On the Stabilizability of Control Constrained Linear Systems

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Abstract. We study the problem of stabilization of time-invariant linear systems with controls which are constrained in a cone. Under small-time local controllability conditions we propose a simple construction of a Lipschitz piece-wise linear stabilizing feedback. Moreover, we show that the stabilizing feedback can be chosen in such a way that the number of switchings from one linear form to another, that may occur along a trajectory of the closed-loop system, is uniformly bounded.

1 Introduction

In this note we consider the control system:

$$\dot{x} \in Ax + U, \tag{1}$$

where A is a matrix of dimension $n \times n$ and U is a closed convex cone.

For a function $k : \mathrm{R}^n \to U$ we consider the closed-loop system

$$\dot{x}(t) = Ax(t) + k(x(t)), \tag{2}$$

Below the feedback $k(\cdot)$ is a Lipschitz function therefore the notion of asymptotic stability for (2) has the standard meaning.

The main result is contained in the next theorem and some of its consequences discussed in Section 4.

Theorem 1. *Let the control system (1) is small-time locally controllable at the origin. Then there exists a continuous positively homogeneous piece-wise linear feedback $k : \mathrm{R}^n \to U$ such that the closed-loop system (2) is asymptotically stable at the origin.*

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The proof of the theorem is constructive. It is based on a reduction of the original system to an auxiliary system with unconstrained controls, which however, contains nonlinearities of sign-type with respect to a part of the controls. Thanks to its simple structure, the auxiliary system allows decomposition to a cascade of single-input systems for which stabilizing controls are easy to obtain, applying the Ackermann formula ([1, 2]), for example. The resulting stabilizing control for the original system is piece-wise linear. More precisely, each component of the stabilizer is a sum of a linear function and absolute values of linear functions. We also show that the stabilizing feedback can be constructed in such a way that the number of switchings from one linear form to another, that may occur along a trajectory of the closed-loop system, is uniformly bounded.

The stabilization problem with cone-constrained controls is studied (under weaker controllability conditions) also in [4], but the constructions proposed there are much more complicated and the resulting feedback lows are not piece-wise linear.

For comparison we also mention that for the multiple integrators on R^n a bounded stabilizing feedback that can be represented as a composition of a "saturate function" and a linear function does not exist ([8]). This was the motivation in [5] for developing a complex construction of a locally Lipschitz stabilizing feedback in R^n for linear systems for which U is a box containing the origin in its interior, the matrix A has not eigenvalues with positive real parts and its all uncontrollable modes have strictly negative real parts. Our result is weaker than that in [5] in the respect that the feed-back that we construct is only locally bounded (piece-wise linear), but on the other hand we consider a set U that need not contain the origin in its interior, and this makes the problem different. But, when the staring point belongs to a bounded set X_0, our main result implies under the same assumptions as in [5] that for every $\varepsilon > 0$ there exists of a linear stabilizing feddback law $u(x)$ such that $\max_{x \in X_0} \|u(x)\| \leq \varepsilon$.

In the next section we introduce some notations and auxiliary results. In Section 3 we prove Theorem 1, which is discussed in more details in the last section.

2 Preliminaries

We remind that admissible trajectories of (1) defined on $[0, T]$ are all absolutely continuous functions on $[0, T]$ that satisfy (1) for almost every t.

In Theorem 1 we assume that the system is small-time local controllable (STLC) at the origin. In the case of a cone U this means that for every $T > 0$ and for every point $x \in \mathrm{R}^n$ there exists an admissible trajectory $x(\cdot)$ such that $x(0) = x$ and $x(T) = 0$. Necessary and sufficient algebraic conditions for STLC of (1) are proved in [3, 7] and will be used in the proof of the theorem.

Below co B denotes the convex hull of the set $B \subset \mathrm{R}^n$, $\mathrm{rec}(V)$ stays for the maximal subspace contained in the convex cone $V \subset \mathrm{R}^n$, $\mathrm{Lin}(Z)$ is the minimal linear subspace of R^n that contains the set Z and $\mathrm{Inv}(Z)$ is the minimal linear subspace of R^n that contains the set Z and is invariant with respect to the matrix A.

In the proof of Theorem 1 we use the following assertions:

Lemma 1. *Suppose that L be a proper subspace of R^n, U is a convex cone and* $\mathrm{rec}(co(L \cup U)) \neq L$. *Then there exists elements $u_i \in U \setminus L$, $i = 1, 2$, such that* $u_1 + u_2 \in L$.

Lemma 2. *Let all eigenvalues of the $n \times n$-matrix D have real parts not greater than the negative number $-\lambda$. Then there exists a constant C such that for every* $\eta(\cdot) \in \mathcal{L}_\infty(0, +\infty)$ *the solution of*

$$\dot{x}(t) = Dx(t) + \eta(t), \quad x(0) = x_0$$

satisfies for all $t \geq 0$ and $T \geq t$ the inequality

$$|x(T)| \leq C \left[e^{-\lambda T}|x_0| + \frac{1}{\lambda} e^{-\lambda(T-t)} \operatorname*{essup}_{s\in[0,t]} |\eta(s)| + \frac{1}{\lambda} \operatorname*{essup}_{s\in[t,+\infty)} |\eta(s)| \right].$$

The proofs of these lemmas is standard.

3 Proof of the Main Result

The following is an equivalent formulation of the STLC condition from [3], [6], [7]: Denote successively the subspaces

$$\begin{aligned} L_1 &= \mathrm{Inv}(\mathrm{rec}(U)), \\ L_2 &= \mathrm{Inv}(\mathrm{rec}(\mathrm{co}(L_1 \cup U))), \\ \ldots &\quad \ldots\ldots\ldots \\ L_{i+1} &= \mathrm{Inv}(\mathrm{rec}(\mathrm{co}(L_i \cup U))), \\ \ldots &\quad \ldots\ldots\ldots \end{aligned}$$

Then the system (1) is STLC if and only if $L_k = \mathrm{R}^n$ for some (minimal) k. In fact, $L_{i+1} = L_i$ for $i \geq n$, therefore $k \leq n$ if the system is STLC. If the set U is a subspace, then $k = 1$. In this case the claim of the theorem is well known. To avoid notational complications we assume below that $k = 2$. Since the construction and the proof are inductive, they go in the same way for $k > 2$.

Let us fix $b_1, \ldots, b_p \in \mathrm{rec}(U)$ and the numbers $n_1, \ldots, n_p$ in such a way that the vectors

$$b_1, Ab_1, \ldots, A^{n_1-1}b_1, \ldots, b_p, Ab_p, \ldots A^{n_p-1}b_p \tag{3}$$

are linearly independent, each of the subspaces

$$L_{11} = \mathrm{Lin}(L_{21}, d_2, Ad_2, \ldots, A^{m_2-1}d_2), \ldots$$

$$L_{1,p} = \mathrm{Lin}(L_{1,p-1}, b_p, Ab_p, \ldots, A^{n_p-1}b_p)$$

is invariant with respect to A, and

$$\mathrm{Inv}(b_1, \dots b_p) = \mathrm{Inv}(\mathrm{rec}(U)) = L_1.$$

Since we have assumed that $L_1 \neq \mathrm{R}^n$, using Lemma 1 we obtain pares

$$d_1, d_1^-, \dots, d_q, d_q^- \in U \quad \text{for which} \quad d_i + d_i^- \in L_1,$$

and appropriate numbers $m_1, \dots, m_q$ such the set of vectors (3) together with

$$d_1, Ad_1, \dots, A^{m_1-1}d_1, \dots, d_q, Ad_q, \dots A^{m_q-1}d_q \tag{4}$$

constitute a basis in L_2 (therefore in R^n, according to the assumption that $k = 2$), and moreover the subspaces

$$L_{21} = \mathrm{Lin}(L_1, d_1, Ad_1, \dots, A^{m_1-1}d_1), \quad L_{22} = \mathrm{Lin}(L_{21}, d_2, Ad_2, \dots, A^{m_2-1}d_2), \dots$$

$$L_{2,q} = \mathrm{Lin}(L_{2,q-1}, d_q, Ad_q, \dots, A^{m_q-1}d_q)$$

are invariant with respect to A. (If $k > 2$ we continue in the same way applying again Lemma 1.)

Let us denote $l_i = -(d_i + d_i^-) \in L_1$, and

$$s(w) = \begin{cases} 0, & \text{if } w \geq 0, \\ w, & \text{if } w < 0. \end{cases}$$

Then for every measurable functions $v_i(t), w_j(t) \in \mathrm{R}$, any trajectory of the system

$$\dot{x} = Ax + \sum_{i=1}^{p} b_i v_i(t) + \sum_{j=1}^{q} d_j w_j(t) + \sum_{j=1}^{q} l_j s(w_j(t)) \tag{5}$$

is also a trajectory of the original system (1). Indeed, $\pm b_i \in U$. Moreover, if $w_j \geq 0$ we have also $d_j w_j + l_j s(w_j) = d_j w_j \in U$, otherwise $d_j w_j + l_j s(w_j) = (d_j + l_j)w_j = -d_j^- w_j = d_j^- |w_j| \in U$. Then for every (open, or closed-loop) control v_i, w_j of (5) we may define the control $u \in U$ by the expression

$$u = \sum_{i=1}^{p} b_i v_i + \sum_{j \in J_+} d_j w_j + \sum_{j \in J_-} d_j^- |w_j|, \tag{6}$$

where $J_+ = \{j : \ w_j \geq 0\}$, $J_- = \{j : \ w_j < 0\}$. When applied to (1) the control u gives the same trajectory as v_i, w_i applied to (5). Thus we reduce the problem of stabilization of (1) to the problem of stabilization of (5), with unconstrained controls v_i, w_j. The latter system, however contains the simple nonlinearity $s(\cdot)$ with respect to a part of the controls.

Now we proceed with the stabilization of (5). Denote by M the matrix with columns (3), (4), which is invertible by the construction. Passing to the new variable $y = M^{-1}x$ the system (5) takes the form

$$\dot{y} = M^{-1}AMy + \sum_{i=1}^{p} M^{-1}b_i v_i(t) + \sum_{j=1}^{q} M^{-1}d_j w_j(t) + \sum_{j=1}^{q} M^{-1}l_j s(w_j(t)). \tag{7}$$

Let us denote by $e_1, e_2, \ldots, e_n$ the canonical basis in R^n. Using the equalities

$$M^{-1}b_1 = e_1, \quad M^{-1}Ab_1 = e_2, \ldots, M^{-1}A^{n_p-1}b_p = e_{k_1}, \quad k_1 = \dim(L_1),$$
$$M^{-1}d_1 = e_{k_1+1}, \quad M^{-1}Ad_1 = e_{k_1+2}, \ldots, M^{-1}A^{m_q-1}d_q = e_n,$$

the invariance of the spaces $L_{1,i}, L_{2,j}$ with respect to A, and the inclusions $l_i \in L_1$, one can check that system (7) has the following 'cascade' form:

$$\begin{array}{ll} \dot{y}_1 = Ey_1 + Fy_2 + Bv & + Cs(w) \\ \dot{y}_2 = \qquad\quad Gy_2 & + Dw \end{array} \tag{8}$$

where y_1 is k_1-dimensional, y_2 is $(n-k_1)$-dimensional, the rest of the matrices have the corresponding dimensions, $s(w)$ should be understood componentwise. Let $\hat{w}(y_2)$ be any linear feedback that stabilizes the second subsystem, and let $\hat{v}(y_1)$ be any linear feedback that stabilizes the first subsystem for fixed $y_2 = 0$, $w = 0$. Then $(\hat{v}(y_1), \hat{w}(y_2))$ stabilizes the overall system (8), according to Lemma 2. Hence, $v(x) = \hat{v}([M^{-1}x]_1)$, $w(x) = \hat{w}([M^{-1}x]_2)$ is a linear stabilizing feedback for (5). The proof is complete.

4 Consequences and Comments

1. According to the proof of Theorem 1 the state space R^n can be split into a finite number of polyhedral convex cones $S_1, \ldots S_M$ (with faces determined by the equations $w_i(x) = 0$), such that the stabilizing feedback $u(x)$ is linear on each S_i and continuous on R^n. Thanks to the continuity of $u(\cdot)$ we may assume all S_i closed without any ambiguity.

Theorem 2. *The stabilizing feedback in Theorem 1 can be chosen in such a way that there is a number N such that for every trajectory $x(\cdot)$ of the closed-loop system (2) there are $0 = t_0 < t_2 < \ldots < t_N = +\infty$ such that on each subinterval (t_i, t_{i+1}) the trajectory $x(t)$ belongs to some of the sets S_i.*

Proof. As in the proof of the main result, we set $y = M^{-1}x$ and consider the control system (8). Let us choose the linear feedback laws $\hat{v}(y_1) = Py_1$ and $\hat{w}(y_2) = Qy_2$ (here P and Q are matrices of appropriate dimensions) such that the eigenvalues λ_i, $i = 1, \ldots, k$ of the matrix R are negative real numbers, where

$$R = \begin{pmatrix} E + BP & 0 \\ 0 & G + DQ \end{pmatrix}.$$

It is well known that each component $q_{ij}(t)$ of the exponent matrix $\exp((G+DQ)t)$ is a quasipolynomial which has at most $n-1$ zeros. Let y_2^0 be an arbitrary point. Then $y_2(t) = \exp((G+DQ)t)y_2^0$ is the solution of the second subsystem of (8) starting from y_2^0 at the moment of time 0. Clearly, each component $y_2^j(t)$ of $y_2(t)$ is also a quasipolinomial with at most $n-1$ zeros (as is a linear combination of quasipolinomials of this type). Let $t_0 = 0$, $t_N = +\infty$ and $0 < t_1 < t_2 < \cdots < t_{N-1}$ be the zeros of all components $y_2^j(t)$, $j = 1, 2, \ldots, n$, i.e.

$$\{t_0, t_1, \ldots, t_{N-1}\} = \{t_0\} \cup_{j=1}^n \{t_1^j, t_2^j, \ldots t_{n_j}^j\}.$$

Thus,

$$N \; < \; n(n-1)$$

and each component of $y_2(t)$ does not change its sign on every open interval (t_{i-1}, t_i), $i = 1, \dots, N$, i.e. the trajectory $x(t) = My(t)$ belongs to some of the sets S_i on each of these open intervals. This completes the proof in the case $k = 2$.

Since each component of $y_2(t)$ does not change its sign on every open interval (t_{i-1}, t_i), $i = 1, \dots, N$, we obtain (according to the Cauchy formula) that each component of $y_1(t)$ is also a quasipolinomial on (t_{i-1}, t_i), $i = 1, \dots, N$, and hence it has also a finite number of zeros (it can be shown that the number of zeros of each component of $y_2(t)$ can be estimated by a constant which does not depend on the starting point y_1^0). Thus, the same argument can be also applied in the case $k > 2$.

2. The stabilizing feedback in Theorem 1 is defined trough the one defined for the cascade system (8). The matrices E, F, G, B, D in this system, however, have its own specific structures. This structure is implied by the invariance of the spaces L_{1i} and L_{2j} with respect to A. One can easily check that the second subsystem in (8) has a cascade form, where each subsystem depends on a single control w_j, and does not depend on the state variables from the upper levels. This makes it possible, utilizing again Lemma 2, to construct the feedback controls $\hat{w}_j$ independently of each other, each of them stabilizing a corresponding part of y_2. The explicit Ackermann formula ([1, 2]) can be utilized for this purpose, since each subsystem has a single control.

3. As shown in [7], in the case of a cone U that is not a subspace the STLC property need not be robust: arbitrarily small perturbations in the data may spoil the controllability. Nevertheless, below we shall show that the stabilizability property of the control system (1) is robust with respect to small perturbations.

We denote by $\mathcal{B}$ the unit ball of R^n centered at the origin, by $||A||$ - the standard norm of the matrix A and by $H(V_1, V_2)$ - the Hausdorff distance between the sets V_1 and V_2.

Proposition 1. *Let the assumptions of Theorem 1 hold true. Then there exists $\delta > 0$ such that for every $n \times n$ matrix $\tilde{A}$ and for every closed convex con $\tilde{U} \subset \mathrm{R}^n$ for which*

$$||\tilde{A} - A|| \leq \delta \quad \textit{and} \quad H(\tilde{U} \cap \mathcal{B}, U \cap \mathcal{B}) \leq \delta, \tag{9}$$

the perturbed system

$$\dot{x}(t) \; = \; \tilde{A}x(t) \; + \; \tilde{U}, \tag{10}$$

is stabilizable by a Lipschitz continuous positively homogeneous feedback control $\tilde{u}(x) \in \tilde{U}$.

Proof. We shall use the notations from the proof of Theorem 1. The stabilizability property does not depend on the choice of the coordinates. Thus, for simplicity we shall consider the original system (2) and the system (10) with respect to the above introduced y-variables, i.e. we consider the following representations of the both systems (10) in y-coordinates:

$$\dot{y}(t) \;=\; Py(t) \;+\; W \quad \text{and} \quad \dot{y}(t) \;=\; \tilde{P}y(t) \;+\; \tilde{W}, \tag{11}$$

where $y = (y_1, y_2)^T$,

$$P \;=\; M^{-1}AMy \;=\; \begin{pmatrix} E & F \\ 0 & G \end{pmatrix}, \quad \|\tilde{P} - P\| \le \delta \quad \text{and} \quad \|\tilde{W} - W\| \le \delta.$$

Let $\tilde{w}(y_2) = Ry_2$ be a linear feedback that stabilizes the second subsystem in (8), and let $\tilde{v}(y_1) = Qy_1$ be a linear feedback that stabilizes the first subsystem in (8) for fixed $y_2 = 0$ and $w = 0$. Then there exist positive definite symmetric matrices L_1 and L_2 and a positive real number ε such that for every point $y = (y_1, y_2)^T$

$$\langle L_1 y_1, Ey_1 + BQy_1 \rangle \le -2\varepsilon \|y_1\|^2 \quad \text{and} \quad \langle L_2 y_2, Gy_2 + DRy_2 \rangle \le -2\varepsilon \|y_2\|^2. \tag{12}$$

Let M be an arbitrary positive number. For every point $y = (y_1, y_2)^T$ of R^n we set

$$L(y_1, y_2) := \frac{1}{2} \langle L_1 y_1, \; y_1 \rangle + \frac{M}{2} \langle L_2 y_2, \; y_2 \rangle,$$

and

$$\mathcal{F}(y) = \begin{pmatrix} BQy_1 \quad + \quad Cs(Ry_2) \\ DRy_2 \end{pmatrix}.$$

Clearly, $\mathcal{F}(\cdot)$ is a positively homogeneous piecewise linear function such that $\mathcal{F}(y) \in W$ for each $y \in \mathrm{R}^n$ for which there exists a conctant $C > 0$ such that

$$\|\mathcal{F}(y)\| \;\le\; C\|y\|. \tag{13}$$

Then for every $y = (y_1, y_2)^T \in \mathrm{R}^n$, we define the feedback $\tilde{\mathcal{F}}(y)$ as the metric projection of $\mathcal{F}(y)$ on the set $\tilde{W} \cap \mathcal{B}$. Clearly, $\tilde{\mathcal{F}}(y)$ is a positively homogeneous Lipschitz continuous functions with values from $\tilde{W}$ ($F(\cdot)$ is a function of this kind and $\tilde{W}$ is a closed convex cone). According to the definition of $\tilde{\mathcal{F}}(y)$ we obtain that

$$\|\tilde{\mathcal{F}}(y) - \mathcal{F}(y)\| \le \|\mathcal{F}(y)\| \left\| \frac{\tilde{\mathcal{F}}(y)}{\|\mathcal{F}(y)\|} - \frac{\mathcal{F}(y)}{\|\mathcal{F}(y)\|} \right\| \le \|\mathcal{F}(y)\| \; \delta. \tag{14}$$

Let $y = (y_1, y_2)^T$ be an arbitrary nonvanishing point. Then

$$\left\langle \text{ grad } L(y), \tilde{P}y + \tilde{\mathcal{F}}(y) \right\rangle = \langle \text{ grad } L(y), Py + \mathcal{F}(y) \rangle +$$

$$\left\langle \text{ grad } L(y), (\tilde{P} - P)y + \tilde{\mathcal{F}}(y) - \mathcal{F}(y) \right\rangle.$$

Taking into account (12) we obtain that

$$\langle \text{ grad } L(y), Py + \mathcal{F}(y) \rangle =$$

$$= \langle L_1 y_1,\ Ey_1 + Fy_2 + BQy_1 + Cs(y_2)\rangle + M\,\langle L_2 y_2,\ Gy_2 + DRy_2\rangle \le$$
$$\le -2\varepsilon\left(\|y_1\|^2 + M\|y_2\|^2\right) + \langle L_1 y_1, Fy_2 + Cs\,(y_2)\rangle \le$$
$$\le -\varepsilon(\|y_1\|^2 + M\|y_2\|^2) \tag{15}$$

for all suffisiently large $M > 0$. According to (14) and (13) we obtain that

$$\left\langle \text{ grad } L(y), (\tilde{P} - P)y + \tilde{\mathcal{F}}(y) - \mathcal{F}(y)\right\rangle \le$$

$$\le \|L\|\ \|y\|\delta(\|y\| + \|\mathcal{F}(y)\|) \le \delta\ \|L\|\ \ \|y\|^2\ (1 + C).$$

The last inequality and (15) imply that for suffisiently large $M > 0$ and sufficiently small $\delta > 0$

$$\left\langle \text{ grad } L(y), \tilde{P}y + \tilde{\mathcal{F}}(y)\right\rangle < 0 \tag{16}$$

for all nonvanishing points y, i.e. the function L is a Lyapunov function for the closed loop control system (16). Thus, the stabilizability of the system (1) is robust with respect to small perturbations.

4. The stabilizing feedback $u(x)$ that we propose is Lipschitz in x and $u(0) = 0$. This makes it possible to apply Theorem 1 also for stabilization of a nonlinear system

$$\dot{x} = f(x, u), \quad u \in W \tag{17}$$

with f having Lipschitz derivatives in x and u and W – a convex cone. By a standard argument, the stabilizing feedback for (1) with $A = f'_x(0)$, $U = f'_u(0)W$ (provided that the linear system is STLC) stabilizes *locally* also (17).

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