Reduction of Controlled Lagrangian and Hamiltonian Systems with Symmetry

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Abstract

We develop reduction theory for controlled Lagrangian and controlled Hamiltonian systems with symmetry. Reduction theory for these systems is needed in many of the fundamental examples, such as a spacecraft with rotors, a heavy top with rotors, and underwater vehicle dynamics. We also show the equivalence of the method of reduced controlled Lagrangian systems and that of reduced controlled Hamiltonian systems for simple mechanical systems with symmetry.

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1 Introduction

We develop the theory of symmetry reduction for the class of controlled Lagrangian (CL) and controlled Hamiltonian (CH) systems with symmetry and show the equivalence of the method of reduced CL systems and that of reduced CH system for simple mechanical systems. The phrases "controlled Lagrangian" and "controlled Hamiltonian" systems were coined in Chang, Bloch, Leonard, Marsden and Woolsey [2002] and their technical definition is recalled below. The concept of a controlled Hamiltonian system used here is the same as that of "port-controlled Hamiltonian system" in van der Schaft [2000]. We prefer to use the term "controlled Hamiltonian" as it is parallel to "controlled Lagrangian". The notion of an "implicit Hamiltonian system" is related to that of a CH system and a type of reduction theory (different from what is given here) in that context was developed in Blankenstein and van der Schaft [2001].

Controlled Lagrangian Systems. We will begin by describing the Lagrangian side. The CL method was developed to help design stabilizing controllers for mechanical systems whose equations are written in Euler–Lagrange form with control forces by a number of authors, including Bloch, Leonard and Marsden[1997, 1998, 1999, 2000, 2001], Bloch, Chang, Leonard and Marsden [2001], Chang, Bloch, Leonard, Marsden and Woolsey [2002], Chang [2002], Hamberg [1999] and Auckly, Kapitanski and White [2000]. In the study of CL systems, it is important to take into account external forces and control forces, including gyroscopic forces. Some related ideas were given in Bloch, Krishnaprasad, Marsden and Ratiu [1996] and Wang and Krishnaprasad [1992].

The main idea of the CL method is to proceed in the following steps.

1. Define a (feedback-transformation) equivalence relation among CL systems.
2. Given a CL system, find an equivalent CL system such that the energy of the second CL system has a minimum at the equilibrium of interest and the system is forced or controlled by a dissipative (plus gyroscopic) force.
3. Using the energy as a Lyapunov function, one obtains asymptotic stability of the equilibrium in the second CL system.
4. Since the two systems are equivalent, the equivalence relation gives an asymptotically stabilizing controller for the first system.

Controlled Hamiltonian Systems. A similar approach has been taken on the Hamiltonian side, too, where the main governing equations are in Hamiltonian form (see Bloch, Krishnaprasad, Marsden and Alvarez [1992], Woolsey and Leonard [1999], van der Schaft [2000], and references therein).

Equivalence. There have been some recent studies concerning the equivalence of both methods. First of all, Blankenstein, Ortega and van der Schaft [2001] showed that the CH method is more general than the CL method for simple mechanical systems and a limited definition of CL systems. Then Chang, Bloch, Leonard, Marsden and Woolsey [2002] extended the theory of controlled Lagrangian systems and showed the full equivalence of the method of controlled Lagrangian systems with the method of controlled Hamiltonian systems for simple mechanical systems.

Symmetry. The main goal of the present paper is to include symmetry into the discussion. We do so in a systematic and general way. In previous works, symmetry was considered in certain cases (see, for instance, Krishnaprasad [1985] and Bloch, Leonard and Marsden [2001]), but a comprehensive theory has been lacking.

In this paper, we define the notion of G-invariant CL/CH systems and develop a reduction theory for them. In addition, we trace through the equivalence of the CL and CH systems under the reduction.

Reduction Theory. Reduction theory for mechanical systems has been studied both on the Lagrangian side and on the Hamiltonian side. It has proven to be a powerful tool for studying the dynamics, stability and control of many mechanical systems, such as rigid body and spacecraft systems, underwater vehicles, heavy tops, fluid systems, etc.

On the Lagrangian side, one reduces variational principles, an idea due to Marsden and Scheurle [1993a,b]. A comprehensive account of this theory is given in Cendra, Marsden and Ratiu [2001].

On the Hamiltonian side, one reduces symplectic and Poisson structures. See, for instance, Abraham and Marsden [1978], Marsden [1992] and Marsden and Ratiu [1999] and references therein.

Using these ideas, we develop the notions and basic properties of a reduced CL and a reduced CH system.

Equivalence and Reduction. Having developed the reduction theory, we define an equivalence relation among G-invariant CL (resp. CH) systems and then show that two reduced CL (resp. CH) systems are equivalent if and only their unreduced G-invariant CL (resp. CH) systems are equivalent. This is given in Theorem 2.10 and Theorem 3.9.

We also show that this equivalence relation is a feedback-transform equivalence relation (see Theorem 2.10 and Theorem 3.9). Based on these results and the equivalence of the methods of CL and CH systems for simple mechanical systems proven in Chang, Bloch, Leonard, Marsden and Woolsey [2002], we prove that the method of reduced CL systems and that of reduced CH systems are equivalent for reduced simple mechanical systems (this is given in Theorem 4.3).

Example. In the final section, we review the example of a satellite controlled with a spinning rotor and show that the Euler–Poincaré matching conditions derived in Bloch, Chang, Leonard, Marsden and Woolsey [2000] and Bloch, Leonard and Marsden [2001] can be obtained in a natural way in the framework of the present paper. We also study the stabilization of the heavy top with rotors using the reduced CL method.

2 Reduction of CL Systems with Symmetry

We begin this section with some general notations and then we introduce the notion of a CL system with symmetry and finally study the theory of reduction for such systems.

We first summarize some general notations that will be used. We refer to Abraham, Marsden, and Ratiu [1988], Marsden and Ratiu [1999], Cendra, Marsden and Ratiu [2001] and Kobayashi and Nomizu [1963] for more details and background information.

Manifolds and Bundles. Let Q be the configuration manifold, and $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ be the tangent bundle projection and the cotangent bundle projection, respectively. The ***second order tangent bundle*** $\tau_Q^{(2)} : T^{(2)}Q \to Q$ is defined as follows. For $\bar{q} \in Q$, elements of $T_q^{(2)}Q$ are equivalence classes of curves in Q; the equivalence relation being defined as follows: two curves $q_i(t)$, $i = 1, 2$, with $q_1(\bar{t}) = q_2(\bar{t}) = \bar{q}$ are equivalent, by definition, if and only if in any local chart we have $q_1^{(l)}(\bar{t}) = q_2^{(l)}(\bar{t})$ for $l = 1, 2$ where $q^{(l)}(t)$ denotes the derivative of order l. The second order bundle has local coordinates given by $(q^i, \dot{q}^i, \ddot{q}^i)$ and may be thought of as a subbundle of the second tangent bundle TTQ via the embedding $(q^i, \dot{q}^i, \ddot{q}^i) \mapsto (q^i, \dot{q}^i, \dot{q}^i, \ddot{q}^i)$. As is explained in Marsden and Ratiu [1999], the second order tangent bundle is the basic space on which the Euler–Lagrange operator of mechanics is defined; in fact, given a Lagrangian L, the associated Euler–Lagrange operator $\mathcal{EL}$ induces a bundle map $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$.

For a manifold M, $\mathcal{F}(M)$ denotes the set of smooth real-valued functions on M.

Symmetry Groups. Let G be a Lie group acting (on the left) on Q freely and properly so that $\pi_G(Q) : Q \to Q/G$ becomes a principal bundle. The tangent (resp. cotangent) lifted action of G on TQ (resp. T^*Q) is automatically free and proper as well, so that the maps $\tau_{/G} : TQ \to TQ/G$ (resp. $\pi_{/G} : T^*Q \to T^*Q/G$) also define principal bundles. When M is a manifold on which G acts, we let $[m]_G$ to denote an equivalence class of $m \in M$ in the quotient space M/G. Even though we do not explicitly denote the manifold M in this notation, it will be clear which manifold is meant from the context. The space TQ/G becomes a vector bundle with base Q/G by inheriting the vector bundle structure of TQ as follows:

$$[u_q]_G + \lambda[v_q]_G = [u_q + \lambda v_q]_G$$

where $\lambda \in \mathbb{R}$, $u_q, v_q \in T_qQ$ and $[u_q]_G, [v_q]_G$ are their equivalence classes in the quotient space TQ/G. The fiber $(TQ/G)_x$ is isomorphic, as a vector space, to T_qQ

for each $x = [q]_G \in Q/G$, $q \in Q$ (see Lemma 2.4.1 in Cendra, Marsden and Ratiu [2001]). In the same manner, the space T^*Q/G becomes a vector bundle with base Q/G.

Vertical Lifts. Let V be a vector bundle over a manifold Q. The ***vertical lift*** of a vector $w_q \in V_q$ ***along the vector*** $v_q \in V_q$ is the vector $\mathrm{vlift}_{v_q}(w_q) \in T_{v_q}V$ defined by

$$\mathrm{vlift}_{v_q}(w_q) = \left.\frac{d}{dt}\right|_{t=0} (v_q + tw_q).$$

The ***vertical lift of a fiber-preserving map*** $F : V \to V$ is a section, $\mathrm{vlift}(F) : V \to TV$ defined by

$$\mathrm{vlift}(F)(v_q) = \mathrm{vlift}_{v_q}(F(v_q)). \tag{2.1}$$

The ***vertical lift of a subbundle*** W of V is the subbundle of TV defined by

$$\mathrm{vlift}(W) = \{\mathrm{vlift}_{v_q}(w_q) \mid v_q \in V_q, w_q \in W_q, q \in Q\}. \tag{2.2}$$

CL Systems. We next recall the definition of a CL system from Chang, Bloch, Leonard, Marsden and Woolsey [2002].

Definition 2.1. *A* ***controlled Lagrangian (CL) system*** *is a triple* (L, F, W) *where the function* $L : TQ \to \mathbb{R}$ *is the* ***Lagrangian****, the fiber-preserving map* $F : TQ \to T^*Q$ *is the (external)* ***force map****, and the subbundle* W *of* T^*Q *is called the* ***control subbundle****. When one chooses a feedback control law* $u : TQ \to W$*, the triple* (L, F, u) *will denote the* ***closed-loop CL system****.*

Euler–Lagrange Operator. The Euler–Lagrange operator $\mathcal{EL}$ assigns to a Lagrangian $L : TQ \to \mathbb{R}$, a bundle map $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$ which may be written in local coordinates (and using the index summation convention) as

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) - \frac{\partial L}{\partial q^i}(q, \dot{q}) \right) \mathbf{d}q^i$$

in which it is understood that one regards the first term on the right hand side as a function on the second order tangent bundle $T^{(2)}Q$ by formally applying the chain rule and replacing everywhere dq/dt by $\dot{q}$. The ***equations of motion*** of a CL system (L, F, W) with a choice of feedback control $u : TQ \to W$ are written as

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = F(q, \dot{q}) + u(q, \dot{q}). \tag{2.3}$$

Of course these may be derived from the Lagrange-d'Alembert principle.

CL systems with Symmetry. We next define the notion of a G-invariant CL system on TQ and the associated notion of a reduced CL system on TQ/G, where G is a Lie group acting on Q.

Definition 2.2. *Let* G *be a Lie group action on* Q*. A* ***G-invariant controlled Lagrangian (G-CL) system*** *is a CL system,* (L, F, W) *where* L *is a* G*-invariant Lagrangian,* F *is a* G*-equivariant force map, and* W *is a* G*-invariant subbundle of* T^*Q*.*

Reduction of CL Systems with Symmetry. Now we can define the notion of a reduced CL system.

Definition 2.3. *A* ***reduced controlled Lagrangian (RCL) system*** *is a triple* (l, f, U) *where the function* $l : TQ/G \to \mathbb{R}$ *is called the* ***reduced Lagrangian****, the fiber-preserving map* $f : TQ/G \to T^*Q/G$ *is called the* ***reduced force map****, and the subbundle* U *of* T^*Q/G *is called the* ***reduced control subbundle****. A* ***feedback control*** *for an RCL system is a fiber-preserving map of* TQ/G *into* U*.*

Suppose that we are given a G-CL system (L, F, W). The G-invariance of L induces the reduced Lagrangian l on TQ/G satisfying

$$l \circ \tau_{/G} = L. \tag{2.4}$$

The equivariance of F induces the reduced force map $[F]_G : TQ/G \to T^*Q/G$ satisfying

$$[F]_G \circ \tau_{/G} = \pi_{/G} \circ F. \tag{2.5}$$

Similarly, a G-invariant control subbundle W induces a reduced control subbundle W/G in a natural way; namely we have $W = \pi_{/G}^{-1}(W/G)$.

These considerations lead to the following definition:

Definition 2.4. *The RCL system of a* G*-CL system* (L, F, W) *is the triple* $(l, [F]_G, W/G)$ *where* l *is the reduced Lagrangian satisfying (2.4), and* $[F]_G$ *is the reduced force satisfying (2.5).*

One may ask a question if there exists a G-CL system on TQ when one is given a RCL system on TQ/G. The following proposition proves its unique existence.

Proposition 2.5. *Given a RCL system* (l, f, U)*, there is a unique* G*-CL system* (L, F, W) *whose RCL system is* (l, f, U)*.*

Proof. Define L by (2.4). Define a force map F on TQ as follow: for $v_q, w_q \in T_qQ$,

$$\langle F(v_q),\, w_q\rangle = \langle f \circ \tau_{/G}(v_q),\, \tau_{/G}(w_q)\rangle. \tag{2.6}$$

One can check the G-equivariance of F. One can also check that relation (2.6) defines the unique *fiber-preserving* map F of TQ to T^*Q. Let $W := \pi_{/G}^{-1}(U)$. By construction, (L, F, W) is the unique G-CL system whose RCL system is (l, f, U). ■

By Proposition 2.5, we can write an arbitrary RCL in the form of the RCL of a G-CL without loss of generality. Besides, the proof of Proposition 2.5 establishes the following assertion: Given a fiber-preserving map $f : TQ/G \to T^*Q/G$, there exists the unique fiber-preserving map $F : TQ \to T^*Q$ satisfying

$$f \circ \tau_{/G} = \pi_{/G} \circ F.$$

Reduced Equations of Motion. Given a G-CL system (L, F, W), the G-invariance of L implies the G-equivariance of the bundle map $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$, which induces a quotient map

$$\mathcal{REL}(l) := [\mathcal{EL}(L)]_G : T^{(2)}Q/G \to T^*Q/G,$$

which depends only on the reduced Lagrangian l on TQ/G induced from L. The operator $\mathcal{REL}$ is called the ***reduced Euler–Lagrange operator***. The ***equations of motion*** of a RCL $(l, [F]_G, W/G)$ with a choice of control $[u]_G : TQ/G \to W/G$ are given by

$$\mathcal{REL}(l)([q, \dot{q}, \ddot{q}]_G) = [F]_G([q, \dot{q}]_G) + [u]_G([q, \dot{q}]_G).$$

To write computable equations of $\mathcal{REL}$, one normally chooses a principal connection on the principal bundle $Q \to Q/G$ to identify the quotient bundles,

$$TQ/G \text{ with } T(Q/G) \oplus \tilde{\mathfrak{g}}$$
$$T^{(2)}Q/G \text{ with } T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}}$$

and

$$T^*Q/G \text{ with } T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*,$$

where $\tilde{\mathfrak{g}} := Q \times_G \mathfrak{g}$ is the associated adjoint bundle, $\tilde{\mathfrak{g}}^* := Q \times_G \mathfrak{g}^*$ is the associated coadjoint bundle, $T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}}$ is the product bundle over Q/G, $2\tilde{\mathfrak{g}} := \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}$, and $\oplus$ is the Whitney sum(see Lemma 2.4.2 and Lemma 3.2.2 in Cendra, Marsden and Ratiu [2001]). For example, a principal connection $A : TQ \to \mathfrak{g}$ on the principal bundle $\pi : Q \to Q/G$ induces the bundle isomorphism $\alpha_A : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}}$ as follows:

$$\alpha_A([q, \dot{q}]_G) = T\pi(q, \dot{q}) \oplus [q, A(q, \dot{q})]_G. \tag{2.7}$$

With these identifications, $\mathcal{REL}$ induces the ***Lagrange–Poincaré operator*** $\mathcal{LP}$ as follows: for a reduced Lagrangian l,

$$\mathcal{LP}(l) : T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*. \tag{2.8}$$

Hence, the reduced Euler–Lagrange operator, $\mathcal{REL}$ may be replaced by the Lagrange–Poincaré operator $\mathcal{LP}$ in this paper as far as one choose a connection on $Q \to Q/G$. More detail may be found in Cendra, Marsden and Ratiu [2001].

Let us give the local coordinate expression of the Lagrange–Poincaré operator $\mathcal{LP}$ induced from a connection A on the principal bundle $Q \to Q/G$. This will be used in § 5. We choose a local trivialization of the bundle $Q \to Q/G$ to be $X \times G \to X$ where X is an open subset of $\mathbb{R}^r$ with $r = \dim(Q/G)$. Then, at any tangent vector $(x, g, \dot{x}, \dot{g}) \in T_{(x,g)}(X \times G)$, we have

$$A(x, g, \dot{x}, \dot{g}) = \mathrm{Ad}_g(A_e(x) \cdot \dot{x} + \xi)$$

where A_e is the $\mathfrak{g}$-valued 1-form on X defined by $A_e(x) \cdot \dot{x} = A(x, e, \dot{x}, 0)$ and $\xi = g^{-1}\dot{g}$. The bundle isomorphism α_A in this case becomes

$$\alpha_A([x, g, \dot{x}, \dot{g}]_G) = (x, \dot{x}) \oplus (x, \Omega)$$

where $\Omega = A_e(x) \cdot \dot{x} + \xi$ and we choose a local trivialization of the associated bundle $\tilde{\mathfrak{g}}$ to be $X \times \mathfrak{g} \to X$. The Lagrange–Poincaré operator $\mathcal{LP}$ for a reduced Lagrangian l on $TQ/G = T(Q/G) \oplus \tilde{\mathfrak{g}}$ gives

$$\mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} + \frac{\partial l}{\partial \Omega^a}(B^a_{\beta\alpha}\dot{x}^\beta + C^a_{db}\Omega^d A^b_\alpha) \\ \frac{d}{dt}\frac{\partial l}{\partial \Omega^b} - \frac{\partial l}{\partial \Omega^a}(C^a_{db}\Omega^d - C^a_{db}A^d_\alpha \dot{x}^\alpha) \end{pmatrix} \tag{2.9}$$

where $B^a_{\beta\alpha}$ is the curvature of the connection A^a_α, C^a_{bd} are the structure constants of the Lie algebra $\mathfrak{g}$ and where $\Omega = (\Omega^a)$ and $x = (x^\alpha)$ with $a = 1, \ldots, \dim \mathfrak{g}$, and $\alpha = 1, \ldots, \dim(Q/G)$. More details of the derivation of (2.9) may be found in § 3.3 and § 4.2 of Cendra, Marsden and Ratiu [2001]. In particular, when the connection is flat, the Lagrange–Poincaré equation in (2.9) is given by

$$\mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt}\frac{\partial l}{\partial \dot{x}^\alpha} - \frac{\partial l}{\partial x^\alpha} \\ \frac{d}{dt}\frac{\partial l}{\partial \xi^b} - C^a_{db}\xi^d \frac{\partial l}{\partial \xi^a} \end{pmatrix} \tag{2.10}$$

One can always choose a flat connection when the bundle is trivial, i.e., $Q = X \times G$.

We study the relation between trajectories of G-CL systems and trajectories of RCL systems. Let (L, F, W) be a G-CL system and $(l, [F]_G, W/G)$ its RCL system. Choose an arbitrary *G-equivariant* feedback control law $u : TQ \to W$ for (L, F, W). The control u induces a reduced map $[u]_G : TQ/G \to W/G$. Then, if $(q(t), \dot{q}(t)) \in TQ$ is a trajectory of the closed-loop system (L, F, u), then $\tau_{/G}(q(t), \dot{q}(t)) \in TQ/G$ is the trajectory of the closed-loop system $(l, [F]_G, [u]_G)$.

Simple CL Systems. We define simple CL systems, which include most mechanical systems in engineering applications.

Definition 2.6. *A CL system (L, F, W) on TQ is called **simple** if its Lagrangian $L : TQ \to \mathbb{R}$ is of the form kinetic minus potential energy as follows:*

$$L(q, \dot{q}) = \frac{1}{2} m_q(\dot{q}, \dot{q}) - V(q) \tag{2.11}$$

*where m is a (generalized) mass tensor, i.e., a nondegenerate symmetric $(0, 2)$-tensor. A reduced CL system $(l, [F]_G, W/G)$ is called **simple** if the reduced Lagrangian l is induced by a G-invariant simple Lagrangian L on TQ. The acronym, (R)SCL, will denote "(reduced) simple controlled Lagrangian".*

When a simple G-invariant Lagrangian L is given by (2.11), its reduced simple Lagrangian $l : TQ/G \to \mathbb{R}$ is given by

$$l([q, \dot{q}]_G) = \frac{1}{2}[m]_G([q, \dot{q}]_G, [q, \dot{q}]_G) - [V]_G([q]_G)$$

where $[m]_G \in \Gamma(Q/G, T^*Q/G \otimes T^*Q/G)$ is the reduced mass tensor induced from the G-invariance of the mass tensor $m \in \Gamma(Q, T^*Q \otimes T^*Q)$ and $[V]_G : Q/G \to \mathbb{R}$ is the reduced potential energy.

SCL-equivalence. We now recall the fundamental definition of CL-equivalence from Chang, Bloch, Leonard, Marsden and Woolsey [2002] as follows:

Definition 2.7. *Two SCL systems* (L_1, F_1, W_1) *and* (L_2, F_2, W_2) *are said to be* ***CL-equivalent****, or simply,* $(L_1, F_1, W_1) \overset{L}{\sim} (L_2, F_2, W_2)$ *if the following* ***Euler–Lagrange matching conditions*** *hold;*

ELM-1 : $W_1 = m_1 m_2^{-1}(W_2)$,

ELM-2 : $\mathrm{Im}[\mathcal{EL}(L_1) - F_1 - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)] \subset W_1$,

where m_i *is the mass tensor of* L_i *and* Im *means the pointwise image of the map in brackets.*

One can check by coordinate computation that the map

$$[\mathcal{EL}(L_1) - F_1 - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)]$$

in **ELM-2** can be regarded as a map defined on TQ because the acceleration terms from the two Euler–Lagrange expressions cancel.

The following proposition from Chang, Bloch, Leonard, Marsden and Woolsey [2002] explains the significance of the CL-equivalence property. It shows that in a very natural sense the two control systems can be made to correspond by using an appropriate choice of control.

Proposition 2.8. *Suppose that two SCL systems* (L_i, F_i, W_i), $i = 1, 2$ *are CL-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two control laws* u_i, $i = 1, 2$ *is given by*

$$u_1 = \mathcal{EL}(L_1) - F_1 - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2) + m_1 m_2^{-1} u_2 \tag{2.12}$$

where m_i *is the mass tensor of* L_i, $i = 1, 2$.

Proof. Recall that the Euler–Lagrange operator is given by

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q})_j = m_{ij}\ddot{q}^i + \frac{\partial m_{ij}}{\partial q^k}\dot{q}^i\dot{q}^k - \frac{1}{2}\frac{\partial m_{ik}}{\partial q^j}\dot{q}^i\dot{q}^k + \frac{\partial V}{\partial q^j}$$

where the Lagrangian L is given by

$$L = \frac{1}{2} m_{ij}\dot{q}^i\dot{q}^j - V(q).$$

One can solve (2.3) for $\ddot{q}$. Denote by $\ddot{q}_{L_i}$ the expression of the acceleration $\ddot{q}$ obtained from the closed-loop SCL system (L_i, F_i, u_i), $i = 1, 2$. Then,

$$m_1(\ddot{q}_{L_1} - \ddot{q}_{L_2}) = u_1 - m_1 m_2^{-1} u_2 - \left[(\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)\right]$$

The conditions **ELM-1** and **ELM-2** imply that (2.12) holds if and only if $\ddot{q}_{L_1} = \ddot{q}_{L_2}$ if and only if they produce the same equations of motion. Notice that the term

$$\mathcal{EL}(L_1) - m_1 m_2^{-1}\mathcal{EL}(L_2)$$

in (2.12) can be regarded as a map defined on TQ because the acceleration $\ddot{q}$ cancels out. ■

RSCL-equivalence We now define an equivalence relation among RSCL systems on TQ/G.

Definition 2.9. *Two RSCL systems* $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$ *are said to be* ***reduced-CL-equivalent (RCL-equivalent)****, or simply,* $(l_1, [F_1]_G, W_1/G) \overset{L}{\sim} (l_2, [F_2]_G, W_2/G)$ *if the following* ***reduced Euler–Lagrange matching conditions*** *hold:*

RELM-1 : $W_1/G = [m_1]_G [m_2]_G^{-1} (W_2/G)$,

RELM-2 : $\operatorname{Im} [\mathcal{REL}(l_1) - [F_1]_G - [m_1]_G [m_2]_G^{-1} (\mathcal{REL}(l_2) - [F_2]_G)] \subset W_1/G$

where $[m_i]_G$ *is the reduced mass tensor of* l_i, $i = 1, 2$ *and* Im *means the pointwise image of the map in brackets.*

Equivalence Commutes with Reduction. The following theorem explains the relationship between the CL-equivalence relation among G-SCL systems and the RCL-equivalence relation among RSCL systems.

Theorem 2.10. *Two G-SCL systems are CL-equivalent if and only if their associated RSCL systems are RCL-equivalent.*

Proof. Let (L, F, W) be a G-SCL system, and $(l, [F]_G, W/G)$ its associated RSCL system. Then, the theorem follows from the G-invariance of W and the following relations:

$$\mathcal{REL}(l) \circ \tau_{/G}^{(2)} = \pi_{/G} \circ \mathcal{EL}(L); \qquad [F]_G \circ \tau_{/G} = \pi_{/G} \circ F$$

where $\tau_{/G}^{(2)} : T^{(2)}Q \to T^{(2)}Q/G$ is the G-quotient map. ■

Hence, one can check the RCL-equivalence of two RSCL systems in two ways; one is to directly check it, and the other is to check the CL-equivalence of their associated *unreduced* G-SCL systems. In practice, it is more convenient to check it directly at the reduced level; we shall see an example of this in §5.

The following theorem explains the property of the RCL-equivalence relation:

Theorem 2.11. *Suppose that two RSCL systems* $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$ *are RCL-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two feedback control laws* $[u_i]_G$, $i = 1, 2$ *is given by*

$$\begin{aligned} [u_1]_G = \mathcal{REL}(l_1) - [F_1]_G - [m_1]_G [m_2]_G^{-1} (\mathcal{REL}(l_2) - [F_2]_G) \\ + [m_1]_G [m_2]_G^{-1} [u_2]_G \end{aligned} \tag{2.13}$$

where $[m_i]_G$ *is the reduced mass tensor of* l_i, $i = 1, 2$.

Proof. Let $[u_i]_G$ be a control for $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$. Let (L_i, F_i, W_i) be the unreduced G-CL system of $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$. By Theorem 2.10, the two G-SCL systems are CL-equivalent. By Proposition 2.8, the two closed-loop G-SCL systems (L_i, F_i, u_i), $i = 1, 2$ produce the same equations of motion when u_1 and u_2 satisfy (2.12). Hence, the two closed-loop RSCL systems $(l_i, [F_i]_G, [u_i]_G)$, $i = 1, 2$ produce the same equations of motion when $[u_1]_G$ and $[u_2]_G$ satisfy (2.13) because each term in (2.12) is G-equivariant. Also notice that for any choice of $[u_i]_G$, one can choose the other $[u_j]_G$ such that (2.13) holds. ■

One can prove Theorem 2.11 by comparing the expressions of "accelerations" of both equations using (2.9), as in the proof of Proposition 2.8. For this purpose, one needs to choose a connection on $Q \to Q/G$ because one has to split the variations to write down the equations of motion in coordinates because the Euler-lagrange equations come from the variational principles (see (2.9) of this paper and Chapter 3 of Cendra, Marsden and Ratiu [2001] for more detail). In the current proof of Theorem 2.11, we were able to bypass this route by Theorem 2.10.

3 Reduction of Controlled Hamiltonian Systems with Symmetry

There is a Hamiltonian counterpart to controlled Lagrangian systems called *controlled Hamiltonian systems*. We study the reduction of controlled Hamiltonian systems with symmetry in this section.

CH Systems. We first recall the definition of controlled Hamiltonian systems from Chang, Bloch, Leonard, Marsden and Woolsey [2002].

Definition 3.1. *A* ***controlled Hamiltonian (CH) system*** *is a quadruple* (H, B, F, W) *where the function* $H : T^*Q \to \mathbb{R}$ *is called the Hamiltonian,* $B \in \Gamma(\wedge^2 TT^*Q)$ *is called the almost Poisson tensor on* T^*Q*, the fiber-preserving map* $F : T^*Q \to T^*Q$ *is called the (external) force map, and the subbundle* W *of* T^*Q *is called the control subbundle. When a feedback control law* $u : T^*Q \to W$ *is chosen, the quadruple* (H, B, F, u) *denotes the closed-loop CH system.*

The vector field $X_{(H,B,F,u)}$ of a CH system (H, B, F, W) with a control law u is given by

$$X_{(H,B,F,u)} = B^\sharp \mathbf{d}H + \mathrm{vlift}(F) + \mathrm{vlift}(u)$$

where $\mathrm{vlift}(F)$ and $\mathrm{vlift}(u)$ are the vertical lifts defined in (2.1).

Reduction of G-invariant CH Systems. We define G-invariant CH systems on T^*Q and reduced CH systems on T^*Q/Q as follows.

Definition 3.2. *Let* G *be a Lie group action on* Q*. A* ***G-invariant controlled Hamiltonian (G-CH) system*** *is a CH system,* (H, B, F, W) *where* H*,* B*,* F *and* W *are all* G*-invariant.*

Definition 3.3. *A* ***reduced controlled Hamiltonian (RCH) system*** *is a quadruple* (h, b, f, U) *where the function* $h : T^*Q/G \to \mathbb{R}$ *is called the* ***reduced Hamiltonian****,* $b \in \Gamma(\wedge^2 T(T^*Q/G))$ *is called the* ***reduced almost Poisson tensor****, the* *fiber-preserving map* $f : T^*Q/G \to T^*Q/G$ *is called the* ***reduced force map****, and the subbundle* U *of* T^*Q/G *is called the* ***reduced control subbundle****.*

Suppose that we are given a G-CH system (H, B, F, W) on T^*Q. The G-invariant $H : T^*Q \to \mathbb{R}$ induces the reduced Hamiltonian $h : T^*Q/G \to \mathbb{R}$ as follows:

$$H = h \circ \pi_{/G}. \tag{3.1}$$

The G-invariance of the almost Poisson tensor $B \in \Gamma(\wedge^2 TT^*Q)$ induces a reduced almost Poisson tensor $[B]_G \in \Gamma(\wedge^2 T(T^*Q/G))$ as follows: for $f_1, f_2 \in \mathcal{F}(T^*Q/G)$,

$$[B]_{G_{[q,p]_G}}(\mathbf{d}f_1, \mathbf{d}f_2) = B_{(q,p)}(\pi^*_{/G}\mathbf{d}f_1, \pi^*_{/G}\mathbf{d}f_2). \tag{3.2}$$

This is well-defined since

$$\begin{aligned} B_{g(q,p)}(\mathbf{d}(f_1 \circ \pi_{/G}), \mathbf{d}(f_2 \circ \pi_{/G})) &= B_{(q,p)}(g^*\mathbf{d}(f_1 \circ \pi_{/G}), g^*\mathbf{d}(f_2 \circ \pi_{/G})) \\ &= B_{(q,p)}(\mathbf{d}(f_1 \circ \pi_{/G} \circ g), \mathbf{d}(f_2 \circ \pi_{/G} \circ g)) \\ &= B_{(q,p)}(\mathbf{d}(f_1 \circ \pi_{/G}), \mathbf{d}(f_2 \circ \pi_{/G})) \end{aligned}$$

for any $g \in G$ where we used the G-invariance of B in the first equality. One can easily check that $[B]_G$ is skew-symmetric. The G-invariance of F induces the reduced force $[F]_G : T^*Q/G \to T^*Q/G$ satisfying

$$[F]_G \circ \pi_{/G} = \pi_{/G} \circ F. \tag{3.3}$$

This discussion motivates the following definition:

Definition 3.4. *The* ***RCH system of a*** G***-CH system*** (H, B, F, W) *is a quadruple* $(h, [B]_G, [F]_G, W/G)$ *where* h *is the reduced Hamiltonian defined in (3.1),* $[B]_G$ *is the reduced almost Poisson tensor defined in (3.2), and* $[F]_G$ *is the reduced force defined in (3.3).*

Similarly to Proposition 2.5, the following proposition explains the relations between G-CH systems and RCH systems.

Proposition 3.5. *Given a RCH system* (h, b, f, U)*, there is a (not necessarily unique)* G*-CH system* (H, B, F, W) *whose RCH system is* (h, b, f, U)*.*

Proof. Define H by $H = h \circ \pi_{/G}$. Define a force map F on T^*Q as follows: for $\alpha_q \in T^*_qQ$, $v_q \in T_qQ$

$$\langle F(\alpha_q),\, v_q\rangle = \langle f \circ \pi_{/G}(\alpha_q),\, \tau_{/G}(v_q)\rangle.$$

Choose a connection on the principal bundle $\tau_{/G} : TQ \to TQ/G$ (see Chapter 2, Theorem 2.1 in Kobayashi and Nomizu [1963] for the proof of the existence). Then, we can split TT^*Q into the vertical space V and the horizontal space H as $TT^*Q = \mathrm{V} \oplus \mathrm{H}$. This induces the decomposition of T^*T^*Q as $T^*T^*Q = \mathrm{H}^\circ \oplus \mathrm{V}^\circ$ where H° and V° are the annihilators of H and V, respectively. Let $\mathrm{hor} : T(T^*Q/G) \to \mathrm{H}$ be the horizontal lift. Then, its dual map $\mathrm{hor}^* : \mathrm{V}^\circ \to T^*(T^*Q/G)$ is an isomorphism.

For simplicity, we use H° (resp. V°) as the projection of T^*T^*Q onto H° (resp. V°). Define an almost Poisson tensor B on T^*Q as follows: for $\alpha, \beta \in T^*_pT^*Q$

$$B(\alpha,\beta) := b(\mathrm{hor}^*\, \mathrm{V}^\circ\, \alpha,\, \mathrm{hor}^*\, \mathrm{V}^\circ\, \beta).$$

One can check that this almost Poisson tensor is G-invariant. We now show that $\pi_{/G} : T^*Q \to T^*Q/G$ is the Poisson map, i.e., $b = [B]_G$. Let h_1, h_2 be two functions on T^*Q/G. Then, $\mathbf{d}(h_i \circ \pi_{/G}) \in V^\circ$, $i = 1, 2$. So, $\mathrm{hor}^*\, \mathbf{d}(h_i \circ \pi_{/G}) = \mathbf{d}h_i$, $i = 1, 2$. Hence,

$$\begin{aligned}[B]_G(\mathbf{d}h_1,\, \mathbf{d}h_2) &= B(\mathbf{d}(h_1 \circ \pi_{/G}),\, \mathbf{d}(h_2 \circ \pi_{/G}))\\ &= b(\mathbf{d}h_1,\, \mathbf{d}h_2).\end{aligned}$$

It follows that $[B]_G = b$. Let $W = \pi^{-1}_{/G}(U)$. Then, one can see that (H, B, F, W) is a G-CH system and its RCH system coincides with (h, b, f, U). This completes the proof. ■

Notice in Proposition 3.5 that there can be more than one B satisfying $[B]_G = b$, which is the source of the non-uniqueness of the G-CH systems (H, B, F, W) in Proposition 3.5. By Proposition 3.5, we can write an arbitrary RCH system in the form of the RCH system of a G-CH system without loss of generality.

Reduced CH Dynamics. Given a G-CH system (H, B, F, W), let $(h, [B]_G, [F]_G, W/G)$ be its RCH system. The (reduced) Hamiltonian vector field $X_{(h,[B]_G,[F]_G,[u]_G)}$ of $(h, [B]_G, [F]_G, W/G)$ with a control $[u]_G \in W/G$ is given by

$$X_{(h,[B]_G,[F]_G,[u]_G)} = [B]^\sharp_G\mathbf{d}h + \mathrm{vlift}([F]_G) + \mathrm{vlift}([u]_G)$$

where $\mathrm{vlift}([F]_G)$ and $\mathrm{vlift}([u]_G)$ are the vertical lifts defined in (2.1). Let $X_{(H,B,F,u)}$ be the vector field of (H, B, F, W) with control $u \in W$. Then, we have

$$X_{(h,[B]_G,[F]_G,[u]_G)} \circ \pi_{/G} = T\pi_{/G} \cdot X_{(H,B,F,u)}. \tag{3.4}$$

The CH-equivalence Relation. We shall first recall the CH-equivalence relation among CH systems on T^*Q from Chang, Bloch, Leonard, Marsden and Woolsey [2002].

Definition 3.6. *Two CH systems (H_i, B_i, F_i, W_i), $i = 1, 2$, are said to be* ***CH-equivalent****, or simply,*

$$(H_1, B_1, F_1, W_1) \overset{H}{\sim} (H_2, B_2, F_2, W_2)$$

if the following ***Hamiltonian matching conditions*** *hold;*

HM-1 : $W_1 = W_2$,

HM-2 : $\mathrm{Im}[B^\sharp\mathbf{d}H_1 + \mathrm{vlift}(F_1) - B^\sharp\mathbf{d}H_2 - \mathrm{vlift}(F_2)] \subset \mathrm{vlift}(W_1)$

where vlift(W_1) *is the vertical lift of* W_1 *defined in (2.2).*

The following proposition explains the significance of the CH-equivalence relation.

Proposition 3.7. *Suppose that two CH systems* (H_i, B_i, F_i, W_i)*,* $i = 1, 2$*, are CH-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two control laws* u_i*,* $i = 1, 2$*, is given by*

$$\mathrm{vlift}(u_1) = -B_1^{\sharp}\mathbf{d}H_1 - \mathrm{vlift}(F_1) + B_2^{\sharp}\mathbf{d}H_2 + \mathrm{vlift}(F_2) + \mathrm{vlift}(u_2).$$

Proof. Just compare $X_{(H_1,B_1,F_1,u_1)}$ and $X_{(H_2,B_2,F_2,u_2)}$ with controls $u_1 : T^*Q \to W_1$ and $u_2 : T^*Q \to W_2$. ■

RCH-equivalence. We now introduce an equivalence relation among RCH systems on T^*Q/G as follows.

Definition 3.8. *Two RCH systems,* $(h_i, [B_i]_G, [F_i]_G, W_i/G)$*,* $i = 1, 2$*, are said to be* ***reduced-CH-equivalent (RCH-equivalent)****, or simply,*

$$(h_1, [B_1]_G, [F_1]_G, W_2/G) \overset{H}{\sim} (h_2, [B_2]_G, [F_2]_G, W_2/G)$$

if the following ***reduced Hamiltonian matching conditions*** *hold:*

RHM-1 : $W_1/G = W_2/G$,

RHM-2 : $\mathrm{Im}[[B_1]_G^{\sharp}\mathbf{d}h_1 + \mathrm{vlift}([F_1]_G) - [B_2]_G^{\sharp}\mathbf{d}h_2 - \mathrm{vlift}([F_2]_G)] \subset \mathrm{vlift}(W_1/G)$

where vlift(W_1/G) *is the vertical lift of the subbundle* W_1/G *defined in (2.2).*

Reduction Commutes with Equivalence. The following theorem explains the relation between the reduced-CH-equivalence relation among RCH systems on T^*Q/G and the CH-equivalence relation among G-CH systems on T^*Q.

Theorem 3.9. *Two* G*-CH systems are CH-equivalent if and only if their associated RCH systems are RCH-equivalent.*

Proof. Use Definition 3.6 and Definition 3.8 as well as the relation (3.4). ■

Theorem 3.10. *Suppose that two RCH systems* $(h_i, [B_i]_G, [F_i]_G, W_i/G)$*,* $i = 1, 2$*, are RCH-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop RCH systems produce the same equations of motion. The explicit relation between the two control laws* $[u_i]_G$*,* $i = 1, 2$*, is given by*

$$\mathrm{vlift}([u_1]_G) = -[B_1]_G^{\sharp}\mathbf{d}h_1 - \mathrm{vlift}([F_1]_G) + [B_2]_G^{\sharp}\mathbf{d}h_2 + \mathrm{vlift}([F_2]_G) + \mathrm{vlift}([u_2]_G).$$

Proof. Mimic the proof of Theorem 2.11 and use Proposition 3.7 and Theorem 3.9 instead. ■

Simple CH Systems. Let us review the definition of simple controlled Hamiltonian systems from Chang, Bloch, Leonard, Marsden and Woolsey [2002].

Definition 3.11. *A CH system (H, B, F, W) on T^*Q is called* ***simple*** *if the Hamiltonian H has the form kinetic plus potential energy:*

$$H(q, p) = \frac{1}{2}\langle p, \, m_q^{-1} p\rangle + V(q)$$

and its almost Poisson tensor B is nondegenerate and is of the form

$$B(q, p) = \begin{bmatrix} 0 & K(q)^T \\ -K(q) & J(q, p) \end{bmatrix} \tag{3.5}$$

in cotangent coordinates (q, p) on T^*Q where $K(q), J(q, p)$ are $n \times n$ matrices with $n = \dim Q$. We call H the simple Hamiltonian and B the simple almost Poisson tensor. The acronym SCH denotes "simple controlled Hamiltonian".

One can check the statement that B has the form (3.5) is independent of the choice of cotangent bundle coordinates on T^*Q.

Let B be a simple almost Poisson tensor. Then the relation

$$\operatorname{vlift}(\psi_B) = B \circ \Theta \tag{3.6}$$

defines a unique $\psi_B \in \Gamma(\operatorname{Aut}(T^*Q))$, where Θ is the canonical one form[1] on T^*Q and B is regarded as a bundle map $B : T^*T^*Q \to TT^*Q$. In other words, there exists a unique $\psi_B \in \Gamma(\operatorname{Aut}(T^*Q))$ such that the following diagram commutes.

$$\begin{tikzcd}
T^*T^*Q \arrow[r, "B"] & TT^*Q \\
T^*Q \arrow[u, "\Theta"'] \arrow[r, "\psi_B"] & T^*Q \arrow[u, "\operatorname{vlift}"']
\end{tikzcd}$$

See Chang, Bloch, Leonard, Marsden and Woolsey [2002] for the proof of the existence of ψ_B. When B is given in coordinates as in (3.5), the map ψ_B is given in coordinates by

$$\psi_B(q, p) = (q, K(q)p).$$

Now, we make the following definition of the reduced simple CH system:

Definition 3.12. *A RCH system $(h, [B]_G, [F]_G, W/G)$ is called a reduced simple CH system (or,* ***RSCH*** *system) if it is the reduced CH system of a G-invariant simple CH system.*

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[1] The canonical one form Θ on T^*Q is given by $\Theta = p_i \mathbf{d}q^i$ in cotangent bundle coordinates $(q^i, p_j) \in T^*Q$.

Simple Almost Poisson Tensors and their Reductions. Recall that we defined simple almost Poisson tensors on T^*Q using local coordinates in Definition 3.11. Here, we characterize the reduced simple almost Poisson tensors using local coordinates, so we may assume that $Q = G \times X$ where G is a Lie group acting on the manifold X trivially. Recall the following identifications by left translation of G:

$$T^*G = G \times \mathfrak{g}^*, \quad TT^*G = (G \times \mathfrak{g}^*) \times (\mathfrak{g} \times \mathfrak{g}^*), \quad T^*T^*G = (G \times \mathfrak{g}^*) \times (\mathfrak{g}^* \times \mathfrak{g}).$$

We use (g_a, μ_a, x_i, p_i) as local coordinates for $T^*Q = G \times \mathfrak{g}^* \times T^*X$, and (μ_a, x_i, p_i) for $T^*Q/G = \mathfrak{g}^* \times T^*X$ where $a = 1, \ldots, \dim G$, and $i = 1, \ldots, \dim X$. We will use $\{e_a\}$ as a basis for $\mathfrak{g}$, and $\{e_a^*\}$ as its dual basis. Let $B \in \Gamma(\wedge^2 TT^*Q)$ be a G-invariant simple almost Poisson tensor. Then, it is of the following form:

$$\begin{aligned} B(g,\mu,x,p) = {} & A_{ab}(x)(e_a \otimes e_b^* - e_b^* \otimes e_a) + E_{ia}(x)(\partial x_i \otimes e_b^* - e_b^* \otimes \partial x_i) \\ & + C_{ai}(x)(e_a \otimes \partial p_i - \partial p_i \otimes e_a) + D_{ij}(x)(\partial_{x_i} \otimes \partial_{p_j} - \partial_{p_j} \otimes \partial_{x_i}) \\ & + R_{ab}(\mu,x,p) e_a^* \otimes e_b^* + S_{ai}(\mu,x,p)(e_a^* \otimes \partial p_i - \partial p_i \otimes e_a^*) \\ & + U_{ij}(\mu,x,p)\partial_{p_i} \otimes \partial_{p_j}. \end{aligned} \tag{3.7}$$

In the matrix form, B is given by

$$B = \begin{bmatrix} O & O & A(x) & C(x) \\ O & O & E(x) & D(x) \\ -A(x)^T & -E(x)^T & R(\mu,x,p) & S(\mu,x,p) \\ -C(x)^T & -D(x)^T & -S(\mu,x,p)^T & U(\mu,x,p) \end{bmatrix}$$

where we used the basis for T_zT^*Q in the following order:

$$e_a,\ \partial_{x_i},\ e_a^*,\ \partial_{p_i}.$$

The non-degeneracy condition for B is given by

$$\operatorname{rank}\begin{bmatrix} A & C \\ E & D \end{bmatrix} = \dim Q.$$

The reduced simple Poisson tensor $[B]_G$ is given by

$$\begin{aligned} [B]_G(\mu,x,p) = {} & E_{ia}(x)(\partial x_i \otimes e_b^* - e_b^* \otimes \partial x_i) + D_{ij}(x)(\partial_{x_i} \otimes \partial_{p_j} - \partial_{p_j} \otimes \partial_{x_i}) \\ & + R_{ab}(\mu,x,p) e_a^* \otimes e_b^* + S_{ai}(\mu,x,p)(e_a^* \otimes \partial p_i - \partial p_i \otimes e_a^*) \\ & + U_{ij}(\mu,x,p)\partial_{p_i} \otimes \partial_{p_j}. \end{aligned} \tag{3.8}$$

In a matrix form,

$$[B]_G = \begin{bmatrix} R(\mu,x,p) & -E(x)^T & S(\mu,x,p) \\ E(x) & O & D(x) \\ -S(\mu,x,p)^T & -D(x)^T & U(\mu,x,p) \end{bmatrix}$$

where we used the basis for $T_{[z]}(T^*Q/G)$ in the following order:

$$e_a^*,\ \partial_{x_i},\ \partial_{p_i}.$$

The non-degeneracy condition for B induces the following rank condition for $[B]_G$:

$$\operatorname{rank}[E \ \ D] = \dim X. \tag{3.9}$$

Remark. One could argue that it might be better if we could characterize all the tensors

$$b \in \Gamma(\wedge^2 T(T^*Q/G))$$

for which there exists a G-invariant simple almost Poisson tensor B such that $b = [B]_G$. Then, we can define reduced simple CH systems without reference to G-invariant simple CH systems. This point has to be studied more and we think that the use of connections is crucial; see Montgomery, Marsden and Ratiu [1984], and Cendra, Marsden and Ratiu [2001].

4 Equivalence Between Reduced Simple Lagrangian and Hamiltonian Systems

In this section we show the equivalence between the method of RSCL systems on TQ/G and that of RSCH systems on T^*Q/G.

The CL-CH Equivalence Theorem. In Chang, Bloch, Leonard, Marsden and Woolsey [2002] and Chang [2002], the equivalence between the method of SCL systems on TQ and that of SCH systems on T^*Q was shown. It is summarized in the following Theorem[2]:

Theorem 4.1. *The method of controlled Lagrangian systems is equivalent to that of controlled Hamiltonian systems for simple mechanical systems in the following sense.*

1. For any two simple CL systems (L_i, F_i^L, W_i^L), $i = 1, 2$, *there exist two associated simple CH systems*[3] (H_i, B_i, F_i^H, W_i^H), $i = 1, 2$,*such that*

$$\begin{aligned}&(L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L)\\ &\qquad \Longleftrightarrow (H_1, B_1, F_1^H, W_1^H) \overset{H}{\sim} (H_2, B_2, F_2^H, W_2^H).\end{aligned} \tag{4.1}$$

2. For any two simple CH systems (H_i, B_i, F_i^H, W_i^H), $i = 1, 2$, *there exist two associated simple CL systems,* (L_i, F_i^L, W_i^L), $i = 1, 2$, *such that*

$$\begin{aligned}&(H_1, B_1, F_1^H, W_1^H) \overset{H}{\sim} (H_2, B_2, F_2^H, W_2^H)\\ &\qquad \Longleftrightarrow (L_1, F_1^L, W_1^L) \overset{L}{\sim} (L_2, F_2^L, W_2^L) \quad \text{and} \quad \psi_{B_2} \circ \psi_{B_1}^{-1} = m_{H_2}(m_{H_1})^{-1}\end{aligned}$$

with m_{H_i} *the mass tensor of* H_i *and* ψ_{B_i} *defined in (3.6) for* $i = 1, 2$.

In statement 1, one does not need the condition $\psi_{B_2} \circ \psi_{B_1}^{-1} = m_{H_2}(m_{H_1})^{-1}$ *along with the CL equivalence condition for the two CL systems because it automatically holds for the associated CH systems.*

---

[2] The statement of Theorem 4.1 above is slightly different from that of Corollary 4.1 in Chang, Bloch, Leonard, Marsden and Woolsey [2002] in that there appears an additional term, $\psi_{B_2} \circ \psi_{B_1}^{-1} = m_{H_2}(m_{H_1})^{-1}$ in statement 2. However, one can still prove Theorem 4.1 of this paper using § 4 of Chang, Bloch, Leonard, Marsden and Woolsey [2002]. For example, see Chang [2002] for the proof.

[3] Refer to § 4 of Chang, Bloch, Leonard, Marsden and Woolsey [2002] to learn how to find the associated CH systems.

This result implies the following: Suppose we want to find all SCL systems which is equivalent to a given SCL system. One can directly search for them on the Lagrangian side using the CL equivalence relation. Alternatively, one firstly find a SCH system which is associated with the given SCL system, secondly search for all SCH systems which are CH-equivalent to this SCH system, and finally transform those SCH systems to SCL systems. Those SCL systems are all SCL systems CL-equivalent to the original SCL system. In the similar way, one can find all the SCH systems which is CH-equivalent to a given SCH system, directly or with CL-equivalence relation. (refer to Chang, Bloch, Leonard, Marsden and Woolsey [2002] and Chang [2002] for more detail). Hence, one can describe a given simple mechanical system as a SCL system or as a SCH system and then apply the CL method or the CH method, correspondingly. Both procedures are equivalent.

We now restrict Theorem 4.1 to G-invariant systems.

Theorem 4.2. *The method of G-invariant simple controlled Lagrangian systems and that of G-invariant simple controlled Hamiltonian systems are equivalent. In other words, Theorem 4.1 restricted to G-invariant simple CL and CH systems holds.*

Proof. Keep track of the G-invariance in the proof of Theorem 4.1 in Chang, Bloch, Leonard, Marsden and Woolsey [2002] and the proof of Theorem 3.2.1 in Chang [2002]. ■

The following theorem explains the equivalence between the method of RSCL systems on TQ/G and that of RSCH systems on T^*Q/G.

Theorem 4.3. *The method of RSCL systems is equivalent to that of RSCH systems in the following sense.*

1. For two given RSCL systems $(l_i, [F_i^L]_G, W_i^L/G)$, $i = 1, 2$, there exist two associated RSCH systems $(h_i, [B_i]_G, [F_i^H]_G, W_i^H/G)$, $i = 1, 2$ such that

$$\begin{aligned}&(l_1, [F_1^L]_G, W_1^L/G) \overset{L}{\sim} (l_2, [F_2^L]_G, W_2^L/G)\\ &\qquad \Longleftrightarrow (h_1, [B_1]_G, [F_1^H]_G, W_1^H/G) \overset{H}{\sim} (h_2, [B_2]_G, [F_2^H]_G, W_2^H/G). \qquad (4.2)\end{aligned}$$

2. For two given RSCH systems $(h_i, [B_i]_G, [F_i^H]_G, W_i^H/G)$, $i = 1, 2$, there exist two associated RSCL systems $(l_i, [F_i^L]_G, W_i^L/G)$, $i = 1, 2$ such that

$$\begin{aligned}&(h_1, [B_1]_G, [F_1^H]_G, W_1^H/G) \overset{H}{\sim} (h_2, [B_2]_G, [F_2^H]_G, W_2^H/G)\\ &\Longleftrightarrow (l_1, [F_1^L]_G, W_1^L/G) \overset{L}{\sim} (l_2, [F_2^L]_G, W_2^L/G) \text{ and } [\psi_{B_2} \circ \psi_{B_1}^{-1}]_G = [m_{H_2}]_G [m_{H_1}]_G^{-1}\end{aligned}$$

where $[m_{H_i}]_G$ is the reduced mass tensor of h_i.

Proof. Let us prove statement 1. For given two RSCL systems $(l_i, [F_i^L]_G, W_i^L/G)$, $i = 1, 2$, consider their unreduced G-SCL systems (L_i, F_i^L, W_i^L), $i = 1, 2$ with $L_i = l_i \circ \tau_{/G}$ (see Proposition 2.5). By Theorem 4.2, there are two G-SCH systems (H_i, B_i, F_i^H, W_i^H), $i = 1, 2$, such that (4.1) holds. Let $(h_i, [B_i]_G, [F_i^H]_G, W_i^H/G)$ be the RSCH system of (H_i, B_i, F_i^H, W_i^H) for $i = 1, 2$. Then, (4.2) follows from Theorem 2.10, Theorem 3.9, and (4.1). Now, we prove statement 2. In this case, use Proposition 3.5 instead of Proposition 2.5, and then proceed in a similar manner. $\blacksquare$

Hence, one can describe a given simple mechanical system as a SCL/SCH system, apply the CL/CH reduction and then apply the reduced CL/CH method, correspondingly. Both procedures are equivalent.

Remark. Notice in (4.2) that $\psi_{B_2} \circ \psi_{B_1}^{-1}$ is G-equivariant even though each of ψ_{B_i} may not be. This equivariance is a consequence of the following commutative diagram:

$$\begin{tikzcd}
TT^*Q & T^*Q^*Q \arrow[l, "B_1"'] \arrow[r, "B_2"] & TT^*Q \\
T^*Q \arrow[u, "\mathrm{vlift}"'] & T^*Q \arrow[l, "\psi_{B_1}"'] \arrow[r, "\psi_{B_2}"] \arrow[u, "\Theta"'] & T^*Q \arrow[u, "\mathrm{vlift}"']
\end{tikzcd}$$

It follows that for $\alpha \in T^*Q$

$$B_2 B_1^{-1}(\mathrm{vlilft}(\alpha)) = \mathrm{vlift}(\psi_{B_2} \circ \psi_{B_1}^{-1}(\alpha)).$$

One can easily check that vlift is G-equivariant, i.e. $\mathrm{vlift}(g\alpha) = g\,\mathrm{vlift}(\alpha)$ for $g \in G$. Then, the G-equivariance of $\psi_{B_2} \circ \psi_{B_1}^{-1}$ follows from the G-equivariance of B_1, B_2 and vlift, and the injectivity of vlift.

5 Examples

We review the example of a satellite with a rotor and the Euler–Poincaré matching conditions presented in Bloch, Leonard and Marsden [2001] in the framework of the current paper. We also apply the reduced CL method to the stabilization of the heavy top with rotors. This will show the application of CL systems only. For an excellent application of CH systems to the problem of underwater vehicle stabilization by internal rotors, refer to Woolsey and Leonard [1999] and Bloch, Leonard and Marsden [2001].

5.1 Satellite with a Rotor and the Euler–Poincaré Matching

We review the satellite with a rotor and the Euler–Poincaré matching conditions presented in Bloch, Leonard and Marsden [2001] in the framework of the current paper.

Satellite with a Rotor. We review the example of a satellite with a rotor aligned along the third principal axis of the body (see Figure 5.1). The configuration space is $Q = G \times X = \mathrm{SO}(3) \times S^1$ with the first factor being the satellite attitude and the second factor being the rotor angle. The Lie group $G = \mathrm{SO}(3)$ acts on the first factor of Q only. We take a trivial (flat) connection on Q such that $TQ/G \simeq \mathfrak{g} \times TX$.

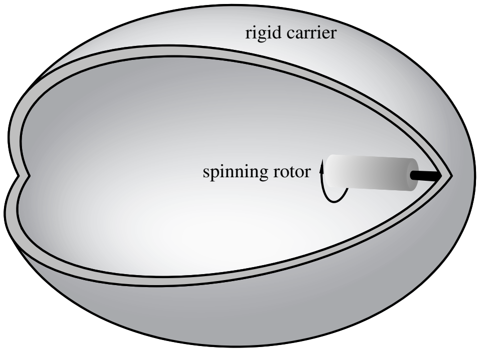

Figure 5.1: The satellite with a rotor along the third body axis.

Use $((\Omega_1, \Omega_2, \Omega_3), (\phi, \dot{\phi}))$ as coordinates for $\mathfrak{so}(3) \times TS^1 \simeq \mathbb{R}^3 \times TS^1$. This system is described by the RSCL system $(l_1, [F_1]_G = 0, W_1/G)$ given by

$$l_1(\Omega, \dot{\phi}) = \frac{1}{2}\big(\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + (I_3 + J_3)\Omega_3^3 + 2J_3 \Omega_3 \dot{\phi} + J_3 \dot{\phi}^2\big)$$

and

$$W_1/G = \operatorname{span}\{\mathbf{d}\phi\} = \operatorname{span}\{(0,0,0,1)\}$$

where $\lambda_1 > \lambda_2 > \lambda_3 := J_3 + I_3$. Notice that l_1 does not depend on ϕ. Recall that the reduced Euler–Lagrange operator $\mathcal{REL}$ induces the Lagrange–Poincaré operator $\mathcal{LP}$ in (2.8) with respect to the trivial connection on $Q \to X$. By (2.10), this Lagrange–Poincaré operator $\mathcal{LP}(l_1)$ is given by

$$\mathcal{LP}(l_1) = \begin{bmatrix} \frac{d}{dt}\frac{\partial l_1}{\partial \Omega} - \frac{\partial l_1}{\partial \Omega} \times \Omega \\ \frac{d}{dt}\frac{\partial l_1}{\partial \dot{\phi}} - \frac{\partial l_1}{\partial \phi} \end{bmatrix} \tag{5.1}$$

where we switched the first and second components of (2.10).

Consider another RSCL system $(l_2, 0, W_2/G)$ with $W_2 = W_1$ and

$$l_2(\Omega, \dot{\phi}) = \frac{1}{2}\big(\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + (I_3 + J_3)\Omega_3^3 + 2J_3 \Omega_3 \dot{\phi} + \rho J_3 \dot{\phi}^2\big)$$

with $\rho \in \mathbb{R}$. Here we allow only one free parameter ρ as in Bloch, Leonard and Marsden [2001], but one can consider a more general form of CL systems. By definition of W_2, **RELM-1** is automatically satisfied and one can check that **RELM-2** also holds for the two RSCL systems. Since these two CL systems are equivalent, one has only to design a controller for the second system, which will give an asymptotically stabilizing controller for the first system by Theorem 2.11. See Bloch, Chang, Leonard, Marsden and Woolsey [2000] for the discussion on asymptotic stabilization of the rotation about the middle axis in the body-fixed frame. There, it is shown how to choose ρ and the dissipative input for stability of the equivalent system $(l_2, 0, W_2/G)$. This leads to an asymptotically stabilizing controller for the original system $(l_1, 0, W_1/G)$.

Euler–Poincaré Matching. Here we briefly sketch the proof that the set of Euler–Poincaré matching conditions in Bloch, Leonard and Marsden [1998] and Bloch, Leonard and Marsden [2001] is a special case of the reduced Euler–Lagrange matching conditions in this paper. This set of matching conditions can handle such examples as a satellite with a rotor and underwater vehicles with internal rotors. Let $Q = G \times X$ be the configuration space where G is a Lie group acting trivially on the manifold X. We choose the trivial (flat) connection on $Q \to X$ to write down the Lagrange–Poincaré equation on $TQ/G \simeq \mathfrak{g} \times TH$ with the Lie algebra $\mathfrak{g}$ of the Lie group G. We use $\eta = (\eta^\alpha)$ as coordinates for $\mathfrak{g}$ and $(\theta, \dot{\theta}) = (\theta^a, \dot{\theta}^a)$ as coordinates for TH. By (2.10), the Lagrange–Poincaré operator $\mathcal{LP}$ with respect to the *trivial* connection is given by

$$\mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt}\frac{\partial l}{\partial \eta^\alpha} - C^\beta_{\gamma\alpha}\eta^\gamma \frac{\partial l}{\partial \eta^\beta} \\ \frac{d}{dt}\frac{\partial l}{\partial \dot{\theta}^a} - \frac{\partial l}{\partial \theta^a} \end{pmatrix} \tag{5.2}$$

for any reduced Lagrangian $l = l(\eta^\alpha, \dot{\theta}^a, \theta)$ where $C^\beta_{\gamma\alpha}$ are the structure constants of the Lie algebra $\mathfrak{g}$. In (5.2) we wrote the vertical part of $\mathcal{LP}(l)$ first while in (2.10) the vertical part was written in the second component.

Let $(l, 0, T^*H)$ be the given RSCL system with the reduced Lagrangian

$$l(\eta^\alpha, \dot{\theta}^a) = \frac{1}{2} g_{\alpha\beta}\eta^\alpha\eta^\beta + g_{\alpha a}\eta^\alpha\dot{\theta}^a + \frac{1}{2} g_{ab}\dot{\theta}^a\dot{\theta}^b$$

where $g_{\alpha\beta}, g_{\alpha a}, g_{ab}$ are constant functions on TQ/G. Notice that this Lagrangian is cyclic in the variables θ^a and the controls act only on the cyclic variables. Let $(l_{\tau,\sigma,\rho}, 0, T^*H)$ be another RCL system with the reduced Lagrangian of the following form:

$$\begin{aligned} l_{\tau,\sigma,\rho} &= l(\eta^\alpha, \dot{\theta}^a + \tau^a_\alpha\eta^\alpha) + \frac{1}{2}\sigma_{ab}\tau^a_\alpha\tau^b_\beta\eta^\alpha\eta^\beta \\ &\quad + \frac{1}{2}(\rho_{ab} - g_{ab})(\dot{\theta}^a + g^{ac}g_{c\alpha}\eta^\alpha + \tau^a_\alpha\eta^\alpha)(\dot{\theta}^b + g^{bc}g_{c\beta}\eta^\beta + \tau^b_\beta\eta^\beta) \end{aligned} \tag{5.3}$$

which is exactly the equation (11) in Bloch, Leonard and Marsden [2001]. See also Bloch, Leonard and Marsden [2000] for the motivation of this choice of the form in (5.3). Bloch, Leonard and Marsden [2001] proposes the following so called Euler–Poincaré matching conditions:

EP-1: $\tau^a_\alpha = -\sigma^{ab}g_{b\alpha}$,

EP-2: $\sigma^{ab} + \rho^{ab} = g^{ab}$.

Then, one can show that the two assumptions of **EP-1** and **EP-2** imply the RCL-equivalence of the two RSCL systems $(l, 0, T^*H)$ and $(l_{\tau,\sigma,\rho}, 0, T^*H)$. Hence, one can equivalently work with the second system to design controllers. Refer to Bloch, Leonard and Marsden [2001] for the method of constructing a Lyapunov function using the energy and Casimir functions.

5.2 Heavy Top with Rotors

It is well known that the upright spinning top is unstable if the angular velocity is small. The motion of the heavy top and the stability of the Lagrange top are well studied in Marsden and Ratiu [1999] and Holm, Marsden and Ratiu [1998]. We use the controlled Lagrangian method to asymptotically stabilize the upright spinning motion of the heavy top with small angular velocity, including zero velocity. See Figure 5.2 for the heavy top system. One can notice that the system dynamics are not SO(3)-invariant because the gravitational force breaks the SO(3) symmetry, so we cannot perform the usual reduction of the system by the SO(3) group. However, there is a way of doing the SO(3)-reduction of this system by considering this system as a system depending on a parameter in $\mathbb{R}^3$.

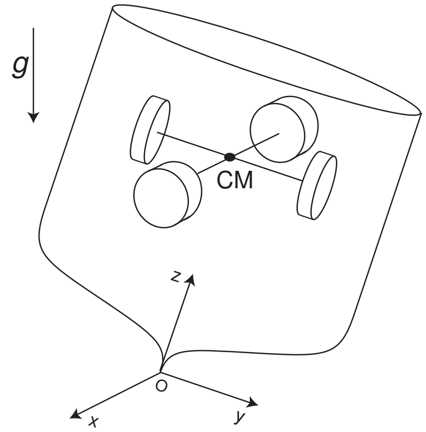

Figure 5.2: Heavy top with two rotors, each consisting of two rigidly coupled disks. The center of mass is at CM.

In this section, we will first review the general theory of reduction of systems depending on a parameter and then apply this reduction theory to the design of a controller for the heavy top system. We will not develop the whole theory of *CL systems depending on a parameter* and the reduction theory for those systems with symmetry because it is a straightforward modification of the theory in § 2 of this paper. Moreover, the complete theory of reduction for (uncontrolled) systems depending on a parameter is in Cendra, Marsden and Ratiu [2001].

Systems Depending on a Parameter. We here review the reduction theory for systems depending on a parameter presented in Cendra, Marsden and Ratiu [2001]. Consider a Lagrangian

$$L : T(G \times X) \times V^* \to \mathbb{R}$$

where G is a Lie group, X is a manifold, and V^* is the dual space of the vector space V. The value of L at the point $(g, x, \dot{g}, \dot{x}, a_0) \in T(G \times X) \times V^*$ will be denoted by $L(g, x, \dot{g}, \dot{x}, a_0)$, as usual, and we will think of a_0 as a parameter that remains fixed along the evolution of the system. Assume that there is an action of G on V, so there is an induced action on V^* such that $\langle ga_0, gb_0\rangle = \langle a_0, b_0\rangle$ for all $a_0 \in V^*$, $b_0 \in V$, $g \in G$. Assume that G acts trivially on X and that L is G-invariant, i.e.,

$$L(hg, x, h\dot{g}, \dot{x}, ha_0) = L(g, x, \dot{g}, \dot{x}, a_0)$$

for $h \in G$, $(g, x, \dot{g}, \dot{x}, a_0) \in T(G \times X) \times V^*$. Define the reduced Lagrangian $l : \mathfrak{g} \times TX \times V^* \to \mathbb{R}$ by

$$l(\xi, x, \dot{x}, a) = L(e, x, \xi, \dot{x}, a) \tag{5.4}$$

for $\xi \in \mathfrak{g}$, $(x, \dot{x}) \in TX$, $a \in V^*$ where $\xi = g^{-1}\dot{g}$ and $a = g^{-1}a_0$.

Fix $a_0 \in V^*$. Let $L_{a_0} : T(G\times X) \to \mathbb{R}$ be the restriction of L to $T(G\times X)\times\{a_0\}$. Then, the Euler-Lagrange operator $\mathcal{EL}(L_{a_0})$ induces the reduced Euler-Lagrange operator $\mathcal{REL}(l) : 2\tilde{\mathfrak{g}} \times T^{(2)}X \times TV^* \to \mathbb{R}$ and the equation in (5.6) as follows:

$$\mathcal{REL}(l) = \begin{pmatrix} \frac{d}{dt}\frac{\partial l}{\partial \xi} - \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi} - \frac{\partial l}{\partial a} \diamond a \\ \frac{d}{dt}\frac{\partial l}{\partial \dot{x}} - \frac{\partial l}{\partial x} \end{pmatrix} \tag{5.5}$$

and

$$\dot{a} = -\xi a \tag{5.6}$$

where the map $\diamond : V \times V^* \to \mathfrak{g}^*$ is defined by

$$\langle b \diamond a, \eta\rangle = -\langle \eta a, b\rangle$$

for $\eta \in \mathfrak{g}$, $a \in V^*$, $b \in V$. One may find the derivation of (5.5) and (5.6) in § 7.4 of Cendra, Marsden and Ratiu [2001].

The equations of motion of the reduced Lagrangian l with a $(\mathfrak{g} \times T^*X)$-valued (reduced) force f, are given by

$$\mathcal{REL}(l) = f, \quad \text{and} \quad \dot{a} = -\xi a \tag{5.7}$$

where f includes external forces and control forces.

Heavy Top with Two Pairs of Rotors. We first describe the heavy top with two pairs of rotors. We mount two pairs of rotors within the top so that each pair's rotation axis is parallel to the first and the second principal axes of the top; see Figure 5.2. Let I_1, I_2, I_3 be the moments of inertia of the top in the body fixed frame. Let J_1, J_2 be the moments of inertia of the rotors around their rotation axes. Let J_{i1}, J_{i2}, J_{i3} be the moments of inertia of the ith rotor with $i = 1, 2$ around the first, the second and the third principal axes, respectively. Let $\bar{I}_1 = I_1 + J_{11} + J_{21}$, $\bar{I}_2 = I_2 + J_{12} + J_{22}$, and $\bar{I}_3 = I_3 + J_{13} + J_{23}$. Let $\lambda_1 = \bar{I}_1 + J_1$ and $\lambda_2 = \bar{I}_2 + J_2$. Let M be the total mass of the system, g the magnitude of the gravitational acceleration, and h the distance from the origin O to the center of mass of the system.

Let $G = \mathrm{SO}(3)$, $X = S^1 \times S^1$, $V^* = \mathbb{R}^3$. We use the following notations for coordinates:

$$\mathbf{R} \in \mathrm{SO}(3), \quad \theta = (\theta_1, \theta_2) \in S^1 \times S^1, \quad a \in \mathbb{R}^3.$$

We will use $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$ as coordinates for the Lie algebra $\mathfrak{so}(3)$ under the Lie algebra isomorphism, $\vee : (\mathfrak{so}(3), [,]) \to (\mathbb{R}^3, \times)$

$$\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}^{\vee} = (x, y, z).$$

Let $L : T(G \times X) \times V^* \to \mathbb{R}$ be the Lagrangian defined by

$$L(\mathbf{R}, \theta, \dot{\mathbf{R}}, \dot{\theta}, a) = \frac{1}{2} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \dot{\theta}_1 \\ \theta_2 \end{pmatrix}^T \begin{pmatrix} \lambda_1 & 0 & 0 & J_1 & 0 \\ 0 & \lambda_2 & 0 & 0 & J_2 \\ 0 & 0 & \bar{I}_3 & 0 & 0 \\ J_1 & 0 & 0 & J_1 & 0 \\ 0 & J_2 & 0 & 0 & J_2 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \dot{\theta}_1 \\ \theta_2 \end{pmatrix} - Mgh\mathbf{R}^{-1} a \cdot \chi$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3) = (\mathbf{R}^{-1}\dot{\mathbf{R}})^{\vee}$ is the body-fixed angular momentum and χ is the body-fixed unit vector on the line of segment connecting the origin O with the body's center of mass, i.e., $\chi = (0, 0, 1)$ in the body-fixed frame.

Fix $\mathbf{k} := (0, 0, 1) \in \mathbb{R}^3$. Let $L_{\mathbf{k}}$ be the restriction of L to $T(G \times X) \times \{\mathbf{k}\}$ as follows:

$$L_{\mathbf{k}}(\mathbf{R}, \theta, \dot{\mathbf{R}}, \dot{\theta}) = L(\mathbf{R}, \theta, \dot{\mathbf{R}}, \dot{\theta}, \mathbf{k}).$$

Then, one can check that $L_{\mathbf{k}}$ is the Lagrangian of the heavy top system in Figure 5.2. The actuation is exerted on each pair of rotors, so the control bundle U is given by $U = T^*X$.

By (5.4), the reduced Lagrangian $l : \mathfrak{so}(3) \times TX \times \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\begin{aligned} l(\Omega, \theta, \dot{\theta}, \Gamma) &= \frac{1}{2}(\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + \bar{I}_3 \Omega_3^2 + 2J_1 \Omega_1 \dot{\theta}_1 + 2J_2 \Omega_2 \dot{\theta}_2) \\ &\quad + \frac{1}{2}(J_1 \dot{\theta}_1^2 + J_2 \dot{\theta}_2^2) - Mgh\Gamma \cdot \chi \end{aligned}$$

where

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) = \mathbf{R}^{-1}\mathbf{k}.$$

Physically, the vector Γ represents the motion of the *unit vector* with the *opposite direction* of gravity as seen from the body. Recall that the reduced equations of motion are derived from (5.7) and (5.5).

Let us consider a new reduced Lagrangian $\tilde{l}$ defined by

$$\begin{aligned} \tilde{l}(\Omega, \theta, \dot{\theta}, \Gamma) &= \frac{1}{2}(\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + \bar{I}_3 \Omega_3^2 + 2J_1 \Omega_1 \dot{\theta}_1 + 2J_2 \Omega_2 \dot{\theta}_2) \\ &\quad + \frac{1}{2}(\rho_1 J_1 \dot{\theta}_1^2 + \rho_2 J_2 \dot{\theta}_2^2) - Mgh\Gamma \cdot \chi \end{aligned} \tag{5.8}$$

where $\rho_1, \rho_2 \in \mathbb{R}$ are free parameters to be chosen later. See Chang and Marsden [2000] for the motivation of this choice of the form in (5.8).

Even though we have not developed the general theory of CL systems depending on a parameter and their reduction, one can check that Definition 2.9 and Theorem 2.11 hold for reduced CL systems depending on a parameter where one should use the reduced Euler-Lagrange operator in (5.5) instead of (2.9) or (2.10). Notice that the equation in (5.6) is common for all reduced CL systems depending on a parameter. With these modifications in mind, one can check that

$$(l, 0, T^*X) \overset{L}{\sim} (\tilde{l}, 0, T^*X).$$

By Theorem 2.11, we can work with the RCL system, $(\tilde{l}, 0, T^*X)$, to design a controller.

Let us define the angular momentum, $\Pi = (\Pi_1, \Pi_2, \Pi_2)$, and the control momentum, $\tilde{J} = (\tilde{J}_1, \tilde{J}_2)$, as follows:

$$\Pi = \frac{\partial \tilde{l}}{\partial \Omega} = (\lambda_1 \Omega_1 + J_1 \dot{\theta}_1,\ \lambda_2 \Omega_2 + J_2 \dot{\theta}_2,\ \bar{I}_3 \Omega_3), \tag{5.9}$$

$$\tilde{J} = \frac{\partial \tilde{l}}{\partial \dot{\theta}} = (J_1 \Omega_1 + J_1 \rho_1 \dot{\theta}_1, J_2 \Omega_2 + J_2 \rho_2 \dot{\theta}_2). \tag{5.10}$$

One can check that Π is the same as the angular momentum, $\frac{\partial l}{\partial \Omega}$, of the original reduced Lagrangian l. By (5.7) and (5.5), the equations of motion of the system $(\tilde{l}, 0, T^*X)$ with a choice of control $v = (v_1, v_2)$, are given as follows:

$$\dot{\Pi} = \Pi \times \Omega + Mgh\Gamma \times \chi, \tag{5.11}$$

$$\dot{\tilde{J}} = v, \tag{5.12}$$

$$\dot{\Gamma} = \Gamma \times \Omega. \tag{5.13}$$

These dynamics have two constants of motion

$$\Pi \cdot \Gamma, \quad \text{and} \quad \|\Gamma\| = 1$$

where $\Pi \cdot \Gamma$ is the vertical component of the space-fixed angular momentum, and $\|\Gamma\| = \|\mathbf{R}^{-1}\mathbf{k}\| = \|\mathbf{k}\| = 1$.

Since the reduced Lagrangian $\tilde{l}$ (or, l) does not depend on the rotor angle, θ, and we are not interested in the angle of rotors but in the angular velocity of rotors, we will remove X from the *phase space* for the sake of simplicity. Hence, we will regard $\mathfrak{so}(3) \times \mathbb{R}^2 \times \mathbb{R}^3$ as a new phase space where $\mathbb{R}^2$ is the velocity component of $TX = \mathbb{R}^2 \times \mathbb{R}^2$.

Let $\Omega(0)$, $\dot{\theta}(0)$, and $\Gamma(0)$ with $||\Gamma(0)||^2 = 1$ be an initial condition with

$$\Omega_3^\circ := \Pi(0) \cdot \Gamma(0) / \bar{I}_3 < \sqrt{Mgh/\bar{I}_3}. \tag{5.14}$$

We are interested in the equilibrium $e = (\Omega_e, \dot{\theta}_e, \Gamma_e)$:

$$\Omega_e = (0, 0, \Omega_3^\circ), \quad \dot{\theta}_e = (0, 0), \quad \Gamma_e = (0, 0, 1), \tag{5.15}$$

or

$$\Omega_e = (0, 0, \Omega_3^\circ), \quad \tilde{J}_e = (0, 0), \quad \Gamma_e = (0, 0, 1),$$

which corresponds to the upright spinning top with the rotors at rest. Notice that this equilibrium lies in the same level set of $(\Pi \cdot \Gamma, ||\Gamma||^2)$ as the initial condition.

We construct a Lyapunov function using the energy-Casimir method (see Bloch, Chang, Leonard, Marsden and Woolsey [2000] for more detail of this method). Set

$$E_{\tilde{\Phi}} = K + U + \Phi(\Pi \cdot \Gamma, ||\Gamma||^2) + \Psi(\tilde{J}_1, \tilde{J}_2) \tag{5.16}$$

where the potential energy U is given by $U(\Gamma) = Mgh\Gamma \cdot \chi = Mgh\Gamma_3$, and the kinetic energy K is given by

$$K = \frac{1}{2}\left(\lambda_1 - \frac{J_1}{\rho_1}\right)\Omega_1^2 + \frac{1}{2}\left(\lambda_2 - \frac{J_2}{\rho_2}\right)\Omega_2^2 + \frac{1}{2}\bar{I}_3\Omega_3^2 + \frac{\tilde{J}_1^2}{2J_1\rho_1} + \frac{\tilde{J}_2^2}{2J_2\rho_2},$$

in the new coordinates $(\Omega, \tilde{J}, \Gamma)$. Choose the function Ψ as follows

$$\Psi(\tilde{J}_1, \tilde{J}_2) = \frac{\tilde{J}_1^2}{2\epsilon_1 J_1} + \frac{\tilde{J}_2^2}{2\epsilon_2 J_2}, \tag{5.17}$$

where coefficients ϵ_i will be determined later. Choose the function Φ of the form

$$\begin{aligned}\Phi(x,y) = &-\Omega_3^\circ(x - \bar{I}_3\Omega_3^\circ) + \frac{1}{2}\left(\bar{I}_3(\Omega_3^\circ)^2 - Mgh\right)(y-1)\\ &+\frac{1}{2}a_1(x - \bar{I}_3\Omega_3^\circ)^2 + \frac{1}{2}a_2(y-1)^2,\end{aligned}$$

where the constants a_1 and a_2 are chosen such that

$$a_1 < -1/\bar{I}_3$$

and

$$4a_2 + a_1(\bar{I}_3\Omega_3^\circ)^2 + \bar{I}_3(\Omega_3^\circ)^2 - Mgh < \frac{\bar{I}_3(a_1\bar{I}_3\Omega_3^\circ - \Omega_3^\circ)^2}{1 + a_1\bar{I}_3}.$$

One can check that the equilibrium e is a critical point of $E_{\tilde{\Phi}}$. We now find conditions under which this critical point is a local *maximum*. First, choose ρ_i satisfying

$$\frac{\bar{I}_3(\Omega_3^\circ)^2 - Mgh}{(\Omega_3^\circ)^2} < \lambda_i - \frac{J_i}{\rho_i} < 0 \tag{5.18}$$

for $i = 1, 2$, and then we can choose ϵ_1 and ϵ_2 such that the second derivative of $E_{\tilde{\Phi}}$ becomes negative definite at e, which implies that $E_{\tilde{\Phi}}$ has a local maximum at e. For later use, we impose an additional condition on ρ_i and ϵ_i as follows:

$$J_i(\Omega_3^\circ)^2 + (\epsilon_i + \rho_i)\left((\Omega_3^\circ)^2(\bar{I}_3 - \lambda_i) - Mgh\right) \neq 0. \tag{5.19}$$

With (5.19), it is still possible to find ρ_i and ϵ_i to ensure negative definiteness of the second derivative of $E_{\tilde{\Phi}}$ at e.

The following choice of $v = (v_1, v_2)$

$$v_i = c_i\left(\dot{\theta}_i + \frac{\tilde{J}_i}{\epsilon_i J_i}\right) \tag{5.20}$$

with $c_i > 0$ for $i = 1, 2$, implies

$$\frac{d}{dt} E_{\tilde{\Phi}} = \sum_{i=1}^{2} c_i \left(\dot{\theta}_i + \frac{\tilde{J}_i}{\epsilon_i J_i} \right)^2 \geq 0, \tag{5.21}$$

which proves the Lyapunov stability of the equilibrium e in the closed-loop system. The complete control law u for the original system $(l, 0, T^*X)$ can be obtained from Proposition 2.11.

Asymptotic stabilization will now be shown by using LaSalle's theorem. Since $E_{\tilde{\Phi}}$ has a local maximum at e, it is nondecreasing in time, and $\Pi \cdot \Gamma$ and $||\Gamma||^2$ are conserved, there is a number c such that the set $S = \{x \in \mathfrak{so}(3) \times \mathbb{R}^2 \times \mathbb{R}^3 | E_{\tilde{\Phi}} \geq c,\ \Pi \cdot \Gamma = \Pi_e \cdot \Gamma_e,\ ||\Gamma||^2 = 1\}$ is non-empty, compact, and positively invariant. Define $\mathcal{E} = \{x \in S | \dot{E}_{\tilde{\Phi}} = 0\} = \{x \in S | v = 0\}$. Let $\mathcal{M}$ be the largest invariant subset of $\mathcal{E}$. One can show $\mathcal{M} = \{e\}$ by (5.19) after shrinking the set S if necessary. Thus, by LaSalle's theorem, e is asymptotically stable.

Here is a brief proof of showing $\mathcal{M} = \{e\}$. Let $(\Omega(t), \dot{\theta}(t), \Gamma(t))$ be a trajectory in $\mathcal{M}$. The condition $v = 0$ and (5.12) imply that $\tilde{J}(t)$ is constant. Hence, $\theta_i(t)$ and $\Omega_i(t)$ are constant for $i = 1, 2$. By (5.9), $\Pi_i(t)$, $i = 1, 2$ are constant. Then the third component of (5.11) becomes $\lambda_3 \dot{\Omega}_3(t) =$ constant. By the Lyapunov stability of the equilibrium, it follows $\dot{\Omega}_3(t) \equiv 0$. Hence, $\Omega_3(t)$ is constant. The first and second component of (5.11) implies that $\Gamma_1(t)$ and $\Gamma_2(t)$ are constant. Then, the third component of (5.13) implies that $\Gamma_3(t)$ is constant. So far we have shown that the trajectory $(\Omega(t), \dot{\theta}(t), \Gamma(t))$, or $(\Pi(t), \dot{\theta}(t), \Gamma(t))$ is constant for all $t \geq 0$. Consider the map $f : \mathbb{R}^8 \to \mathbb{R}^{10}$ defined by

$$f(\Omega, \dot{\theta}, \Gamma) = \begin{pmatrix} \Pi \times \Omega + Mgh\Gamma \times \chi \\ \Gamma \times \Omega \\ J_1 \Omega_1 + (\epsilon_1 + J_1 \rho_1)\dot{\theta}_1 \\ J_2 \Omega_2 + (\epsilon_2 + J_2 \rho_2)\dot{\theta}_2 \\ \Pi \cdot \Gamma - \Pi_e \cdot \Gamma_e \\ ||\Gamma||^2 - 1 \end{pmatrix} = \begin{pmatrix} \dot{\Pi} \\ \dot{\Gamma} \\ \epsilon_1 v_1 \\ \epsilon_2 v_2 \\ \Pi \cdot \Gamma - \Pi_e \cdot \Gamma_e \\ ||\Gamma||^2 - 1 \end{pmatrix}$$

where Π is expressed in terms of $(\Omega, \dot{\theta})$ as in (5.9). Then, one can see that all the trajectories lying in $\mathcal{M}$ are contained in the set $f^{-1}(O)$. In particular, the equilibrium $(\Omega_e, \dot{\theta}_e, \Gamma_e)$ in (5.15) is also contained in $f^{-1}(O)$. One can check that the rank of the Jacobian matrix Df at the equilibrium is the full rank 8 by (5.19). Thus, f is locally one-to-one around the equilibrium by Theorem 4.12 in Boothby [1986]. Therefore, the only possible trajectory totally lying in $\mathcal{M}$ is the equilibrium only. It follows from LaSalle's theorem that the equilibrium is asymptotically stable.

Remarks. 1 . The above procedure shows that the choice of control gains depends on the initial condition. This is unavoidable because we need to know the value of the constant of motion $\Pi \cdot \Gamma$, which the internal actuation cannot change; however, our suggested controller is robust to small errors in the measurement of the initial condition. Let $\tilde{e}$ be the equilibrium of the form (5.15) with $\tilde{\Omega}_3^\circ$ instead of Ω_3°. Suppose the Ω_3° used in constructing the control law is very close to the value $\tilde{\Omega}_3^\circ$. Let $\tilde{E}_{\tilde{\Phi}}$ be the function of the form (5.16), with Ω_3° replaced by $\tilde{\Omega}_3^\circ$. Then $\tilde{e}$ is a critical point of $\tilde{E}_{\tilde{\Phi}}$. By continuity, the second derivative of $\tilde{E}_{\tilde{\Phi}}$ at $\tilde{e}$ will remain negative definite, proving Lyapunov stability of $\tilde{e}$.

2. The same form of controller works for the asymptotic stabilization of the upright spinning top with $\Omega_3^\circ > \sqrt{Mgh/\bar{I}_3}$, which is the opposite of (5.14). All that needs to be done is to choose ρ_i and ϵ_i to make $E_{\tilde{\Phi}}$ have a local minimum at the equilibrium and to choose negative c_i such that $E_{\tilde{\Phi}}$ decreases in time. LaSalle's theorem argument guarantees asymptotic stability.

6 Conclusions

In this paper we have studied the reduction of controlled Lagrangian (CL) and controlled Hamiltonian (CH) systems. We have shown that the notion of equivalence of controlled systems is preserved by the reduction procedure. This leads to a natural derivation of the Bloch-Leonard-Marsden Euler–Poincaré matching conditions and shows in a precise sense how they are related to the unreduced Euler–Lagrange matching conditions. The theory also shows how to do the equivalent matching on the Hamiltonian side. We studied the examples of a rigid body with rotors (a spacecraft) as well as the heavy top with rotors to illustrate the theory.

In the future we will study more examples and also see to what extent this theory applies to controlled nonholonomic systems, following Cendra, Marsden, and Ratiu [2001a].

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