx_1 + 2cx_n) \,, \\ & \operatorname{ad}_{g_i} \operatorname{ad}_{g_1}^2 f^{[3]} = (0, \dots, 2cx_i) \,, \quad \operatorname{ad}_{g_i} \operatorname{ad}_{g_j} f^{[3]} = 0 \,. \end{aligned}$$

Using again equation (5), we conclude that the linearized averaged system is also in companion form. Indeed, $\overline{A}$ equals

$$\operatorname{Comp}\left(a_n + \frac{3br_1^2}{2\omega^2}, a_{n-1} - \frac{r_2 r_{n-1}}{2\omega^2}, \dots, a_2 - \frac{r_{n-1}^2}{2\omega^2}, a_1 + \frac{cr_1^2}{2\omega^2} \right) .$$

Selecting $c < -a_1\omega^2/r_1^2$, we have that $\operatorname{tr}\overline{A} < 0$. Finally, an appropriate selection of the coefficients $b, r_1, \ldots, r_{n-1}$ makes the linearization of the averaged system a Hurwitz matrix. As before, this implies that $y = 0$ is asymptotically stable for the averaged system. The application of Theorem III.2 concludes the proof. ■

Remark VI.6: In summary, stabilization has been achieved building on strong nilpotency assumptions of the type $B_iB_j = 0$. In the following, we show that the less restrictive nilpotency properties discussed in Section V may also be instrumental to provide alternative stabilization schemes. The interesting observation in this case is that the *full* series expansion is taken into account, not just a truncated version. Consider the bilinear system $\dot{x} = Ax + (1/\epsilon)u(t/\epsilon)Bx$ with $x \in \mathbb{R}^2$. Assume A is diagonal and $\operatorname{tr}(A) < 0$. Write $A = A_1 + A_2$, where

$$A_1 = \begin{pmatrix} \operatorname{tr}(A)/2 & 0 \\ 0 & \operatorname{tr}(A)/2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}.$$

Note that A_1 is a stable matrix. Take $B = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$, and compute

$$\operatorname{ad}_B^2 A = -A_2\,, \quad \operatorname{ad}_B^4 A = A_2 = -\operatorname{ad}_B^2 A\,.$$

Let $u = a\cos\omega t$ and, following Lemma V.3 with $(p,q) = (4,2)$, compute

$$\overline{F}(x) = (A_1 + A_2)x - \operatorname{ad}_B^2 A\left(-1 + I_0\left(\frac{a}{\omega}\sqrt{-1}\right)\right)x = \left(A_1 + A_2 I_0\left(\frac{a}{\omega}\sqrt{-1}\right)\right)x\,.$$

The modified Bessel function with imaginary arguments possesses an infinite number of positive zeros. The smallest $z_1 \in \mathbb{R}_+$ such that $I_0(z_1\sqrt{-1}) = 0$ is an irrational number belonging to the interval $[2.424,\ 2.425]$. Therefore, the averaged system is asymptotically stable provided the input parameters satisfy $a = z_1\omega$.

VII. On tracking via oscillatory controls

We present an application of our averaging analysis to the problem of trajectory tracking via oscillatory controls. We consider the tracking problem for second order systems, and we design control laws inspired by the inversion algorithm in [51]. In what follows, let i, j, k take values in $\{1, \ldots, m\}$ unless otherwise stated.

Consider the control system

$$\ddot{x} + f_1(x)\dot{x} + f_0(x) = \sum_i w_i \, g_i(x) \,, \tag{28}$$

and the following tracking problem: given a smooth desired curve $x^d : [0,T] \to \mathbb{R}^n$ with initial conditions $x^d(0) = x(0)$, $\dot{x}^d(0) = \dot{x}(0)$, find controls laws $w_i : \mathbb{R}^{2n} \times [0,T] \to \mathbb{R}^m$ such that the solution x to equation (28) approximates x^d up to an error of order ϵ.

We make the following controllability assumption:

(A) The distribution $\operatorname{span}\{g_i \,, \langle g_j : g_k \rangle\}$ is full rank, and $\langle g_j : g_j \rangle$ belongs to $\operatorname{span}\{g_i\}$. Accordingly,

(i) there exist functions $z_i^d, z_{jk}^d : [0,T] \to \mathbb{R}$, for $j < k$, such that

$$\ddot{x}^d + f_1(x^d)\dot{x}^d + f_0(x^d) = \sum_i z_i^d g_i(x^d) + \sum_{j<k} z_{jk}^d \langle g_j : g_k \rangle (x^d) \,,$$

(ii) there exist smooth functions $\alpha_{ij} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\langle g_i : g_i \rangle (x) = \sum_j \alpha_{ij}(x) g_j(x) \,, \qquad \forall x \in \mathbb{R}^n \,.$$

There are $N = m(m-1)/2$ pairs of integers (j,k), with $j < k$. Let $(j,k) \mapsto a(j,k) \in \{1, \dots, N\}$ be a enumeration of these pairs, and define the scalar functions

$$\psi_{a(j,k)}(t) = \sqrt{2}\, a(j,k) \cos(a(j,k)t) \,.$$

Proposition VII.1: Let $x^d : [0,T] \to \mathbb{R}^n$ be a desired curve with initial conditions $x^d(0) = x(0)$, $\dot{x}^d(0) = \dot{x}(0)$. The solution x to equation (28) equals x^d up to an error of order ϵ over the time scale 1 when the control laws w_i are

$$\begin{aligned}
w_i &= v_i(t,x) + \frac{1}{\epsilon} u_i\left(\frac{t}{\epsilon}, t\right) \,, \\
v_i(t,x) &= z_i^d(t) + \frac{1}{2} \sum_j \alpha_{ji}(x) \left(j - 1 + \sum_{\ell=j+1}^{m} (z_{j\ell}^d(t))^2 \right) \,, \\
u_i(\tau, t) &= -\sum_{\ell=1}^{i-1} \psi_{a(\ell,i)}(\tau) + \sum_{\ell=i+1}^{m} z_{i\ell}^d(t) \psi_{a(i,\ell)}(\tau) \,.
\end{aligned}$$

Proof: The control system (28) is written as

$$\ddot{x} + f_1(x)\dot{x} + f_0(x) = \sum_i v_i(t,x) g_i(x) + \frac{1}{\epsilon} \sum_i u_i\left(\frac{t}{\epsilon}, t\right) g_i(x) \,,$$

and, according to Proposition V.2, its averaged system is

$$\begin{aligned}\ddot{y} + f_1(y)\dot{y} + f_0(y) &= \sum_i v_i(t,y)g_i(y) + \sum_i \left(\frac{1}{2}\overline{U}^2_{\{i\}}(t) - \overline{U}_{\{i,i\}}(t)\right)\langle g_i : g_i\rangle(y) \\ &\quad + \sum_{i<j}\left(\overline{U}_{\{i\}}(t)\overline{U}_{\{j\}}(t) - \overline{U}_{\{i,j\}}(t) - \overline{U}_{\{j,i\}}(t)\right)\langle g_i : g_j\rangle(y)\,,\end{aligned}$$

with initial conditions $(y(0), \dot{y}(0)) = (x(0), \dot{x}(0) + \sum_i \overline{U}_{\{i\}}(0)g_i(x(0))$. We compute the iterated integrals of the given oscillatory inputs u_i as

$$\begin{aligned}&\overline{U}_{\{i\}}(t) = \frac{1}{T}\int_0^T u_i(\tau,t)d\tau = 0\,,\\ &\overline{U}_{\{i,j\}}(t) + \overline{U}_{\{j,i\}}(t) = \overline{U_{\{i\}}U_{\{j\}}}(t) = \frac{1}{T}\int_0^T\left(\int_0^\tau u_i(s,t)ds\right)\left(\int_0^\tau u_j(s,t)ds\right)d\tau = -z^d_{ij}(t)\,,\end{aligned}$$

for $i<j$, so that the averaged system reads

$$\ddot{y} + f_1(y)\dot{y} + f_0(y) = \sum_i v_i(t,y)g_i(y) - \sum_i \overline{U}_{\{i,i\}}(t)\langle g_i : g_i\rangle(y) + \sum_{i<j} z^d_{ij}(t)\langle g_i : g_j\rangle(y)\,.$$

Next, we examine the definition of the v_i inputs. Note that

$$\overline{U}_{\{j,j\}}(t) = \frac{1}{2T}\int_0^T\left(\int_0^\tau u_j(s,t)ds\right)^2 d\tau = \frac{1}{2}\left(j - 1 + \sum_{\ell=j+1}^m (z^d_{j\ell}(t))^2\right)$$

and therefore

$$\begin{aligned}\sum_i v_i(t,y)g_i(y) &= \sum_i z^d_i(t)g_i(y) + \sum_{i,j}\overline{U}_{\{i,i\}}(t)\alpha_{ij}(y)g_j(y)\\ &= \sum_i z^d_i(t)g_i(y) + \sum_i \overline{U}_{\{i,i\}}(t)\langle g_i : g_i\rangle(y)\end{aligned}$$

where we have exploited the property of the functions α_{ij}. In summary, we have shown that

$$\ddot{y} + f_1(y)\dot{y} + f_0(y) = \sum_i z^d_i g_i(y) + \sum_{i<j} z^d_{ij}\langle g_i : g_j\rangle(y)\,,$$

with initial conditions $y(0) = x^d(0)\,, \dot{y}(0) = \dot{x}^d(0)$. Since y and x^d are the solution to the same initial value problem, they are identical. Finally, from Proposition V.2, we conclude that $x(t) = y(t) + O(\epsilon) = x^d(t) + O(\epsilon)$. ■

Remark VII.2 (Lagrangian systems on manifolds)

Proposition VII.1 can be extended to a large class of Lagrangian control systems and written in a coordinate-free setting within the so-called affine connection formalism [26], [52]. Let q be the system's configuration on the n-dimensional manifold Q, and let $\{\Gamma^a_{bc}, a, b, c \in \{1, \dots, n\}\}$ be the n^3 Christoffel functions associated to the system's kinetic energy. Define the operation of symmetric product between the vector fields g_i, g_j on Q according to

$$\langle g_i : g_j \rangle^a = \frac{\partial g_i^a}{\partial q^b} g_j^b + \frac{\partial g_j^a}{\partial q^b} g_i^b + \Gamma^a_{bc}\left(g_i^b g_j^c + g_i^c g_j^b\right) ,$$

and define the quantity $(\nabla_{\dot q}\dot q)^a = \ddot q^a + \Gamma^a_{bc}(q)\dot q^b \dot q^c$. Then, the Euler-Lagrange equations read

$$\nabla_{\dot q}\dot q + f_1(q)\dot q + f_0(q) = \sum_i w_i g_i(q) .$$

Under the controllability assumption as (A), the result in Proposition VII.1 holds verbatim.

We end this section by considering two examples.

A second-order nonholonomic integrator

There are many interesting dynamical extensions of Brockett's nonholonomic integrators [53] (see the discussion in [41]). We consider

$$\ddot x_1 = w_1 , \quad \ddot x_2 = w_2 , \quad \ddot x_3 = w_1 x_2 + w_2 x_1 ,$$

and note that this system fulfills the controllability assumption (A). We design control inputs to track a desired trajectory, $(x_1^d(t), x_2^d(t), x_3^d(t))$, following Proposition VII.1,

$$\begin{aligned} w_1 &= \ddot x_1^d + \frac{1}{\sqrt{2}\epsilon}\left(\ddot x_3^d - \ddot x_1^d x_2^d - \ddot x_2^d x_1^d\right)\cos\left(\frac{t}{\epsilon}\right) \\ w_2 &= \ddot x_2^d - \frac{\sqrt{2}}{\epsilon}\cos\left(\frac{t}{\epsilon}\right) \end{aligned} \tag{29}$$

An illustration of the performance of these controls is shown in Figure 2.

A PVTOL model

We consider the model of a simple planar vertical takeoff and landing aircraft model based upon that of [54] with added viscous damping forces; see Figure 3. We parametrize

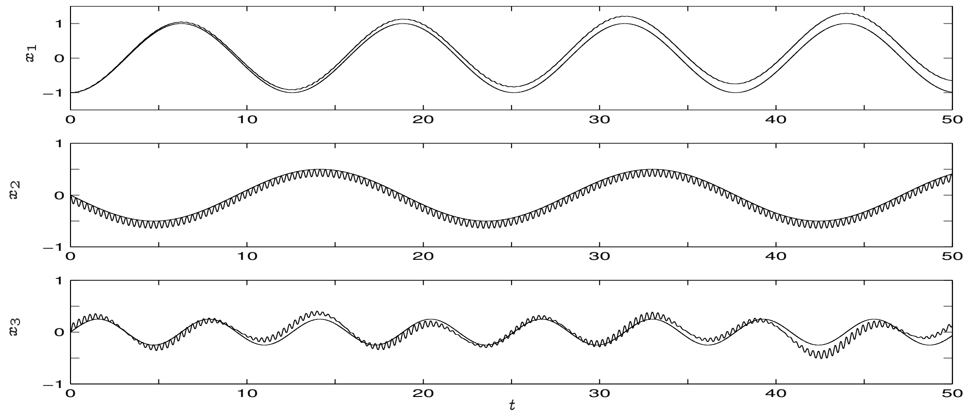

Fig. 2. Tracking for the modified nonholonomic integrator with the controls defined in equation (29) and with $\epsilon = .05$.

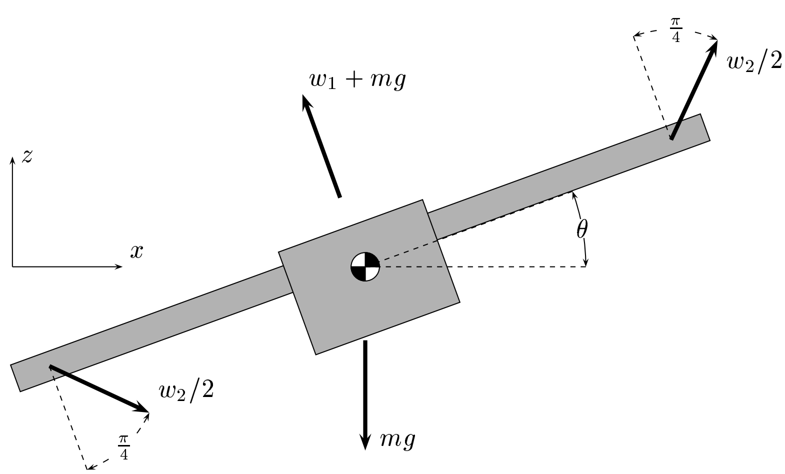

Fig. 3. Diagram of the PVTOL model.

its configuration and velocity space via the state variables $\{x, z, \theta, v_x, v_z, \omega\}$. We let x and z be the horizontal and vertical displacement of the aircraft, and θ be its roll angle. The angular velocity is ω and the linear velocities in the body-fixed x (respectively z)

axis are v_x (respectively v_z). The equations are written as:

$$
\begin{aligned}
\dot{x} &= \cos\theta v_x - \sin\theta v_z \\
\dot{z} &= \sin\theta v_x + \cos\theta v_z \\
\dot{\theta} &= \omega \\
\dot{v}_x &= (-k_1/m)v_x - g\sin\theta + v_z\omega + (1/m)w_2 \\
\dot{v}_z &= (-k_2/m)v_z - g(\cos\theta - 1) - v_x\omega + (1/m)w_1 \\
\dot{\omega} &= (-k_3/J)\omega + (h/J)w_2
\end{aligned} \tag{30}
$$

Control w_1 corresponds to the body vertical force minus gravity, while w_2 corresponds to coupled forces on the wingtips with a net horizontal component. The other forces depend upon the constants k_i, which parametrize a linear damping force, and g, the gravity constant. The constant h is the distance from the center of mass to the wingtip, while m and J are mass and moment of inertia, respectively.

Equations (30) can be written as a second order system in the variables (x, z, θ) and the model fulfills the controllability assumption (A). We design control inputs to track a desired trajectory $(x^d(t), z^d(t), \theta^d(t))$ as

$$
\begin{aligned}
w_1 &= \frac{J}{h}\ddot{\theta}^d + \frac{k_3}{h}\dot{\theta}^d - \frac{\sqrt{2}}{\epsilon}\cos\left(\frac{t}{\epsilon}\right) \\
w_2 &= \frac{h}{J} - f_1\sin\theta^d + f_2\cos\theta^d - \frac{J\sqrt{2}}{h\epsilon}\left(f_1\cos\theta^d + f_2\sin\theta^d\right)\cos\left(\frac{t}{\epsilon}\right),
\end{aligned} \tag{31}
$$

where we let $c = \frac{J}{h}\ddot{\theta}^d + \frac{k_3}{h}\dot{\theta}^d$ and

$$
\begin{aligned}
f_1 &= m\ddot{x}^d + \left(k_1\cos^2\theta^d + k_2\sin^2\theta^d\right)\dot{x}^d + \frac{\sin(2\theta^d)}{2}(k_1 - k_2)\dot{z}^d + mg\sin\theta^d - c\cos\theta^d, \\
f_2 &= m\ddot{z}^d + \frac{\sin(2\theta^d)}{2}(k_1 - k_2)\dot{x}^d + \left(k_1\sin^2\theta^d + k_2\cos^2\theta^d\right)\dot{z}^d + mg(1 - \cos\theta^d) - c\sin\theta^d.
\end{aligned}
$$

The simulations are run with $m = 20$, $J = 10$, $h = 5$, $k_1 = 12$, $k_2 = 11$, $k_3 = 10$, $g = 9.8$. Figure 4 shows an example of the behavior of the controls (31). Figure 5 illustrates the linear decay of the tracking error. Finally, Figure 6 shows how the convergence to the desired trajectory improves as ϵ decreases.

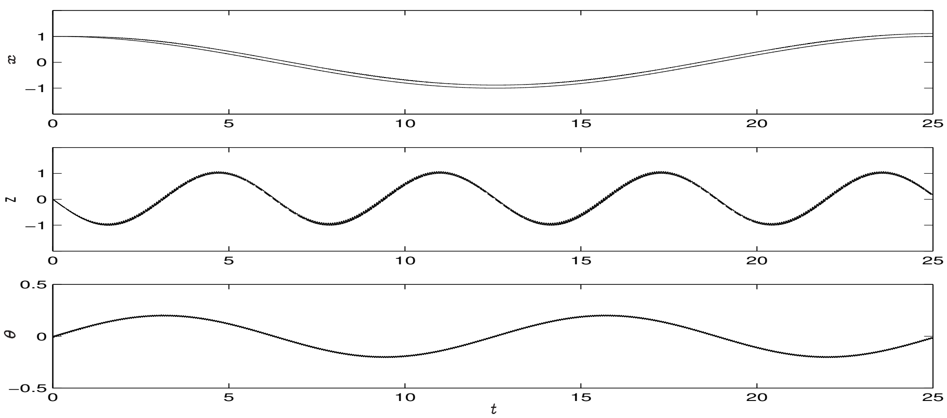

Fig. 4. Tracking for the PVTOL model with the controls defined in equation (31) and with $\epsilon = .01$.

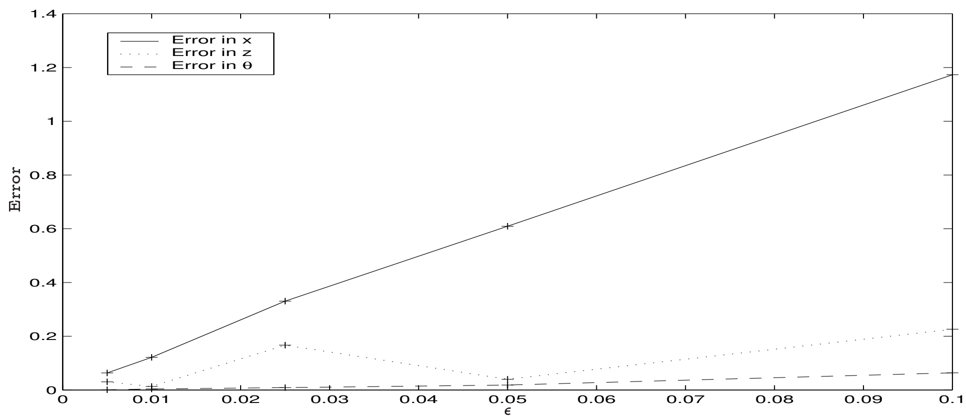

Fig. 5. Illustration of the tracking errors for the PVTOL model at $t = 10$ with the controls defined in equation (31).

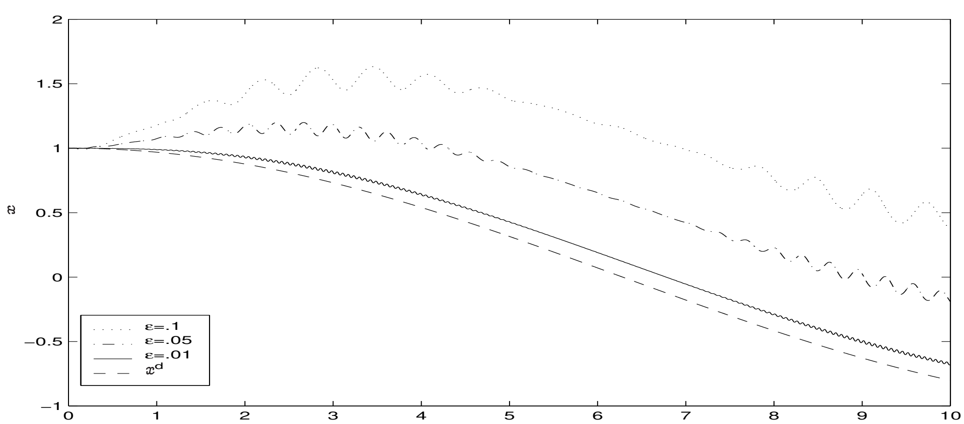

Fig. 6. Illustration of the tracking in the horizontal displacement of the PVTOL model with the controls defined in equation (31).

VIII. Conclusions

We have presented a coordinate-free averaging analysis for a class of control systems subject to oscillatory inputs. Based on this analysis, we have developed design tools for stabilization and trajectory tracking in certain classes of nonlinear systems. Future directions of research include deriving extensions of these results to the case of higher-order averaging, distributed parameter systems, time-delayed systems, and systems with resonances.

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