Remarks on the Strong Invariance Property for Non-Lipschitz Control Systems with Set-Valued Disturbances*

Mikhail Krastanov, Michael Malisoff, and Peter Wolenski

***Abstract*— We announce a new sufficient condition for strong invariance for differential inclusions in terms of a Hamiltonian inequality. In lieu of the usual Lipschitzness assumption on the multifunction, we assume a feedback realization condition that can in particular be satisfied for measurable dynamics that are neither upper nor lower semicontinuous. Our condition is based on H. Sussmann's unique limiting property. We apply our result to a broad class of nonlinear control systems with general measurable set-valued disturbances. As a consequence, we also prove a new strong invariance characterization for feedback realizable lower semicontinuous differential inclusions.**

***Index Terms*— Set-valued disturbances, non-Lipschitz systems, state constraints, strong invariance, nonsmooth analysis**

I. Introduction

Consider a nonlinear control system $\dot{x} = g(x, \alpha)$ where $\alpha \in \mathcal{M}(A) := \{\text{measurable functions } [0, \infty) \to A\}$, $g : \mathbb{R}^n \times A \to \mathbb{R}^n : (x, a) \mapsto g(x, a)$ is locally Lipschitz in x uniformly in $a \in A$ and continuous, and $A \subseteq \mathbb{R}^n$ is compact. The inputs α represent either controls or disturbances acting on the system. Using Filippov's selection theorem, this system can be represented as a *differential inclusion* $\dot{x} \in F(x)$, where $F(x) := \{g(x, a) : a \in A\}$. Then an absolutely continuous function $\phi : [0, T] \to \mathbb{R}^n$ is a trajectory of the system if and only if $\dot{\phi}(t) \in F(\phi(t))$ for almost all (a.a.) $t \in [0, T]$, i.e., if and only if ϕ is a trajectory of F. Oftentimes, it is important to know sufficient conditions under which all trajectories of such a system that start in a given closed set $S \subseteq \mathbb{R}^n$ remain in S; when this is the case, we say that (F, S) is strongly invariant. Specifically, given a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ (i.e., a mapping from $\mathbb{R}^n$ into the subsets of $\mathbb{R}^n$) and a closed set $S \subseteq \mathbb{R}^n$ defining state constraints, we say that (F, S) is *strongly invariant (in* $\mathbb{R}^n$) provided for each $\bar{x} \in S$, each trajectory $t \mapsto \phi(t)$ of F starting at $\bar{x}$ remains in S on each interval $[0, T]$ on which ϕ is defined.

Sufficient conditions for strong invariance usually invoke a Lipschitz condition on the dynamics (cf. section III-B for a survey of results in this direction). For example, if F is locally Lipschitz and nonempty, compact, and convex valued with linear growth, then it is well known (cf. [6, Chapter 4]) that (F, S) is strongly invariant in $\mathbb{R}^n$ if and only if $F(x) \subseteq T^C_S(x)$ for all $x \in S$, where T^C_S denotes the Clarke tangent cone (cf. §III below or [6] for the definition of T^C_S). This characterization can be applied to the Lipschitz control system $\dot{x} = g(x, \alpha)$ above. However, this cone characterization can fail for a general *non*-Lipschitz dynamic F, as illustrated in the following example: Take $n = 1$, $S = \{0\}$, $F(0) = [-1, +1]$, and $F(x) = \{-\text{sign}(x)\}$ for $x \neq 0$. Then $T^C_S(0) = \{0\}$, even though (F, S) is strongly invariant. This example satisfies our dynamic assumptions (cf. Example 2.3 below). It is also covered by our main theorem (see section III).

More generally, consider a control system of the form

$$\dot{x} \in g(x, \alpha) U(x), \tag{1}$$

where g is as above and $U : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is a (Borel) measurable *set-valued disturbance* perturbation (cf. [14, Chapter 2] for control problems with set-valued disturbances). As before, the trajectories of (1) are absolutely continuous functions ϕ satisfying $\dot{\phi}(t) \in F(\phi(t))$ for a.a. t, but in this case we now have $F(x) = \{g(x, a)b : a \in A, b \in U(x)\} \subseteq \mathbb{R}^n$. In this context, the values $t \mapsto \beta(t) \in U(\phi(t))$ of the disturbance perturbation are unknown to the controller; one only knows that $\beta(t)$ takes *some value* in $U(\phi(t))$ for each t. However, the multifunction U, the perturbation $t \mapsto \alpha(t) \in A$, and the current state $t \mapsto \phi(t)$ are known and can be measured. The dynamics (1) include the example from the previous paragraph by taking $n = 1$ and $g \equiv 1$. The objective is to find sufficient conditions, in terms of g and U, under which all the trajectories of (1) starting in a given closed set $S \subseteq \mathbb{R}^n$ remain in S, i.e., such that (F, S) is strongly invariant in $\mathbb{R}^n$. Since F will not in general be Lipschitz, or even continuous, the usual invariance criteria for (locally) Lipschitz systems (cf. [6], [14], [16]) do not give such conditions.

Such topics in flow invariance provide the foundation for many important applications in control theory and optimization (cf. [5], [6], [7], [9], [12], [14], [15]). Starting from strong invariance and its Hamiltonian characterizations, one can develop uniqueness and regularity theory for solutions of Hamilton-Jacobi-Bellman equations, stability theory, nonsmooth characterizations of monotonicity in systems biology, and much more (cf. [1], [6], [7], [14], [16]). On the other hand, it is well appreciated that many important dynamics are non-Lipschitz and may even be discontinuous (e.g., system (1) above), and therefore are beyond the scope of the usual strong invariance characterizations. Therefore, the development of conditions guaranteeing strong invariance under less restrictive assumptions is a problem that is of considerable ongoing research interest.

This motivates the search for sufficient conditions for strong invariance for non-Lipschitz differential inclusions, which is the focus of this note. Donchev, Rios and Wolenski

*Corresponding Author: Peter Wolenski. Date: March 1, 2004.

M. Krastanov is with the Department of Biomathematics, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev str., bl 8, 1113 Sofia BULGARIA, krast@math.bas.bg

M. Malisoff and P. Wolenski are with the Department of Mathematics, 304 Lockett Hall, Louisiana State University and A. & M. College, Baton Rouge LA 70803-4918 USA, {malisoff,wolenski}@math.lsu.edu

[9], [12] recently developed necessary and sufficient conditions for strong invariance under more general conditions on the structure of the dynamics such as dissipativity and one-sided Lipschitzness (cf. section III for further details). In this note, we pursue a very different approach. Rather than restricting the structure of the dynamics, we provide a sufficient condition for strong invariance under an appropriate feedback realization hypothesis. Our hypothesis is related to H. Sussmann's 'unique limiting' property that was introduced in [13]; it is a less restrictive hypothesis than those of the known strong invariance characterizations because it is satisfied by a broad class of differential inclusions with measurable, but possibly neither upper nor lower semicontinuous, right-hand sides (cf. section II for examples).

In section II, we state our realization hypothesis precisely and provide the necessary background from nonsmooth analysis. We also illustrate the applicability of our hypothesis to the general control system (1) and other discontinuous dynamics that are beyond the scope of the well known strong invariance results. In section III, we announce our main strong invariance result and discuss its relationship to the known theorems in invariant system theory. In section IV, we sketch the proof of this result (cf. [11] for the complete proof), and we close in section V by proving a new necessary and sufficient Hamiltonian condition for strong invariance for general lower semicontinuous feedback realizable dynamics.

II. Realization Hypothesis and Preliminaries

A. Basic Hypothesis

Our main object of study in this note is an autonomous differential inclusion $\dot{x} \in F(x)$. In this subsection, we state our realization hypothesis on F and illustrate its relevance using several non-Lipschitz applications. We require the following definitions. By a *trajectory* of $\dot{x} \in F(x)$ on an interval $[0, T]$ starting at a point $x_o \in \mathbb{R}^n$, we mean an absolutely continuous function $\phi : [0, T] \to \mathbb{R}^n$ for which $\phi(0) = x_o$ and $\dot{\phi}(t) \in F(\phi(t))$ for (Lebesgue) almost all (a.a.) $t \in [0, T]$. We let $\mathrm{Traj}_T(F, x)$ denote the set of all trajectories $\phi : [0, T] \to \mathbb{R}^n$ for F starting at x on all possible intervals $[0, T]$, and we set $\mathrm{Traj}(F, x) := \cup_{T\geq 0}\mathrm{Traj}_T(F, x)$ and $\mathrm{Traj}(F) := \cup_{x\in\mathbb{R}^n}\mathrm{Traj}(F, x)$.

A multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to have *linear growth* provided there exist positive constants c_1 and c_2 such that $||v|| \leq c_1 + c_2||x||$ for all $v \in F(x)$ and $x \in \mathbb{R}^n$, where $|| \cdot ||$ denotes the Euclidean norm. For any interval I, a function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is said to have *linear growth (on I)* provided $x \mapsto F(x) := \{f(t, x) : t \in I\}$ has linear growth. For any sets $D, M \subseteq \mathbb{R}^n$ and any constant $\eta \in \mathbb{R}$, we set $M + \eta D := \{m + \eta d : m \in M, d \in D\}$ and we set $\mathrm{cone}\,\{D\} := \cup\{\eta D : \eta \geq 0\}$. Also,

$$\mathcal{B}_n(p) := \{x \in \mathbb{R}^n : \|x - p\| \leq 1\}$$

for all $p \in \mathbb{R}^n$ and $\mathcal{B}_n := \mathcal{B}_n(0)$. A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is said to be *upper* (resp., *lower*) *semicontinuous* provided for each $x \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x') \subseteq F(x) + \varepsilon\mathcal{B}_n$ (resp., $F(x') + \varepsilon\mathcal{B}_n \supseteq F(x)$) for all $x' \in \delta\mathcal{B}_n(x)$; it is said to be *closed* (resp., *compact*, *convex*, *nonempty*) *valued* provided $F(x)$ is closed (resp., compact, convex, nonempty) for each $x \in \mathbb{R}^n$. A continuous function $\omega(\cdot) : [0, \infty) \to [0, \infty)$ is called a *modulus* provided it is nondecreasing with $\omega(0) = 0$. For each $T \geq 0$, we let $\mathcal{C}[0, T]$ denote the set of all $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfy

- (C_1) For each $x \in \mathbb{R}^n$, the map $t \mapsto f(t, x)$ is measurable;
- (C_2) For each compact set $K \subseteq \mathbb{R}^n$, there exists a modulus $\omega_{f,K}(\cdot)$ such that, for all $t \in [0, T]$ and $x_1, x_2 \in K$, $\|f(t, x_1) - f(t, x_2)\| \leq \omega_{f,K}(\|x_1 - x_2\|)$; and
- (C_3) f has linear growth on [0,T].

For each $\bar{x} \in \mathbb{R}^n$, denote by $\mathcal{C}_F([0, T], \bar{x})$ those $f \in \mathcal{C}[0, T]$ that are also selections of the cone of F for a.a. $t \in [0, T]$ and all $x \in \mathbb{R}^n$ sufficiently near $\bar{x}$; that is,

$$\mathcal{C}_F([0, T], \bar{x}) := \left\{ \begin{array}{l} f \in \mathcal{C}[0, T] : \exists\gamma > 0 \text{ such that} \\ f(t, x) \in \mathrm{cone}\,\{F(x)\} \text{ for a.a.} \\ t \in [0, T] \text{ and all } x \in \gamma\mathcal{B}_n(\bar{x}). \end{array} \right\}$$

Notice that while the elements $f \in \mathcal{C}_F([0, T], \bar{x})$ are defined on all of $[0, T] \times \mathbb{R}^n$, they need only satisfy the requirement $f(t, x) \in \mathrm{cone}\,\{F(x)\}$ on *part of* their domain. Let $\mathcal{C}_F[0, T]$ denote those $f \in \mathcal{C}[0, T]$ such that $f(t, x) \in \mathrm{cone}\,\{F(x)\}$ for almost all $t \in [0, T]$ and all $x \in \mathbb{R}^n$. We will assume:

- (U) For each $\bar{x} \in \mathbb{R}^n$, $T \geq 0$, and $\phi \in \mathrm{Traj}_T(F, \bar{x})$, there exists $f \in \mathcal{C}_F([0, T], \bar{x})$ for which ϕ is the unique solution of the initial value problem $\dot{y}(t) = f(t, y(t))$, $y(0) = \bar{x}$ on $[0, T]$.

Our uniqueness hypothesis (U) is less restrictive than requiring a continuous selection from the dynamics F that realizes the trajectory. This is because f is allowed to depend on time as well as the state, and need only be a *local* selection. Moreover, f is allowed to depend on the choice of the trajectory ϕ, and need not be continuous. In practice, hypothesis (U) can be checked using open or closed loop controls, and may be satisfied for non-Lipschitz dynamics. The following examples illustrate these points and also show how to use cones to check condition (U).

Example 2.1: Choose the dynamics $F(x) = g(x, A)U(x)$ where $A \subseteq \mathbb{R}^m$ is compact, and $g : \mathbb{R}^n \times A \to \mathbb{R}^n$ is continuous and satisfies

- (H) For each compact $K \subseteq \mathbb{R}^n$, there exists $L_K > 0$ such that $(g(x_1, a) - g(x_2, a)) \cdot (x_1 - x_2) \leq L_K\|x_1 - x_2\|^2$ for all $x_1, x_2 \in K$ and $a \in A$. Also, $x \mapsto g(x, A)$ has linear growth.

and $U : \mathbb{R}^n \rightrightarrows \mathbb{R}$ is locally bounded, (Borel) measurable, closed and nonempty valued, and satisfies $U(x) \cap (0, \infty) \neq \emptyset$ and $U(x) \cap (-\infty, 0) \neq \emptyset$ for all $x \in \mathbb{R}^n$. (The argument we are about to give still applies if instead of assuming that $U(x) \cap (0, \infty) \neq \emptyset$ and $U(x) \cap (-\infty, 0) \neq \emptyset$ for all $x \in \mathbb{R}^n$, we assume either $U : \mathbb{R}^n \rightrightarrows (0, \infty)$ or $U : \mathbb{R}^n \rightrightarrows (-\infty, 0)$.) This includes systems of the form (1) from the introduction with set-valued disturbances, as special cases. One can easily check (cf. [2]) that condition (H) guarantees the existence of a unique trajectory $\phi : [0, T] \to \mathbb{R}^n$ of $\dot{x} = g(x, \alpha)\beta$ for each initial condition, $T > 0$, and (essentially) bounded measurable functions $\alpha : [0, T] \to A$ and $\beta : [0, T] \to \mathbb{R}$.

To check condition (U), let $\phi \in \mathrm{Traj}(F)$. Applying the (generalized) Filippov lemma (cf. [14, p. 72]), we find a measurable pair (α, β) for which $\alpha(t) \in A$, $\beta(t) \in U(\phi(t))$, and $\dot{\phi}(t) = g(\phi(t), \alpha(t))\beta(t)$ for almost all t. We now show that condition (U) holds with $f(t,x) := g(x, \alpha(t))\beta(t)$. In general, we will not have $\beta(t) \in U(x)$ for all t and x. In fact, it could be that $U(\phi(t)) \cap U(x) = \emptyset$ for some t and x, so we may not have $f(t,x) \in F(x)$ for a.a. t and all x. On the other hand, one can easily check that $\beta(t) \in \text{cone}\,\{U(x)\}$ for a.a. t and x, so $f(t,x) \in g(x,A)\text{cone}\{U(x)\} = \text{cone}\{F(x)\}$ for a.a. t and all x, and this gives the desired result.

Example 2.2: Assume $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is (locally) Lipschitz and nonempty, compact, and convex valued. We claim that F satisfies condition (U). To see why, let $\bar{x} \in \mathbb{R}^n$, $T > 0$, and $\phi \in \text{Traj}_T(F, \bar{x})$ be given, and set

$$f(t,x) = \text{proj}_{F(x)}(\dot{\phi}(t))$$

(i.e., $f(t,x)$ is the closest point to $\dot{\phi}(t)$ in $F(x)$, which is well defined by the convexity of $F(x)$). Then $f \in \mathcal{C}_F[0,T]$ satisfies the requirement. If on the other hand we instead define the mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by $F(x) = \{1\}$ for $x < 0$, $F(0) = \{0\} \cup [1,2]$, and $F(x) = [0,2]$ for $x > 0$, and if $\phi \in \text{Traj}(F)$, then $f(t,x) \equiv \dot{\phi}(t) \in \text{cone}\{F(x)\}$ for almost all t and all $x \in \mathbb{R}^n$. In this case, condition (U) is satisfied with this choice of f, even though F is neither upper nor lower semicontinuous nor convex valued.

Example 2.3: Consider the example from the introduction in which $n = 1$, $F(0) = [-1, +1]$, and $F(x) = \{-\text{sign}(x)\}$ for $x \neq 0$. We claim that (U) is again satisfied. To see why, let $T > 0$, $\bar{x} \in \mathbb{R}$, and $\phi \in \text{Traj}_T(F, \bar{x})$ be given. Note that $(F, \{0\})$ is strongly invariant in $\mathbb{R}$. Therefore, either (i) ϕ starts at some $\bar{x} \neq 0$ and then moves to 0 at unit speed and then stays at 0 or (ii) $\phi \equiv 0$. If $\bar{x} \neq 0$, then the requirement is satisfied using $f(t,x) \equiv -\text{sign}(\bar{x})\beta(t)$, where $\beta(t) = 1$ if $t \in [0, |\bar{x}|]$ and 0 otherwise. In this case, we then have $f(t,x) \in \text{cone}\{F(x)\}$ for all $t \in [0,T]$ and $x \in (|\bar{x}|/2)\mathcal{B}_1(\bar{x})$. If instead $\bar{x} = 0$, then the requirement is instead satisfied with $f(t,x) \equiv 0 \in \text{cone}\{F(x)\}$ for all $t \in [0,T]$ and $x \in \mathbb{R}$.

B. Preliminaries in Nonsmooth Analysis

The principal nonsmooth objects used in this note are the proximal subgradient and normal cone, and here we review these concepts; see [6] for a complete treatment. Let $S \subseteq \mathbb{R}^n$ be closed and $x \in S$. A vector $\zeta \in \mathbb{R}^n$ is called a *proximal normal* vector of S at x provided there exists a constant $\sigma = \sigma(\zeta, x) > 0$ so that

$$\langle \zeta, x' - x \rangle \leq \sigma ||x' - x||^2 \;\; \forall x' \in S. \tag{2}$$

The set of all proximal normal vectors of S at x is denoted by $N_S^P(x)$ and is a convex cone. Notice that for each $\delta > 0$ and $x \in S$, $\zeta \in N_S^P(x)$ if and only if there exists $\sigma = \sigma(\zeta, x) > 0$ so that $\langle \zeta, x' - x \rangle \leq \sigma ||x' - x||^2$ for all $x' \in S \cap \delta\mathcal{B}_n(x)$.

Next assume $f : \mathbb{R}^n \to (-\infty, \infty]$ is lower semicontinuous and let $x \in \text{domain}(f) := \{x' : f(x') < \infty\}$. Then $\zeta \in \mathbb{R}^n$ is called a *proximal subgradient* for f at x provided there exist $\sigma > 0$ and $\eta > 0$ such that

$$f(x') \geq f(x) + \langle \zeta, x' - x \rangle - \sigma \|x' - x\|^2$$

for all $x' \in \eta\mathcal{B}_n(x)$. The (possibly empty) set of all proximal subgradients for f at x is denoted by $\partial_P f(x)$.

We next state the version of the Clarke-Ledyaev Mean Value Inequality needed for our strong invariance results (cf. [6, p. 117] for its proof). Let $[x, Y]$ denote the closed convex hull of $x \in \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$.

Theorem 1: Assume $x \in \mathbb{R}^n$, $Y \subseteq \mathbb{R}^n$ is compact and convex, and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous. Then for any $\delta < \min_{y \in Y} \Psi(y) - \Psi(x)$ and $\lambda > 0$, there exist $z \in [x, Y] + \lambda\mathcal{B}_n$ and $\zeta \in \partial_P \Psi(z)$ so that $\delta < \langle \zeta, y - x \rangle$ for all $y \in Y$.

The following is a variant of the well known "compactness of trajectories" lemma. Its proof is a special case of the compactness of trajectories proof in [6].

Lemma 2.4: Let $\bar{x} \in \mathbb{R}^n$, $T > 0$, $\tilde{f} \in \mathcal{C}[0,T]$ be also continuous in t, and $\{y_i : [0,T] \to \mathbb{R}^n\}$ be a sequence of uniformly bounded absolutely continuous functions satisfying $y_i(0) = \bar{x}$ for all i. Assume

$$\dot{y}_i(t) \in \tilde{f}(\tau_i(t), y_i(t) + r_i(t)) + \delta_i(t)\mathcal{B}_n \tag{3}$$

for a.a. $t \in [0,T]$ and all i, where $\{\delta_i(\cdot)\}$ is a sequence of nonnegative measurable functions that converges to 0 in L^2 as $i \to \infty$, $\{r_i(\cdot)\}$ is a sequence of measurable functions converging uniformly to 0 as $i \to \infty$, and $\{\tau_i(\cdot)\}$ is a sequence of nonnegative measurable functions converging uniformly to t on $[0,T]$ as $i \to \infty$. Then there exists a trajectory y of $\dot{y} = \tilde{f}(t, y)$, $y(0) = \bar{x}$ such that a subsequence of $\{y_i\}$ converges to y uniformly on $[0,T]$.

We will apply Lemma 2.4 to continuous mollifications of our feedback maps $f \in \mathcal{C}[0,T]$. More precisely, set

$$\eta(t) = \begin{cases} C\exp\left(\frac{1}{t^2-1}\right), & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}} \eta(s)ds = 1$. For each $\varepsilon > 0$ and $t \in \mathbb{R}$, set $\eta_\varepsilon(t) := \eta(t/\varepsilon)/\varepsilon$. Notice that $\int_{\mathbb{R}} \eta_\varepsilon(t)dt = 1$ for all $\varepsilon > 0$. Define the following convolutions of $f \in \mathcal{C}[0,T]$ in the t-variable:

$$f_\varepsilon(t,x) := \int_{\mathbb{R}} f(s,x)\eta_\varepsilon(t-s)ds \tag{4}$$

with the convention that $f(s,x) = 0$ for all $s \notin [0,T]$. Then $f_\varepsilon \in \mathcal{C}[0,T]$ and is continuous for all $\varepsilon > 0$. (See [10, Appendix C] for the theory of convolutions and mollifiers.) We will apply Lemma 2.4 to a sequence $\tilde{f} := f_{\varepsilon(i)}$ with $\varepsilon(i) > 0$ converging to zero, using ideas from the standard proof that $f_{\varepsilon(i)}(\cdot, x) \to f(\cdot, x)$ in L^1 for each x as $i \to \infty$.

Remark 2.5: Note for later use (cf. (20)) that if $\tau_i(t) \equiv t$ in Lemma 2.4, then the conclusions of the lemma remain true even if the t-continuity hypothesis on $f \in \mathcal{C}[0,T]$ is omitted. This follows from the proof of the compactness of trajectories lemma in [6].

III. Strong Invariance Theorem

A. Statement of Theorem and Remarks

Let $H_F : \mathbb{R}^n \times \mathbb{R}^n \to [-\infty, +\infty]$ denote the *(upper) Hamiltonian* for our dynamics F; i.e.,

$$H_F(x,p) := \sup_{v \in F(x)} \langle v, p \rangle.$$

For any subset $D \subseteq \mathbb{R}^n$, we write $H_F(x, D) \leq 0$ to mean that $H_F(x, d) \leq 0$ for all $d \in D$. By definition, this inequality holds vacuously if $D = \emptyset$.

Theorem 2: Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy (U), $\Psi : \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous, and $\mathcal{S} := \{x \in \mathbb{R}^n : \Psi(x) \leq 0\}$. If there exists an open set $\mathcal{U} \subseteq \mathbb{R}^n$ containing $\mathcal{S}$ for which $H_F(x, \partial_P \Psi(x)) \leq 0$ for all $x \in \mathcal{U}$, then $(F, \mathcal{S})$ is strongly invariant in $\mathbb{R}^n$.

We sketch the proof of this theorem in section IV. Note that we require the Hamiltonian inequality in a *neighborhood* $\mathcal{U}$ of $\mathcal{S}$. The result is not true in general if the Hamiltonian condition is placed only on $\mathcal{S}$, even if Ψ and F are smooth. For example, take $n = 1$, $\Psi(x) = x^2$, and $F(x) \equiv \{1\}$. In this case, $\mathcal{S} = \{0\}$ and $H_F(0, \partial_P \Psi(0)) = 0$, but $(F, \mathcal{S})$ is not strongly invariant. On the other hand, Example 2.3 is covered by Theorem 2, once we choose the verification function $\Psi(x) = x^2$. In this case, the Hamiltonian condition reads $H_F(x, \Psi'(x)) = -2|x| \leq 0$ for all $x \in \mathbb{R}$, so our sufficient condition for strong invariance is satisfied.

Theorem 2 contains the usual sufficient condition for strong invariance for an arbitrary closed set $S \subseteq \mathbb{R}^n$ by letting Ψ be the characteristic function I_S of F; that is, $I_S(x) = 0$ if $x \in S$ and is 1 otherwise. Then $\partial_P \Psi(x) = \{0\}$ for all $x \notin \text{boundary}\,(S)$, and $\partial_P \Psi(x) = N_S^P(x)$ for all $x \in \text{boundary}\,(S)$, by the local characterization of $N_S^P(x)$ in section II-B. This implies the following special case of Theorem 2:

Corollary 3.1: Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy (U) and $S \subseteq \mathbb{R}^n$ be closed. If $H_F(x, N_S^P(x)) \leq 0$ for all $x \in \text{boundary}\,(S)$, then (F, S) is strongly invariant in $\mathbb{R}^n$.

Remark 3.2: The converse of Corollary 3.1 does not hold, as illustrated by the data in Example 2.3; there, $(F, \{0\})$ is strongly invariant in $\mathbb{R}$ and $N_{\{0\}}^P(0) = \mathbb{R}$, so the Hamiltonian condition fails. This means that the converse of Theorem 2 does not hold. On the other hand, see section V for a necessary and sufficient condition for strong invariance under certain additional hypotheses on F.

B. Relationship to Known Strong Invariance Results

Theorem 2 improves on the known strong invariance results because it does not require the usual Lipschitz or other structural assumptions on the dynamics. The papers [4], [5] provide strong invariance results for locally Lipschitz dynamics (see also [6, Chapter 4]). For locally Lipschitz F, Clarke showed (cf. [4]) that the strong invariance property for (F, S) is equivalent to $F(x) \subseteq T_S^C(x)$ for all $x \in S$, where T_S^C denotes the Clarke tangent cone (cf. [6]). Recall that $v \in T_S^C(x)$ if and only if for each sequence $x_i \in S$ converging to x and each sequence $t_i > 0$ decreasing to 0, there exists a sequence $v_i \in \mathbb{R}^n$ converging to v such that $x_i + t_i v_i \in S$ for all i. In particular, if $S = \{0\}$, then $T_S^C(0) = \{0\}$. See [5] for Hilbert space versions, and [14] for other strong invariance results for Lipschitz dynamics and nonautonomous versions. For strong invariance characterizations under more general structural conditions on F (e.g., dissipativity and one-sided Lipschitzness), see [8], [9], [12].

On the other hand, Theorem 2 does not make any structural assumptions on the dynamics, and allows general set-valued disturbances, as in (1). In particular, our feedback realizability hypothesis (U) can be satisfied for non-Lipschitz dynamics that are not tractable by the well known strong invariance results. For instance, see the examples in section II.

IV. Sketch of Proof of Theorem 2

This section is devoted to a sketch of the proof of Theorem 2. For a complete proof, see [11].

Fix $T > 0$ and $\bar{x} \in \mathcal{S}$. We first develop some properties that hold for all $f \in \mathcal{C}_F([0, T], \bar{x})$. Fixing $f \in \mathcal{C}_F([0, T], \bar{x})$ and $\varepsilon > 0$, and fixing $\gamma > 0$ such that $f(t, x) \in \text{cone}\{F(x)\}$ for all $x \in \gamma \mathcal{B}_n(\bar{x})$ and almost all $t \in [0, T]$, set

$$G_f^\varepsilon[t, x, k] \;=\; \overline{co}\left\{ f_\varepsilon(t, y) : \|y - x\| \leq \frac{1}{k} \right\} \;\subseteq\; \mathbb{R}^n \quad (5)$$

for each $t \in [0, T]$, $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, where f_ε is the regularization (4) of f and $\overline{co}$ denotes the closed convex hull. By reducing $\gamma > 0$, we can assume that $\gamma \mathcal{B}_n(\bar{x}) \subseteq \mathcal{U}$. We also set $g_f^\varepsilon[t, x, k] = 1 + \sup\{\|p\| : p \in G_f^\varepsilon[t, x, k]\}$ for all $t \in [0, T]$, $x \in \mathbb{R}^n$, and $k \in \mathbb{N}$. Note that

$$g_f^\varepsilon[t, x, k] \leq g_f[t, x, k] := 1 + c_1 + c_2 \left(||x|| + \frac{1}{k} \right) \quad (6)$$

for all $t \in [0, T]$, $x \in \mathbb{R}^n$, and $k \in \mathbb{N}$, where c_1 and c_2 are the constants from the linear growth requirement (C_3) on f, so the sets $G_f^\varepsilon[t, x, k]$ are compact. The following estimate is based on Theorem 1 from section II:

Lemma 4.1: If $x \in \frac{\gamma}{2}\mathcal{B}_n(\bar{x})$, $t \geq 0$, $k \in \mathbb{N}$, $h > 0$, and if

$$h \leq \frac{1}{2k\, g_f[t, x, k]} \quad \text{and} \quad x + h g_f[t, x, k]\mathcal{B}_n \subseteq \frac{2\gamma}{3}\mathcal{B}_n(\bar{x}), \quad (7)$$

then

$$\Psi(x + hv) \leq \Psi(x) + \frac{h}{k} \quad (8)$$

holds for some $v \in G_f^\varepsilon[t, x, k]$.

Proof: Suppose the contrary. Since Ψ is lower semicontinuous, there must then exist $x \in \frac{\gamma}{2}\mathcal{B}_n(\bar{x})$, $t \geq 0$, $k \in \mathbb{N}$, and $h > 0$ satisfying (7) but such that

$$\delta := \frac{h}{k} < \min_{y \in Y} \Psi(y) - \Psi(x), \quad Y := x + h G_f^\varepsilon[t, x, k]. \quad (9)$$

Let $\lambda \in (0, \frac{1}{2k})$ be such that

$$x + h g_f[t, x, k]\mathcal{B}_n + \lambda \mathcal{B}_n \subseteq \gamma \mathcal{B}_n(\bar{x}). \quad (10)$$

Next we apply Theorem 1 with the choices Y and δ defined by (9). It follows that there exist $z \in [x, Y] + \lambda \mathcal{B}_n$ and $\zeta \in \partial_P \Psi(z)$ for which

$$\delta < \min_{y \in Y} \langle \zeta, y - x \rangle = \min_{v \in G_f^\varepsilon[t, x, k]} \langle \zeta, hv \rangle. \quad (11)$$

Note that $z \in \gamma \mathcal{B}_n(\bar{x}) \subseteq \mathcal{U}$, by (10). Since we have $z \in [x, Y] + \lambda \mathcal{B}_n$, (7) combined with the choice of λ gives $\|z - x\| \leq h g_f[t, x, k] + \lambda \leq \frac{1}{k}$. Therefore $f_\varepsilon(t, z) \in G_f^\varepsilon[t, x, k]$, by the definition (5) of $G_f^\varepsilon[t, x, k]$. Since $f(s, z) \in \text{cone}\{F(z)\}$ for a.a. $s \in [0, T]$ (by our choice of $\gamma > 0$), our Hamiltonian hypothesis gives $\langle \zeta, f(s,z)\rangle \le 0$ for almost all $s \in [0,T]$. Therefore, (11) gives

$$\delta \;\le\; h\langle \zeta, f_\varepsilon(t,z)\rangle \;=\; h\int_{\mathbb{R}} \eta_\varepsilon(t-s)\langle \zeta, f(s,z)\rangle ds \;\le\; 0.$$

This contradiction concludes the proof of Claim 4.1. ■

Now set $D := \frac{\gamma}{2}\mathcal{B}_n(\bar{x}) \subseteq \mathcal{U}$. Let $\omega_{f,K}$ be a modulus of continuity for $x \mapsto f(t,x)$ on $K := D + \mathcal{B}_n$ for all $t \in [0,T]$. Such a modulus exists by condition (C_2). Then $\omega_{f,K}$ is also a modulus of continuity of $K \ni x \mapsto f_\varepsilon(t,x)$ for all $t \in [0,T]$ and $\varepsilon > 0$. The following estimate follows from Carathéodory's Lemma (cf. [11] for its proof):

Lemma 4.2: Let $(t,x,k) \in [0,T]\times D\times\mathbb{N}$ and assume that $v \in G^\varepsilon_f[t,x,k]$. Then $\|v - f_\varepsilon(t,x)\| \le \omega_{f,K}(1/k) + 1/k$.

Next define $\delta(D) := 1 + c_1 + c_2 + c_2\max\{\|v\| : v \in D\}$. It follows from the estimate (6) that

$$G^\varepsilon_f[t,x,k] \subseteq \delta(D)\mathcal{B}_n \;\; \forall\, t \in [0,T], \;\; x \in D, \;\; k \in \mathbb{N}. \quad (12)$$

Next set

$$\tilde{T} := \min\left\{T, \frac{\gamma}{8\delta(D)}\right\} \quad \text{and} \quad h_k := \frac{\gamma}{4k\delta(D)} \qquad (13)$$

for all $k \in \mathbb{N}$. Choose $N > 2$ such that

$$D + h_k\delta(D)\mathcal{B}_n \;\;\subseteq\;\; \frac{2\gamma}{3}\mathcal{B}_n(\bar{x}) \;\;\; \forall k \ge N. \qquad (14)$$

We can assume $\gamma < 1$. By the choices of γ and $\delta(D)$,

$$0 \;<\; h_k \;\le\; \frac{1}{2kg_f[t,x,k]} \;\;\; \forall t \in [0,T], x \in D, k \in \mathbb{N}. \quad (15)$$

Next we define $c(k) = \text{Ceiling}(\tilde{T}/h_k)$, i.e., $c(k)$ is the smallest integer $\ge \tilde{T}/h_k$. For each $k \ge N$, we then define a partition $\pi(k) : 0 = t_{0,k} < t_{1,k} < \ldots < t_{c(k),k} = \tilde{T}$ by setting $t_{i,k} = t_{i-1,k} + h_k$ for $i = 1,2,\ldots,c(k)-1$.

We next define sequences $x_{0,k}, x_{1,k}, x_{2,k}, \ldots, x_{c(k),k}$ for $k \ge N$. We set $x_{0,k} = \bar{x}$ and $x_{1,k} = \bar{x} + (t_{1,k} - t_{0,k})v_{o,k}$, where $v = v_{o,k} \in G^\varepsilon_f[0,\bar{x},k]$ satisfies the requirement from Claim 4.1 for the pair $(t_{0,k}, x_{0,k}) = (0,\bar{x})$ and $h = h_k$. By (12), we get

$$\|x_{1,k} - \bar{x}\| \;\le\; h_k\delta(D) \;=\; \frac{\gamma}{4k}, \qquad (16)$$

so $x_{1,k} \in D$. If $c(k) \ge 2$, then we set

$$x_{2,k} = x_{1,k} + (t_{2,k} - t_{1,k})v_{1,k},$$

where $v_{1,k} \in G^\varepsilon_f[t_{1,k}, x_{1,k}, k]$ satisfies the requirement from Claim 4.1 for the pair $(t_{1,k}, x_{1,k})$. Reapplying (12) gives $\|x_{2,k} - x_{1,k}\| \le h_k\delta(D) = \frac{\gamma}{4k}$, so $\|x_{2,k} - \bar{x}\| \le \frac{\gamma}{2k}$, by (16). Therefore $x_{2,k} \in D$. We now repeat this process except with $x_{2,k} \in D$ instead of $x_{1,k}$. Proceeding inductively gives sequences $v_{i,k} \in G^\varepsilon_f[t_{i,k}, x_{i,k}, k]$ and sequences $\{x_{i,k}\}$ that satisfy $x_{i+1,k} = x_{i,k} + (t_{i+1,k} - t_{i,k})v_{i,k}$ for each index $i = 0,1,\ldots,c(k)-1$. The choices of $\tilde{T}$ and $k \ge 2$ give

$$\|x_{i,k} - \bar{x}\| \le \frac{c(k)\gamma}{4k} \le \frac{\gamma}{2}$$

for all i and k. It follows that the sequences $\{x_{i,k}\}$ lie in D.

For each $k \ge N$, we then choose $x_{\pi(k)}$ to be the unique polygonal arc satisfying $x_{\pi(k)}(0) = \bar{x}$ and

$$\dot{x}_{\pi(k)}(t) = f_\varepsilon(\tau_k(t), x_{\pi(k)}(t) + r_k(t)) + z_k(\tau_k(t)) \qquad (17)$$

for all $t \in [0,\tilde{T}] \setminus \pi(k)$, where $\tau_k(t)$ is the partition point $t_{i,k} \in \pi(k)$ immediately preceding t for each $t \in [0,\tilde{T}]$,

$$z_k(t_{i,k}) := v_{i,k} - f_\varepsilon(t_{i,k}, x_{\pi(k)}(t_{i,k})) \;\; \forall i,k \qquad (18)$$

the $v_{i,k} \in G^\varepsilon_f[t_{i,k}, x_{\pi(k)}(t_{i,k}), k]$ satisfy the conclusions from Claim 4.1 for the pairs $(t,x) = (t_{i,k}, x_{i,k})$ and $h = h_k$, and

$$r_k(t) := x_{\pi(k)}(\tau_k(t)) - x_{\pi(k)}(t) \quad \forall t \in [0,\tilde{T}], \forall k.$$

Then $x_{\pi(k)}$ is the polygonal arc connecting the points $x_{i,k}$ for $i = 0,1,2,\ldots,c(k)$. In particular, $x_{i,k} \equiv x_{\pi(k)}(t_{i,k})$.

Since f_ε is continuous, one can check that (17) satisfies the requirements from our compactness of trajectories lemma, so we can find a subsequence of $\{x_{\pi(k)}(\cdot)\}$ that converges uniformly to a trajectory y_ε of $\dot{y} = f_\varepsilon(t,y)$, $y(0) = \bar{x}$. By possibly passing to a subsequence without relabelling, we can assume that $x_{\pi(k)} \to y_\varepsilon$ uniformly on $[0,\tilde{T}]$. Moreover, since $x_{i+1,k} = x_{i,k} + (t_{i+1,k} - t_{i,k})v_{i,k} \in D$ for all $i = 0,1,\ldots,c(k)-1$ and $k \ge N$, conditions (14) and (15) along with Claim 4.1 give $\Psi(x_{i,k}) - \Psi(x_{i-1,k}) \le \frac{h_k}{k}$ for $i = 1,2,\ldots,c(k)$. Summing these inequalities over i and recalling that $h_k \le \gamma$ gives $\Psi(x_{i,k}) \le \Psi(\bar{x}) + \frac{1}{k}(\tilde{T}+\gamma)$ for all i and k . Hence,

$$\Psi(x_{\pi(k)}(\tau_k(t))) \le \Psi(\bar{x}) + \frac{1}{k}(\tilde{T}+\gamma) \qquad (19)$$

for all $t \in [0,\tilde{T}]$. Since $|\tau_k(t) - t| \le h_k \to 0$ as $k \to +\infty$ for all $t \in [0,\tilde{T}]$, it follows that $x_{\pi(k)}(\tau_k(t)) \to y_\varepsilon(t)$ for all $t \in [0,\tilde{T}]$ as $k \to +\infty$. Since Ψ is lower semicontinuous, it follows from (19) that $\Psi(y_\varepsilon(t)) \le \Psi(\bar{x})$ for all $t \in [0,\tilde{T}]$.

Now let $y_{1/i} : [0,\tilde{T}] \to \mathbb{R}^n$ be the trajectory obtained by the preceding argument with the choice $\varepsilon = 1/i$ for each $i \in \mathbb{N}$. Note that $y_{1/i}(t) \in D$ for all i and t, because each of the polygonal arcs $x_{\pi(k)}$ constructed above joins points in D and D is closed and convex. Moreover,

$$\begin{array}{rcl}\dot{y}_{1/i}(t) &=& f(t, y_{1/i}(t)) \\ &+& \left[f_{1/i}(t, y_{1/i}(t)) - f(t, y_{1/i}(t))\right]\end{array} \qquad (20)$$

for all i and almost all $t \in [0,\tilde{T}]$. One can check (cf. [11]) that the $y_{1/i}$ are uniformly bounded and equicontinuous, so we can assume (possibly by passing to a subsequence) that there is a continuous function $y : [0,\tilde{T}] \to D$ such that $y_{1/i} \to y$ uniformly on $[0,\tilde{T}]$ (by the Ascoli-Arzelà lemma). We show that y is a trajectory of f, using the following lemma (cf. [11] for its proof):

Lemma 4.3: $f_{1/i}(t, y_{1/i}(t)) - f(t, y_{1/i}(t)) \to 0$ in L^2 as $i \to \infty$.

It therefore follows from Remark 2.5 and the form of the dynamics (20) that a subsequence of $\{y_{1/i}\}$ converges to a trajectory of f uniformly on $[0,\tilde{T}]$. This must be the aforementioned function y, as desired. Again using the lower semicontinuity of Ψ, we can therefore conclude that $\Psi(y(t)) \le \liminf_{i\to\infty}\Psi(y_{1/i}(t)) \le \Psi(\bar{x})$ for all $t \in [0,\tilde{T}]$.

Finally, we show the strong invariance asserted in the theorem. Let $x_o \in \mathcal{S}$, $T \ge 0$, and $\phi \in \text{Traj}_T(F, x_o)$ be given. Set

$$\bar{t} := \sup\left\{t \ge 0 : \Psi(\phi(s)) \le \Psi(x_o) \; \forall s \in [0,t]\right\}. \qquad (21)$$

We next show that $\bar{t} = T$ (by contradiction), which would imply that ϕ remains in $\mathcal{S}$ on $[0, T]$. To this end, note that the lower semicontinuity of Ψ gives

$$\Psi(\phi(\bar{t})) \leq \Psi(x_o). \tag{22}$$

In particular, this implies that $\bar{x} := \phi(\bar{t}) \in \mathcal{S} \subseteq \mathcal{U}$. Next let $f \in \mathcal{C}_F([0,T], \bar{x})$ satisfy the requirement (U) for F and the trajectory $[0, T - \bar{t}] \ni t \mapsto y(t) := \phi(t + \bar{t})$, and $\gamma \in (0,1)$ satisfy $f(t,x) \in \mathrm{cone}\{F(x)\}$ for a.a. $t \in [0, T - \bar{t}]$ and all $x \in \gamma\mathcal{B}_n(\bar{x})$. We can assume $\gamma\mathcal{B}_n(\bar{x}) \subseteq \mathcal{U}$.

By uniqueness of solutions of the initial value problem $\dot{y} = f(t,y)$, $y(0) = \phi(\bar{t})$ on $[0, T - \bar{t}]$, the above argument applied to f and $\bar{x} = \phi(\bar{t})$ gives $\tilde{t} \in (0, T - \bar{t})$ such that

$$\Psi(\phi(\bar{t} + t)) - \Psi(\phi(\bar{t})) \leq 0 \quad \forall t \in [0, \tilde{t}]. \tag{23}$$

Here we use the fact that the trajectory on $[0, \tilde{T}]$ constructed above for f starting at $\bar{x}$ can be extended to $[0, T - \bar{t}]$, by the linear growth assumption (C_3), and so coincides with y by our uniqueness assumption in (U). Since ϕ remains in $\mathcal{S}$ on $[0, \bar{t}]$, summing (22) and (23) then contradicts the definition (21) of the supremum $\bar{t}$. This proves the theorem.

V. Strong Invariance Characterization

As we saw in Example 2.3, the Hamiltonian condition that $H_F(x, N_S^P(x)) \leq 0$ for all $x \in \mathrm{boundary}(S)$ is not necessary for strong invariance for (F, S); there, $(F, \{0\})$ is strongly invariant in $\mathbb{R}^n$, but the Hamiltonian condition is not satisfied, and F is upper semicontinuous but not lower semicontinuous. On the other hand, if we strengthen our assumption on F to

$(U^\sharp)$ Condition (U) holds; and F is lower semicontinuous, and closed, convex, and nonempty valued.

then we get the following strong invariance characterization:

Theorem 3: Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy $(U^\sharp)$ and $S \subseteq \mathbb{R}^n$ be closed. Then (F, S) is strongly invariant in $\mathbb{R}^n$ if and only if $H_F(x, N_S^P(x)) \leq 0$ for all $x \in \mathrm{boundary}\,(S)$.

Proof: We showed the sufficiency of the Hamiltonian condition for strong invariance in Theorem 2, so it remains to show the necessity. We do this by extending an argument from the appendix of [1] to non-Lipschitz F. Assume (F, S) is strongly invariant. Fix $x \in \mathrm{boundary}(S)$, $v \in F(x)$, and $\zeta \in N_S^P(x)$. Using Michael's Selection Theorem (cf. [3, p. 219]), we can find a continuous selection $s : \mathbb{R}^n \to \mathbb{R}^n$ of F for which $s(x) = v$. Choose $\sigma > 0$ satisfying the condition (2) for ζ to be in $N_S^P(x)$.

We can now use the local existence property to find $\bar{t} > 0$ and a trajectory $\phi : [0, \bar{t}] \to \mathbb{R}^n$ of the continuous dynamics $y \mapsto s(y)$ starting at x, so $\dot{\phi}(0) = s(\phi(0)) = s(x) = v$. Since $\phi \in \mathrm{Traj}_{\bar{t}}(F, x)$ and (F, S) is strongly invariant in $\mathbb{R}^n$, it follows that $\phi(t) \in S$ for all $t \in [0, \bar{t}]$. Condition (2) then gives $\langle \zeta, \phi(t) - x \rangle \leq \sigma ||\phi(t) - x||^2$ for all $t \in [0, \bar{t}]$. Dividing this inequality by $t \in (0, \bar{t}]$, and letting $t \to 0$ gives

$$\langle \zeta, v \rangle \leq \sigma \lim_{t \to 0} t||(\phi(t) - x)/t||^2 = 0.$$

Taking the supremum over all $v \in F(x)$ and noting that $x \in \mathrm{boundary}(S)$ was arbitrary gives the desired result. ■

Theorem 3 is no longer true if the requirement that F be lower semicontinuous is dropped, as shown by Example 2.3.

VI. Acknowledgements

This work was funded in part by the US National Academy of Sciences under the Collaboration in Basic Science and Engineering (COBASE) Program, supported by Contract No. INT-0002341 from the US National Science Foundation. The first author (Krastanov) was also supported by Swiss NSF Contract 7 IP 65642 (Program SCOPES) and by the Bulgarian Ministry of Science and Higher Education - National Fund for Science Research under contracts MM-807/98 and MM-1104/01. The second author (Malisoff) was also supported by Louisiana Board of Regents Grant LEQSF(2003-06)-RD-A-12, and he thanks Eduardo Sontag for commenting on an earlier version of this work.

VII. References

[1] D. Angeli and E. Sontag, *Monotone Control Systems*. IEEE Trans. Autom. Control **48** (2003), 1684–1698.

[2] M. Bardi and I. Capuzzo–Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations*. Birkhäuser, Boston, 1997.

[3] J. Borwein and A. Lewis, *Convex Analysis and Nonlinear Optimization. Theory and Examples.* Springer-Verlag, New York, 2000.

[4] F. Clarke, *Generalized gradients and applications*. Trans. Amer. Math. Soc. **205** (1975), 247–262.

[5] F. Clarke, Y. Ledyaev and M. Radulescu, *Approximate invariance and differential inclusions in Hilbert spaces*. J. Dynam. Control Syst. **3** (1997), 493–518.

[6] F. Clarke, Y. Ledyaev, R. Stern and P. Wolenski, *Nonsmooth Analysis and Control Theory*. Springer-Verlag, New York, 1998.

[7] F. Clarke and R. Stern, *State constrained feedback stabilization*. SIAM J. Control Optim. **42** (2003), 422–441.

[8] T. Donchev, *Properties of the reachable set of control systems*. Systems and Control Letters **46** (2002), 379–386.

[9] T. Donchev, V. Rios and P. Wolenski, *A characterization of strong invariance for perturbed dissipative systems*. Optimal Control, Stabilization, and Nonsmooth Analysis (M. de Queiroz, M. Malisoff and P. Wolenski, Eds.), Lecture Notes in Control and Information Sciences, Springer-Verlag, Heidelberg, to appear.

[10] L. Evans, *Partial Differential Equations.* American Mathematical Society, Providence, RI, 1998.

[11] M. Krastanov, M. Malisoff and P. Wolenski, *On the strong invariance property for non-Lipschitz dynamics.* LSU Mathematics Electronic Preprint Series,[0] and submitted.

[12] V. Rios and P. Wolenski, *A characterization of strongly invariant systems for a class of non-Lipschitz multifunctions*. Proceedings of the 42nd IEEE Conference on Decision and Control (Maui, HI, December 2003), 2593-2594.

[13] H. Sussmann, *Uniqueness results for the value function via direct trajectory-construction methods*. Proceedings of the 42nd IEEE Conference on Decision and Control (Maui, HI, December 2003), 3293-3298.

[14] R. Vinter, *Optimal Control*. Birkhäuser, Boston, 2000.

[15] P. Wolenski and Y. Zhuang, *Proximal analysis and the minimal time function*. SIAM J. Control Optim. **36** (1998), 1048–1072.

[16] J. Ye, *Discontinuous solutions of the Hamilton-Jacobi equation for exit time problems*. SIAM J. Control Optim. **38** (2000), 1067–1085.

[0] Available at `http://www.math.lsu.edu/~preprint/`