Further Results on the Bellman Equation for Optimal Control Problems with Exit Times and Nonnegative Lagrangians[*]

Michael Malisoff [†]

Department of Mathematics

Louisiana State University

Baton Rouge, LA 70803-4918 USA

malisoff @ math.lsu.edu

Abstract

In a series of papers, we proved theorems characterizing the value function in exit time optimal control as the unique viscosity solution of the corresponding Bellman equation that satisfies appropriate side conditions. The results applied to problems which satisfy a positivity condition on the integral of the Lagrangian. This positive integral condition assigned a positive cost to remaining outside the target on any interval of positive length. In this note, we prove a new theorem which characterizes the exit time value function as the unique bounded-from-below viscosity solution of the Bellman equation that vanishes on the target. The theorem applies to problems satisfying an asymptotic condition on the trajectories, including cases where the positive integral condition is not satisfied. Our results are based on an extended version of "Barbălat's lemma". We apply the theorem to variants of the Fuller Problem and other examples where the Lagrangian is degenerate.

Key Words: viscosity solutions, optimal control, Fuller Problem, asymptotics of trajectories

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1 Introduction

This note is devoted to the study of Hamilton-Jacobi-Bellman equations (HJBE's) for a large class of unbounded optimal control problems for fully nonlinear systems

$$y'(t) \ = \ f(y(t), \alpha(t)) \quad \text{a.e. } t \geq 0, \quad \alpha(t) \in A, \quad y(0) \ = \ x \tag{1}$$

for compact sets A. The optimal control problems are of the form

$$\text{Minimize} \quad \int_0^{t_x(\alpha)} \ell(y_x(t, \alpha), \alpha(t)) \, dt \quad \text{over} \quad \alpha \in \mathcal{A}(x), \tag{2}$$

where $\mathcal{A} := \{ \text{ measurable functions } [0, +\infty) \to A \}$, $y_x(\cdot, \alpha)$ is the solution of (1) for each $\alpha \in \mathcal{A}$, $\mathcal{T} \subseteq \mathbb{R}^N$ is a fixed closed set which we refer to as the target,

$$t_x(\alpha) \ := \ \inf \{ t \geq 0 : y_x(t, \alpha) \in \mathcal{T} \} \ \in \ [0, +\infty] \quad \forall \alpha \in \mathcal{A}, \ \forall x \in \mathbb{R}^N$$

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and $\mathcal{A}(x)$ is the set of all inputs $\beta \in \mathcal{A}$ for which $t_x(\beta) < \infty$. We will refer to ℓ as the Lagrangian of (2), A as the control set, and $t_x(\beta)$ as an exit time. Our hypotheses also allow exit time problems with possibly unbounded A (cf. §6), including cases where the target $\mathcal{T}$ is unbounded.

The value function of (2) will be denoted by v, and $\mathcal{R}$ denotes the set of all points that can be brought to $\mathcal{T}$ in finite time using the dynamics (2). Therefore,

$$\mathcal{R} := \{x \in \mathbb{R}^N : \mathcal{A}(x) \neq \emptyset\},$$

and

$$v(x) := \inf_{\alpha \in \mathcal{A}(x)} \left(\int_0^{t_x(\alpha)} \ell(y_x(t, \alpha), \alpha(t)) \, dt \right) \tag{3}$$

In particular, $v \equiv +\infty$ outside $\mathcal{R}$. Our results extend to problems with variable discount rates from [1] (cf. §4.2). Notice that if A is bounded, then it need not be the case that $\mathcal{R}$ is all of $\mathbb{R}^N$, even if the dynamics (1) has the form $\dot{x} = Ax + Ba$ and (A, B) is a controllable pair (cf. [18]). The Hamilton-Jacobi-Bellman equation (HJBE) of (2) is

$$\sup_{a \in A} \{-f(x, a) \cdot Dv(x) - \ell(x, a)\} = 0, \quad x \notin \mathcal{T} \tag{4}$$

We will study solutions w of (4) on open sets of the form $\Omega \setminus \mathcal{T}$ for which $w \equiv 0$ on $\mathcal{T}$ (cf. §2 for the precise conditions we put on the data, the solutions of (4), and their domains).

We prove a new uniqueness theorem which characterizes v as the unique bounded-from-below solution of (4) on $\mathcal{R} \setminus \mathcal{T}$ that satisfies appropriate side conditions. [1] These side conditions reduce to the nullness of the solution on $\mathcal{T}$ if $\mathcal{R} = \mathbb{R}^N$. Since v will not be differentiable in general (cf. [1, 14]), we will use the definition of solution from viscosity solutions theory (cf. [1, 11] and §2 below). Our new result applies to variants of the Fuller Problem (cf. [17], [24], and §4.1 below) and problems with possibly unbounded targets which satisfy a certain asymptotics condition (namely, (A_4) below). This asymptotic assumption is more stringent than the asymptotic condition used to prove uniqueness of bounded-from-below solutions in [14]. However, [14] also requires a positive integral condition on the integrated costs (cf. (12) below) which will not be needed in our new theorem. In particular, our new theorem applies to examples with non-Lipschitz, "very degenerate Lagrangians" with general null sets, which are not tractable using the known results (cf. §4). (For a discussion of the uniqueness results of [12, 13, 14], and other known uniqueness results, see Remark 3.2.) The hypotheses of our new theorem can be checked from the data of the HJBE using a variant of "Barbălat's lemma" (cf. Lemma 2.5 and Remark 2.6).

Value function characterizations of this kind have been studied and applied by many authors for a large number of stochastic and deterministic optimal control problems and differential games. Also, uniqueness characterizations form the bases for convergence proofs for numerical schemes for approximating value functions. One generally proves the convergence of such schemes by showing that the value function v_n of the n-th discretization of the problem converges uniformly on compact sets to a viscosity solution of the HJBE as the mesh of the discretization converges to zero, and then one uses the uniqueness characterization to show that the viscosity solution being approximated is actually the desired value function. Recent accounts of work in these areas are in [1, 3, 11]. The papers [4, 19] give uniqueness characterizations for the HJBE for problems whose Lagrangians satisfy [2]

$$\forall \varepsilon > 0, \ \exists C_\varepsilon > 0 \ \text{ such that } \ \ell(x, a) \geq C_\varepsilon \ \text{ for all } \ x \notin B(\mathcal{T}, \varepsilon) \ \text{ and all } \ a \in A. \tag{5}$$

Since we are allowing $\ell(\cdot, a)$ to vanish on open sets outside $\mathcal{T}$ for some choices of $a \in A$, these earlier results will not in general apply to (4) (cf. [13] for examples where (5) fails and the HJBE has several bounded-from-below solutions vanishing on $\mathcal{T}$). For theorems characterizing v as the unique nonnegative viscosity solution of the HJBE (4) on $\mathcal{R} \setminus \mathcal{T}$ that satisfies appropriate boundary conditions, see [9, 21].

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[1] A function w is said to be **bounded-from-below** provided there exists a finite constant $\underline{b}$ such that $w(x) \geq \underline{b}$ for all x in the domain of w.

[2] We set $B(S, \varepsilon) := \{p \in \mathbb{R}^N : \inf\{||p - s|| : s \in S\} < \varepsilon\}$ for all $S \subseteq \mathbb{R}^N$ and $\varepsilon > 0$. For singleton $S = \{p\}$, we write $B_\varepsilon(p)$ to mean $B(\{p\}, \varepsilon)$.

Our work is part of a larger project which generalizes uniqueness characterizations for HJBE's to well-known applied problems with non-Lipschitz dynamics and general nonnegative Lagrangians, such as linear quadratic (LQ) problems. Earlier work can be found in [2] (which covers LQ problems on a fixed finite horizon), [9] (which covers infinite horizon LQ problems), and [12] (which gives a uniqueness characterization for the HJBE for a class of exit time problems with non-Lipschitz dynamics, including Sussmann's Reflected Brachystochrone Problem from [23]). For a stronger definition of solution for a subclass of these problems, leading to a characterization of the maximal solution as a unique solution, see [6].

This note is organized as follows. In §2, we list assumptions used in most of the sequel, and we state our main lemmas. In §3, we state our main theorem, and we show how the theorem improves what was known about solutions of the exit time HJBE. This is followed in §4 by applications, including cases with "very degenerate Lagrangians" which are not tractable using the earlier results. We prove our theorem in §5, and §6 shows how to extend our results to problems with unbounded control sets whose Lagrangians take both positive and negative values.

2   Definitions, Assumptions, and Main Lemmas

For any $M \in \mathbb{N}$ and $S \subseteq \mathbb{R}^M$, let $C(S) = \{$continuous functions $S \to \mathbb{R}\}$, and, if S is also open, let $C^1(S)$ denote those functions in $C(S)$ with one continuous derivative. The boundary of S will be denoted by $\mathrm{bd}(S)$, and $\bar{S}$ denotes the closure of S. Recall the following definition of viscosity solutions:

Definition 2.1 Assume $\mathcal{G} \subseteq \mathbb{R}^N$ is open, $S \supseteq \mathcal{G}$, $F \in C(\mathbb{R}^N \times \mathbb{R}^N)$, and $w \in C(S)$. We call w a **(viscosity) solution** of $F(x, Dw(x)) = 0$ on $\mathcal{G}$ provided the following conditions are satisfied:

(C_1) If $\gamma \in C^1(\mathcal{G})$ and x_o is a local minimizer of $w - \gamma$, then $F(x_o, D\gamma(x_o)) \geq 0$.

(C_2) If $\lambda \in C^1(\mathcal{G})$ and x_1 is a local maximizer of $w - \lambda$, then $F(x_1, D\lambda(x_1)) \leq 0$.

This definition is equivalent to the definition of viscosity solutions based on semidifferentials used in [9] (cf. [1]). The extension of our results to *dis*continuous viscosity solutions (cf. Chapter 5 of [1]) is easily done, so we limit our discussion to continuous solutions, in the sense of Definition 2.1. We next list our main assumptions (but see §6 for the case of unbounded A and possibly negative Lagrangians ℓ). Recall that for any closed set $\mathcal{T}$, $STC\mathcal{T}$ is defined to be the following property: For all $\varepsilon > 0$, $\mathcal{T} \subset \mathrm{Interior}\, \mathcal{R}(\varepsilon)$, where $\mathcal{R}(\varepsilon) := \{x \in \mathbb{R}^N : \exists t \in [0, \varepsilon) \,\&\, \alpha \in \mathcal{A} \text{ such that } y_x(t, \alpha) \in \mathcal{T}\}$, and $y_x(\cdot, \alpha)$ is the unique trajectory for (1). The condition $STC\mathcal{T}$ holds under suitable assumptions on the directions of the vector field f and its Lie brackets at $\partial \mathcal{T}$ (cf. [22]). Our main assumptions are

(A_0) The control set A is a nonempty compact metric space.

(A_1) The dynamics $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ is continuous, and there is a constant $L > 0$ such that $||f(x, a) - f(y, a)|| \leq L||x - y||$ for all $x, y \in \mathbb{R}^N$ and $a \in A$.

(A_2) The target $\mathcal{T} \subseteq \mathbb{R}^N$ is closed and nonempty, and $STC\mathcal{T}$ holds.

(A_3) $\ell : \mathbb{R}^N \times A \to [0, \infty)$ is continuous.

(A_4) If $\beta \in \mathcal{A}$ and $x \in \mathbb{R}^N$, and if $\int_0^\infty \ell(y_x(s, \beta), \beta(s))\, ds < \infty$, then $\lim_{s \to \infty} y_x(s, \beta) \in \mathcal{T}$.

The meaning of (A_0)-(A_4) is as follows. Conditions (A_0)-(A_1) imply that for each $\alpha \in \mathcal{A}$ and each $x \in \mathbb{R}^N$, there is a unique trajectory $y_x(\cdot, \alpha)$ for the dynamics (1) which starts at x and which is defined on $[0, \infty)$. Moreover, one can show (cf. [1], Chapter 3) that if (A_1) holds, then for all $\alpha \in \mathcal{A}$ and $q \in \mathbb{R}^N$, $y_q(\cdot, \alpha)$ satisfies the condition

$$||y_q(t, \alpha) - q|| \leq M_q t \quad \text{for all} \quad t \in [0, 1/M_q], \tag{6}$$

where

$$M_q := \max\{\|f(z,a)\| : a \in A, \|z - q\| \le 1\}$$

if the maximum is nonzero and $M_q = 1$ otherwise. Notice that $\mathcal{T}$ is allowed to be unbounded, that $\ell(\cdot, a)$ is allowed to vanish outside $\mathcal{T}$ for some choices of $a \in A$, and that the positive costs condition (cf. (12) below) required in [12, 13, 14] is no longer assumed. In particular, $\ell(\cdot, a)$ is allowed to vanish on *open* sets outside $\mathcal{T}$ for some values $a \in A$, which is not allowed under (5) or under the hypotheses of [12, 13, 14] (cf. §4.1 below). On the other hand, condition (A_4) is stronger than the asymptotics condition used in [14] (cf. §4.1 for applications which are tractable by [14] where (A_4) does not hold). Note also that ℓ is not required to be convex in the state or locally Lipschitz (cf. [20, 21] for complementary results for compact control sets A and locally Lipschitz ℓ). For sufficient conditions for (A_4) in terms of the data f and ℓ, see Lemma 2.5, Remark 2.6, and Example 6.1.

For given open sets Ω, we will give conditions guaranteeing that if w is a solution of the HJBE on $\Omega \setminus \mathcal{T}$, and if there is a value ω_o satisfying the boundary condition

$$BC(\Omega, \omega_o) \qquad w : \Omega \to \mathbb{R} \quad \text{is bounded below,} \quad \omega_o \in \mathbb{R} \cup \{+\infty\}, \quad w(x) < \omega_o$$
$$\text{on } \Omega, \, w \equiv 0 \text{ on } \mathcal{T}, \text{ and } \lim_{x \to x_o} w(x) = \omega_o \text{ for all } x_o \in \partial\Omega.$$

then $w \equiv v$ on Ω. Notice that when $\mathcal{R} = \mathbb{R}^N$, the limit condition in $BC(\mathcal{R}, \omega_o)$ is satisfied vacuously. In particular, there are no growth restrictions on functions w which satisfy $BC(\mathbb{R}^N, +\infty)$. Our uniqueness results are based on the following representation lemma from Chapter 3 of [1]:

Lemma 2.2 Let (A_0)-(A_3) hold, let $B \subseteq \mathbb{R}^N$ be a bounded open set, and let $w \in C(\bar{B})$ be a viscosity solution of the HJBE (4) on B. For each $q \in B$, $\beta \in \mathcal{A}$, and $\delta > 0$, set

$$\tau_q(\beta) = \inf\{t \ge 0 : y_q(t, \beta) \in \text{bd}(B)\} \quad \text{and} \quad T_\delta(q) = \inf\{t : \text{dist}(y_q(t, \alpha), \text{bd}(B)) \le \delta, \alpha \in \mathcal{A}\}.^{3}$$

Then the inequalities

$$w(q) \quad \le \quad \int_0^r \ell(y_q(s, \beta), \beta(s)) \, ds \; + \; w(y_q(r, \beta)) \tag{7}$$

$$w(q) \quad \ge \quad \inf_{\alpha \in \mathcal{A}} \left[\int_0^t \ell(y_q(s, \alpha), \alpha(s)) \, ds \; + \; w(y_q(t, \alpha)) \right] \tag{8}$$

hold for all $q \in B$, $\beta \in \mathcal{A}$, $r \in [0, \tau_q(\beta))$, $\delta \in \left(0, \frac{1}{2}\text{dist}(q, \text{bd}(B))\right)$, and $t \in [0, T_\delta(q))$.

We use the following definition from [16]:

Definition 2.3 We say that a continuous function $\gamma : \mathbb{R} \to [0, \infty)$ is of **class** $\mathcal{MK}$, and write $\gamma \in \mathcal{MK}$, provided $\gamma(0) = 0$ and γ is strictly increasing on $[0, \infty)$ and even.

Remark 2.4 Notice that any function g of class $\mathcal{K}_\infty$ (cf. [10]) gives rise to a function $\gamma \in \mathcal{MK}$ defined by $\gamma(x) = g(|x|)$. Also $\gamma(x) = |x|^q$ is of class $\mathcal{MK}$ for all $q > 0$. In some of our applications, we will use the Lagrangians $\ell(x, a) \equiv \gamma(x)$ for $\gamma \in \mathcal{MK}$. Notice that functions of class $\mathcal{MK}$ need not be bounded, convex, or locally Lipschitz.

From the point of view of PDE, it is desirable to be able to check asymptotics hypotheses such as (A_4) by verifying conditions on the data f and ℓ, rather than assuming complete information about the trajectories $y_x(\cdot, \alpha)$. To develop ways of checking (A_4), we will use the following generalization of "Barbălat's lemma", which can also be used to check the asymptotics condition from [14] (cf. Remark 2.6). Introduce the notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

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^{3}We set $\text{dist}(q, P) := \inf\{\|q - p\| : p \in P\}$ for all $q \in \mathbb{R}^N$ and $P \subseteq \mathbb{R}^N$.

Lemma 2.5 Assume we are given the following:

1. a compact topological space A

2. continuous functions $g : \mathbb{R}^2 \times A \to \mathbb{R}$, $h : \mathbb{R} \times A \to \mathbb{R}$, and $\gamma \in \mathcal{MK}$

3. constants $R \geq 0$ and $\delta > 0$ for which

   3a. $\inf\{g(x, z, a) : x \in \mathbb{R}, a \in A\} \geq \delta z$ for all $z \geq R$

   3b. $\sup\{g(x, z, a) : x \in \mathbb{R}, a \in A\} \leq \delta z$ for all $z \leq -R$

4. a trajectory $\tau = (\tau_1, \tau_2)$ of $\dot{x} = g(x, z, a)$, $\dot{z} = h(z, a)$ for which $\int_0^\infty \gamma(\tau_1(s)) \mathrm{d}s < \infty$

Then $\lim\limits_{s \to +\infty} \tau_1(s) = 0$ and $\limsup\limits_{s \to +\infty} |\tau_2(s)| \leq R$.

Proof. We assume $\gamma(x) \equiv x^2$, the general case being similar. Let $\varepsilon \in (0, 1/2)$, and let $\alpha \in \mathcal{A}$ be the input for τ. First suppose there is a sequence $t_n \to +\infty$ for which $R + \varepsilon \leq |\tau_2(t_n)| \leq R + 1$ for all n. By passing to a subsequence without relabeling, we may assume $t_{n+1} - t_n \geq 1$ for all n. Introduce the constants $C = \max\{|h(z, a)| : a \in A, |z| \leq R + 2\}$ and $\bar{p} = \varepsilon/(8[C + 1])$, and, for each $n \in \mathbb{N}$, set $a_n = t_n - \bar{p}$, $b_n = t_n + \bar{p}$, and $I_n = [a_n, b_n]$. By the choice of $\varepsilon > 0$, the intervals I_n are disjoint. We claim that $|\tau_2(s)| \leq R + 2$ for all $s \in I_n$ and all $n \in \mathbb{N}$. To verify this claim, suppose the contrary. Let $t \in I_n$ be the time closest to t_n at which $|\tau_2(t)| \geq R + 3/2$. Since $|\tau_2(s)| \leq R + 2$ for all $s \in [t \wedge t_n, t \vee t_n]$,

$$|\tau_2(t) - \tau_2(t_n)| \leq \int_{t \wedge t_n}^{t \vee t_n} |h(\tau_2(s), \alpha(s))| \mathrm{d}s \leq (2\bar{p})C < \varepsilon/2 \tag{9}$$

Therefore, $|\tau_2(t)| < |\tau_2(t_n)| + \varepsilon/2 \leq R + 1 + \varepsilon/2 < R + 3/2$. This contradiction verifies the claim. It follows that (9) holds for all $t \in I_n$ and $n \in \mathbb{N}$.

Claim: If $\tau_2(t_n) \geq R + \varepsilon$, then $\tau_1'(t) \geq \frac{\delta}{2}(R + \varepsilon)$ for almost all (a.a.) $t \in I_n$. If $\tau_2(t_n) \leq -R - \varepsilon$, then $\tau_1'(t) \leq -\frac{\delta}{2}(R + \varepsilon)$ for a.a. $t \in I_n$.

Indeed, if $\tau_2(t_n) \geq R + \varepsilon$, then, by (9), $\tau_2(t) \geq R + \frac{\varepsilon}{2}$ for all $t \in I_n$. Therefore, by 3a. and (9),

$$\tau_1'(t) = g(\tau(t), a(t)) \geq \delta\tau_2(t) \geq -\delta|\tau_2(t) - \tau_2(t_n)| + \delta\tau_2(t_n) \geq -\delta\varepsilon/2 + \delta(R + \varepsilon) \geq \delta(R + \varepsilon)/2$$

a.e. $t \in I_n$. The second assertion of the claim is proven similarly, using 3b. and (9). We may now assume that $R + \varepsilon \leq \tau_2(t_n) \leq R + 1$ for all n, so $\tau_1'(t) \geq \frac{\delta}{2}(R + \varepsilon) > 0$ a.e. $t \in I_n$ for all $n \in \mathbb{N}$, possibly by passing to a further subsequence without relabeling. (Otherwise, we can apply the argument we are about to give to $-\tau_1$, instead of to τ_1, to get the same contradiction.) Set $v_n = \inf\{|\tau_1(t)| : t \in I_n\}$, and choose $s_n \in I_n$ so that $|\tau_1(s_n)| = v_n$ for all $n \in \mathbb{N}$. If $\tau_1(s_n) > 0$, then (since $\tau_1' \geq \frac{\delta}{2}(R + \varepsilon) > 0$ a.e. on I_n) $s_n = a_n$, so we get $\tau_1(t) \geq (\delta/2)(R + \varepsilon)(t - a_n)$ for all $t \in I_n$. Therefore,

$$\int_{I_n} \tau_1^2(s) \mathrm{d}s \geq \frac{\delta^2}{8}(R + \varepsilon)^2 \int_{I_n} (s - a_n)^2 \mathrm{d}s = \frac{1}{3}\delta^2(R + \varepsilon)^2 \bar{p}^3 > 0$$

The preceding inequalities remain true if $\tau_1(s_n) < 0$ (in which case $s_n = b_n$, $\tau_1(b_n) < 0$, and we have $-\tau_1(t) \geq \frac{\delta}{2}(R + \varepsilon)(b_n - t)$ on I_n) or if $\tau_1(s_n) = 0$ (in which case we argue as in the $\tau_1(s_n) > 0$ case on $[s_n, b_n]$ and as in the $\tau_1(s_n) < 0$ case on $[a_n, s_n]$, and then we use $[b_n - s_n] \vee [s_n - a_n] \geq \bar{p}$). This contradicts the 4. and the disjointedness of the I_n's. Therefore, there exists $T_\varepsilon > 0$ such that either (S1) $|\tau_2(t)| \geq R + 1$ for all $t \geq T_\varepsilon$, or else (S2) $|\tau_2(t)| \leq R + \varepsilon$ for all $t \geq T_\varepsilon$. If situation (S1) occurs with $\tau_2(t) \geq R + 1$ for all $t \geq T_\varepsilon$, then we get $\tau_1'(t) \geq (R + 1)\delta$ for all $t \geq T_\varepsilon$ (by 3a.) so $\tau_1(t) \geq \delta(R + 1)(t - T_\varepsilon) + \tau_1(T_\varepsilon)$ for all $t \geq T_\varepsilon$. This contradicts 4.. A similar contradiction arises if $\tau_2(t) \leq -(R + 1)$ for all $t \geq T_\varepsilon$. Therefore, (S2) occurs, so the arbitrariness of $\varepsilon > 0$ implies $\limsup\limits_{t \to +\infty} |\tau_2(t)| \leq R$.

A similar argument which we now sketch shows that $\tau_1(t) \to 0$ as $t \to +\infty$. Given $\varepsilon \in (0, 1/2)$, assume there exist $t_n \to \infty$ with $\varepsilon \leq |\tau_1(t_n)| \leq 1$ for all n. We can assume that $t_{n+1} - t_n \geq 1$ for

all n and $|\tau_2(t)| \leq R + 1$ for all $t \geq 0$. Set $K = \max\{|g(x, z, a)| : a \in A, |x| \leq 2, |z| \leq R + 1\}$ and $\bar{r} = \varepsilon/[8(K+1)]$, and, for each $n \in \mathbb{N}$, set $\alpha_n = t_n - \bar{r}$, $\beta_n = t_n + \bar{r}$, and $J_n = [\alpha_n, \beta_n]$. A slight variant of the above argument shows that $|\tau_1(t) - \tau_1(t_n)| \leq \frac{\varepsilon}{2}$ for all $t \in J_n$ and $n \in \mathbb{N}$. This gives

$$\int_{J_n} \tau_1^2(s)\, \mathrm{d}s \;\; \geq \;\; \left(\frac{\varepsilon}{2}\right)^2 (2\bar{r}) \;\; > 0$$

for all large n, which contradicts the integrability of τ_1^2 and the disjointness of the intervals J_n. By again invoking the integrability of τ_1^2, it follows that there is a $T_\varepsilon > 0$ for which $|\tau_1(t)| \leq \varepsilon$ for all $t \geq T_\varepsilon$. Since ε was arbitrary, it follows that $\tau_1(t) \to 0$ as $t \to +\infty$, which completes the proof of the lemma. ∎

Remark 2.6 We will use Lemma 2.5 to verify assumption (A_4) for two-dimensional dynamics. However, this lemma also gives a criterion for the less stringent asymptotics condition

$$\left[\limsup_{t \to \infty} ||y_x(t, \alpha)|| = +\infty\right] \;\; \Rightarrow \;\; \left[\int_0^\infty \ell(y_x(s, \alpha), \alpha(s))\mathrm{d}s = +\infty\right] \qquad (10)$$

required in [14]. In particular, given 1.-3. of the lemma, (10) holds for the dynamics $f(x, z, a) = (g(x, z, a), h(z, a))$ and any Lagrangian ℓ that satisfies $\ell(x, z, a) \geq \gamma(x)$ for all $(x, z) \in \mathbb{R}^2$ and $a \in A$. If, in addition, 3. holds for $R = 0$, and if $\mathcal{T} = \{\vec{0}\}$, then (A_4) holds. For an alternative approach to verifying (A_4), in which the compactness assumption for A is replaced by a penalizing term in the Lagrangian, see Example 6.1. For criteria for (A_4) under additional boundedness assumptions on f, see [20], §4.

Remark 2.7 It is worth noticing how Lemma 2.5 extends "Barbălat's lemma". In the special case where the conditions of the lemma hold with $g(x, z, a) = z$ and globally bounded h, it follows that $\tau_2(\cdot) = \tau_1'(\cdot)$ is globally Lipschitz. In that case, (A_4) follows from the following lemma in [16]:

$$\left[\gamma \in \mathcal{MK}, \ \phi : [0, \infty) \to \mathbb{R}, \ \phi' \text{ Lipschitz}, \int_0^\infty \gamma(\phi(s))\mathrm{d}s < \infty\right] \Rightarrow \left[\lim_{s \to +\infty} \phi(s) = \lim_{s \to \infty} \phi'(s) = 0\right] \ (11)$$

This is the special case of Lemma 2.5 where $g(x, z, a) = z$ and $h(z, a) = a$, with α chosen to be the bounded a.e. derivative of ϕ'. However, (11) cannot in general be applied to the perturbed cascades of Lemma 2.5 since τ_1' might not be Lipschitz. The classical version of "Barbălat's lemma" is the following: If $\phi : [0, \infty) \to \mathbb{R}$ is Lipschitz, and if $\int_0^\infty \phi^2(s)\, ds < \infty$, then $\phi(t) \to 0$ as $t \to +\infty$. This follows from Lemma 2.5 by picking $g(x, z, a) = a + z$, $h(z, a) = 0$, $z(0) = 0$, and α to be the a.e. derivative of ϕ.

Remark 2.8 Under the hypotheses of Lemma 2.5, it may or may not be the case that $\tau_2(t) \to 0$ as $t \to +\infty$ when $R > 0$. For example, consider the cascade $\dot{x} = z^3$, $\dot{z} = a \in [-1, +1]$. If $\phi = (\phi_1, \phi_2)$ is a trajectory of this cascade for which $\int_0^\infty \phi_1^2(s)ds < \infty$, then Lemma 2.5 and (11) give $\phi(t) \to 0$ as $t \to +\infty$. On the other hand, consider the dynamics

$$\dot{x} = z + a_1, \ \dot{z} = a_2, \ |a_1| \leq 1, \ |a_2| \leq 1,$$

which has the trajectory $\tau(t) = (0, \sin(t))$ for the input $\alpha(t) = (-\sin(t), \cos(t))$. In this case, the hypotheses of the lemma hold with $\delta = 1/2$, $R = 2$, and $\gamma(x) \equiv x^2$, but $\lim_{t \to +\infty} \tau_2(t)$ does not exist.

3 Statement of Main Result and Remarks

3.1 Main Result

We will prove the following theorem:

Theorem 1 *Assume conditions (A_0)-(A_4) are satisfied. If $w \in C(\mathcal{R})$ is a viscosity solution of (4) on $\mathcal{R} \setminus \mathcal{T}$ which satisfies $BC(\mathcal{R}, \omega_o)$ for some $\omega_o \in \mathbb{R} \cup \{+\infty\}$, then $w \equiv v$ on $\mathcal{R}$.*

Remark 3.1 If $v \in C(\mathcal{R})$ satisfies $BC(\mathcal{R}, \nu_o)$ for some $\nu_o \in \mathbb{R} \cup \{+\infty\}$, then this theorem becomes a uniqueness characterization for v. This follows from the fact (cf. [1]) that v is then a viscosity solution of the HJBE (4) on $\mathcal{R} \setminus \mathcal{T}$. If we also have $\mathcal{R} = \mathbb{R}^N$, then the limit in $BC(\mathcal{R}, +\infty)$ holds vacuously, so v is the unique continuous bounded-from-below solution of the HJBE that is null on $\mathcal{T}$.

Remark 3.2 Theorem 1 remains true if the hypothesis $w \in C(\mathcal{R})$ is relaxed to the local boundedness of w and all the other hypotheses are kept the same (cf. [1] for the definition of viscosity solutions in this more general case). We then get a characterization of v as the unique locally bounded solution on $\mathcal{R} \setminus \mathcal{T}$ which in null on $\mathcal{T}$, as long as $v \in C(\mathcal{R})$ satisfies $v(x) \to +\infty$ as $x \to x_o$ for all $x_o \in \mathrm{bd}(\mathcal{R})$. For versions of Theorem 1 for unbounded A and possibly negative ℓ, see §6. The statement of Theorem 1 remains true if $\mathcal{R}$ is replaced by any open set $\Omega \subseteq \mathbb{R}^N$ containing $\mathcal{T}$ (by the same proof). Therefore, we also get local uniqueness characterizations for the HJBE.

3.2 Comparison of Theorem 1 With Other Uniqueness Characterizations

Theorem 1 applies to exit time problems which are not tractable by means of the known uniqueness results in [1, 12, 13, 14]. The (undiscounted) exit time results in [1, 4, 19] assume (5), i.e., positive uniform lower bounds for ℓ outside neighborhoods of $\mathcal{T}$, and deduce that v is the only possible viscosity solution of the HJBE (4) on $\mathcal{R} \setminus \mathcal{T}$ which is continuous on $\mathcal{R}$ and which satisfies $BC(\mathcal{R}, \omega_o)$ for some $\omega_o \in \mathbb{R} \cup \{+\infty\}$. On the other hand, condition (5) will not in general be satisfied in the applications considered below (cf. §4.1). Therefore, our uniqueness characterization does not follow from the well-known results. In [6], a stronger definition of solution is given for a *subclass* of the problems we are considering in this note, and then a maximal solution is characterized as a unique solution.

The papers [12, 13] give conditions under which v is the unique solution of the HJBE (14) on $\mathcal{R} \setminus \mathcal{T}$ in the class of functions w which vanish at $\mathcal{T}$, are continuous on $\mathbb{R}^N$, and satisfy a generalization of properness. (Properness of a function w is the condition that $w(x) \to +\infty$ as $||x|| \to +\infty$.) This generalized properness condition can be satisfied by functions which are not bounded-from-below. The main argument of [12, 13] is a generalization of arguments for free boundary problems from Chapter 4 of [1]. The results in [12, 13, 14] *do not* require (A_4), but they *do* require

$$\int_0^t \ell^r(y_x^r(s, \alpha), \alpha(s)) \, ds > 0 \quad \forall t > 0, x \notin \mathcal{T}, \text{ and } \alpha \in \mathcal{A}^r, \tag{12}$$

where $\mathcal{A}^r := \{\text{measurable } \alpha : [0, \infty) \to A^r\}$, A^r is the set of Radon probability measures on A viewed as a subset of the dual of $C(A)$, $y_x^r(\cdot, \alpha)$ is the trajectory of the relaxed dynamics f^r starting at x, and $h^r(x, m) := \int_A h(x, a) dm(a)$ for all $x \in \mathbb{R}^N$, $m \in A^r$, and $h = f, \ell$. (The set $\mathcal{A}^r$ is called the set of relaxed controls on A. See [1] for discussions of relaxed controls.) Condition (12) is less restrictive than (5), since it allows cases where f immediately moves the state out of the null sets of ℓ to produce positive integrated costs (e.g., the Fuller Problem below). Also, (12) allows $a \in A$ for which $\ell(x, a) \to 0$ as $||x|| \to +\infty$, which is not allowed under (5) if $\mathcal{T}$ is bounded. The novelty of Theorem 1 is that (i) it gives a uniqueness characterization for v within a class of functions which includes functions which violate the growth and nonnegativity conditions of the known uniqueness characterizations, and that (ii) it applies to cases of "very degenerate Lagrangians" where neither (5) nor (12) is satisfied, e.g., to cases where the vector fields $\ell(\cdot, a)$ vanish on an open set outside $\mathcal{T}$ (cf. §4.1 for examples). In particular, we do not require any regularity such as Lipschitzness for ℓ. Our results can therefore be regarded as an extension of the results [12] on exit time problems with non-Lipschitz dynamics. (For uniqueness results for cases where ℓ satisfies Lipschitz-like conditions, leading to a characterization of v as a unique nonnegative solution of the HJBE, see [20, 21].)

Remark 3.3 The paper [14] proves that if $\mathcal{R} = \mathbb{R}^N$ and $v \in C(\mathbb{R}^N)$, and if (A_0)-(A_3), (10), and (12) all hold, then v is the unique bounded-from-below solution of the HJBE on $\mathbb{R}^N \setminus \mathcal{T}$ in $C(\mathbb{R}^N)$ that is null on $\mathcal{T}$. For bounded targets, (10) is a less restrictive condition than (A_4) and could be satisfied by problems violating (A_4), e.g., the 'Shifted FP' in Remark 4.3 below, which is tractable by [14]. For

example, (10) allows trajectories that oscillate around $\mathcal{T}$ without ever approaching any target point. However, we will not need to assume (12). The arguments of [14] are based on an iteration of arguments from [13]. Condition (A_4) is used in [20] to establish uniqueness of *nonnegative* solutions of the HJBE.

4 Applications

4.1 Variations of the Fuller Problem and Its Perturbations

We first show how to apply Theorem 1 to variants of the Fuller Problem (FP) from [17, 24]. The classical FP is an exit time problem with the following data:

$$N = 2, \quad A = [-1, +1], \quad \ell(x, z, a) = x^2, \quad \mathcal{T} = \{\vec{0}\} \subseteq \mathbb{R}^2, \quad f(x, z, a) = (z, a) \tag{13}$$

As shown in [14], the FP value function v_{FP} is the unique continuous bounded-from-below viscosity solution of the corresponding HJBE

$$- z(Dw((x, z)'))_1 \; + \; |(Dw((x, z)'))_2| \; - \; x^2 \; = \; 0, \quad (x, z) \in \mathbb{R}^2 \setminus \{\vec{0}\} \tag{14}$$

that is null at $\vec{0}$. This also follows from Theorem 1, since (A_4) is satisfied.

Let us now show how Theorem 1 also applies to generalized 'flattened' versions of the FP which are not tractable using the known uniqueness characterizations. Set $\mathbf{n} = (0, 4) \in \mathbb{R}^2$. For each $\delta \in [0, 1]$, let $\Phi_\delta : \mathbb{R}^2 \to [0, 1]$ be any $C^1(\mathbb{R}^2)$ function which is null on $B_\delta(\mathbf{n})$ and 1 on $\mathbb{R}^2 \setminus B_{2\delta}(\mathbf{n})$. In particular, $\Phi_o \equiv 1$. For given $\gamma \in \mathcal{MK}$ and $\delta \in [0, 1]$, the exit time problem data for the "Flattened FP" is then

$$N = 2, \quad A = [-1, +1], \quad \ell(x, z, a) \equiv \Phi_\delta(x, z)\gamma(x), \quad \mathcal{T} = \{\vec{0}\} \subseteq \mathbb{R}^2, \quad f(x, z, a) = (z, a) \tag{15}$$

These data reduce to (13) when $\delta = 0$ and $\gamma(x) = x^2$. The data differ from (13), because when $\delta > 0$, ℓ is null on an open set containing a portion of the y-axis. Consequently, for $\delta \in (0, 1]$, these data are not tractable by the known uniqueness results, because neither (5) nor (12) is satisfied. Let $v_{\delta\gamma}$ denote the exit time value function (3) for data (15). A slight modification of the proof of Lemma 2.5 shows that the data (15) satisfy (A_4).[4] Since $STC\mathcal{T}$ holds as well (cf. [1]), we conclude as follows.

Corollary 4.1 Let $\gamma \in \mathcal{MK}$ and $\delta > 0$ be given, and choose the exit time problem data (15). If $w : \mathbb{R}^2 \to \mathbb{R}$ is a continuous, bounded-from-below viscosity solution of the corresponding HJBE

$$- z(Dw((x, z)'))_1 \; + \; |(Dw((x, z)'))_2| \; - \; \Phi_\delta(x, z)\gamma(x) \; = \; 0 \tag{16}$$

on $\mathbb{R}^2 \setminus \{\vec{0}\}$ that is null at $\vec{0}$, then $w \equiv v_{\delta\gamma}$ on $\mathbb{R}^2$.

Remark 4.2 In particular, since $v_{FP} \in C(\mathbb{R}^2)$ (cf. [24]), v_{FP} is the unique continuous bounded-from-below solution of the corresponding HJBE that vanishes at $\vec{0}$. This PDE characterization for v_{FP} was

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[4]The modified proof is as follows. Fix $\delta \in (0, 1]$. One can easily find constants $\eta_1, \eta_2 > 0$ (both possibly depending on $\delta > 0$) for which

- for all $\alpha \in \mathcal{A}$ and $x \in B_{3\delta}(\mathbf{n})$, there exists $t \in [0, \eta_1]$ such that $y_x(t, \alpha) \in \mathrm{bd}(B_{3\delta}(\mathbf{n}))$.
- for all $\alpha \in \mathcal{A}$, all $x \in \mathbb{R}^2 \setminus B_{3\delta}(\mathbf{n})$, and all $t \in [0, \eta_2]$, $y_x(t, \alpha) \notin B_{2\delta}(\mathbf{n})$.

Now let $x \in \mathbb{R}^2$, $\alpha \in \mathcal{A}$, $\varepsilon > 0$, and $\int_0^\infty \ell(y_x(t, \alpha), \alpha(t)) \, dt < \infty$. Set $y_x(t, \alpha) = \tau(t) = (\tau_1(t), \tau_2(t))$. Suppose $t_n \to +\infty$ is such that $|\tau_2(t_n)| \geq \varepsilon$ for all n. We can assume as before that $\tau_2(t_n) \geq \varepsilon$ and $t_{n+1} - t_n \geq \max\{1, 3\eta_1\}$ for all n. We can also assume that for all n, $\tau(t_n) \notin B_{3\delta}(\mathbf{n})$, possibly by replacing t_n with

$$\hat{t}_n \; := \; t_n \; + \; \inf\{t \geq 0 : \tau(t_n + t) \in \mathrm{bd}(B_{3\delta}(\mathbf{n}))\} \; \leq \; t_n + \eta_1$$

without relabeling. The argument of Lemma 2.5 gives disjoint intervals $I_n = [t_n - \bar{p}, t_n + \bar{p}]$ such that $\tau_2(t) \geq \varepsilon/2$ for all $t \in I_n$. By replacing I_n with the subinterval $[t_n, t_n + \bar{p} \wedge (\eta_2/2)]$ without relabeling, we can assume that for all $t \in I_n$ and $n \in \mathbb{N}$, $\tau(t) \notin B_{2\delta}(\mathbf{n})$. We then get the same contradiction as before, since ℓ agrees with the $\gamma \in \mathcal{MK}$ outside $B_{2\delta}(\mathbf{n})$. It follows that $\tau_2(t) \to 0$ as $t \to +\infty$, so τ_1' is Lipschitz, so (A_4) holds by (11).

shown in [14]. However, the data (15) is not tractable by [14], since the required positive integral condition (12) from [14] is not satisfied when $\delta > 0$. More generally, we get a uniqueness result for

$$\text{For each} \quad p \in \mathbb{R}^2, \quad \text{infimize} \quad \int_0^{t_p(\alpha)} \sum_{j=1}^{n-1} \gamma_j \left[x_j(t, \alpha) \right] dt \quad \text{subject to} \quad \alpha \in \mathcal{A}(p), \ x(0) = p, \quad \text{and}$$

$$\dot{x}_1(t, \alpha) = x_2(t, \alpha), \quad \dot{x}_2(t, \alpha) = x_3(t, \alpha), \quad \ldots, \quad \dot{x}_{n-1}(t, \alpha) = x_n(t, \alpha), \quad \dot{x}_n(t, \alpha) = \alpha(t) \in A$$

with the target $\{\vec{0}\} \subset \mathbb{R}^n$ for any nonempty compact set $A \subset \mathbb{R}$ and $\gamma_j \in \mathcal{MK}$. This follows from a repeated application of (11) which we omit.

Remark 4.3 As shown in [14], if we perturb the FP data to

$$N = 2, \quad \mathcal{T} = \{(k, k)\}, \quad A = [-1, +1], \quad f(x, z, a) = (z - k\Phi(x, z), a), \quad \ell(x, z, a) = x^2 + k(1 - |a|)^2 \quad (17)$$

where $k \geq 0$ and $\Phi : \mathbb{R}^2 \to [0, 1]$ is any $C^1(\mathbb{R}^2)$ function which is 1 at all points of $B_{k/4}((k, k))$ but identically 0 on $\mathbb{R}^2 - B_{k/2}((k, k))$, then Theorem 1 above no longer applies, since (A_4) would not hold for $k > 0$. In fact, [14] shows that for all $k > 0$, we can construct trajectories for (17) such that

$$\int_0^\infty \ell(y_x(s, \beta), \beta(s)) \, ds < \infty, \quad \text{but} \quad y_x(s, \beta) \to \vec{0} \notin \mathcal{T} \quad \text{as} \quad s \to +\infty.$$

However, one shows (cf. [14]) that the hypotheses from [14] are all satisfied for (17) for all choices of $k \geq 0$. Therefore, [14] shows that any continuous bounded-from-below solution $w : \mathbb{R}^2 \to \mathbb{R}$ of the HJBE that vanishes on $\mathcal{T}$ must coincide with the corresponding value function v. The results of [14] generalize to discontinuous solutions of the HJBE in neighborhoods of $\mathcal{T}$. Therefore, while Theorem 1 applies to problems with "very degenerate Lagrangians" which violate (12), including cases where the Lagrangian ℓ can vanish on open sets outside $\mathcal{T}$ (e.g., the "Flattened FP"), the results of [14] apply under less stringent asymptotics, e.g., to perturbations of the FP which are not tractable by Theorem 1.

4.2 Other Applications

Condition (A_4) holds automatically when all trajectories asymptotically approach $\mathcal{T}$. In particular, Theorem 1 can be used to give a uniqueness characterization on any strongly invariant robust domain of attraction if we choose $\mathcal{T}$ to be the attractor. (See Theorem 3 below, and also [15], which uses Theorem 1 to give PDE characterizations for maximal cost type Lyapunov functions for asymptotically stable dynamics and sublevel set characterizations for the domains of attraction). A totally different application of Theorem 1 is as follows. Notice that (A_4) also holds automatically if $\int_0^\infty \ell(y_x(s, \beta), \beta(s)) \, ds = \infty$ for all $x \in \mathbb{R}^N$ and $\beta \in \mathcal{A}$. In particular, it applies to the shape-from-shading equation

$$I(x) \left[1 + ||Du(x)||^2 \right]^{1/2} - 1 = 0, \quad I(x) = \frac{||x||}{1 + ||x||}, \quad x \in \mathbb{R}^2 \setminus \mathcal{T} \quad (18)$$

from image processing for cases where $\vec{0} \notin \mathcal{T}$, since (18) can also be written in the HJBE form

$$\max_{||a|| \leq 1} \left\{ I(x) a \cdot Du(x) - \left[1 - \frac{||x||}{1 + ||x||} \left(1 - ||a||^2 \right)^{1/2} \right] \right\} = 0 \quad (19)$$

(See [14] for the verification of (A_4) for the shape-from-shading data in (18) and further applications to image processing PDE's). Notice that (5) does not in general hold for (19). This example was studied in [14, 20]. In a similar way, Theorem 1 also applies to the eikonal equations studied in [14]. The natural extension of our result to the variable discount rate equation

$$\sup_{a \in A} \left\{ -f(x, a) \cdot Dv(x) - \ell(x, a) + \rho(x, a) \cdot v(x) \right\} = 0, \quad x \in \mathbb{R}^N \setminus \mathcal{T} \quad (20)$$

is straightforward. For further applications to problems with unbounded control sets, see §6 below.

5 Proof of Main Result

Let w satisfy the hypotheses of Theorem 1, and let $\bar{x} \in \mathcal{R} \setminus \mathcal{T}$ be given. The fact that $v(\bar{x}) \geq w(\bar{x})$ is an easy consequence of (7) of Lemma 2.2 which is obtained by fixing an input $\alpha \in \mathcal{A}(\bar{x})$, taking B to be an appropriate open tube containing the trace of $y_{\bar{x}}(\cdot, \alpha)$ on $[0, t_{\bar{x}}(\alpha)]$, and then infimizing over all $\alpha \in \mathcal{A}(\bar{x})$ (cf. [13] for a generalization of this argument). Such a tube exists since trajectories that reach $\mathcal{T}$ in finite time remain in $\mathcal{R}$ on $[0, t_{\bar{x}}(\alpha)]$ (because points which are outside $\mathcal{R}$ cannot be brought to $\mathcal{T}$ in finite time). We therefore only show that $w(\bar{x}) \geq v(\bar{x})$.[5]

Choose $\varepsilon \in (0, 1)$ such that
$$w(\bar{x}) \quad < \quad \omega_o - \varepsilon. \tag{21}$$
This is possible since $BC(\mathcal{R}, \omega_o)$ is satisfied. (We allow $\omega_o = +\infty$, in which case (21) holds for all $\varepsilon > 0$.) The inequality $w(\bar{x}) \geq v(\bar{x})$ will follow if we find an input $\bar{\alpha} \in \mathcal{A}(\bar{x})$ such that
$$w(\bar{x}) \quad \geq \quad \int_0^{t_{\bar{x}}(\bar{\alpha})} \ell(y_{\bar{x}}(s, \bar{\alpha}), \bar{\alpha}(s)) \, ds \quad - \quad \varepsilon, \tag{22}$$
since that would give $w(\bar{x}) \geq v(\bar{x}) - \varepsilon$, by definition (3) of v, and then we can let $\varepsilon \downarrow 0$. We now prove the existence of such an $\bar{\alpha}$. In what follows, we define the functions $E_1, E_2, \ldots$ by
$$E_j(t) \quad \equiv \quad \frac{\varepsilon}{4} \left[e^{-(j-1)} - e^{-(t+j-1)} \right] \quad \text{for each} \quad t > 0.$$
Note for future use that $E_1(1) + E_2(1) + \ldots + E_n(1) = \frac{\varepsilon}{4}[1 - e^{-n}]$ for all $n \in \mathbb{N}$. Set
$$\mathcal{Z}_1 \quad = \quad \left\{ \begin{array}{l} (t, \gamma): \quad 0 \leq t \leq 1, \quad \text{and} \quad \gamma : [0, t] \to \mathcal{R} \quad \text{is a trajectory} \\[4pt] \text{for} \quad \dot{x} = f(x, \alpha) \quad \text{for some} \quad \alpha \in \mathcal{A} \quad \text{with} \quad \gamma(0) = \bar{x} \\[4pt] \text{so that} \quad w(\bar{x}) \geq \int_0^t \ell(\gamma(s), \alpha(s)) \, ds + w(\gamma(t)) - E_1(t). \end{array} \right\}.$$

Then $\mathcal{Z}_1$ is partially ordered by the relation
$$(t_1, \gamma_1) \quad \sim \quad (t_2, \gamma_2) \quad \text{iff} \quad [\ t_1 \leq t_2 \quad \text{and} \quad \gamma_2 \lceil [0, t_1] \equiv \gamma_1 \ ] \tag{23}$$
Moreover, one can check that every totally ordered subset of $\mathcal{Z}_1$ has an upper bound in $\mathcal{Z}_1$.

Indeed, let $\{(s_j, \mu_j)\}_j$ be a totally ordered subset of $\mathcal{Z}_1$, set $\bar{s} = \sup_j s_j$, and define the trajectory $\bar{\mu} : [0, \bar{s}) \to \mathcal{R}$ by $\bar{\mu}(p) = \mu_j(p)$ for any j so that $s_j > p$. (This is valid since the s_j's increase and μ_j extends μ_k for $k \leq j$. We assume $\bar{s} > 0$, since otherwise (s_1, μ_1) is the desired upper bound.) Let $\beta_j \in \mathcal{A}$ be a control for the trajectory μ_j, fix $\bar{a} \in A$, and define $\beta^\dagger : (0, \infty) \to A$ by $\beta^\dagger(s) := \beta_j(s)$ on $(s_{j-1}, s_j]$ for all $j \in \mathbb{N}$ and $\beta^\dagger(s) = \bar{a}$ for $s > \bar{s}$ (where $s_0 = 0$). Then $\beta^\dagger$ is measurable and therefore admits a corresponding trajectory $y_{\bar{x}}(\cdot, \beta^\dagger)$ on $[0, \infty)$. Moreover, $y_{\bar{x}}(\cdot, \beta^\dagger)$ coincides with $\bar{\mu}$ on $[0, \bar{s})$. Setting $\bar{\mu}(\bar{s}) = y_{\bar{x}}(\bar{s}, \beta^\dagger)$, it follows that $(\bar{s}, \bar{\mu})$ is an upper bound for $\{(s_j, \mu_j)\}_j$ in the relation (23). Also, the choice of $\varepsilon > 0$ guarantees that $\bar{\mu}(\bar{s}) \in \mathcal{R}$, which implies that $(\bar{s}, \bar{\mu}) \in \mathcal{Z}_1$. To see why $\bar{\mu}(\bar{s}) \in \mathcal{R}$, suppose $\bar{\mu}(\bar{s}) \in \partial \mathcal{R}$. Then $\mathcal{R} \ni \mu_j(s_j) \to \bar{\mu}(\bar{s}) \in \partial \mathcal{R}$. It would then follow from $BC(\mathcal{R}, \omega_o)$ and the nonnegativity of ℓ that
$$w(\bar{x}) \quad \geq \quad \int_0^{s_j} \ell(\mu_j(s), \beta_j(s)) \, ds \quad + \quad w(\mu_j(s_j)) - E_1(s_j) \tag{24}$$
$$\geq \quad w(\mu_j(s_j)) - E_1(s_j) \quad \geq \quad w(\mu_j(s_j)) - \varepsilon \quad \to \quad \omega_o - \varepsilon,$$
so $w(\bar{x}) \geq \omega_o - \varepsilon$. This contradicts (21), so we conclude that $\bar{\mu}(\bar{s}) \in \mathcal{R}$. Since w is continuous at $\bar{\mu}(\bar{s}) \in \mathcal{R}$, it follows that $(\bar{s}, \bar{\mu}) \in \mathcal{Z}_1$ is an upper bound for $\{(s_j, \mu_j)\}_j$.

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[5]The proof that $v(\bar{x}) \geq w(\bar{x})$ is slightly more complicated if $\mathcal{R}$ is replaced by a more general open set Ω containing $\mathcal{T}$ (cf. §3.1), since one must consider trajectories starting at $\bar{x}$ but exiting Ω before the first time they reach $\mathcal{T}$. The proof of this inequality for such cases is the same as the proof of this inequality in [13]. On the other hand, the proof of the reverse inequality we are about to give remains valid if $\mathcal{R}$ is replaced by any open neighborhood Ω of $\mathcal{T}$, as long as w satisfies $BC(\Omega, \omega_o)$ for some $\omega_o \in \mathbb{R} \cup \{+\infty\}$.

It follows from Zorn's Lemma that $\mathcal{Z}_1$ contains a maximal element, which we will denote by $(\bar{t}, \bar{\gamma})$. We can assume that $\bar{\gamma}(\bar{t}) \notin \mathcal{T}$ (since otherwise, we can satisfy (22) with the input for $\bar{\gamma}$). We will now show that $\bar{t} = 1$. Let B be an open set containing $\bar{\gamma}(\bar{t})$ whose closure lies in $\mathcal{R} \setminus \mathcal{T}$. Such a set exists since our hypothesis $STC\mathcal{T}$ implies that $\mathcal{R}$ is open (cf. [1]) and $\mathcal{T}$ is closed. Suppose that $\bar{t} < 1$, and set $q = \bar{\gamma}(\bar{t})$, $\delta = \frac{1}{4} \mathrm{dist}(q, \mathrm{bd}(B))$, and $T_\delta(q) = \inf\{t : \mathrm{dist}(y_q(t, \alpha), \mathrm{bd}(B)) \leq \delta, \alpha \in \mathcal{A}\}$. By (6), $T_\delta(q) > 0$. Indeed, for each $\mu > 0$, we can use (6) to find a $\tilde{t} > 0$ so that $y_q(t, \alpha) \in B(\{q\}, \mu)$ for all $\alpha \in \mathcal{A}$ and all $t \in [0, \tilde{t}]$. Choosing $\mu = 1/4 \mathrm{dist}(q, \mathrm{bd}(B))$, we get $\mathrm{dist}(y_q(t, \alpha), \mathrm{bd}(B)) \geq \mathrm{dist}(q, \mathrm{bd}(B)) - ||y_q(t, \alpha) - q|| \geq \frac{1}{2}\mathrm{dist}(q, \mathrm{bd}(B)) > \delta$ for all $t \in [0, \tilde{t}]$ and $\alpha \in \mathcal{A}$. In particular, $T_\delta(q) \geq \tilde{t} > 0$.

By Lemma 2.2, there exist a $t \in (0, 1 - \bar{t})$ and a $\beta \in \mathcal{A}$ so that

$$w(\bar{\gamma}(\bar{t})) \;\geq\; \int_0^t \ell(y_{\bar{\gamma}(\bar{t})}(s, \beta), \beta(s))\,ds \;+\; w(y_{\bar{\gamma}(\bar{t})}(t, \beta)) \;-\; E_1(\bar{t} + t) \;+\; E_1(\bar{t}) \tag{25}$$

and so that $y_{\bar{\gamma}(\bar{t})}(\cdot, \beta)$ remains in B on $[0, t]$. Let c_1 denote the control for $\bar{\gamma}$. Since $(\bar{t}, \bar{\gamma}) \in \mathcal{Z}_1$, it follows that $\bar{\gamma}(s) \in \mathcal{R}$ for all $s \in [0, \bar{t}]$, and that

$$w(\bar{x}) \;-\; w(\bar{\gamma}(\bar{t})) \;\geq\; \int_0^{\bar{t}} \ell(\bar{\gamma}(s), c_1(s))\,ds \;-\; E_1(\bar{t}). \tag{26}$$

Let $\beta^\sharp$ denote the concatenation of the restriction of the input c_1 to $[0, \bar{t}]$ followed by the input β. If we now add the inequalities (25) and (26), then we get

$$w(\bar{x}) \;\geq\; \int_0^{\bar{t}+t} \ell(y_{\bar{x}}(s, \beta^\sharp), \beta^\sharp(s))\,ds \;+\; w(y_{\bar{x}}(\bar{t} + t, \beta^\sharp)) \;-\; E_1(\bar{t} + t). \tag{27}$$

Since t was chosen so that $\bar{t} + t < 1$ and $B \subseteq \mathcal{R}$, we conclude from (27) that $(\bar{t} + t, y_{\bar{x}}(\cdot, \beta^\sharp)) \in \mathcal{Z}_1$. Since $y_{\bar{x}}(\cdot, \beta^\sharp)$ is an extension of $\bar{\gamma}$, this contradicts the maximality of the pair $(\bar{t}, \bar{\gamma})$. Therefore, $\bar{t} = 1$.

We will now extend $\bar{\gamma} : [0, 1] \to \mathcal{R}$ to get the desired trajectory. Set

$$\mathcal{Z}_2 \;=\; \left\{ \begin{array}{l} (t, \gamma): \quad 0 \leq t \leq 1, \;\; \text{and}\; \gamma: \; [0, t] \to \mathcal{R} \;\; \text{is a trajectory} \\[4pt] \text{for}\; \dot{x} \;=\; f(x, \alpha) \;\; \text{for some}\; \alpha \in \mathcal{A} \;\; \text{with}\; \gamma(0) \;=\; \bar{\gamma}(1) \\[4pt] \text{so that}\; w(\bar{\gamma}(1)) \geq \int_0^t \ell(\gamma(s), \alpha(s))\,ds + w(\gamma(t)) - E_2(t). \end{array} \right\},$$

and partially order $\mathcal{Z}_2$ by (23). By the preceding, $\mathcal{Z}_2$ contains a maximal element $(1, \tau_2)$. Let $a_2 \in \mathcal{A}$ denote the input for τ_2. We can assume that $\tau_2(1) \notin \mathcal{T}$, since otherwise we can satisfy (22) by using the concatenation of $c_1 \lceil [0, 1]$ followed by a_2. Let c_2 denote the concatenation of $c_1 \lceil [0, 1]$ followed by a_2, and let ϕ_2 denote the corresponding concatenated trajectory for c_2 starting from $\bar{x}$ and ending at $\tau_2(1)$. We can assume $\phi_2(2) \notin \mathcal{T}$, as argued before. Now reapply the procedure to $\phi_2(2)$ to get a trajectory $\phi_3 : [0, 3] \to \mathcal{R}$ that starts at $\bar{x}$, and which reaches $\mathcal{T}$ at some time $t \in [0, 3]$ or runs for three time units without leaving $\mathcal{R} \setminus \mathcal{T}$. The procedure is iterated, and it results in a sequence of trajectories

$$s \mapsto \phi_n(s) = y_{\bar{x}}(s, c_n), \quad \phi_n \lceil [0, n-1] \equiv \phi_{n-1}, \quad c_n \lceil [0, n-1] \equiv c_{n-1}, \quad n \geq 2$$

which are defined on $[0, n]$, and which we can assume remain in $\mathcal{R} \setminus \mathcal{T}$. Therefore, for all $n \in \mathbb{N}$,

$$w(\bar{x}) \;\geq\; \int_0^n \ell(\phi_n(s), c_n(s))\,ds \;+\; w(\phi_n(n)) \;-\; \frac{\varepsilon}{2}(1 - e^{-n}). \tag{28}$$

Setting $\hat{\alpha}(s) = c_n(s)$ if $n - 1 < s \leq n$ for all $n \in \mathbb{N}$, letting $\hat{\phi}$ denote the trajectory for $\hat{\alpha}$ starting at $\bar{x}$, and letting $\underline{b}$ denote a lower bound for w, a passage to the limit as $n \to \infty$ in (28) gives

$$\int_0^\infty \ell(\hat{\phi}(s), \hat{\alpha}(s))\,ds \;\leq\; w(\bar{x}) \;-\; \underline{b} \;+\; \frac{\varepsilon}{2} \;<\; \infty,$$

since $\hat{\phi} \equiv \phi_n$ on $(0, n]$. It follows from (A_4) that $\hat{\phi}(s) \to \bar{p}$ as $s \to +\infty$ for some point $\bar{p} \in \mathcal{T}$.

Set $p_n = \phi_n(n)$ for all n, so p_n converges to $\bar{p}$. By (A_0), (A_3), and (6), it follows that

$$\sup_{n \in \mathbb{N}, 0 \leq s \leq 1, \alpha \in \mathcal{A}, a \in A} \ell(y_{p_n}(s, \alpha), a) < \infty \tag{29}$$

Since $STCT$ is satisfied and $w \in C(\mathcal{R})$, the argument of [13] gives $n \in \mathbb{N}$ and $\tilde{\beta} \in \mathcal{A}(p_n)$ such that

$$w(p_n) > -\frac{\varepsilon}{4}, \qquad \int_0^{t_{p_n}(\tilde{\beta})} \ell(y_{p_n}(s, \tilde{\beta}), \tilde{\beta}(s)) \, ds < \frac{\varepsilon}{4}. \tag{30}$$

Fixing $n \in \mathbb{N}$ and $\tilde{\beta}$ satisfying (30), and letting $\bar{\alpha}$ denote the concatenation of the input $c_n \lceil [0, n]$ followed by the input $\tilde{\beta}$, we combine (28) and (30) to get

$$
\begin{aligned}
w(\bar{x}) &\geq \int_0^n \ell(\phi_n(s), c_n(s)) \, ds + w(p_n) \\
&\quad + \int_0^{t_{p_n}(\tilde{\beta})} \ell(y_{p_n}(s, \tilde{\beta}), \tilde{\beta}(s)) \, ds - \frac{3\varepsilon}{4} \\
&\geq \int_0^{t_{\bar{x}}(\bar{\alpha})} \ell(y_{\bar{x}}(s, \bar{\alpha}), \bar{\alpha}(s)) \, ds - \varepsilon.
\end{aligned}
\tag{31}
$$

Therefore, $\bar{\alpha}$ satisfies the requirement (22). This proves Theorem 1.

6 Unbounded Control Sets and Negative Lagrangians

In [13], uniqueness characterizations for proper viscosity solutions of the HJBE were given, under the additional integral condition (12). The methods of [13] can be extended to cases where A is unbounded and ℓ can assume negative values (cf. [13], §6). In this section, we show how Theorem 1 can also be extended to problems with unbounded A and Lagrangians ℓ that take both positive and negative values.

6.1 Unbounded Control Sets

If we add the regularity assumptions on ℓ required in [20, 21], then Theorem 1 can be extended to problems with unbounded control sets A and unbounded Lagrangians. Indeed, assume that $A \subseteq \mathbb{R}^M$ is closed and nonempty but possibly unbounded, and that there are positive constants L, L_δ, C_δ and $1 \leq q < p$ so that the following conditions from [20, 21] hold:

(A_5) $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ is continuous, and the estimates $||f(x, a) - f(z, a)|| \leq L(1 + ||a||^q)||x - z||$ and $||f(x, a)|| \leq L(1 + ||x|| + ||a||^q)$ hold for all $x, z \in \mathbb{R}^N$ and $a \in A$.

(A_6) $\ell : \mathbb{R}^N \times A \to [0, \infty)$ is continuous, and the estimates $|\ell(x, a) - \ell(z, a)| \leq L_\delta[1 + ||a||^p]||x - z||$ and $C_\delta ||a||^p - L_\delta \leq \ell(x, a) \leq L_\delta(1 + ||a||^p)$ hold for all $x, z \in B_\delta(0)$ and $a \in A$ and for all $\delta > 0$.

(A_7) $\exists$ compact set $C \subseteq A$ so that $f \lceil [\mathbb{R}^N \times C]$ satisfies $STCT$, where $\mathcal{T} \subseteq \mathbb{R}^N$ is closed and nonempty.

These assumptions imply the continuity of $(x, p) \mapsto \max\{-f(x, a) \cdot p - \ell(x, a) : a \in A\}$ (cf. [21], p.278). Moreover, Lemma 2.2 remains valid if the hypotheses (A_1)-(A_3) are replaced by (A_5)-(A_7), except that we choose as our set of admissible controls $\mathcal{A} = L_{loc}^p([0, \infty), A)$ (cf. [21]). By (A_5),

$$||y_x(t, \alpha) - y_z(t, \alpha)|| \leq \exp\left(L \int_0^t [1 + ||\alpha(s)||^q] ds \right) ||x - z|| \tag{32}$$

for all $t \geq 0$, $\alpha \in \mathcal{A}$, and $x, z \in \mathbb{R}^N$. By (32) and (A_7) and a standard argument (cf. [1], Chapter 4), $\mathcal{R} \setminus \mathcal{T}$ is open. Also, if (A_5)-(A_6) hold, then any viscosity solution w of the HJBE on $\mathcal{R} \setminus \mathcal{T}$ satisfies

$$w(x) = \inf_{\alpha \in \mathcal{A}} \sup_{t \in [0, e_x(\alpha))} \left[\int_0^t \ell(y_x(s, \alpha), \alpha(s)) ds + w(y_x(t, \alpha)) \right] \quad \forall x \in \mathcal{R} \setminus \mathcal{T}, \tag{33}$$

where

$$e_x(\alpha) = \inf\{t \geq 0 : y_x(t, \alpha) \in \mathrm{bd}(\mathcal{R} \setminus \mathcal{T})\}$$

(cf. [21]). By (A_5), it follows that $e_x(\alpha) > 0$ for all $x \in \mathcal{R} \setminus \mathcal{T}$ and $\alpha \in \mathcal{A}$. Letting v denote the value function (3) with this modified $\mathcal{A}$, this allows the construction of §5 in a single iteration. [6] This gives the following:

Theorem 2 *Let $\emptyset \neq A \subseteq \mathbb{R}^M$ be closed. Assume (A_4)-(A_7). Let $w : \mathcal{R} \to \mathbb{R}$ be a viscosity solution of (4) on $\mathcal{R} \setminus \mathcal{T}$ that satisfies $BC(\mathcal{R}, \omega_o)$ for some $\omega_o \in \mathbb{R} \cup \{+\infty\}$. Then $w \equiv v$.*

In the linear quadratic (LQ) case, the data has the form

$$f(x, a) = Px + Qa, \;\; \ell(x, a) = x'Rx + a'Sa, \;\; x \in \mathbb{R}^N, \;\; A = \mathbb{R}^M, \;\; \mathcal{T} = \{\vec{0}\}$$

for constant matrices P, Q, R, and S with $R \geq 0$ and $S > 0$. In that case, (A_5)-(A_7) hold with $C := [-1, +1]^M \subseteq \mathbb{R}^M$ if the controllability matrix $[Q \; PQ \; P^2Q \; \ldots \; P^{N-1}Q]$ has rank N (cf. [1]). Then Theorem 2 applies if (A_4) holds. Here is an application of Theorem 2 that includes an LQ problem.

Example 6.1 Take as before the dynamics $\dot{x} = z$, $\dot{z} = a \in \mathbb{R}$, but now let $A \subseteq \mathbb{R}$ be any nonempty closed (but possibly unbounded) control set that satisfies $A = -A$ (i.e., symmetry). Fix $c > 1$, $\mathcal{T} = \{\vec{0}\} \subseteq \mathbb{R}^2$, and $\ell(x, z, a) = x^2 + |a|^c$. We check (A_4). Let $\tau = (\tau_1, \tau_2)$ be a trajectory satisfying $\int_0^\infty \ell(\tau(t), \dot{\tau}_2(t))dt =: \bar{k} < \infty$. Given $0 \leq s < t$, it follows that

$$\frac{1}{(t-s)^c}\left[\int_s^t |\dot{\tau}_2(p)|dp\right]^c \;\leq\; \frac{1}{t-s}\int_s^t |\dot{\tau}_2(p)|^c dp \;\leq\; \frac{\bar{k}}{t-s}$$

so

$$|\tau_2(t) - \tau_2(s)| \leq \bar{k}^{1/c}|t-s|^{1-\frac{1}{c}} \quad \forall t, s \geq 0. \tag{35}$$

To prove $\tau_2(t) \to 0$ as $t \to +\infty$, we can suppose as in the proof of Lemma 2.2 that there exist a sequence $t_n \to +\infty$, with $t_{n+1} - t_n \geq 1$ for all n, and $\varepsilon \in (0, 1/2)$, for which $\tau_2(t_n) \geq \varepsilon$ for all n. Set

$$I_n = [t_n - \bar{t}, t_n + \bar{t}] \;\; \forall n \in \mathbb{N}, \quad \text{where} \quad \bar{t} = \frac{1}{2}\min\left\{1, \bar{k}^{\frac{1}{1-c}}\left(\frac{\varepsilon}{2}\right)^{\frac{c}{c-1}}\right\} \tag{36}$$

It follows from (35)-(36) that

$$\tau_1'(t) = \tau_2(t) = \tau_2(t) - \tau_2(t_n) + \tau_2(t_n) \geq -\bar{k}^{1/c}|t - t_n|^{1-\frac{1}{c}} + \varepsilon \geq \varepsilon/2 \quad \text{a.e.} \quad t \in I_n \;\; \forall n \in \mathbb{N}.$$

Condition (A_4) now follows from the proof of Lemma 2.5. [7] In particular, by taking $A = \mathbb{R}$, Theorem 2 therefore proves uniqueness of bounded-from-below solutions $w \in C(\mathbb{R}^2)$ of the HJBE

$$-4z(Dw)_1 + |(Dw)_2|^2 - 4x^2 = 0, \;\; (x, z) \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

that are null at the origin.

---

[6]The construction is as follows. Fix $x \in \mathcal{R} \setminus \mathcal{T}$ and $0 < \varepsilon < \omega_o - w(x)$. Use (33) to find $\beta \in \mathcal{A}$ such that

$$w(x) + \frac{\varepsilon}{4} \geq \int_0^t \ell(y_x(s, \beta), \beta(s))\mathrm{d}s + w(y_x(t, \beta)) \quad \forall t \in (0, e_x(\beta)) \tag{34}$$

First assume $E := e_x(\beta) < \infty$. If $y_x(E, \beta) \in \mathcal{T}$, then we can let $t \uparrow E$ in (34) to get $w(x) + \varepsilon/4 \geq v(x)$. If instead $y_x(E, \beta) \in \mathrm{bd}(\mathcal{R})$, then a passage to this limit and the nonnegativity of ℓ give $w(x) + \varepsilon/4 \geq \omega_o$, contradicting the choice of ε. If $E = +\infty$, then (34) gives $\int_0^\infty \ell(y_x(s, \beta), \beta(s))\mathrm{d}s < \infty$, since w is bounded-from-below. By (A_4), there exists $\bar{p} \in \mathcal{T}$ such that $y_x(s, \beta) \to \bar{p}$ as $s \to +\infty$. Since $STC\mathcal{T}$ holds for the compact control set $C \subseteq A$, (30) holds for large enough n, where now $p_n = y_x(n, \beta)$ and $\tilde{\beta} \in \mathcal{C}$. This again gives $w(x) + \varepsilon \geq v(x)$ and the desired inequality $w(x) \geq v(x)$, by (31) with the choices $\phi_n(s) \equiv y_x(s, \beta)$ and $c_n(s) \equiv \beta(s)$. The reverse inequality is shown as before.

[7]More generally, we get the following analogue of (11) for all $\gamma \in \mathcal{MK}$:

$$\left[\phi : [0, \infty) \to \mathbb{R}, \phi'' \in L^2([0, \infty), \mathbb{R}), \int_0^\infty \gamma(\phi(t))dt < \infty\right] \Rightarrow \left[\lim_{t \to +\infty} \phi(t) = \lim_{t \to +\infty} \phi'(t) = 0\right].$$

Remark 6.2 In [2, 7, 9], uniqueness results for viscosity solutions of the HJBE were given for problems with unbounded control sets A. The conclusions of those results were that the value function is the unique nonnegative solution of the HJBE that has locally bounded subdifferentials and satisfies certain side conditions. As shown in Theorem I.7.3 in [8], the requirement that the solutions have locally bounded subdifferentials is equivalent to assuming that the solutions are locally Lipschitz. On the other hand, Theorem 2 extends these earlier results by proving uniqueness of solutions within a more general class of functions whose subdifferentials are not necessarily locally bounded, and which can take negative values.

6.2 Extension to Negative Lagrangians

As shown in §6 in [13], the uniqueness results for proper viscosity solutions in [13] can be applied to problems whose Lagrangians take negative values, under additional assumptions on the positivity set

$$P := \left\{ x \in \mathbb{R}^N : \int_0^t \ell^r(y_x^r(s, \alpha), \alpha(s)) \, ds > 0 \quad \forall \alpha \in \mathcal{A}^r \ \forall t > 0 \right\}$$

One of these required assumptions was that ℓ was "not very negative", meaning

(NVN) For each $x \in \mathcal{R} \setminus [P \cup \mathcal{T}]$, there is a bounded open set $B \subseteq \mathcal{R}$ containing x so that $\bar{B} \subseteq \mathcal{R} \setminus \mathcal{T}$ and a positive number

$$\Psi \ < \ \inf_{\alpha \in \mathcal{A}} \left\{ t > 0 : \ \text{dist}(y_x(t, \alpha), \text{bd}(B)) \leq \frac{1}{2} \, \text{dist}(x, \text{bd}(B)) \right\}$$

such that $y_x(\Psi, \alpha) \in P \cap \mathcal{R}$ and $\int_0^\Psi \ell(y_x(s, \alpha), \alpha(s)) \, ds \geq 0$ for all $\alpha \in \mathcal{A}$.

Roughly speaking, (NVN) says each point $q \notin P$ admits an escape time $T(q)$ such that $y_q(T(q), \alpha) \in P$ for all $\alpha \in \mathcal{A}$. In practice, (NVN) may not be easy to verify. An alternative approach to uniqueness of HJBE solutions for possibly negative Lagrangians ℓ, based on Theorem 1, is as follows. We set

$$\tilde{\mathcal{A}}(x) := \left\{ \alpha \in \mathcal{A}^r : \lim_{T \to +\infty} \int_0^T \ell^r(y_x^r(t, \alpha), \alpha(t)) \mathrm{d}t \ \text{ exists in } \ \mathbb{R} \right\}$$

and write as usual

$$\int_0^\infty \ell^r(y_x^r(t, \alpha), \alpha(t)) \mathrm{d}t \ = \ \lim_{T \to +\infty} \int_0^T \ell^r(y_x^r(t, \alpha), \alpha(t)) \mathrm{d}t \quad \forall \alpha \in \tilde{\mathcal{A}}(x).$$

Also, for a given dynamics f, an open set $\mathcal{G} \subseteq \mathbb{R}^N$ containing the origin is called **asymptotically null for f** provided (i) $y_x^r(t, \alpha) \in \mathcal{G}$ for all $x \in \mathcal{G}$, $t \geq 0$, and $\alpha \in \mathcal{A}^r$ and (ii) $y_x^r(t, \alpha) \to 0$ as $t \to +\infty$ for all $x \in \mathcal{G}$ and $\alpha \in \mathcal{A}^r$.

Theorem 3 *Assume we are given the following.*

1. *a compact metric space A.*

2. *a dynamics $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ satisfying (A_1).*

3. *a continuous function $\ell : \mathbb{R}^N \times A \to \mathbb{R}$.*

4. *an open set $\mathcal{G} \subseteq \mathbb{R}^N$ which is asymptotically null for f.*

5. *a continuous viscosity solution of (4) on $\mathcal{G} \setminus \{\vec{0}\}$ satisfying $w(\vec{0}) = 0$*

Then $w(x) \equiv \inf \left\{ \int_0^\infty \ell^r(y_x^r(t, \alpha), \alpha(t)) \mathrm{d}t : \alpha \in \tilde{\mathcal{A}}(x) \right\}$.

Proof. We show how to modify the proof of Theorem 1. Let $V(x)$ denote the infimum in the conclusion of Theorem 3. Let $\bar{x} \in \mathcal{G} \setminus \{\vec{0}\}$. For each $\varepsilon > 0$, the proof of Theorem 1 gives a pair $(1, \alpha_\varepsilon) \in \mathcal{Z}_1$ such that

$$w(\bar{x}) \quad \geq \quad \int_0^1 \ell(y_{\bar{x}}(s, \alpha_\varepsilon), \alpha_\varepsilon(s)) \, \mathrm{d}s + w(y_{\bar{x}}(1, \alpha_\varepsilon)) - \varepsilon \quad \geq \quad w(\bar{x}) - \varepsilon \tag{37}$$

with the last inequality following from Lemma 2.2. Using the classical Compactness Lemma for relaxed controls (cf. [1]), we can find a sequence of the α_ε's (which we do not relabel) and $\beta \in \mathcal{A}^r$ such that (i) $\alpha_\varepsilon \to \beta$ weak-$\star$ on $[0, 1]$ and (ii) $\max\{||y_{\bar{x}}(t, \alpha_\varepsilon) - y_{\bar{x}}^r(t, \beta)|| : 0 \leq t \leq 1\} \to 0$ as $\varepsilon \downarrow 0$. Since $\mathcal{G}$ is asymptotically null, it follows that $y_{\bar{x}}^r(t, \beta) \in \mathcal{G}$ for all $t \in [0, 1]$. Letting $\varepsilon \downarrow 0$ in (37) and using the continuity of w now gives

$$w(\bar{x}) = \int_0^1 \ell^r(y_{\bar{x}}^r(s, \beta), \beta(s)) \, \mathrm{d}s + w(y_{\bar{x}}^r(1, \beta)) \tag{38}$$

The argument, which is based on the continuity of the maps $a \mapsto \ell(x, a)$ for all x, is similar to the argument in the appendix of [12]. Now repeat the preceding construction, but with the starting point $\bar{x}$ replaced by $y_{\bar{x}}^r(1, \beta)$, and substitute the resulting expression for $w(y_{\bar{x}}^r(1, \beta))$ into (38). This procedure is iterated and gives $\hat{\alpha} \in \mathcal{A}^r$ such that

$$w(\bar{x}) = \int_0^M \ell^r(y_{\bar{x}}^r(s, \hat{\alpha}), \hat{\alpha}(s)) \, \mathrm{d}s + w(y_{\bar{x}}^r(M, \hat{\alpha})) \quad \forall M \in \mathbb{N} \tag{39}$$

Since $\mathcal{G}$ is asymptotically null for f and w is continuous,

$$w(y_{\bar{x}}^r(M, \hat{\alpha})) \to 0 \quad \text{as} \quad M \to +\infty. \tag{40}$$

By (39)-(40), $\hat{\alpha} \in \tilde{\mathcal{A}}(\bar{x})$, so (39) gives $w(\bar{x}) \geq V(\bar{x})$. The proof that $w(\bar{x}) \leq V(\bar{x})$ is similar to the proof of the corresponding inequality in Theorem 1, except with $\mathcal{A}$ replaced by $\mathcal{A}^r$, using the fact (cf. [1]) that $\sup\{-f^r(x, a) \cdot p - \ell^r(x, a) : a \in A^r\} \equiv \sup\{-f(x, a) \cdot p - \ell(x, a) : a \in A\}$ for all $x, p \in \mathbb{R}^N$. ∎

Remark 6.3 Notice that in Theorem 3, it was not necessary to assume that w was bounded-from-below, or that ℓ assumed any positive values. By applying the methods of Theorem 1 to negative ℓ, one can also prove PDE characterizations for maximum cost type robust Lyapunov functions, which generalize Zubov's method for calculating domains of attraction (cf. [5]). These PDE characterizations will be the subject of [15].

References

[1] Bardi, M., and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Birkhäuser, Boston, 1997.

[2] Bardi, M., and F. Da Lio, "On the Bellman equation for some unbounded control problems," *NoDEA Nonlinear Differential Equations Appl.*, **4**(1997), pp. 491-510.

[3] Bardi, M., T.E.S. Raghavan, and T. Parthasarathy, Eds., *Stochastic and Differential Games: Theory and Numerical Methods*, Birkhäuser, Boston, 1999.

[4] Bardi, M., and P. Soravia, "Hamilton-Jacobi equations with singular boundary conditions on a free boundary and applications to differential games," *Trans. Amer. Math. Soc.*, **325**(1991), pp. 205-229.

[5] Camilli, F., L. Grüne, and F. Wirth, "A generalization of Zubov's method to perturbed systems," *SIAM J. Control Optim.*, **40**(2001), pp. 496-515.

[6] Camilli, F., and A. Siconolfi, "Maximal subsolutions for a class of degenerate Hamilton-Jacobi problems," *Indiana Univ. Math. Journal*, **48**(1999), pp. 1111-1131.

[7] Cannarsa, P., and G. Da Prato, "Nonlinear optimal control with infinite horizon for distributed parameter systems and stationary Hamilton-Jacobi equations," *SIAM J. Control Optim.*, **27**(1989), pp. 861-875.

[8] Clarke, F., Yu. Ledyaev, R. Stern, and P. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.

[9] Da Lio, F., "On the Bellman equation for infinite horizon problems with unbounded cost functional," *Appl. Math. Optim.*, **41**(1999), pp. 171-197.

[10] Desch, W., H. Longemann, E.P. Ryan, and E.D. Sontag, "Meagre functions and asymptotic behavior of dynamical systems," *Nonlinear Analysis TMA*, **44**(2001), pp. 1087-1109.

[11] Fleming, W.H., and H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, 1993.

[12] Malisoff, M., "Viscosity solutions of the Bellman equation for exit time optimal control problems with non-Lipschitz dynamics," *ESAIM: Control, Optimisation and Calculus of Variations*, **6**(2001), pp. 415-441.

[13] Malisoff, M., "Viscosity solutions of the Bellman equation for exit time optimal control problems with vanishing Lagrangians," *SIAM Journal on Control and Optimization*, **40**(2002), pp. 1358-1383.

[14] Malisoff, M., "Bounded-from-below solutions of the Hamilton-Jacobi equation for optimal control problems with exit times: Vanishing Lagrangians, eikonal equations, and shape-from-shading," *NoDEA Nonlinear Differential Equations and Applications*, to appear. (Preprint at `http : //www.math.lsu.edu/ ∼ malisoff/research.html`.)

[15] Malisoff, M., "Further results on Lyapunov functions and domains of attraction for perturbed asymptotically stable systems," Louisiana State University Mathematics Electronic Preprint Series 2002-1, submitted. (Preprint at `http : //www.math.lsu.edu/ ∼ malisoff/research.html`.)

[16] Malisoff, M., and H.J. Sussmann, "Further results on the Bellman equation for exit time optimal control problems with nonnegative Lagrangians: The case of Fuller's Problem," in *Proc. 39th IEEE Conf. on Decision and Control, Sydney, Australia, December 2000*, pp. 2308-2310.

[17] Piccoli, B., and H. Sussmann, "Regular synthesis and sufficient conditions for optimality," *SIAM J. Control Optim.*, **39**(2000), pp. 359-410.

[18] Sontag, E., *Mathematical Control Theory, Second Edition*, Springer, New York, 1998.

[19] Soravia, P., "Pursuit evasion problems and viscosity solutions of Isaacs equations," *SIAM J. Control Opt.*, **31**(1993), pp. 604-623.

[20] Soravia, P., "Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations I: Equations of unbounded and degenerate control problems without uniqueness," *Adv. Differential Equations*, **4**(1999), pp. 275-296.

[21] Soravia, P., "Optimality principles and representation formulas for viscosity solutions of Hamilton-Jacobi equations II: Equations of control problems with state constraints," *Differential Integral Equations*, **12**(1999), pp. 275–293

[22] Sussmann, H.J., "A general theorem on local controllability," *SIAM J. Control Optim.*, **25**(1987), pp. 158-194.

[23] Sussmann, H.J., "From the Brachystochrone problem to the maximum principle," in *Proc. 35th IEEE CDC*, IEEE, New York, 1996, pp. 1588-1594.

[24] Zelikin, M.I., and V.F. Borisov, *Theory of Chattering Control with Applications to Astronautics, Robotics, Economics, and Engineering*, Birkhäuser, Boston, 1994.