A new result on the Bellman equation for exit time control problems with critical growth dynamics

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§1. Introduction

This note proves that the value function of Sussmann's Reflected Brachystochrone Problem (RBP) for an arbitrary singleton target $\{B\}$ is the unique viscosity solution of the corresponding Bellman equation on $\mathbb{R}^2 \setminus \{B\}$ among functions which vanish at B, are continuous in the plane, and are bounded below. The RBP is the time-optimal problem of bringing points in the plane to B using the dynamics $\dot{x} = u_1\sqrt{|y|}$, $\dot{y} = u_2\sqrt{|y|}$, subject to $(u_1, u_2) \in C := \{z \in \mathbb{R}^2 : ||z||_2 \leq 1\}$. For an analogous uniqueness characterization for a *general* class of exit time problems whose dynamics do not admit unique trajectories for some choices of inputs and initial positions, see [2]. We denote the value function of the RBP for the target $\{B\}$ by T_B (so $T_B(x)$ is the infimum of the times t for which some RBP trajectory brings x to B in time t). The corresponding Bellman equation is

$$|Dv((x,y))|\sqrt{|y|} - 1 = 0. \tag{1}$$

(If $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous, then a *viscosity subsolution* (resp., *supersolution*) of $F(q, Dv(q)) = 0$ on an open subset $\Omega \subset \mathbb{R}^N$ is a continuous function $w : \Omega \to \mathbb{R}$ so that $F(p, D\gamma(p)) \leq$ (resp., $\geq$) 0 for all C^1 functions $\gamma : \Omega \to \mathbb{R}$ and all local maxima (resp., minima) p of $w - \gamma$. A *viscosity solution* of $F(q, Dv(q)) = 0$ on Ω is then a function which is both a viscosity sub- and supersolution of that equation on Ω.)

§2. Proof that T_B is a viscosity solution of (1) on $\mathbb{R}^2 \setminus \{B\}$

Consider the problems obtained from the RBP by replacing the RBP dynamics with $\dot{x} = u_1\sqrt[4]{y^2 + 1/n}$, $\dot{y} = u_2\sqrt[4]{y^2 + 1/n}$ for each $n \in \mathbb{N}$ (with the same constraint on (u_1, u_2)). Call the n-th such problem P_n. By standard results from [1], we know that the value function $T_{n,B}$ for P_n is a viscosity solution of the corresponding Bellman equation

$$|Dv((x,y))|\sqrt[4]{y^2 + 1/n} - 1 = 0 \tag{2}$$

on $\mathbb{R}^2 \setminus \{B\}$. Standard stability results would then let us conclude that T_B is a viscosity solution of (1) on $\mathbb{R}^2 \setminus \{B\}$ if we show $T_{n,B} \to T_B$ uniformly on compacta on $\mathbb{R}^2 \setminus \{B\}$. We prove this convergence by the Ascoli-Arzelá Theorem. First note that $T_{n,B} \leq T_B$ pointwise (since if (ϕ_1, ϕ_2) is a trajectory for the RBP for the input u, then it is also a trajectory for the P_n dynamics for $u|\phi_2|^{1/2}/(\phi_2^2 + 1/n)^{1/4}$). The definition of the infimum gives $|T_{n,B}(x) - T_{n,B}(y)| \leq T_y(x) \vee T_x(y)$ for $x, y \in \mathbb{R}^2$, and an elementary consideration of vertical and horizontal movements along RBP trajectories which we omit establishes that $(p,q) \mapsto T_p(q)$ is continuous. Fixing $p \in \mathbb{R}^2 \setminus \{B\}$, we can therefore let v_p denote a uniform limit of a subsequence of the $T_{n,B}$'s (which we do not relabel) on $B_\delta(p)$ for some $\delta > 0$ (depending on p). Again using stability, v_p is a viscosity solution of (1) on $B_\delta(p)$, so the result follows if we show $T_{n,B} \to T_B$ pointwise on $B_\delta(p)$. Fixing $\varepsilon \in (0,1)$ and $x \in B_\delta(p)$, this amounts to showing that $T_{n,B}(x) + \varepsilon > T_B(x)$ for n large enough. In what follows, $f_n, f : \mathbb{R}^2 \times C \to \mathbb{R}^2$ denote the dynamics for P_n and the RBP, respectively.

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For each $n \in \mathbb{N}$, there is an input α_n so that the corresponding f_n trajectory $\phi_n = (\phi_{n,1}, \phi_{n,2})$ drives x to B in time $t_n \leq T_{n,B}(x) + \varepsilon/2$. Note that ϕ_n is also a trajectory for f_1 (using the input $\beta_n := \alpha_n(\phi_{n,2}^2 + 1/n)^{1/4}/(\phi_{n,2}^2 + 1)^{1/4}$). Now apply the classical compactness theorem for relaxed controls (cf. [1] and [4]). Passing to a weak-$\star$, Radon measure valued (subsequential) limit $\bar{\beta}$ of the β_n's, we obtain a relaxed trajectory ϕ^r for f_1 (meaning $\phi^{r\prime}(t) = \int_C f_1(\phi^r(t), a) d(\bar{\beta}(t))(a)$ for all t) so that $\phi_n \to \phi^r$ uniformly on compacts (along a subsequence). Assuming wlog that $\alpha_n \to \bar{\alpha}$ weak-$\star$, it follows that

$$\phi^r(t) \;\leftarrow\; \phi_n(t) \;=\; x + \int_0^t f_n(\phi_n(s), \alpha_n(s))\, ds \;\to\; x + \int_0^t \int_C f(\phi^r(s), a) d(\bar{\alpha}(s))(a)\, ds$$

for all $t \geq 0$. Since f is affine in the input, it follows that there is an input α so that (α, ϕ^r) is an input-trajectory pair of f. Since $t_n \leq T_B(x) + 1$ for all n, we assume $t_n \to \mu \in \mathbb{R}$. It follows that $\phi^r(\mu) = B$, so $T_{n,B}(x) + \varepsilon \geq T_B(x)$ for large n, as desired.

§3. Proof of uniqueness characterization

Let $w \in C(\mathbb{R}^2)$ be a viscosity solution of (1) on $\mathbb{R}^2 \setminus \{B\}$ which is bounded below and vanishes at B. By standard comparison theorems (e.g., Theorem IV.4.3 of [1]), we get $w \geq T_{n,B}$ pointwise for all n, since w is also a viscosity supersolution of (2) on $\mathbb{R}^2 \setminus \{B\}$, so $w \geq T_B$ pointwise, since $T_{n,B} \to T_B$ pointwise. These theorems apply since the f_n's are *uniformly Lipschitz*, i.e., Lipschitz in the state variable uniformly in the control value.

To prove the reverse inequality, first note that any RBP trajectory ψ starting at a $p \in \mathbb{R}^2$ that first reaches B at time $s > 0$ admits $\mu_1, \mu_2 \in (0, s)$ with $\mu_2 \leq \mu_1$ and an RBP subtrajectory $\tilde{\psi}$ on $[0, \mu_2]$ with $\tilde{\psi}(\mu_2) = \psi(\mu_1)$ such that $\tilde{\psi}\lceil[1/n, \mu_2] \cap \{y = 0\} = \emptyset$ for large n. To see why, assume wlog that $p = (p_1, 0)$. Note that f does not allow movement along the x-axis. It follows that $\psi(\tilde{t}\,)$ lies in $\{y > 0\}$ or $\{y < 0\}$ at some $\tilde{t} \in (0, s)$. Assume $\psi(\tilde{t}\,) \in \{y > 0\}$ wlog. Reflect the subtrajectories of ψ which lie in $\{y < 0\}$ over the x-axis to get another RBP trajectory $\hat{\psi}$ that first reaches $\psi(\tilde{t}\,)$ at some time $\hat{t} \in (0, \tilde{t}\,]$ and lies in the closed upper half plane. From [3], one optimal RBP trajectory joining points P and Q in the closed upper half plane is a cycloid arc passing from P to Q without hitting the x-axis in between. Replace $\hat{\psi}\lceil[0, \hat{t}\,]$ with such an arc $\tilde{\psi}$ for $P = p$ and $Q = \hat{\psi}(\hat{t}\,)$ which first reaches $\psi(\tilde{t}\,)$ at μ_2 and set $\mu_1 = s$.

Fix $x \in \mathbb{R}^2 \setminus \{B\}$. Pick an RBP trajectory ϕ starting at x and first reaching B at time τ, and let τ' be the supremum of those times $t \leq \tau$ for which $w(x) \leq t + w(\phi(t))$. Supposing $\tau' < \tau$, use the continuity of w to conclude that $w(x) \leq \tau' + w(\phi(\tau'))$. Note that $f\lceil(\mathbb{R}^2 \setminus \{|y| \leq 1/n\}) \times C$ is uniformly Lipschitz for all n. Setting $\psi(\cdot) = \phi(\cdot + \tau')$ and $p = \phi(\tau')$ in the argument above and using the well-known local suboptimality principle for viscosity subsolutions for problems with uniformly Lipschitzian dynamics (e.g., Theorem III.2.32 of [1]), we get $w(\tilde{\psi}(1/n)) \leq \mu_2 - 1/n + w(\phi(\tau' + \mu_1))$ for large enough n, so $w(\phi(\tau')) \leq \mu_2 + w(\phi(\tau' + \mu_1))$. Hence, $w(x) \leq \tau' + \mu_1 + w(\phi(\tau' + \mu_1))$, which is a contradiction. Since $w(B) = 0$, it follows from infimizing that $w \leq T_B$ pointwise.

§4. References

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