A remark on the Bellman equation for optimal control problems with exit times and noncoercing dynamics [1]

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1 Introduction

This note continues our work (cf. [3]) on uniqueness questions for viscosity solutions of Hamilton-Jacobi-Bellman equations (HJB's) arising from deterministic control problems with exit times (cf. [1]). We prove a general uniqueness theorem characterizing the value functions for a class of problems of this type for nonlinear systems as the unique solutions of the corresponding HJB's among continuous functions with appropriate boundary conditions when the dynamical law is non-Lipschitz and noncoercing. The class includes Sussmann's Reflected Brachystochrone Problem (RBP), as well as problems with unbounded nonlinear running cost functionals. We show that the RBP value function is the unique viscosity solution of the corresponding HJB among the continuous functions which vanish on the target and which are bounded below.

Value function characterizations of this kind have been studied by many authors for a large number of stochastic and deterministic optimal control problems (cf. [1] and [2]). However, these earlier characterizations assume the dynamics are coercing and positive lower bounds on the running cost functionals and therefore do not apply to many standard problems. Our work is part of a larger research program which extends uniqueness results from viscosity theory to versions covering well-known optimal control problems with unbounded cost functionals or dynamics that do not have uniqueness of solutions. A uniqueness characterization for a class including the Fuller Problem is in [3].

2 Statement of Main Result

Let A be a compact metric space, and let $\mathcal{A}$ denote the set of measurable functions $[0,\infty)\to A$. For any closed $S\subset\mathbb{R}^N$, any continuous function $f:\mathbb{R}^N\times A\to\mathbb{R}^N$, and any $\alpha\in\mathcal{A}$, we let $\text{Traj}_\alpha(x,f,S)$ denote the set of trajectories of $\dot{y}=f(y,\alpha)$ defined on $[0,\infty)$ which start at x and reach S in finite time. We also set $\text{Traj}(x,f,A,S)=\cup_{\alpha\in\mathcal{A}}\text{Traj}_\alpha(x,f,S)$, and, for each $x\in\mathbb{R}^N$ and $\alpha\in\mathcal{A}$, $y^f_x(\cdot,\alpha)$ denotes the unique solution of $\dot{y}=f(y,\alpha)$ starting at x when $\text{Traj}_\alpha(x,f,\mathbb{R}^N)$ is a singleton for all $\alpha\in\mathcal{A}$ and $x\in\mathbb{R}^N$.

From now on, $\mathcal{T}$ denotes a fixed closed subset of $\mathbb{R}^N$. Let $\mathcal{R}_f(A)$ denote those x for which $\text{Traj}(x,f,A,\mathcal{T})$ is nonempty, and $\tau(\phi):=\inf\{t\geq 0:\phi(t)\in\mathcal{T}\}$ for each $\phi:[0,\infty)\to\mathbb{R}^N$. If $U\subset\mathbb{R}^N$ is open, we call a continuous function $g:U\times B\to\mathbb{R}^N$ **coercing** if there is an $L>0$ such that, for all $x,y\in U$ and $a\in B$, $(g(x,a)-g(y,a))\cdot(x-y)\leq L\,||\,x-y\,||^2$. The coercing functions $\mathbb{R}^N\times B\to\mathbb{R}^N$ will be denoted by $C_{co}(\mathbb{R}^N\times B,\mathbb{R}^N)$. We let $C_{ar}(\mathbb{R}^N\times B,\mathbb{R})$ denote those B-uniformly continuous functions[1] $g:\mathbb{R}^N\times B\to\mathbb{R}$ which radially increase in B, i.e., such that if $x\in\mathbb{R}^N$ and if $||\,a\,||\leq||\,a'\,||$ in B, then $g(x,a)\leq g(x,a')$.

When $S\supseteq A$, $f:\mathbb{R}^N\times S\to\mathbb{R}^N$ and $\ell:\mathbb{R}^N\times S\to\mathbb{R}$ are continuous, $\alpha\in\mathcal{A}$, and $x\in\mathcal{R}_f(A)\setminus\mathcal{T}$, we let $\text{Traj}^{co}_\alpha(x,f,\mathcal{T},\ell)$ denote those $\phi\in\text{Traj}_\alpha(x,f,\mathcal{T})$ that admit $t,t'\in(0,\tau(\phi))$, an input $\beta\in\mathcal{A}$, a trajectory $\psi\in\text{Traj}_\beta(x,f,\mathcal{T})$, an $N\in\mathbb{N}$, and open subsets $\{S_n\}_{n=N}^\infty$ of $\mathbb{R}^N$ such that $\phi(t)=\psi(t')$, $S_n\supseteq\text{Trace}\,\psi\lceil[1/n,t']$ and $f\lceil S_n\times A$ is coercing for all $n\geq N$, and

$$\int_0^t \ell(\phi(s),\alpha(s))\,ds \;\geq\; \int_0^{t'}\ell(\psi(s),\beta(s))\,ds.$$

For $x\in\mathcal{T}$, we set $\text{Traj}^{co}_\alpha(x,f,\mathcal{T},\ell)\equiv\text{Traj}_\alpha(x,f,\mathcal{T})$. We call $\bigcup_{\alpha\in\mathcal{A}}\text{Traj}^{co}_\alpha(x,f,\mathcal{T},\ell)$ the **coercifiable trajectories at** x (**relative to** f, $\mathcal{T}$, and ℓ). We call a pair (f,ℓ) **(coercively) transient (relative to** A**)** if for all $x\in\mathcal{R}_f(A)$ and $\alpha\in\mathcal{A}$, each $\phi\in\text{Traj}_\alpha(x,f,\mathcal{T})$

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[1] For each $a\in A$, we set $||a||=d(a,0)$, where d is the metric on A and 0 is some distinguished point. Recall that if $(X,\|\cdot\|)$ is a normed space, then a function $h:X\times Y\to\mathbb{R}$ is called Y-uniformly continuous if there is a modulus ω_h such that, for all $x,x'\in X$ and $y\in Y$, $|h(x,y)-h(x',y)|\leq\omega_h(||x-x'||)$. A modulus is a function $\omega:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ such that for all $R>0$, $\omega(\cdot,R)$ is continuous and nondecreasing and $\omega(0,R)=0$.

is a coercifiable trajectory at x. For each $x, y \in \mathcal{R}_f(B)$ and compact metric space B, $T_{B,f}(x,y)$ denotes the infimum of those $t \geq 0$ for which there is a measurable $[0,\infty) \to B$ and a $\phi \in \text{Traj}_{\alpha}(x, f, \mathbb{R}^N)$ such that $\phi(t) = y$. For each $x \in \mathcal{R}_f(A)$, let $v_{f,\ell,A}(x)$ denote

$$\inf_{\alpha \in \mathcal{A}} \left\{ \int_0^{\tau(\phi)} \ell(\phi(s), \alpha(s))\, ds : \phi \in \text{Traj}_{\alpha}(x, f, \mathcal{T}) \right\}. \quad (1)$$

The HJB of (1) is

$$\sup_{a \in A} \{-f(x,a) \cdot Dv(x) - \ell(x,a)\} = 0, \quad (2)$$

and we study uniqueness questions for viscosity solutions of (2) on $\mathcal{R}_f(A) \setminus \mathcal{T}$ among functions v satisfying

$$\begin{cases} \lim_{x \to x_0} v(x) = +\infty \ \text{ for all } \ x_o \in \partial(\mathcal{R}_f), \\ v \equiv 0 \ \text{ on } \ \mathcal{T}, \ \text{and } v \ \text{ is bounded below} \end{cases} . \quad (3)$$

We will be interested in cases where (1) is an 'upper envelope' of coercing problems, as follows:

Definition A We call (f, ℓ) an $\{(f_n, \ell_n)\}$**-coercing upper envelope** if there exists a $B \supseteq A$, a sequence $\{(f_n, \ell_n)\}$ in $C_{co}(\mathbb{R}^N \times B, \mathbb{R}^N) \times C_{ar}(\mathbb{R}^N \times B, \mathbb{R})$, a sequence $\{M_n\}$ of positive numbers, and compact sets $A_n \subseteq B$ such that the following conditions hold:

1. $M_1 \leq \ell_n(x,a) \leq M_{n+1}$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}^N$, and $a \in A$, and B is compact.
2. $\mathcal{R} := \mathcal{R}_f(A) \equiv \mathcal{R}_{f_n}(A_n)$ is open, $f_n \to f$ and $\ell_n \uparrow \ell$ uniformly on compacta, and $A_n \downarrow A$.[2]
3. $\text{Traj}(x, f_n, A_n, \mathcal{T}) \subseteq \text{Traj}(x, f_1, A_1, \mathcal{T})$ for each $n \in \mathbb{N}$ and $x \in \mathcal{R}$.
4. For each $p \in \mathbb{N}$, each q and x in $\mathcal{R}$, and each ϕ in $\text{Traj}_{\alpha}(x, f, \{q\})$, there is a measurable function $\beta : [0,\infty) \to A_p$ so that $\phi \in \text{Traj}_{\beta}(x, f_p, \{q\})$ and so that $||\,\alpha\,|| \geq ||\,\beta\,||$ a.e..

If in addition $\{f(x,a) \times \ell(x,a) : a \in A\}$ is convex for all x, then we call (f, ℓ) a **convex coercing upper envelope (cocue)**. We also call the sequence $\{(f_n, \ell_n)\}$ the **associated enveloping sequence**.

We will prove the following uniqueness theorem:

Theorem A *Let A be a compact metric space $\mathcal{T} \subset \mathbb{R}^N$ be closed. Assume (f, ℓ) is a coercively transient cocue with associated enveloping sequence $\{(f_n, \ell_n)\}$, f_n is uniformly Lipschitz in A for all n,* [3] *$T_{A,f}$ is continuous, and the following:*

(TC) There is a $K \in \mathbb{N}$ such that: If $x \in \mathcal{R}$, $p \in \mathbb{R}^N$, then $\exists\, a^\star \in \arg\sup_{a \in A}\{-f(x,a) \cdot p - \ell(x,a)\}$ s.t. $(f(x,a^\star) - f_n(x,a^\star)) \cdot p \geq 0$ for all $n \geq K$.

Then $v_{f,\ell,A}$ is the unique viscosity solution of (2) on $\mathcal{R} \setminus \mathcal{T}$ among functions $v \in C(\mathcal{R})$ that satisfy (3).

[2]The convergence is in the Hausdorff sense.

[3]This means there is a constant $L > 0$ such that $||f(x,a) - f(z,a)|| \leq L||x - z||$ for all a, x, and z.

Remark The method of our proof can also be used to give analogs of Theorem A for not-necessarily-convex coercing upper envelopes, for cases where (TC) is not required, and for data violating the coercive transience condition. One can also drop the condition $\ell_n \uparrow \ell$. Some of these analogs give uniqueness in classes of locally Lipschitz viscosity solutions of (2) on $\mathcal{R} \setminus \mathcal{T}$ satisfying (3). For details, see [4].

3 Definitions and Main Lemmas

This section reviews definitions and earlier results from viscosity theory. Let $\Omega \subseteq \mathbb{R}^N$ be open, let $\Omega \subseteq S$, and assume that $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $w : S \to \mathbb{R}$ are continuous. We call w a **viscosity solution** of the equation $F(x, Dw(x)) = 0$ on Ω if the following hold:

(i) If $\gamma : \Omega \to \mathbb{R}$ is C^1 and x_o is a local minimum of $w - \gamma$, then $F(x_o, D\gamma(x_o)) \geq 0$.

(ii) If $\lambda : \Omega \to \mathbb{R}$ is C^1 and x_1 is a local maximum of $w - \lambda$, then $F(x_1, D\lambda(x_1)) \leq 0$.

When condition (i) (resp., (ii)) holds, we say that w is a **viscosity supersolution (resp., subsolution)** of $F(x, Dw(x)) = 0$ on Ω. These conditions are equivalent to $F(x,p) \geq 0$ for all $x \in \Omega$ and p in the set of subdifferential points $D^- w(x)$ and $F(y,q) \leq 0$ for all $y \in \Omega$ and q in the set of superdifferential points $D^+ w(y)$, respectively (cf. [1]).

We also say that w is the **complete (viscosity) solution** of $F(x, Dw(x)) = 0$ on Ω in a class $\mathcal{C}$ of functions in $C(\Omega, \mathbb{R})$ if it is the maximal subsolution and minimal supersolution of this equation in $\mathcal{C}$, i.e., if $\tilde{w} \in \mathcal{C}$ is a viscosity supersolution (resp., subsolution) of $F(x, D\tilde{w}(x)) = 0$ on Ω, then $w(x) \leq \tilde{w}(x)$ (resp., $w(x) \geq \tilde{w}(x)$) for all $x \in \Omega$. For A a compact metric space and $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ and $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ both continuous, $H(x,p) := \sup_{a \in A}\{-f(x,a) \cdot p - \ell(x,a)\}$. Recall the following results (cf. [1]):

Lemma 3.1 Let A be a compact topological space, let $f \in C_{co}(\mathbb{R}^N \times A, \mathbb{R}^N)$, and let $\ell \in C_{ar}(\mathbb{R}^N \times A, \mathbb{R})$ admit positive constants m and M such that $m \leq \ell(x,a) \leq M$ for all $x \in \mathbb{R}^N$ and $a \in A$. Assume $\mathcal{T} \subseteq \mathbb{R}^N$ is closed, that the dynamics f satisfies STC$\mathcal{T}$, and that $\mathcal{R}$ is open.[4] Then $v_{f,\ell,A}$ is the complete solution of $H(x, Dv(x)) = 0$ on $\mathcal{R} \setminus \mathcal{T}$ in the class of functions $v \in C(\mathcal{R})$ that satisfy (3).

[4] Recall that STC$\mathcal{T}$ is the condition that for any $\varepsilon > 0$, $\mathcal{T}$ lies in the interior of the set of points that can be brought to $\mathcal{T}$ in time $< \varepsilon$ using the dynamics f.

Lemma 3.2 Let A, f, ℓ, m, and M satisfy the assumptions of Lemma 3.1, except allow $m=0$. Assume $u \in C(\bar{\Omega})$ is viscosity subsolution of $H(\,x, Du(x)\,) = 0$ on Ω, where $\Omega \subset \mathbb{R}^N$ is bounded and open. If we set $\tau_x(\alpha) := \inf\{\, t \geq 0 : \quad y_x(t,\alpha) \in \partial\Omega\,\}$, then $u(x) \leq \int_0^t \ell(y_x(s,\alpha),\alpha(s))\,ds + u(y_x(t,\alpha))$ for all $\alpha \in \mathcal{A}$, $x \in \Omega$, and $0 \leq t < \tau_x(\alpha)$. [5]

We often relax the requirements of Lemma 3.2 by assuming that $f\lceil S \times A$ is coercing for some open $S \supseteq \bar{\Omega}$. Since A is a compact metric space, we can view our inputs as members of the class of relaxed controls (cf. [1]). For $B \supseteq A$ compact and $f : \mathbb{R}^N \times B \to \mathbb{R}^N$ and $\ell : \mathbb{R}^N \times B \to \mathbb{R}$ continuous, define ℓ^r and f^r by $\ell^r(x,m) := \int_A \ell(x,a)\ dm(a)$ and $f^r(x,m) := \int_A f(x,a)\ dm(a)$ for each $(x,a) \in \mathbb{R}^N \times B^r$, where B^r is the set of Radon probability measures on B. When f is coercing, we can let $y_x^{r,f}(\cdot,\alpha)$ denote the unique solution of $y'(s) = f^r(y(s),\alpha(s))$ starting at x for each $x \in \mathbb{R}^n$ and $\alpha \in \mathcal{B}^r$. Recall the following (cf. [1]):

Lemma 3.3 Let A be a compact metric space, and let $\{\alpha_n\}_{n=1}^{\infty}$ in $\mathcal{A}^r$ and $c > 0$ be given. Assume $f : \mathbb{R}^N \times A \to \mathbb{R}^n$ is continuous and uniformly Lipschitz in A. There is a subsequence of $\{\alpha_n\}_{n=1}^{\infty}$ (which we do not relabel) and an $\alpha \in \mathcal{A}^r$ such that $\alpha_n \to \alpha$ weak-star on $[0,c]$ and such that $y_{x_n}^{r,f}(\cdot,\alpha_n) \to y_x^{r,f}(\cdot,\alpha)$ uniformly on $[0,c]$ whenever $x_n \to x$ in $\mathbb{R}^N$.

Lemma 3.4 Let A be a compact metric space and let $f : \mathbb{R}^N \times A \to \mathbb{R}^N$ and $\ell : \mathbb{R}^N \times A \to \mathbb{R}$ be continuous and such that $\{f(x,a) \times \ell(x,a) : a \in A\}$ is convex for all $x \in \mathbb{R}^N$. Let (ϕ^r, μ^r) be a trajectory-input pair for f^r. There is a measurable mapping $\alpha : [0,\infty) \to A$ so that

$$\int_0^t \ell^r(\phi^r(s),\mu^r(s))\,ds = \int_0^t \ell(\phi^r(s),\alpha(s))\,ds \quad \forall t \geq 0$$

and so that (ϕ^r, α) is a trajectory-input pair for f.

4 Proof of Theorem A

We set $f_\infty := f$, $\ell_\infty := \ell$, and $v := v_{f,\ell,A}$. Consider, for each $x \in \mathcal{R}$, the problem

$$\inf\left\{\int_0^{t_{x,n}(\alpha)} \ell_n(y_x^{f_n}(s,\alpha),\alpha(s))\,ds : \alpha \in \mathcal{A}_n(x)\right\}, \quad (4)$$

where $\mathcal{A}_n(x)$ is the set of A_n-valued measurable inputs that drive x to $\mathcal{T}$ using the dynamics f_n in finite time and $t_{x,n}(\alpha)$ is the first time $y_x^{f_n}(\cdot,\alpha)$ reaches $\mathcal{T}$. The corresponding HJB is

$$\sup_{a \in A_n}\{\,-f_n(x,a)\cdot Du(x) - \ell_n(x,a)\,\} = 0 \quad (5)$$

[5] The glb of an empty set of real numbers is $+\infty$.

Let v_n denote the value function for this problem. By Lemma 3.1, the continuity of $T_{A,f}$, and Condition 2 of Definition A, v_n is the complete viscosity solution of (5) on $\mathcal{R} \setminus \mathcal{T}$ among the functions satisfying (3). Since the F_n's defined by $F_n(x,p) = \sup_{a \in A_n}\{\,-f_n(x,a)\cdot p - \ell_n(x,a)\,\}$ are continuous and $\ell_n \to \ell_\infty$ and $f_n \to f_\infty$ locally uniformly, we could conclude from standard stability results (cf. [1]) that v is a viscosity solution of (2) on $\mathcal{R} \setminus \mathcal{T}$ if we prove $v_n \to v$ uniformly on compacts. We prove this using the Ascoli-Arzelá Theorem.

For each $x, p \in \mathcal{R}$ and $n \in \mathbb{N}$, let $\hat{v}_n(x,p)$ denote the value (4) but with $\mathcal{T}$ replaced by the singleton $\{p\}$, and $\hat{v}_\infty(x,p)$ is the value (1) with the same target replacement. If $r \in \mathbb{N}$, then $\hat{v}_r \leq \hat{v}_\infty$ pointwise (in x and p). Indeed, if $x, p \in \mathcal{R}$ and $\phi \in \mathrm{Traj}_\alpha\,(x, f_\infty, \{p\})$, then we can use the definition of coercing upper envelopes to find an A_r-valued input β (depending on r), with $||\,\beta\,|| \leq ||\,\alpha\,||$ a.e., so that $\phi \in \mathrm{Traj}_\beta\,(x, f_r, \{p\})$. Since $\ell_n \uparrow \ell_\infty$ pointwise and $\ell_r \in C_{ar}(\mathbb{R}^N \times A, \mathbb{R})$, we get

$$\begin{aligned}\int_0^{\tau_p(\phi)} \ell_\infty(\phi(s),\alpha(s))\,ds &\geq \int_0^{\tau_p(\phi)} \ell_r(\phi(s),\alpha(s))\,ds \\ &\geq \int_0^{\tau_p(\phi)} \ell_r(\phi(s),\beta(s))\,ds \\ &\geq \hat{v}_r(x,p),\end{aligned}$$

where $\tau_p(\phi)$ is the first time ϕ reaches p, so the claim follows by infimizing over $\mathrm{Traj}_\alpha\,(x, f_\infty, \{p\})$ for all α on the left side. Therefore, for all $x \in \mathcal{R}$ and $n \in \mathbb{N}$,

$$v_n(x) = \inf_{p \in \mathcal{T}} \hat{v}_n(x,p) \leq \inf_{p \in \mathcal{T}} \hat{v}_\infty(x,p) = v(x). \quad (6)$$

Now fix $x \in \mathcal{R}$, and let $\delta > 0$ be such that $B_\delta(x) \subset \mathcal{R}$. By the continuity of $T_{A,f}$, we can travel between any points $y, z \in B_\delta(x)$ using the dynamics f and A-valued controls. Therefore, $v_n(z) - v_n(y) \leq \hat{v}_n(z,y) \leq \hat{v}_\infty(z,y)$ for all $y, z \in B_\delta(x)$ and $n \in \mathbb{N} \cup \{\infty\}$, where we use the definition of the infimum, the fact that $y \in \mathcal{R}$, and $A \subseteq A_n$ for all n to get the first inequality. Arguing symmetrically, we get, for all $y, z \in B_\delta(x)$,

$$|v_n(z) - v_n(y)| \;\leq\; \hat{v}_\infty(z,y) \;\vee\; \hat{v}_\infty(y,z). \quad (7)$$

Notice also that $\tilde{v}_\infty$ is continuous on $\mathcal{R} \times \mathcal{R}$. Indeed, let ε, $\delta > 0$ and $x, y, \tilde{x}, \tilde{y} \in \mathcal{R}$ be given. Since $T_{A,f}$ is continuous, there is a $\mu = \mu(\delta) > 0$ so that if $||x - \tilde{x}|| \vee ||y - \tilde{y}|| < \mu(\delta)$, then there are inputs α_1 and α_2 in $\mathcal{A}$, trajectories $\phi_1 \in \mathrm{Traj}_{\alpha_1}(x, f_\infty, \{\tilde{x}\})$ and $\phi_2 \in \mathrm{Traj}_{\alpha_2}(\tilde{y}, f_\infty, \{y\})$, and numbers $t_1, t_2 \in [0,\delta)$ such that $\phi_1(t_1) = \tilde{x}$ and $\phi_2(t_2) = y$. Using the definition of the infimum and concatenating, we get

$$\begin{aligned}\tilde{v}_\infty(x,y) - \tilde{v}_\infty(\tilde{x},\tilde{y}) - \varepsilon/2 &\leq \int_0^{t_1} \ell(\phi_1(s),\alpha_1(s))\,ds \\ &+ \int_0^{t_2} \ell(\phi_2(s),\alpha_2(s))\,ds.\end{aligned}$$

Since (f_∞, ℓ_∞) is a coercing upper envelope, ϕ_j is a trajectory for f_1 for $j = 1, 2$. But if γ is any trajectory for a coercing dynamical law $g : \mathbb{R}^N \times B \to \mathbb{R}^N$ with controls in a compact set B which starts at p, and if L is as in the definition of coercing, then (cf. [1]) for all $s \geq 0$, $||\gamma(s)|| \leq \left(||p|| + \sqrt{2Ks}\right) e^{Ks}$, where $K := L + \max_{a\in B} ||\, g(0, a)\, ||$. It follows that $||\, \gamma(s)\, || \vee ||\, \zeta(s)\, ||$ is uniformly bounded over the restriction to $[0, 1]$ of all trajectory-input pairs (γ, ζ) of f_∞ starting at points p with $||\, p - x\, || \wedge ||\, p - y\, || \leq 1$. Let κ be such a uniform bound, set $\hat{B} := \sup_{B_\kappa(0)\times A} \ell$, and pick $\delta > 0$ such that $\delta < \frac{\varepsilon}{4[\hat{B}+1]} \wedge 1$ and a corresponding $\mu = \mu(\delta) < 1$. This gives $\tilde{v}_\infty(x, y) - \tilde{v}_\infty(\tilde{x}, \tilde{y}) - \varepsilon/2 \leq \frac{\varepsilon}{4[\hat{B}+1]} (\hat{B} + \hat{B})$, so $\tilde{v}_\infty(x, y) - \tilde{v}_\infty(\tilde{x}, \tilde{y}) \leq \varepsilon$. Using symmetry, we get $||x - \tilde{x}|| \vee ||y - \tilde{y}|| < \delta_\varepsilon \Rightarrow |\tilde{v}_\infty(x, y) - \tilde{v}_\infty(\tilde{x}, \tilde{y})| \leq \varepsilon$ for a suitable $\delta_\varepsilon > 0$, so $\hat{v}_\infty$ is continuous on $\mathcal{R} \times \mathcal{R}$, as claimed.

It now follows from (6) and (7) that the v_n's are equicontinuous and pointwise bounded on $\mathcal{R}$, so we can apply the Ascoli-Arzelà Theorem (on $\overline{B_{\delta/2}(x)}$, for example) to get a $\bar{v}$ such that $v_n \to \bar{v}$ uniformly on $B_{\delta/2}(x)$, at least along a subsequence, and the locally defined functions $\bar{v}$ are (local) viscosity solutions of (2).

Fixing $x \in B_{\delta/2}(x) \subseteq \mathcal{R}$ and letting $\bar{v}$ denote the locally defined function on $\overline{B_{\delta/2}(x)}$, we now show that $\bar{v} = v$. Since $v_n(x) \leq v(x)$ for all n, $\bar{v}(x) \leq v(x)$. We assume that $v_n(x) \to \bar{v}(x)$ wlog. Now let $\varepsilon > 0$, and choose α_n's and trajectories $\phi_n \in \mathrm{Traj}_{\alpha_n}(x, f_n, \mathcal{T})$ so that $v_n(x) + \varepsilon \geq \int_0^{\tau(\phi_n)} \ell_n(\phi_n(s), \alpha_n(s))\, ds$ for all n. Since $M_1 > 0$ and $v_n(x) \leq v(x)$ for all n, the exit times of the ϕ_n are bounded above, so we can assume (by passing to a subsequence, if necessary, without relabeling) that $\tau(\phi_n) \to \mu \in \mathbb{R}$. We can also find A_1-valued inputs β_n so that (ϕ_n, β_n) is a trajectory control pair for the dynamics f_1, by Condition 3 of the cocue definition. Also note that $||\, \phi_n(s)\, ||$ is bounded as s varies over a compact interval and n varies in $\mathbb{N}$ (since f_1 is coercing).

From Lemma 3.3, we know there is a weak-$\star$ limit of a subsequence of the β_n's, which we call $\bar{\beta}$, so that the ϕ_n's converge uniformly on compact intervals to a relaxed trajectory for $f_1^r(x, \bar{\beta})$. Let ϕ^r denote this relaxed trajectory. Passing to a further subsequence if necessary (without relabeling), we can also assume that $\alpha_n \to \bar{\alpha}$ weak-$\star$, where now $\bar{\alpha}(s) \in A_1^r$ for all s. Since $A_n \downarrow A$, one easily checks that $\bar{\alpha} \in \mathcal{A}$. Therefore, for all $t \geq 0$,

$$\begin{aligned}\phi^r(t) \quad \leftarrow \quad \phi_n(t) \;=\; x + \int_0^t f_n(\phi_n(s), \alpha_n(s))\, ds \\ \to \quad x + \int_0^t f_\infty^r(\phi^r(s), \bar{\alpha}(s))\, ds\end{aligned}$$

and $(\phi^r, \bar{\alpha})$ is a trajectory-input pair for f_∞^r. Denoting as in Lemma 3.4, we get

$$\begin{aligned} v_n(x) + \varepsilon \quad &\geq \quad \int_0^{\tau(\phi_n)} \ell_n(\phi_n(s), \alpha_n(s))\, ds \\ &\to \quad \int_0^{\mu} \ell_\infty^r(\phi^r(s), \bar{\alpha}(s))\, ds \\ &= \quad \int_0^{\mu} \ell_\infty(\phi^r(s), \alpha(s))\, ds \quad \geq \quad v(x), \end{aligned}$$

since $\mathcal{T} \ni \phi_n(\tau(\phi_n)) \to \phi^r(\mu)$. Since $v_n(x) \to v(x)$ and $v_n(x) \to \bar{v}(x)$ subsequentially on $\overline{B_{\delta/2}(x)}$, $\bar{v} = v$, so v is a viscosity solution of (2) on $\mathcal{R} \setminus \mathcal{T}$.

Let $w \in C(\mathcal{R})$ be another viscosity solution of (2) on $\mathcal{R} \setminus \mathcal{T}$ satisfying (3). Then, w is a viscosity supersolution of the HJB for the data (f_n, ℓ_n) and A_n on $\mathcal{R} \setminus \mathcal{T}$ for n large enough. Indeed, let $x \in \mathcal{R} \setminus \mathcal{T}$ and $p \in D^- w(x)$ be given. Pick $a^\star \in \arg\sup_{a\in A} \{\, f_\infty(x, a) \cdot p - \ell_\infty(x, a)\, \}$. For n large enough, condition (TC) and $\ell_n \uparrow \ell_\infty$ gives

$$\begin{aligned} 0 \quad &\leq \quad -f_\infty(x, a^\star) \cdot p - \ell_\infty(x, a^\star) \\ &\leq \quad -f_n(x, a^\star) \cdot p - \ell_\infty(x, a^\star) \\ &\leq \quad -f_n(x, a^\star) \cdot p - \ell_n(x, a^\star), \end{aligned}$$

(with n not depending on $x \in \mathcal{R}$ or $p \in \mathbb{R}^N$) so since $A_n \supseteq A$ for all n, the result follows. Again using the completeness of v_n as the solution of the corresponding HJB for $n \in \mathbb{N}$, we get $w \geq v_n$ pointwise, so $w \geq v$ pointwise, since $v_n \to v$ pointwise along a subsequence.

Now let $x \in \mathcal{R}$, $\alpha \in \mathcal{A}$, and $\phi \in \mathrm{Traj}_\alpha(x, f_\infty, \mathcal{T})$. Let $\bar{t}$ denote the supremum of those $t \in [0, \tau(\phi)]$ for which $w(x) \leq \int_0^t \ell_\infty(\phi(s), \alpha(s))\, ds + w(\phi(t))$, and suppose $\bar{t} < \tau(\phi)$, for the sake of obtaining a contradiction. Since ϕ and ℓ_∞ are continuous, $w \in C(\mathcal{R} \setminus \mathcal{T})$, and $\phi(\bar{t}) \in \mathcal{R} \setminus \mathcal{T}$, we know that

$$w(x) \quad \leq \quad \int_0^{\bar{t}} \ell_\infty(\phi(s), \alpha(s))\, ds + w(\phi(\bar{t})). \qquad (8)$$

Since (f, ℓ) is coercively transient, we know that $\phi(\cdot + \bar{t})$ is a coercifiable trajectory at $\phi(\bar{t})$. Therefore, we can find $t, t' \in (0, \tau(\phi(\cdot + \bar{t})))$, $\beta \in \mathcal{A}$, and $\psi \in \mathrm{Traj}_\beta(\phi(\bar{t}), f_\infty, \mathcal{T})$ so that $\psi(t') = \phi(t + \bar{t})$ and

$$\int_0^{t'} \ell_\infty(\psi(s), \beta(s))\, ds \leq \int_{\bar{t}}^{t+\bar{t}} \ell_\infty(\phi(s), \alpha(s))\, ds, \quad (9)$$

and a number $N \in \mathbb{N}$ and open sets $\{S_n\}_{n=N}^\infty$ so that $f_\infty \lceil S_n \times A \in C_{co}(S_n \times A)$ and $S_n \supset \psi\lceil [1/n, t']$ for all $n \geq N$. Applying Lemma 3.2, we get

$$w(\psi(1/n)) \quad \leq \quad \int_{1/n}^{t'} \ell_\infty(\psi(s), \beta(s))\, ds + w(\phi(t + \bar{t})).$$

Letting $n \to \infty$ and using $w \in C(\mathcal{R})$, we get

$$w(\phi(\bar{t})) \;\leq\; \int_0^{t'} \ell_\infty(\psi(s), \beta(s))\, ds + w(\phi(t+\bar{t})). \; (10)$$

Combining (8), (9), and (10), we get

$$\begin{aligned} w(x) &\leq \int_0^{\bar{t}} \ell_\infty(\phi(s),\alpha(s))\,ds \\ &+ \int_0^{t'} \ell_\infty(\psi(s),\beta(s))\,ds + w(\phi(\,t+\bar{t}\,)) \\ &\leq \int_0^{t+\bar{t}} \ell_\infty(\phi(s),\alpha(s))\,ds + w(\phi(\,t+\bar{t}\,)) \end{aligned}$$

Since $\bar{t} < t+\bar{t} < \tau(\phi)$, this contradicts the definition of $\bar{t}$. Therefore, $\bar{t} = \tau(\phi)$. The inequality $w \leq v$ now follows by infimizing over all α's that drive x to $\mathcal{T}$ using f_∞ and the corresponding ϕ's in (8), since $w \equiv 0$ on $\mathcal{T}$. Thus, v is the unique viscosity solution of (2) in the class of functions $v \in C(\mathcal{R})$ satisfying (3), as desired.

5 Reflected Brachystochrone Problem

Sussmann's RBP is an optimal time control problem with the dynamical law $f = f_\infty$ given by $\dot{x} = u_x\sqrt{|y|}$, $\dot{y} = u_y\sqrt{|y|}$, the controls $(u_x,u_y)'$ being subject to $(u_x,u_y) \in \overline{B_1(0)}$. The objective is to bring an initial point to some fixed single point target $\mathcal{T} = \{B\}$ in minimal time. For the origins of this problem, see [5]. Since the RBP dynamical law is not coercing, it is beyond the scope of the previously known results characterizing value functions as the unique viscosity solutions of the corresponding HJB's among functions with suitable boundary conditions. We sketch the proof that the RBP satisfies Theorem A's hypotheses.

The continuity of $T_{\overline{B_1(0)},f}$ for each target $\{B\}$ follows from an elementary consideration of vertical and horizontal movements along RBP trajectories and the fact that RBP trajectories can be reversed. Set $\ell_n \equiv 1$ and $A_n \equiv A = B = \overline{B_1(0)}$, and define f_n by $f_n(x,y,u_x,u_y) = (y^2+1/n)^{1/4}\,(u_x,u_y)'$ for all $n \in \mathbb{N}$. These functions are Lipschitz continuous in the state variable, uniformly in the control variable, and therefore coercing, and $f_n \to f$ uniformly on compacta. Since $\mathcal{R}_{f_n}(A) \equiv \mathbb{R}^2$, Conditions 1 and 2 of Definition A are satisfied. For each initial point $p \in \mathbb{R}^2$ and each f_n input $(u_x^\star, u_y^\star)$, the corresponding trajectory $(\phi_x^\star,\phi_y^\star)$ is the trajectory for f_1 using the input $({\phi_y^\star}^2(t)+1/n)^{1/4}/({\phi_y^\star}^2(t)+1)^{1/4}(u_x^\star(t),u_y^\star(t))'$. The trajectory (μ_x,μ_y) corresponding to any f input (β_x,β_y) is a trajectory for f_n using the input $\sqrt{|\mu_y(t)|}/(\mu_y^2(t)+1/n)^{1/4}(\beta_x(t),\beta_y(t))'$, so Conditions 3 and 4 of Definition A are also satisfied. One easily verifies (TC).

That each $\phi \in \mathrm{Traj}_\alpha(p,f,\{B\})$ is coercifiable at $p = (x,y)'$ when $y \neq 0$ follows from the fact that f_∞ is coercing away from $\{(p,q)' :\ q = 0\} \times \overline{B_1(0)}$, so we assume $p = (x,0)' \neq B$ in the sequel. Fix $\alpha \in \mathcal{A}$ and $\phi = (\phi_1,\phi_2)' \in \mathrm{Traj}_\alpha(p,f_\infty,\{B\})$. Since $p \neq B$, and since the RBP dynamics does not allow horizontal movement along the x-axis, it follows that for some $s \in (0,\tau(\phi))$, $\phi(s)$ is not in the x axis. We assume $\phi(s) = (\mu,\nu)$, with $\nu > 0$, wlog. Then $\phi\lceil[0,s]$ is strictly below the x axis on each interval in $\{(u_{j,-},u_{j,+})\}_{j\in S}$, where S is at most countable. Set $\mathcal{D}_\phi = \bigcup_j (u_{j,-},u_{j,+})$. We reflect the subtrajectories $\phi\lceil(u_{j,-},u_{j,+})$ over the x-axis.

Define $\tilde{\phi} = (\tilde{\phi}_1,\tilde{\phi}_2)'$ and $\tilde{w} = (\tilde{u}_x,\tilde{u}_y)'$ on $[0,s]$ by $\tilde{\phi}_1(t) \equiv \phi_1(t)$, $\tilde{u}_x(t) \equiv u_x(t)$,

$$\tilde{\phi}_2(t) = \begin{cases} +\phi_2(t), & t \notin \mathcal{D}_\phi \\ -\phi_2(t), & \text{otherwise} \end{cases} \quad \text{and}$$

$$\tilde{u}_y(t) = \begin{cases} +u_y(t), & t \notin \mathcal{D}_\phi \\ -u_y(t), & \text{otherwise} \end{cases}.$$

Then, $\tilde{\phi}$ reaches $\phi(s)$ at some time $\tilde{s} \in (0,s]$ and lies completely in the closed upper half plane. Also, $(\tilde{\phi},\tilde{w})$ is an RBP trajectory-control pair. Recall from [5] that the time optimal trajectories for joining points P and Q using the dynamics $\dot{x} = u_x\sqrt{y}$, $\dot{y} = u_y\sqrt{y}$ (with $(u_x,u_y) \in \overline{B_1(0)}$) are arcs of cycloids which pass from P to Q without hitting the x-axis in between. Since p lies in the x-axis and $\phi(s)$ is above the axis, we can replace $\tilde{\phi}\lceil[0,\tilde{s}]$ with a cylcoid arc $\hat{\phi}$ to get a trajectory that reaches $\phi(s)$ at some time $\hat{t} \in (\,0,\tilde{s}\,]$ and lies in the open upper half plane along $(\,0,\hat{t}\,]$. Moreover, since the RBP motion is Lipschitz in (x,y) on sets of the form $\{y \geq b\} \times \overline{B_1(0)}$ for $b > 0$, the RBP law is coercing on $S_n \times \overline{B_1(0)}$ for open sets S_n containing Trace $\hat{\phi}\lceil[\,1/n,\hat{t}\,]$ for n large enough. Thus, we take $\psi = \hat{\phi}$, $t = s$, and $t' = \hat{t}$ in the condition defining coercifiability of $\phi \in \mathrm{Traj}_\alpha(p,f_\infty,\{B\})$ to satisfy the requirement. Applying Theorem A, we obtain

Corollary: *The RBP value function is the unique solution of the corresponding HJB in the class of functions $w \in C(\mathbb{R}^2)$ which are bounded below and which vanish on the target.*

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