The peaking phenomenon and singular perturbations : an extension of Tikhonov's theorem *

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We study the asymptotic behaviour, when the parameter γ tends to infinity, of a class of singularly perturbed triangular systems $\dot{x} = f(x, y)$, $\dot{y} = G(y, \gamma)$. We assume that all solutions of the second equation tend to zero arbitrarily fast when γ tends to infinity. We assume that the origin of equation $\dot{x} = f(x, 0)$ is globally asymptotically stable. Some states of the second equation may peak to very large values, before they rapidly decay to zero. Such peaking states can destabilize the first equation. The paper introduces the concept of *instantaneous stability*, to measure the fast decay to zero of the solutions of the second equation, and the concept of *uniform infinitesimal boundedness* to measure the effects of peaking on the first equation. Whe show that all the solutions of the triangular system tend to zero when γ and t tend to infinity. Our results are a generalization of the classical Tikhonov's theorem of singular perturbation theory, concerning the asymptotic behaviour of the solutions in the particular case where the second equation is of the form $\dot{y} = \gamma G(y)$. Our results are formulated in both classical mathematics and nonstandard analysis.

Key Words: peaking phenomenon, singular perturbations, nonstandard analysis

* Supported by the GdR CNRS Automatique.

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1. INTRODUCTION

In this paper, we are concerned with the asymptotic behavior, when the parameter γ tends to infinity, of nonlinear triangular systems

$$\dot{x} = f(x, y), \qquad \dot{y} = G(y, \gamma), \tag{1}$$

where $\dot{} = d/dt$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\gamma \in \mathbb{R}$ and the mappings

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \quad \text{and} \quad G : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$$

are of class $\mathcal{C}^1$. Notice that the limit of G is not assumed to be defined when $\gamma \to +\infty$, so that system (1) is a singular perturbation. We can think of the first equation in system (1) as a controlled system taking its inputs from the second equation in system (1). The *zero-input system* is the system

$$\dot{x} = f(x, 0). \tag{2}$$

We assume that

H1 : system (2) has 0 as a globally asymptotically stable (GAS) equilibrium.

The condition H1 implies that the solutions of (2) tend to 0 as $t \to +\infty$. Our aim is to state conditions on f and G such that the solutions of system (1) tend to zero when $\gamma \to +\infty$ and $t \to +\infty$. If we require that

H2 : the solutions of $\dot{y} = G(y, \gamma)$ tend to 0 arbitrarily fast in t when $\gamma \to +\infty$,

then, the idea that the solutions of system (1) ought to tend to zero appears plausible for the following heuristics. Since H2 holds, the second equation in system (1) drives any initial condition very fast in t near the manifold $y = 0$, where the zero-input system takes over and drives x to zero. Due to the peaking phenomenon, this idea fails. Of course, some solutions of the second equation in system (1) may peak to very large values before they decay to zero. The interaction of this peaking with the nonlinear growth in the first equation in system (1) could destabilize system (1). In general the origin of system (1) is not GAS, even if there is no peaking. The best result one can expect for system (1) is that its origin *seems to be GAS* when $\gamma \to +\infty$ (see Definition 5.1).

Intuitively, assumption H2 means that the "convergence to 0 of the solutions of the second equation in system (1) is increasingly fast as $\gamma \to +\infty$". Our first objective is to give a precise meaning to this sentence. For this purpose we shall use *Internal Set Theory* (IST) which is an axiomatic approach, given by Nelson [15] of the *Nonstandard Analysis* (NSA) of Abraham Robinson [16]. The idea of using NSA in perturbation theory of differential equations goes back to the seventies with the Reebian school (cf. [13, 14] and the references therein). It gave birth to the *nonstandard perturbation theory of differential equations* which has become today a well-established tool in asymptotic theory (see the special five-digits classification 34E18 of the 2000 Mathematics Subject Classification). To have an idea of the rich literature on the subject the reader is referred to [1, 3, 5, 10, 11, 19].

In NSA, instead of considering a family of systems (1) depending on the parameter γ and dealing with its asymptotic propreties when $\gamma \to +\infty$, we consider just one (nonstandard) system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(y), \tag{3}$$

where $g(y)$ plays the role of $G(y, \gamma)$ with γ a *fixed positive unlimited real number* and we look to the (external) properties of system (3).

In this paper, we introduce the concepts of *instantaneous stability*, and ω-*instantaneous stability* which quantify the fast decay to zero of the solutions of the second equation in system (1) or (3) as well as the concepts of *uniform infinitesimal boundedness*, and ω-*uniform infinitesimal boundedness* which measure the effects of the solutions of the second equation in system (1) or (3) on the first equation in this system. All these concepts are defined in both standard and nonstandard terms. Also, we introduce the concept of α-*boundedness* which measure the magnitude of the peaking solutions of system (3) as well as the concept of *steadiness* which measure the effects of a peaking control on a system. The notion of *instantaneous stabilizabilty* was previously defined (see [2]) for linear systems within the IST framework. Our concept of instantaneous stability for a general system is a straightforward extension of their definition.

The paper is organized as follows. In Section 2 we recall Tikhonov's theorem that describes the case where G is of the form $G(y, \gamma) = \gamma G(y)$. In Section 3 we consider the linear case, where G is of the form $G(y, \gamma) = G(\gamma)y$ and we explain why the peaking phenomenon can destabilize the first equation. In Section 4, we give our two main (nonstandard) results (Theorems 4.2 and 4.4) on the global asymptotic stability of the origin of system (3). In Section 5, we provide the standard formulations (Theorems 5.1 and 5.2) of the results of the previous section. Typical examples of applications of these results are given in Section 6. In Section 7, we give the definition of a *peaking of order* α for a general nonlinear system and investigate the particular case of linear systems in the Luenberger-Brunowsky canonical form. In Section 8, we motivate the case where the mapping g in (3) depends also on x. This case arises in control problems where the feedback is computed, not on the state, but on an estimation of the state given by an observer. In Appendix A, we give a short description of IST and some concepts of stability revisited in NSA . In Appendix B, we give some useful results of nonstandard asymptotic theory. In Appendix C, we prove the equivalence between the standard and the nonstandard definitions of the concepts introduced in Sections 4 and 5. In Appendix D, we give the proofs of Theorems 2.3, 5.1 and 5.2.

Theorems 5.1 and 5.2 are formulated in classical mathematics, but their proofs need IST. We don't give classical proofs because IST is a conservative extension of ordinary mathematics, that is, every classical statement which is proved in IST is a theorem of ordinary mathematics. We prefer the IST language in this kind of study because the nonstandard statements are more appealing for us in view of applications. Nonetheless, it's a matter of taste.

A *control* u is a function $t \to u(t)$ defined for $0 \leq t < \infty$ which takes values on $\mathbb{R}^m$ and is bounded and mesurable. A *control system* f is a system

$$\dot{x} = f(x,u), \qquad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \tag{4}$$

such that the map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class $\mathcal{C}^1$. If u is a control, we use $x^f(t, t_0, x_0, u(\cdot))$ to denote the solution of $\dot{x} = f(x, u(t))$ with initial condition $x(t_0) = x_0$. When $t_0 = 0$, we denote the solution simply by $x^f(t, x_0, u(\cdot))$. When there is no risk of confusion, f, t_0, x_0 or(and) u could be omitted. Let $(x(t), y(t))$ be the solution of system (1) with initial condition $x(0) = x_0$, $y(0) = y_0$. According to our notations, we have

$$x(t) = x^{(f,G)}(t, (x_0, y_0), \gamma) = x^f(t, x_0, y(\cdot)) \quad \text{and} \quad y(t) = y^G(t, y_0, \gamma).$$

Throughout the paper, we use the following abbreviations : 'GAS' is read 'Global Asymptotic Stability' (or 'Globally Asymptotically Stable'), 'IS' is read 'Instantaneous Stability' (or 'Instantaneously Stable'), 'UIB' is read 'Uniform Infinitesimal Boundedness' (or 'Uniformly Infinitesimally Bounded'), 'ω-IS' is read 'ω-Instantaneous Stability' (or 'ω-Instantaneously Stable'), 'ω-UIB' is read 'ω-Uniform Infinitesimal Boundedness' (or 'ω-Uniformly Infinitesimally Bounded').

2. TIKHONOV'S THEOREM

Let us consider the particular case where system (1) is of the form

$$\dot{x} = f(x,y), \qquad \dot{y} = \gamma G(y), \tag{5}$$

where $G : \mathbb{R}^m \to \mathbb{R}^m$ is of class $\mathcal{C}^1$. Condition H2 becomes :

H3 : the vector field $y \to G(y)$ has 0 as a GAS equilibrium.

System (5) is very well understood in singular perturbation theory. The main tool is Tikhonov's theorem (cf. [8, 9, 10]) on the infinite interval of time.

THEOREM 2.1. (Tikhonov) *Assume that f and G are standard, γ is unlimited positive and H3 holds. Then for all limited x_0 and y_0, the x-component $x(t)$ of the solution of system (5) with initial condition (x_0, y_0) satisfies $x(t) \simeq x_0(t)$ as long as t and $x_0(t)$ are limited, where $x_0(t)$ is the solution of the zero-input system (2), with initial condition x_0. Moreover, if H1 also holds then $x(t) \simeq x_0(t)$ for all $t \geq 0$.*

Proof. Define $\xi(\tau) = x(\tau/\gamma)$ and $\eta(\tau) = y(\tau/\gamma)$, where $(x(t), y(t))$ is the solution of (5) with initial condition (x_0, y_0). System (5) becomes

$$\xi' = f(\xi, \eta)/\gamma, \qquad \eta' = G(\eta),$$

where $' = d/d\tau$. Since f is limited for all (ξ, η) limited and γ is unlimited positive, this system is a regular perturbation of system

$$\xi' = 0, \qquad \eta' = G(\eta).$$

According to Lemma B.1, for all limited initial condition (x_0, y_0) we have $\xi(\tau) \simeq x_0$ as long as τ is limited. By Lemma A.2, there exists an unlimited s such that $\xi(\tau) \simeq x_0$ for all $\tau \in [0, s]$. Since γ is unlimited, we can chose s such that $t^* = s/\gamma$ is infinitesimal. By H3 (and Proposition 4.1), we have $\eta(\tau) \simeq 0$ for all $\tau \simeq +\infty$, so that $y(t) = \eta(\gamma t) \simeq 0$ for all $t \geq t^*$. Let $x_0^* = x(t^*)$. By Proposition B.3, we have $x(t) \simeq x^f(t, x_0^*, 0)$ as long as t and $x^f(t, x_0^*, 0)$ are limited. Since $x_0^* \simeq x_0$, by Lemma B.1, we have $x^f(t, x_0^*, 0) \simeq x^f(t, x_0, 0) =: x_0(t)$ as long as t and $x_0(t)$ are limited. This proves the first part of the Theorem. Assume now that assumption H1 holds. Then $x_0(t)$ is limited for all $t \geq 0$ and $x_0(t) \simeq 0$ for all $t \simeq +\infty$. Thus, $x(t) \simeq x_0(t)$ for all limited positive t. By Lemma A.2, there exists t_1 unlimited such that $x(t) \simeq x_0(t)$ for all $t \in [0, t_1]$. Thus, we have $x(t) \simeq x_0(t) \simeq 0$ for all unlimited $t \leq t_1$. By H3, we have $y(t) \simeq 0$ for all $t \geq t_1$. By Proposition B.4, we have $x(t) \simeq 0$ for all $t \geq t_1$. Hence $x(t) \simeq x_0(t)$ for all $t \geq 0$. ∎

THEOREM 2.2. *Assume that f and G are standard, γ is unlimited positive, and H1 and H3 hold. Then for all limited x_0 and y_0, the solution $(x(t), y(t))$ of (5) with initial condition (x_0, y_0) satisfies $x(t) \simeq 0$ and $y(t) \simeq 0$ for all positive unlimited t.*

Proof. Let x_0 and y_0 be limited. Let $(x(t), y(t))$ be the solution of (5) with initial condition (x_0, y_0). By Theorem 2.1, we have $x(t) \simeq x_0(t)$ for all $t \geq 0$. By H1, we have $x_0(t) \simeq 0$ for all $t \simeq +\infty$. By H3, we have $y(\tau/\gamma) \simeq 0$ for all $\tau \simeq +\infty$. Hence $x(t) \simeq 0$ and $y(t) \simeq 0$ for all $t \simeq +\infty$. ▮

THEOREM 2.3. *Assume that H1 and H3 hold. For all (x_0, y_0), the solution $(x(t,\gamma), y(t,\gamma))$ of (5) with initial condition (x_0, y_0) tends to 0 as $t \to +\infty$ and $\gamma \to +\infty$. For every $A > 0$, the convergence is uniform with respect to x_0 and y_0 for $\|x_0\| \leq A$ and $\|y_0\| \leq A$.*

Theorem 2.3 is the standard formulation of Theorem 2.2. Its proof is given in Appendix D.1.

EXAMPLE 2.1. Consider the system

$$\dot{x} = -x + x^2 y, \qquad \dot{y} = -\gamma y. \tag{6}$$

Conditions H1 and H3 are satisfied. Hence, the solutions of (6) tend to 0 as $\gamma \to +\infty$ and $t \to +\infty$. Since $\frac{d}{dt}(xy) = xy(xy - 1 - \gamma)$, we see that the hyperbola $xy = 1 + \gamma$ consists of two trajectories. Thus the origin of (6) is not GAS. Tikhonov's theorem asserts that the x-component of the solution $x(t,\gamma)$ of (6) with initial condition (x_0, y_0) is such that, for all $t \geq 0$

$$\lim_{\gamma \to +\infty} x(t,\gamma) = x_0(t) := x_0 \mathrm{e}^{-t} \tag{7}$$

where $x_0(t)$ is the solution of the zero-input system $\dot{x} = -x$, with initial condition x_0. By explicit solutions, it is easy to show that the basin of attraction of the origin of (6) is the set $\{xy < 1 + \gamma\}$.

3. THE PEAKING PHENOMENON

Let us consider now the particular case where system (1) is of the form

$$\dot{x} = f(x, y), \qquad \dot{y} = G(\gamma) y, \tag{8}$$

where $G(\gamma)$ is a square matrix of order m. Condition H2 becomes :

H4 : the real parts of the eigenvalues of $G(\gamma)$ tend to $-\infty$ when $\gamma \to +\infty$.

System (8) was considered in automatic control literature. In this context the second equation in system (8) is understood as a "high gain" dynamic feedback [18]. More precisely, we start with a state feedback-partially linear system of the form

$$\dot{x} = f(x, y), \qquad \dot{y} = Ay + Bu,$$

where the pair (A, B) is controllable. The second equation can be easily stabilized by $u = Ky$, where K is designed such that the matrix $G := A + BK$ is Hurwitz. Asymptotic stability of the origin of the full closed-loop system

$$\dot{x} = f(x, y) \qquad \dot{y} = Gy \tag{9}$$

will now folow from assumption H1 and the concept of *input to state stability* (see [9], p. 548). What about global stabilization ? If the linear feedback control $u = Ky$ is designed to assign the eigenvalues of G far to the left in the complex plane so that the solutions of $\dot{y} = Gy$ decay to zero arbitrarily fast, one might think that the origin of the triangular system (9) can be GAS. It turns out that such a strategy may fail and (8) is not reductible to Tikhonov's theorem because of the so called "peaking phenomenon". The following example of Sussmann and Kokotovic (cf. [18] Example 1.1) shows a system of type (8) such that the solution $y(t, \gamma)$ becomes unbounded when $\gamma \to +\infty$, even if its decay to zero is arbitrary fast. In that case, the origin of (8) is not GAS. Even worse, for some initial conditions, the solution escapes to infinity in finite time.

Example 3.1. Consider the three dimensional system

$$\dot{x} = (1 + y_2)\varphi(x), \qquad \dot{y} = G(\gamma)y, \tag{10}$$

where $\varphi(x) = -x^3/2$ and

$$G(\gamma) = \begin{pmatrix} 0 & 1 \\ -\gamma^2 & -2\gamma \end{pmatrix}.$$

Condition H1 is satisfied. Since both eigenvalues of $G(\gamma)$ are equal to $-\gamma$, condition H4 is also satisfied. The exponential matrix

$$\mathrm{e}^{tG(\gamma)} = \begin{pmatrix} 1 + \gamma t & t \\ -\gamma^2 t & 1 - \gamma t \end{pmatrix} \mathrm{e}^{-\gamma t} \tag{11}$$

shows that as $\gamma \to +\infty$, the solution $y(t, \gamma) = \mathrm{e}^{tG(\gamma)} y_0$ will decay to zero arbitrarily fast. The component $-\gamma^2 t \mathrm{e}^{-\gamma t}$ of (11) reaches the value $-\gamma/\mathrm{e}$ at $t = 1/\gamma$. Then some solutions have a transient behaviour with a peak of order γ before they rapidly decay to 0. This phenomenon is known as the *peaking phenomenon*. The interaction of this peaking with the nonlinear growth in the first equation in system (10) could destabilize the system. Let $y_0 = (y_{10}, y_{20})$, then we have

$$x(t,\gamma) = x_0 \left(1 + x_0^2 \left[t - y_{10} + (y_{10}(\gamma t + 1) + y_{20} t)\mathrm{e}^{-\gamma t}\right]\right)^{-1/2}.$$

If $x_0^2 y_{10} > 1$ the solution will have a finite escape time $t_e > 0$ and t_e tends to zero when $\gamma \to +\infty$. For γ large enough, the solutions are attracted to 0 as soon as $x_0^2 y_{10} < 1$ and for all $t > 0$ we have

$$\lim_{\gamma\to\infty} x(t,\gamma) = x_0(t) := x_0^* \left(1 + x_0^{*2} t\right)^{-1/2}. \tag{12}$$

Here, $x_0(t)$ is the solution of the zero-input system $\dot{x} = -x^3/2$, with initial condition $x_0^* = x_0/\sqrt{1 - x_0^2 y_{10}}$. Since the limit (12) does not hold for $t = 0$, there is a boundary layer at $t = 0$ which quickly drives the state $x(t)$ from x_0 to x_0^*. The peaking phenomenon explains both this boundary layer for the solutions attracted to 0 and the destabilizing effect of the second equation in system (10) on the first equation in this system.

EXAMPLE 3.2. If $\varphi(x) = -x$ in (10) then the origin is GAS in spite of peaking. However, the solutions still have a boundary layer at $t = 0$. Since

$$x(t,\gamma) = x_0 \mathrm{e}^{y_{10} - t - (y_{10}(\gamma t + 1) + y_{20} t)\mathrm{e}^{-\gamma t}}$$

we see that, all the solutions are attracted to 0 and for all $t > 0$ we have

$$\lim_{\gamma\to\infty} x(t,\gamma) = x_0(t) := x_0^* \mathrm{e}^{-t}. \tag{13}$$

Here, $x_0(t)$ is the solution of the zero-input system $\dot{x} = -x$, with initial condition $x_0^* = x_0 \mathrm{e}^{y_{10}}$. The boundary layer at $t = 0$ quickly drives the state $x(t)$ from x_0 to x_0^*.

The limit (7) was obtained from Tikhonov's theorem. The limits (12) and (13) were obtained by direct computation, because Tikhonov's theorem does not apply to Examples 3.1 and 3.2.

4. NONTANDARD RESULTS

4.1. S-Global Asymptotic Stability

The results in Section 2 show that the origin of (5) cannot be GAS in general. The best result one can expect for system (5) is the one given by Theorem 2.2, which asserts that when γ is unlimited, then for every limited initial condition the solution of (5) is infinitesimal, as long as $t \simeq +\infty$. This notion deserves a definition. Consider the (possibly nonstandard) system

$$\dot{z} = F(z) \tag{14}$$

where $z \in \mathbb{R}^p$ and $F : \mathbb{R}^p \to \mathbb{R}^p$.

DEFINITION 4.1. The origin of (14) is S-GAS if for every limited z_0 and for every positive unlimited t, we have $z^F(t, z_0) \simeq 0$.

We can reformulate Theorem 2.2 by stating that the origin of system (5) is S-GAS when $\gamma \simeq +\infty$. In NSA we use the prefix S to avoid confusion between the standard and the nonstandard corresponding notions. The notion of S-GAS corresponds effectively to the usual GAS and we have :

PROPOSITION 4.1. *Assume that F is standard. The origin of (14) is GAS if and only if it is* S-*GAS.*

The proof is given in Appendix A. This property does not hold for a nonstandard system. Indeed, if γ is unlimited, then the origin of system $\dot{z} = z(z/\gamma - 1)$ is S-GAS but not GAS.

4.2. Instantaneous Stability and Uniform Infinitesimal Boundedness

In the previous particular cases (5) and (8) of system (3), the fast attractivity of the origin was guaranteed by the special form of the second equation in the system and by conditions H3 or H4. In the general case, there is no such criteria and we must introduce fast attractivity as a hypothesis. For this purpose, we need the following definitions.

DEFINITION 4.2. A function $t \to u(t)$ is said to be an inpulse if $u(t) \simeq 0$, for all positive non infinitesimal t.

DEFINITION 4.3. The origin of system $\dot{y} = g(y)$ is instantaneously stable (IS) if for every limited y_0, the function $t \to y^g(t, y_0)$ is an impulse.

EXAMPLE 4.1. If γ is unlimited positive, the origin of the second equation in system (10), considered in Example 3.1, is IS.

The effects of any eventual peaking of the solutions of the second equation in system (3) on the first equation in this system must be controlled. Thus, we need a hypothesis on the behaviour of the x-component of the solutions of (3) during the very short time where the peaking can destabilize the system. For this purpose, we need the following definition.

Definition 4.4. The system (3) is uniformly infinitesimally bounded (UIB) if for all limited x_0 and y_0 and for all positive infinitesimal t, the x-component $x(t) = x^{(f,g)}(t,(x_0,y_0))$ of the solution of (3) with initial condition (x_0,y_0) is limited.

We consider the condition below :

H5 : system (3) is UIB and the origin of system $\dot{y} = g(y)$ is IS.

Theorem 4.1. *Assume that f is standard and H5 holds. Then for all limited x_0 and y_0, there exists a limited x_0^* such that the x-component $x(t)$ of the solution of (3) with initial condition (x_0,y_0) satisfies $x(t) \simeq x_0(t)$, as long as $x_0(t)$ is limited and t is appreciable positive, where $x_0(t)$ is the solution of the zero-input system (2), with initial condition x_0^*. Moreover, if H1 also holds then $x(t) \simeq x_0(t)$ for all non infinitesimal positive t.*

Proof. We have $x(t) = x^{(f,g)}(t,(x_0,y_0)) = x^f(t,x_0,y(\cdot))$, where $y(t) = y^g(t,y_0)$. Let x_0 and y_0 be limited. Since the origin of $\dot{y} = g(y)$ is IS, $y(t) \simeq 0$ for every $t \gg 0$. By Lemma A.2, there exists $t^* \simeq 0$ such that $y(t) \simeq 0$ for every $t \geq t^*$. Since (3) is UIB, $x(t)$ is limited for $0 \leq t \leq t^*$. Let $x_0^* = x(t^*)$. By Proposition B.3, for all $t \geq t^*$ we have $x(t) \simeq x^f(t,x_0^*,0) =: x_0(t)$, as long as t and $x_0(t)$ are limited. This proves the first part of the Theorem. Assume now that H1 holds, then $x_0(t)$ is limited for all $t \geq 0$ and $x_0(t) \simeq 0$ for all unlimited positive t. Thus, $x(t) \simeq x_0(t)$ for all appreciable positive t. By Lemma A.2, there exists t_1 unlimited such that $x(t) \simeq x_0(t)$ for all non infinitesimal positive $t \leq t_1$. Thus, we have $x(t) \simeq x_0(t) \simeq 0$ for all unlimited $t \leq t_1$. By IS, we have $y(t) \simeq 0$ for all $t \gg 0$ and hence for all $t \geq t_1$. By Proposition B.4, we have $x(t) \simeq 0$ for all $t \geq t_1$. Hence, $x(t) \simeq x_0(t)$ for all $t \gg 0$. ▮

This result shows that a solution of system (3) starting from a limited point (x_0,y_0) is approximated by a solution of the zero-input system (2) starting from a limited point x_0^*. It is a Tikhonov-like result (compare with Theorem 2.1). In Tikhonov's case we had $x_0^* \simeq x_0$. However in the general case, due to the eventual peaking of the solutions of the second equation in system (3), $x_0^* \not\simeq x_0$ and there is a boundary layer at $t = 0$.

Theorem 4.2. *Assume that f is standard, and that H1 and H5 hold. Then the origin of system (3) is* S-*GAS.*

Proof. Let x_0 and y_0 be limited. Let $(x(t),y(t))$ be the solution of (3) with initial condition (x_0,y_0). By Theorem 4.1, we have $x(t) \simeq x_0(t)$ for all $t \gg 0$. By H1, we have $x_0(t) \simeq 0$ for all $t \simeq +\infty$. By IS, we have $y(t) \simeq 0$

for all $t \gg 0$. Hence, $x(t) \simeq 0$ and $y(t) \simeq 0$ for all $t \simeq +\infty$, that is to say, the origin of (3) is S-GAS. ▌

4.3. ω-Instantaneous Stability and ω-Uniform Infinitesimal Boundedness

In the case of the partially linear system (8), the order of magnitude of the velocity with which the solutions tend to 0 was measured in terms of the spectrum of the matrix $G(\gamma)$, which is no longer possible in the nonlinear case. In this section, we quantify more precisely the behaviour of the solutions of system (3). For this purpose, we need the following definitions. Let ω be an unlimited positive real number.

DEFINITION 4.5. A function $t \to u(t)$ is said to be an ω-inpulse if $u(\tau/\omega) \simeq 0$ for all positive unlimited τ.

DEFINITION 4.6. The origin of system $\dot{y} = g(y)$ is ω-IS if for every limited y_0, the function $t \to y^g(t, y_0)$ is an ω-inpulse.

EXAMPLE 4.2. Let γ be unlimited. The origin of the second equation in system (10) is not γ-IS. Indeed, the components of the exponential matrix (11) are infinitesimal as long as $t = \tau/\gamma$ and $\tau \simeq +\infty$, apart from the component $h(t) = -\gamma^2 t e^{-\gamma t}$, for which we have $h(\ln\gamma/\gamma) = -\ln\gamma$, which is not limited. The origin of the second equation in system (10) is ω-IS with $\omega = \gamma/\ln\gamma$. Indeed, we have $h(\tau/\omega) = \tau\ln\gamma/\gamma^{\tau-1}$, which is infinitesimal for all $\tau \simeq +\infty$.

DEFINITION 4.7. The system (3) is ω-UIB if for all limited x_0 and y_0, there exists a positive unlimited s such that the x-component $x(t) = x^{(f,g)}(t, (x_0, y_0))$ is limited for all $t \in [0, s/\omega]$.

We consider the condition below.

H6 : there exists a positive unlimited real number ω such that the origin of system $\dot{y} = g(y)$ is ω-IS, and system (3) is ω-UIB.

THEOREM 4.3. *The conclusion of Theorem 4.1 holds if assumption H5 is replaced by assumption H6.*

Proof. We have $x(t) = x^{(f,g)}(t, (x_0, y_0)) = x^f(t, x_0, y(\cdot))$ where $y(t) = y^g(t, y_0)$. Let x_0 and y_0 be limited. Since (3) is ω-UIB, there exists $s \simeq +\infty$ such that $x^f(t, x_0, y(\cdot))$ is limited for $0 \leq t \leq s/\omega$. Since ω is unlimited, we can chose s such that $t^* = s/\omega$ is infinitesimal. Since the origin of the second equation in system (3) is ω-IS, we have $y(t) \simeq 0$, for all $t \geq t^*$. Let $x_0^* = x(t^*)$. By Proposition B.3, for all $t \geq t^*$ we have $x(t) \simeq x^f(t, x_0^*, 0) =: x_0(t)$, as long as t and $x_0(t)$ are limited. This proves the first part of the Theorem. Assume now that assumption H1 holds. Then, $x_0(t)$ is limited for all $t \geq 0$ and $x_0(t) \simeq 0$ for all unlimited positive t. Thus, $x(t) \simeq x_0(t)$ for all appreciable positive t. By Lemma A.2, there exists t_1 unlimited such that $x(t) \simeq x_0(t)$ for all non infinitesimal positive $t \leq t_1$. Thus, we have $x(t) \simeq x_0(t) \simeq 0$ for all unlimited $t \leq t_1$. By ω-IS, we have $y(\tau/\omega) \simeq 0$ for all $\tau \simeq +\infty$ and hence, $y(t) \simeq 0$ for all $t \geq t_1$. By Proposition B.4, we have $x(t) \simeq 0$ for all $t \geq t_1$. Hence, $x(t) \simeq x_0(t)$ for all $t \gg 0$. ▮

THEOREM 4.4. *Assume that f is standard and that H1 and H6 hold. Then the origin of (3) is* S*-GAS.*

Proof. Let x_0 and y_0 be limited. Let $(x(t), y(t))$ be the solution of (3) with initial condition (x_0, y_0). By Theorem 4.3, we have $x(t) \simeq x_0(t)$ for all $t \gg 0$. By H1, we have $x_0(t) \simeq 0$ for all $t \simeq +\infty$. By ω-IS, we have $y(\tau/\omega)$ for all $\tau \simeq +\infty$. Hence, $x(t) \simeq 0$ and $y(t) \simeq 0$ for all $t \simeq +\infty$.Thus, the origin of (3) is S-GAS. ▮

5. STANDARD RESULTS

5.1. Seeming Global Asymptotic Stability

It is well known (see [7], Theorem 38.1 or [17], Exercise 2.9, p. 9)) that uniform attractivity with respect to the initial condition implies asymptotic stability. Thus we have :

PROPOSITION 5.1. *The origin of (14) is GAS if and only if it is uniformly attractive with respect to the initial condition on every ball centered in the origin.*

A nonstandard proof of Proposition 5.1 is given in Appendix A.2. Hence, the origin of (14) is GAS if and only if for every $A > 0$, $z^F(t, z_0) \to 0$ as $t \to +\infty$, the convergence being uniform with respect to z_0 for $\|z_0\| \leq A$. Consider now a system depending on a parameter :

$$\dot{z} = F(z, \gamma) \tag{15}$$

where $z \in \mathbb{R}^p$ and $F : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$.

DEFINITION 5.1. The origin of (15) seems to be GAS when $\gamma \to +\infty$ if for every $A > 0$, $z^F(t, z_0, \gamma) \to 0$ as $t \to +\infty$ and $\gamma \to +\infty$, the convergence being uniform with respect to z_0 for $\|z_0\| \leq A$.

We can formulate Theorem 2.3 by saying that the origin of system (5) seems to be GAS when $\gamma \to +\infty$. Notice that a point that seems to be GAS is not GAS. Indeed, the origin of system $\dot{z} = z(z/\gamma - 1)$ seems to be GAS when $\gamma \to +\infty$. However, it is a locally asymptotically stable but not GAS equilibrium. The origin of system $\dot{z} = 1/\gamma - z$ seems to be GAS when $\gamma \to +\infty$. However, it is never an equilibrium. The origin of system $\dot{z} = -z^2(1/\gamma - z)$ seems to be GAS when $\gamma \to +\infty$. However, it is an unstable equilibrium for every γ.

5.2. Instantaneous Stability and Uniform Infinitesimal Boundedness

In Section 4.2 we introduced the nonstandard concepts of IS and UIB for system (3). Let us give their standard formulation for the system (1) depending on the parameter γ.

Definition 5.2. The origin of system $\dot{y} = G(y, \gamma)$ is IS when $\gamma \to +\infty$, if for every $\delta > 0$, $A > 0$ and $t_0 > 0$, there exists $\gamma_0 > 0$ such that whenever an initial condition y_0 satisfies $\|y_0\| \leq A$, it follows that $\|y^G(t, y_0, \gamma)\| < \delta$ for all $t \geq t_0$ and all $\gamma > \gamma_0$.

Definition 5.3. The system (1) is UIB when $\gamma \to +\infty$, if for every $A > 0$, there exist $B > 0$, $t_0 > 0$ and $\gamma_0 > 0$ such that whenever an initial condition (x_0, y_0) satisfies $\|x_0\| \leq A$ and $\|y_0\| \leq A$, it follows that $\|x^{(f,G)}(t, (x_0, y_0), \gamma)\| \leq B$ for all $t \in [0, t_0]$ and all $\gamma > \gamma_0$.

We consider the condition below

H7 : the origin of system $\dot{y} = G(y, \gamma)$ is IS when $\gamma \to +\infty$, and system (1) is UIB when $\gamma \to +\infty$.

Theorem 5.1. *Assume that H1 and H7 hold. Then the origin of system (1) seems to be GAS when* $\gamma \to +\infty$.

This result is a corollary of Theorem 4.2 (see Appendix D).

5.3. ω-Instantaneous Stability and ω-Uniform Infinitesimal Boundedness

In Section 4.3 we introduced the nonstandard concepts of ω-IS and ω-UIB for system (3), where ω is some unlimited positive real number. Let us give their standard formulation for the system (1) depending on the parameter γ. Let $\omega(\gamma)$ be a positive function satisfying $\lim_{\gamma \to +\infty} \omega(\gamma) = +\infty$.

Definition 5.4. The origin of system $\dot{y} = G(y, \gamma)$ is ω-IS when $\gamma \to +\infty$, if for every $\delta > 0$ and $A > 0$, there exist $\gamma_0 > 0$ and $\tau_0 > 0$ such that whenever an initial condition y_0 satisfies $\|y_0\| \leq A$, it follows that $\|y^G(\tau/\omega(\gamma), y_0, \gamma)\| < \delta$ for all $\tau > \tau_0$ and $\gamma > \gamma_0$.

DEFINITION 5.5. The system (1) is ω-UIB when $\gamma \to +\infty$, if for every $A > 0$ and $s > 0$, there exist $B > 0$ and $\gamma_0 > 0$ such that whenever an initial condition (x_0, y_0) satisfies $\|x_0\| \leq A$ and $\|y_0\| \leq A$, it follows that $\|x^{(f,G)}(t,(x_0,y_0),\gamma)\| \leq B$ for all $t \in [0, s/\omega(\gamma)]$ and $\gamma > \gamma_0$.

We consider the condition below.

H8 : there exists a positive function $\omega(\gamma)$ such that $\lim_{\gamma\to+\infty}\omega(\gamma) = +\infty$, (1) is ω-UIB when $\gamma \to +\infty$ and the origin of system $\dot{y} = G(y,\gamma)$ is ω-IS when $\gamma \to +\infty$.

THEOREM 5.2. *Assume that H1 and H8 hold. Then the origin of (1) seems to be GAS when $\gamma \to +\infty$.*

This result is a corollary of Theorem 4.4 (see Appendix D).

6. APPLICATIONS

6.1. Equivalence between H5 and H6

The difference between Theorems 4.2 and 4.4 is that the first result requires assumption H5, that is (IS)+(UIB), while the second result requires assumption H6, that is (ω-IS)+(ω-UIB). Apparently these two assumptions are not equivalent since we have

$$\omega\text{-}IS \Rightarrow IS \quad \text{and} \quad UIB \Rightarrow \omega\text{-}UIB. \tag{16}$$

Notice that in general ω-UIB does not imply UIB. Indeed consider the system

$$\dot{x} = y \qquad \dot{y} = \omega$$

where $\omega \simeq +\infty$. Let $x(t) = x_0 + \omega t^2/2 + y_0 t$ be the solution with limited initial condition (x_0, y_0). Then $x(t)$ is limited for all $t \in [0, s/\omega]$ where $s = \sqrt{\omega}$, that is, the system is ω-UIB. However, $x(1/\omega^{1/4})$ is not limited, that is, the system is not UIB.

We have $IS \Rightarrow \omega\text{-}IS$, for some $\omega \simeq +\infty$, and $H6 \Rightarrow UIB$ as it is shown in the following Lemmas.

LEMMA 6.1. *If the origin of $\dot{y} = g(y)$ is IS then there exists ω unlimited such that it is ω-IS.*

Proof. Since $y(t, y_0) \simeq 0$ for all limited y_0 and all non infinitesimal positive t, the set

$$E = \{l > 0 : \|y(t,y_0)\| \leq 1/l \text{ for all } t \geq 1/l \text{ and } \|y_0\| \leq l\}$$

contains all limited $l > 0$. By the Cauchy Principle (see Appendix A), it contains some unlimited ω. Hence, for all $\|y_0\| \leq \omega$ (and *a fortiori* for all limited x_0), for all $t \geq 1/\omega$ (and *a fortiori* for all $t = \tau/\omega$ with τ unlimited), the solution satisfies $\|y(t, y_0)\| \leq 1/\omega$ (and *a fortiori* $y(t, y_0) \simeq 0$). That is, the origin of (5) is ω-IS. ▮

LEMMA 6.2. *If H6 holds then system (3) is UIB.*

Proof. Let (x_0, y_0) be limited and let $x(t) = x^{(f,g)}(t, (x_0, y_0))$. By the first part of Theorem 4.3, we have that $x(t)$ is limited for all infinitesimal positive t. ▮

From (16) and Lemma 6.1 we deduce that $H5 \Rightarrow H6$. From (16) and Lemma 6.2 we deduce that $H6 \Rightarrow H5$. Hence, $H5 \Leftrightarrow H6$. Let us show how to use both conditions H5 and H6 in the proof of Theorem 2.2. Recall that this theorem was already proven in Section 2 as a corollary of Tikhonov's theorem.

Proof (of Theorem 2.2). The result follows from Theorem 4.2. Since $\gamma \simeq +\infty$ and H3 holds, the origin of the second equation in system (5) is IS. Let us prove that system (1,5) is UIB. Let $(x(t), y(t))$ be the solution of system (5) with initial condition (x_0, y_0). By H3, $y(s)$ is limited for every $s \geq 0$. By Proposition B.1, $x(t)$ is limited for every positive infinitesimal t. Hence, (5) is UIB. The result also follows from Theorem 4.4. Since $\gamma \simeq +\infty$ and H3 holds, the origin of the second equation in system (5) is γ-IS. Let us prove that (1,5) is γ-UIB. Let (x_0, y_0) be limited. Define $\xi(\tau) = x(\tau/\gamma)$ and $\eta(\tau) = y(\tau/\gamma)$, where $(x(t), y(t))$ is the solution of system (5). System (5) becomes

$$\xi' = f(\xi, \eta)/\gamma \qquad \eta' = G(\eta)$$

where $' = d/d\tau$. By Lemma B.1, we have $\xi(\tau) \simeq x_0$ as long as τ is limited. By Lemma A.2, there exists an unlimited s such that $\xi(\tau) \simeq x_0$ for all $0 \leq \tau \leq s$. Thus, $x(t) = \xi(\gamma t)$ is limited for all $t \in [0, s/\gamma]$. Hence, (5) is γ-UIB. ▮

6.2. Steadiness of control systems

The difficulty in applying Theorem 4.2 or 4.4 is to decide whether system (3) is UIB (resp. ω-UIB) or not. Let us give some sufficient conditions for UIB. When f is a standard function and the control u is limited, we know, from Proposition B.1, that for every limited initial condition x_0 and every infinitesimal t, the solution $x^f(t, x_0, u(\cdot))$ of the control system (4) is defined and limited. This property does not hold if the control u is not limited. Consider, for example, the control system $\dot{x} = u$ and take $u(t) = \gamma$, with γ

unlimited. Here, we have $x(1/\sqrt{\gamma}, 0, u(\cdot)) = \sqrt{\gamma}$ which is not limited. This is the reason why we introduce the definition below.

DEFINITION 6.1. Let u be a control. A control system f is u-steady if for every limited x_0 and every positive infinitesimal t, $x^f(t, x_0, u(\cdot))$ is limited.

Hence, (3) is UIB if and only if the control system $\dot{x} = f(x, y)$ is y-steady for every control $y(t) = y^g(t, y_0)$ produced by $\dot{y} = g(y)$, where y_0 is limited.

EXAMPLE 6.1. The control system $\dot{x} = -ux^3/2$ is not u-steady for the control u defined by

$$u(t) = 1 - \gamma^2 t \mathrm{e}^{-\gamma t} \tag{17}$$

where γ is unlimited. Indeed, we have

$$x(t, x_0, u(\cdot)) = x_0 \left(1 + x_0^2 \left[t - 1 + (\gamma t + 1)\mathrm{e}^{-\gamma t}\right]\right)^{-1/2}$$

and $x(t, x_0, u(\cdot))$ escapes to infinity in an infinitesimal time t_e as long as $x_0 > 1$.
The control system $\dot{x} = -ux$ is u-steady for the control u defined by (17). Indeed, we have

$$x(t, x_0, u(\cdot)) = x_0 \mathrm{e}^{1 - t - (\gamma t + 1)\mathrm{e}^{-\gamma t}}$$

and $x(t, x_0, u(\cdot))$ is limited for all limited x_0 and all $t \simeq 0$.

PROPOSITION 6.1. *1) If $\|f\|$ is bounded on $\mathbb{R}^n \times \mathbb{R}^m$ by a limited value M then f is u-steady for every control u.*
2) If there exists a limited real number C, such that for every $(x, u) \in \mathbb{R}^{n+m}$ we have $< x, f(x, u) > \leq C(1 + \|x\|^2)$, then f is u-steady for every control u.

Proof. 1) From $x(t) = x_0 + \int_0^t f(x(s), u(s))ds$ we obtain that $\|x(t)\| \leq \|x_0\| + tM$. Thus $x(t)$ is limited for all limited x_0 and all $t \simeq 0$.
2) Let $\delta(t) = \|x(t)\|^2/2$ then $\delta'(t) =< x(t), f(x(t), u(t)) > \leq C(1 + 2\delta(t))$. Thus $\delta(t) \leq \delta(0)\mathrm{e}^{2Ct} + (\mathrm{e}^{2Ct} - 1)/2$ and $\delta(t)$ is limited for all limited x_0 and all $t \simeq 0$.

∎

DEFINITION 6.2. Let $p \in \mathbb{N}$. A control u is infinitesimally of class $\mathcal{L}^p$ if $\int_0^t \|u(s)\|^p ds$ is limited for all infinitesimal positive t. A control u is of class $\mathcal{L}^p$ if $\int_0^t \|u(s)\|^p ds$ is limited for all limited positive t.

PROPOSITION 6.2. *Assume that $f(x,u) = uX(x)$ where $u \in \mathbb{R}$ and X is a standard complete vector field on $\mathbb{R}^n$. Then f is u-steady for every control u which is infinitesimally of class $\mathcal{L}^1$.*

Proof. We have $x^f(t, u(\cdot)) = X_{v(t)}(x_0)$ where X_t is the flow of the vector field X and $v(t) = \int_0^t u(s)ds$. Thus $x(t)$ is limited for all limited x_0 and all positive infinitesimal t. ▮

PROPOSITION 6.3. *Assume that $f(x,u) = u_1X_1(x) + u_2X_2(x)$ where $u = (u_1, u_2) \in \mathbb{R}^2$ and X_1, X_2 are vector fields on $\mathbb{R}^n$. Assume there exists a limited real number L such that for all $x \in \mathbb{R}^n$ we have $< x, X_1(x) >\leq L$ and $< x, X_2(x) >\leq L$. Then f is u-steady for every control u which is infinitesimally of class $\mathcal{L}^1$.*

Proof. Let $\delta(t) = \|x(t)\|^2/2$ then we have

$$\delta'(t) =< x(t), f(x(t), u(t)) >\leq L(|u_1(t)| + |u_2(t)|).$$

Thus $\delta(t) \leq \delta(0) + L\int_0^t \|u(s)\|ds$ is limited for all limited x_0 and all positive infinitesimal t. ▮

THEOREM 6.1. *Assume that f is standard, and satisfies that f is bounded on $\mathbb{R}^n \times \mathbb{R}^m$ or there exists a real number C such that for every $(x,u) \in \mathbb{R}^{n+m}$ we have $< x, f(x,u) >\leq C(1 + \|x\|^2)$. Assume that H1 holds and the origin of $\dot{y} = g(y)$ is IS. Then the origin of (1,3) is S-GAS.*

Proof. The result follows from Theorem 4.2 By Proposition 6.1, f is u-steady for every control u. Hence (3) is UIB and since the origin of $\dot{y} = g(y)$ is IS it follows that H5 holds. ▮

THEOREM 6.2. *Assume that f is standard, and satisfies that $f(x,y) = \varphi(y)X(x)$ where $\varphi : \mathbb{R}^m \to \mathbb{R}$ is standard and X is a standard complete vector field on $\mathbb{R}^n$. Assume that H1 holds, the origin of $\dot{y} = g(y)$ is IS and for every limited y_0 the control $\varphi(y^g(\cdot, y_0))$ is infinitesimally of class $\mathcal{L}^1$. Then the origin of (3) is S-GAS.*

Proof. The result follows from Theorem 4.2. By Proposition 6.2, f is u-steady for every control $u(t) = \varphi(y^g(t, y_0))$ where y_0 is limited. Hence (3) is UIB and since the origin of $\dot{y} = g(y)$ is IS it follows that H5 holds. ▮

THEOREM 6.3. *Assume that* $f(x,y) = \varphi_1(y)X_1(x) + \varphi_2(y)X_2(x)$ *where* $\varphi = (\varphi_1, \varphi_2) : \mathbb{R}^m \to \mathbb{R}^2$ *is a standard function, and* X_1, X_2 *are standard vector fields on* $\mathbb{R}^n$. *Assume there exists a real number* L *such that for all* $x \in \mathbb{R}^n$ *we have* $< x, X_1(x) > \leq L$ *and* $< x, X_2(x) > \leq L$. *Assume that H1 holds, the origin of* $\dot{y} = g(y)$ *is IS and for every limited* y_0 *the control* $\varphi(y^g(\cdot, y_0))$ *is infinitesimally of class* $\mathcal{L}^1$. *Then the origin of system (3) is* S-*GAS.*

Proof. The result follows from Theorem 4.2. By Proposition 6.3, f is u-steady for every control $u(t) = \varphi(y^g(t, y_0))$ where y_0 is limited. Hence (3) is UIB and since the origin of $\dot{y} = g(y)$ is IS it follows that H5 holds. ▌

Let us end this section by stating a few properties of controls and some criteria that insure that a control is infinitesimally of class $\mathcal{L}^1$.

LEMMA 6.3. *If an impulse control* u *is infinitesimally of class* $\mathcal{L}^p$ *then it is of class* $\mathcal{L}^p$.

Proof. Let u be an impulse control. Then $u(t) \simeq 0$ for all $t \gg 0$. By Lemma A.2, there exists $t_1 \simeq 0$ such that $u(t) \simeq 0$ for all $t \geq t_1$. Let t be positive limited. Then we have

$$\int_0^t \|u(s)\|^p ds = \int_0^{t_1} \|u(s)\|^p ds + \int_{t_1}^t \|u(s)\|^p ds.$$

Since u is infinitesimally of class $\mathcal{L}^p$, the first integral in the right hand side is limited. Since $t - t_1$ is limited and $u(s) \simeq 0$ on $[t_1, t]$, the second integral in the right hand side is infinitesimal. Thus $\int_0^t \|u(s)\|^p ds$ is limited for all limited positive t. ▌

DEFINITION 6.3. Let α be a positive constant. A function $t \to u(t)$ is said to be α-bounded if $u(t)/\alpha$ is limited for all $t \geq 0$.

LEMMA 6.4. *Let* u *be a control. Assume that* u *is an* ω*-inpulse and that it is* ω*-bounded. Then,* u *is of class* $\mathcal{L}^1$.

Proof. Let $a > 0$ be standard. Since $u(\tau/\omega) \simeq 0$ for every $\tau \simeq +\infty$, the set $\{s : \forall \tau \geq s \ \|u(\tau/\omega)\| \leq a\}$ contains every $s \simeq +\infty$. By the Cauchy Principle (see Appendix A.1), it contains some limited s_0. Let t be limited.

Then we have

$$\int_0^t \|u(s)\|ds = \int_0^{s_0/\omega} \|u(s)\|ds + \int_{s_0/\omega}^t \|u(s)\|ds.$$

Since u is ω-bounded, there exists A limited such that $\|u(s)\| \leq \omega A$ for all $s \in [0, s_0/\omega]$ and hence the first integral in the right hand side is limited. Since $t - s_0/\omega$ is limited, the second integral in the right hand side is also limited. Thus, $\int_0^t \|u(s)\|ds$ is limited. ▮

7. α-BOUNDEDNESS AND PEAKING

DEFINITION 7.1. Let α be a positive constant. The system $\dot{y} = g(y)$ is α-bounded if for every limited initial condition y_0, the function $t \to y^g(t, y_0)$ is α-bounded.

PROPOSITION 7.1. *If the system $\dot{y} = g(y)$ is α-bounded and its origin is* S-*GAS then, for all $\gamma \simeq +\infty$, the system $\dot{y} = \gamma g(y)$ is α-bounded and its origin is γ-IS. In particular, if G is standard and H3 holds then system $\dot{y} = \gamma G(y)$ is 1-bounded and its origin is γ-IS.*

Proof. Multiplying the vector field by γ does not change the trajectories, but only the velocity. Moreover, if G is standard and H3 holds then, according to Proposition 4.1, the system $\dot{y} = G(y)$ is 1-bounded and its origin is S-GAS. ▮

DEFINITION 7.2. The system $\dot{y} = g(y)$ exhibits a peaking of order α if it is α-bounded and there exists a limited y_0 and a positive t such that $y^g(t, y_0)/\alpha$ is non infinitesimal.

EXAMPLE 7.1. The second equation in system (10), considered in Example 3.1, exhibits a peaking of order γ.

PROPOSITION 7.2. *Consider the system*

$$\dot{y}_1 = y_2,\ \dot{y}_2 = y_3, \cdots,\ \dot{y}_{n-1} = y_n,\ \dot{y}_n = h(y, \gamma),$$

where $h(0, \gamma) = 0$. Assume that the origin is ω-IS. Then the peaking is at least of order ω^{n-1}.

Proof. We do the proof for $n = 2$. The general case is similar. Assume that $y_1(0) = 1$ and for every positive unlimited τ we have $y_1(\tau/\omega) \simeq 0$. Let $M = \sup_{t\geq 0} |y_2(t)|$. Since $y_1(t) = 1 + \int_0^t y_2(s)ds$, we have, for every positive unlimited τ,

$$\frac{1}{2} \leq \int_0^{\tau/\omega} |y_2(s)|ds \leq M\tau/\omega.$$

Thus $M \geq \omega/(2\tau)$ for all positive unlimited τ. By the Cauchy Principle (see Appendix A), there exists an appreciabe a such that $M \geq a\omega$, that is, the peaking is at least of order ω. ▌

Let us now examine the peaking phenomenon in a ninlinear system.

Example 7.2. Consider the singularly perturbed forced Van der Pol equation

$$\varepsilon\ddot{y} + \dot{y}(y^2 - 1) + y = a \tag{18}$$

where $a > 1$. It is well known that $(y = a, \dot{y} = 0)$ is a GAS equilibrium. In the Lienard space $(y_1 = y - a, v = \varepsilon\dot{y} + y^3/3 - y - a^3/3 + a)$, equation (18) becomes

$$\varepsilon\dot{y}_1 = v - (y_1 + a)^3/3 + (y_1 + a) + a^3/3 - a, \qquad \dot{v} = -y_1. \tag{19}$$

According to Tikhonov's theorem, system (19) is 1-bounded and its origin is S-GAS. In the phase space $(y_1 = y - a, y_2 = \dot{y})$, equation (18) becomes

$$\dot{y}_1 = y_2, \qquad \varepsilon\dot{y}_2 = -y_1 - y_2\left((y_1 + a)^2 - 1\right). \tag{20}$$

From the above analysis in the Lienard plane, we see that system (20) is γ-bounded and its origin is S-GAS, where $\gamma = \varepsilon^{-1}$. Consider now the system

$$\dot{y}_1 = \gamma y_2, \qquad \dot{y}_2 = -\gamma^2\left(y_1 + y_2\left((y_1 + a)^2 - 1\right)\right), \tag{21}$$

obtained by multiplying by γ the vector field associated to system (20). Applying Proposition 7.1, we see that system (21) is γ-bounded and its origin is γ-IS. This system exhibits a peaking of order γ.

Let us now investigate the properties of a linear system

$$\dot{y} = G(\gamma)y, \tag{22}$$

under condition H4. If $G(\gamma) = \gamma G$, where G is a standard Hurwitz matrix, then, by Proposition 7.1, system (22) is 1-bounded and its origin is γ-IS. However, it must be noted that condition H4 does not imply that the origin of system (22) is IS, as it is shown by the following examples.

EXAMPLE 7.3. Consider system (22) where

$$G(\gamma) = \begin{pmatrix} \alpha - \gamma & 1 \\ -\gamma^2 & -\alpha - \gamma \end{pmatrix}, \qquad \alpha \in \mathbb{R}$$

Since both eigenvalues of $G(\gamma)$ are equal to $-\gamma$, condition H4 is satisfied. The exponential matrix

$$\mathrm{e}^{tG(\gamma)} = \begin{pmatrix} 1 + \alpha t & t \\ -\alpha^2 t & 1 - \alpha t \end{pmatrix} \mathrm{e}^{-\gamma t}$$

shows that as $\gamma \to +\infty$, the solution $y(t,\gamma) = \mathrm{e}^{tG} y_0$ will decay to zero arbitrarily fast. However, if α is sufficiently large, the origin is not IS, even if γ is unlimited positive.

EXAMPLE 7.4. Consider system (22) where

$$G(\gamma) = \begin{pmatrix} 0 & 1 \\ -\gamma^2 - \sigma^2 & -2\gamma \end{pmatrix}, \qquad \sigma \in \mathbb{R}$$

Since the eigenvalues of $G(\gamma)$ are equal to $-\gamma \pm \mathrm{i}\sigma$, condition H4 is satisfied. The exponential matrix

$$\mathrm{e}^{tG(\gamma)} = \begin{pmatrix} \cos(\sigma t) + \gamma \sin(\sigma t)/\sigma & \sin(\sigma t)/\sigma \\ -(\gamma^2 + \sigma^2)\sin(\sigma t)/\sigma & \cos(\sigma t) - \gamma \sin(\sigma t)/\sigma \end{pmatrix} \mathrm{e}^{-\gamma t}$$

shows that as $\gamma \to \infty$, the solution $y(t,\gamma) = \mathrm{e}^{tG} y_0$ will decay to zero arbitrarily fast. However, if σ is sufficiently large, the origin is not IS, even if γ is unlimited positive.

The system considered in Example 7.4 is in the Luenberger-Brunowsky (LB) canonical form. It turns out that if we make a supplementary assumption on the spectrum of a system in the LB canonical form, then its origin is IS and we can evaluate the peaking of the system. Consider a system in the LB canonical form:

$$\dot{y}_1 = y_2,\ \dot{y}_2 = y_3, \cdots,\ \dot{y}_{n-1} = y_n,\ \dot{y}_n = a_1 y_1 + a_2 y_2 + \cdots + a_n y_n. \quad (23)$$

THEOREM 7.1. *We assume that the spectrum of system (23) is of order $-\gamma$, that is, every eigenvalue λ_k is of the form $\lambda_k = \gamma c_k$ where c_k is a lim-*

ited complex number with appreciable negative real part. Then the system is γ^{n-1}-bounded and if γ is unlimited positive, its origin is IS.

Proof. The eigenvalues are the roots of the polynomial

$$P(z) = \prod_{i=1}^{n}(z-\lambda_i) = z^n - a_n z^{n-1} - a_{n-1}z^{n-2} - a_2 z - a_1 .$$

The numbers a_i are given in terms of the λ_i by means of the formula

$$a_k = (-1)^{n-k} \sum_{j_1<\cdots<j_{n+1-k}} \lambda_{j_1}\cdots\lambda_{j_{n+1-k}} .$$

Since $\lambda_k = c_k\gamma$ for all k, we have $a_k = \gamma^{n+1-k}b_k$ where

$$b_k = (-1)^{n-k} \sum_{j_1<\cdots<j_{n+1-k}} c_{j_1}\cdots c_{j_{n+1-k}}$$

are limited. Let $\eta_k(\tau) = \gamma^{1-k}y_k(\tau/\gamma)$, for $k = 1\cdots n$, so that (23) becomes

$$\eta_1' = \eta_2,\ \eta_2' = \eta_3, \cdots,\ \eta_{n-1}' = \eta_n,\ \eta_n' = b_1\eta_1 + b_2\eta_2 + \cdots + b_n\eta_n \qquad (24)$$

where $' = d/d\tau$. The eigenvalues of (24) are $\lambda_k/\gamma = c_k$. Therefore there exist two appreciable positive numbers K and α such that

$$\|\eta(\tau)\| \le K\mathrm{e}^{-\alpha\tau}\|\eta(0)\| \qquad \text{for } \tau \ge 0$$

where $\|\eta\| = \sum_{k=1}^{n}|\eta_k|$. Since $\|\eta(\tau)\| \le \|y(\tau/\gamma)\| \le \gamma^{n-1}\|\eta(\tau)\|$ we have

$$\|y(t)\| \le K\gamma^{n-1}\mathrm{e}^{-\gamma\alpha t}\|y(0)\| \qquad \text{for } t \ge 0.$$

Hence (23) is γ^{n-1}-bounded and if γ is unlimited positive, its origin is IS. ∎

8. FURTHER DEVELOPMENTS

Thus far, we considered only triangular systems, in order to focus on the new concepts of IS and UIB without being burdened by technicalities. The second equation does not contain the first variable and the behavior of y does not depend on x. There are many applications for this particular case. There are also more realistic problems where the second equation in system (3) depends on x also

$$\dot{x} = f(x,y), \qquad \dot{y} = g(x,y), \qquad (25)$$

and the stability of system (25) is investigated under condition H1 and the assumption that the origin of the second equation in (25) is IS in some sense to be precised.

For instance, consider the "high gain" observer problem which is well known in automatic control. Let us begin with the linear case which is well understood. We consider the system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{26}$$

with the usual assumptions of controllability of the pair (A, B) and observability of the pair (A, C). System (26) can be stabilized by $u = Rx$ where R is designed such that the matrix $A + BR$ is Hurwitz. Consider now the system

$$\dot{\hat{x}} = A\hat{x} + Bu + KC(\hat{x} - x),$$

which is called a Luenberger observer [12]. The error between the state $x(t)$ and its observation $\hat{x}(t)$ is $y = \hat{x} - x$ and y is solution of the differential equation

$$\dot{y} = (A + KC)y. \tag{27}$$

If K is taken such that the matrix $G := A + KC$ is Hurwitz, then the error tends to 0. Moreover, if we assign the eigenvalues of G far to the left in the complex plane, then the origin of system (27) is IS.

Consider now the case where the feedback is based on the estimation given by the Luenberger observer. The full system is

$$\dot{x} = Ax + BR\hat{x}, \qquad \dot{\hat{x}} = A\hat{x} + BR\hat{x} + KC(\hat{x} - x),$$

which can be rewritten using the variables x and y as

$$\dot{x} = Ax + BR(x + y), \qquad \dot{y} = (A + KC)y. \tag{28}$$

This system turns out to be GAS. This can be seen by elementary algebraic considerations, but also through the previous theory, in the case where we chose K such that the origin of the second equation in (28) is IS.

For the nonlinear case, there are circumstances where one can built Luenberger like observers. Let us consider the problem of stabilization of the control system

$$\dot{x} = \phi(x, u)$$

under some feedback law $u = R(x)$. Assume that we have designed some R such that system

$$\dot{x} = \phi(x, R(x))$$

has the origin as a GAS equilibrium. Assume now that the state vector x is not accessible to measurement, which means that only a certain function $\xi = \varphi(x)$ of the state is available. Is it possible to recover the state $x(t)$ from $\xi(t)$? The answer is yes, to some extent. Namely, under some assumptions that we do not detail here [6], there is a system of the form

$$\dot{\hat{x}} = \vartheta(\hat{x}, \varphi(x))$$

such that the error $y = \hat{x} - x$ tends to zero as $t \to +\infty$. Unlike in the linear case (28), the differential equation of the error does contain the variable x. Actually one has

$$\dot{x} = \phi(x, R(\hat{x})), \qquad \dot{\hat{x}} = \vartheta(\hat{x}, \varphi(x)),$$

which can be rewritten in the variables x and y as

$$\dot{x} = \phi(x, R(x+y)) = f(x,y) \tag{29}$$

$$\dot{y} = \vartheta(x+y, \varphi(x)) - \phi(x, R(x+y)) = g(x,y) \tag{30}$$

Condition H1 is satisfied. If the origin of (30) is IS in some sense, we may hope that the origin of (29,30) is S-GAS. This case is not covered by the results of this paper and thus deserves further attention.

APPENDIX A
Nonstandard Analysis

In IST we adjoin to ordinary mathematics (say ZFC) a new undefined unary predicate *standard* (st). The axioms of IST are the usual axioms of ZFC plus three others which govern the use of the new predicate. Hence, *all theorems of* ZFC *remain valid*. What is new in IST is an addition, not a change. We call a formula of IST *external* in the case where it involves the *new* predicate "st"; otherwise, we call it *internal*. Thus, internal formulas are the formulas of ZFC. The theory IST is a *conservative extension* of ZFC, that is, every internal theorem of IST is a theorem of ZFC. Some of the theorems which are proved in IST are external and can be reformulated so that they become internal. Indeed, there is a *reduction algorithm* which reduces any external formula A of IST to an internal formula A', with the same free variables, which satisfies $A \Leftrightarrow A'$ for all standard values of the parameters (i.e. the free variables). Here, is the reduction of some frequently occuring formulas :

$$\begin{array}{rcl} \forall x(\forall^{\mathrm{st}} yP \Rightarrow \forall^{\mathrm{st}} zQ) & \Leftrightarrow & \forall z \exists^{\mathrm{fin}} y' \forall x(\forall y \in y' P \Rightarrow Q) \\ \forall x(\exists^{\mathrm{st}} w \forall^{\mathrm{st}} yP \Rightarrow \forall^{\mathrm{st}} zQ) & \Leftrightarrow & \forall w \forall z \exists^{\mathrm{fin}} y' \forall x(\forall y \in y' P \Rightarrow Q) \\ \forall x(\exists^{\mathrm{st}} w \forall^{\mathrm{st}} yP \Rightarrow \exists^{\mathrm{st}} zQ) & \Leftrightarrow & \forall w \exists^{\mathrm{fin}} z' \exists^{\mathrm{fin}} y' \forall x(\forall y \in y' P \Rightarrow \exists z \in z' Q) \\ \forall x(\exists^{\mathrm{st}} w \forall^{\mathrm{st}} yP \Rightarrow \forall^{\mathrm{st}} t \exists^{\mathrm{st}} zQ) & \Leftrightarrow & \forall w \forall t \exists^{\mathrm{fin}} z' \exists^{\mathrm{fin}} y' \forall x(\forall y \in y' P \Rightarrow \exists z \in z' Q) \end{array} \tag{A.1}$$

where P and Q are internal formulas whose all parameters are standard.

A real number x is called *infinitesimal* when $|x| < a$ for all standard $a > 0$, *limited* when $|x| \leq a$ for some standard a, *appreciable* when it is limited and not infinitesimal, and *unlimited*, when it is not limited. We use the following notations : $x \simeq 0$ for x infinitesimal, $x \simeq +\infty$ for x unlimited positive, $x \gg 0$ for x non infinitesimal positive. Thus we have

$$\begin{array}{rcl} x \simeq 0 & \Leftrightarrow & \forall^{\mathrm{st}} a > 0 \ |x| < a \\ x \gg 0 & \Leftrightarrow & \exists^{\mathrm{st}} a > 0 \ x \geq a \\ x \text{ limited} & \Leftrightarrow & \exists^{\mathrm{st}} a \ |x| \leq a \\ x \simeq +\infty & \Leftrightarrow & \forall^{\mathrm{st}} a \ x > a \end{array} \tag{A.2}$$

Let (E, d) be a standard metric space. Two points x and y in E are called *infinitely close*, denoted by $x \simeq y$, when $d(x, y) \simeq 0$. A vector in $\mathbb{R}^n$ (n standard) is said to be *infinitesimal* (respectively *limited*) if its norm $\|x\|$ is infinitesimal (resp. limited).

We may not use external formulas to define subsets. The notations $\{x \in \mathbb{R} : x \text{ is limited}\}$ or $\{x \in \mathbb{R} : x \simeq 0\}$ are not allowed. Moreover, we can prove that

LEMMA A.1. *There do not exist subsets L and I of $\mathbb{R}$ such that, for all x in $\mathbb{R}$, x is in L if and only if x is limited, or x is in I if and only if x is infinitesimal.*

This result is frequently used in proofs. Suppose that we have shown that a certain internal property A holds for every limited r; then we know that A holds for some unlimited r, for otherwise we could let $L = \{x \in \mathbb{R} : A\}$. This is called the *Cauchy principle*. It has the following consequence

LEMMA A.2. (Robinson's Lemma). *Let $r(t)$ be a real function such that $r(t) \simeq 0$ for all limited $t \geq 0$, then there exists an unlimited positive number ν such that $r(t) \simeq 0$ for all $t \in [0, \nu]$.*

Proof. The set of all s such that for all $t \in [0, s]$ we have $|r(t)| < 1/s$ contains all limited $s \geq 1$. By the Cauchy principle, it must contain some unlimited ν. ▌

Lemma A.3. *Assume that F is standard. The origin of (14) is stable if and only if $z^F(t, z_0) \simeq 0$ for all $t \geq 0$ and all $z_0 \simeq 0$.*

Proof. The characterization of stability given in the lemma is :

$$\forall z_0 (z_0 \simeq 0 \Rightarrow \forall t \geq 0 \; z^F(t, z_0) \simeq 0).$$

Using (A.2) this formula becomes

$$\forall z_0 (\forall^{\mathrm{st}} \delta > 0 \; \|z_0\| < \delta \Rightarrow \forall^{\mathrm{st}} \varepsilon > 0 \forall t \geq 0 \; \|z^F(t, z_0)\| < \varepsilon).$$

In this formula, F is a standard parameter. By (A.1), this is equivalent to

$$\forall \varepsilon > 0 \exists^{\mathrm{fin}} \delta' \subset]0, +\infty[\forall z_0 (\forall \delta \in \delta' \; \|z_0\| < \delta \Rightarrow \forall t \geq 0 \; \|z^F(t, z_0)\| < \varepsilon).$$

For a finite set $\delta' \subset]0, +\infty[$, $\forall \delta \in \delta' \; \|z_0\| < \delta$ is the same as $\|z_0\| < \delta$ for $\delta = \min \delta' > 0$, and so our formula is equivalent to

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z_0 (\|z_0\| < \delta \Rightarrow \forall t \geq 0 \; \|z^F(t, z_0)\| < \varepsilon).$$

This is the usual definition of stability. ∎

Lemma A.4. *Assume that F is standard. The origin of (14) is globally attractive if and only if $z^F(t, z_0) \simeq 0$ for all positive unlimited t and all standard $z_0 \in \mathbb{R}^p$.*

Proof. The characterization of global attractivity given in lemma is that for any standard $z_0 \in \mathbb{R}^p$ we have :

$$\forall t (t \simeq +\infty \Rightarrow z^F(t, z_0) \simeq 0).$$

Using (A.2) this formula becomes

$$\forall t (\forall^{\mathrm{st}} T \; t > T \Rightarrow \forall^{\mathrm{st}} \varepsilon > 0 \; \|z^F(t, z_0)\| < \varepsilon).$$

In this formula, F and z_0 are standard parameters. By (A.1), this is equivalent to

$$\forall \varepsilon > 0 \exists^{\mathrm{fin}} T' \forall t (\forall T \in T' \; t > T \Rightarrow \|z^F(t, z_0)\| < \varepsilon).$$

For T' a finite set, $\forall T \in T' \; t > T$ is the same as $t > T$ for $T = \max T'$, and so our formula is equivalent to

$$\forall \varepsilon > 0 \exists T \forall t (t > T \Rightarrow \|z^F(t, z_0)\| < \varepsilon).$$

We have shown that for all standard $z_0 \in \mathbb{R}^p$ (and consequently, by transfer, for all $z_0 \in \mathbb{R}^p$) we have $\lim_{t\to+\infty} z^F(t,z_0) = 0$. This is the usual definition of global attractivity. ▮

Proof (of Proposition 4.1). Assume that F is standard and that the origin of (14) is S-GAS. Let $z_0 \simeq 0$. By the continuity of solutions with respect to initial conditions, we have $z(t,z_0) \simeq 0$ for all limited $t \geq 0$. By S-GAS, we have $z(t,z_0) \simeq 0$ for all unlimited $t \geq 0$. Thus, $z(t,z_0) \simeq 0$ for all $t \geq 0$, that is (see Lemma A.3), the origin of (14) is stable. Let z_0 be standard then z_0 is limited and by S-GAS, we have $z(t,z_0) \simeq 0$ for all positive unlimited t. Thus (see Lemma A.4), the origin of (14) is globally attractive. Hence, it is GAS. Conversely, assume that the origin of (14) is GAS. Let z_0 be limited. Let z_{00} be standard in $\mathbb{R}^p$ such that $z_0 \simeq z_{00}$. By the continuity of solutions with respect to initial conditions, we have $z(t,z_0) \simeq z(t,z_{00})$ for all limited $t \geq 0$. By Lemma A.2, there exists an unlimited t_1 such that $z(t,z_0) \simeq z(t,z_{00})$ for all $0 \leq t \leq t_1$. By attractivity (see Lemma A.4), we have $z(t,z_{00}) \simeq 0$ for all positive unlimited t. Thus $z(t,z_0) \simeq 0$ for all unlimited $t \leq t_1$. By stability (see Lemma A.3), we have $z(t,z_0) \simeq z(t_1,z_0) \simeq 0$ for all $t \geq t_1$. Hence, $z(t,z_0) \simeq 0$ for all positive unlimited t, that is, the origin of (14) is S-GAS. ▮

Proof (of Proposition 5.1). By the transfer principle [15], we can assume that F is standard. By Proposition 3.1, we have to prove that the origin of (14) is S-GAS if and only if for all $A > 0$, the origin is uniformly attractive with respect to z_0 for $\|z_0\| \leq A$. This can be seen using the reduction algorithm of external formulas. The definition of S-GAS is :

$$\forall z_0 \forall t (z_0 \text{ limited } \& \ t \simeq +\infty \Rightarrow z^F(t,z_0) \simeq 0).$$

Using (A.2) this formula becomes

$$\forall z_0 \forall t (\exists^{\mathrm{st}} A \ \|z_0\| \leq A \ \& \ \forall^{\mathrm{st}} T \ t > T \Rightarrow \forall^{\mathrm{st}} \varepsilon > 0 \ \|z^F(t,z_0)\| < \varepsilon).$$

In this formula, F is a standard parameter. By (A.1), this is equivalent to

$$\forall A \forall \varepsilon > 0 \exists^{\mathrm{fin}} T' \forall z_0 \forall t (\|z_0\| \leq A \ \& \ \forall T \in T' \ t > T \Rightarrow \|z^F(t,z_0)\| < \varepsilon).$$

For T' a finite set, $\forall T \in T' \ t > T$ is the same as $t > T$ for $T = \max T'$ and so our formula is equivalent to

$$\forall A \forall \varepsilon > 0 \exists T \forall z_0 \forall t (\|z_0\| \leq A \ \& \ t > T \Rightarrow \|z^F(t,z_0)\| < \varepsilon).$$

This is the usual definition of uniform attractivity, with respect to the initial condition z_0 for $\|z_0\| \leq A$. ▮

PROPOSITION A.1. *Assume that F is standard. The origin of (15) seems to be GAS when $\gamma \to +\infty$ if and only if it is S-GAS for all positive unlimited γ.*

Proof. To say that the origin of (15) is S-GAS for all positive unlimited γ is the same as saying that

$$\forall\gamma\forall z_0\forall t(z_0 \text{ limited } \& \ t \simeq +\infty \ \& \ \gamma \simeq +\infty \Rightarrow z^F(t, z_0, \gamma) \simeq 0).$$

Using (A.2) this formula becomes

$$\forall\gamma\forall z_0\forall t(\exists^{\mathrm{st}} A\forall^{\mathrm{st}} T\forall^{\mathrm{st}}\gamma_0 \ P \Rightarrow \forall^{\mathrm{st}}\varepsilon > 0 \ Q)$$

where P is the internal formula $\|z_0\| \leq A \ \& \ t > T \ \& \ \gamma > \gamma_0$ and Q is the internal formula $\|z^F(t, z_0, \gamma)\| < \varepsilon$. Here, F is a standard parameter. By (A.1), this is equivalent to

$$\forall A\forall\varepsilon > 0\exists^{\mathrm{fin}}\gamma_0'\exists^{\mathrm{fin}}T'\forall\gamma\forall z_0\forall t(\forall T \in T'\forall\gamma_0 \in \gamma_0' \ P \Rightarrow Q).$$

For T' a finite set, $\forall T \in T' \ t > T$ is the same as $t > T$ for $T = \max T'$, for γ_0' a finite set, $\forall\gamma_0 \in \gamma_0' \ \gamma > \gamma_0$ is the same as $\gamma > \gamma_0$ for $\gamma_0 = \max\gamma_0'$ and so our formula is equivalent to

$$\begin{aligned}&\forall A\forall\varepsilon > 0\exists\gamma_0\exists T\forall\gamma\forall z_0\forall t\\ &\qquad\qquad(\|z_0\| \leq A \ \& \ T \ t > T \ \& \ \gamma > \gamma_0 \Rightarrow \|z^F(t, z_0, \gamma)\| < \varepsilon).\end{aligned}$$

This is the usual definition of the convergence of $z^F(t, z, \gamma_0)$ to 0 when $t \to +\infty$ and $\gamma \to +\infty$, the convergence being uniform with respect to the initial condition z_0 for $\|z_0\| < A$. Thus the origin seems to be GAS when $\gamma \to +\infty$. ▮

APPENDIX B

Nonstandard asymptotic theory

The classical continuous dependance of solutions with respect to initial conditions and parameters has a beautiful nonstandard translation known in the nonstandard asymptotic theory as the *Short Shadow Lemma* [4, 11]. The problem is to compare the solutions of systems

$$\dot{z} = h_1(z, t) \quad \text{and} \quad \dot{z} = h_0(z, t)$$

under the assumptions that the vector field $h_0 : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ is standard continuous and has the unicity of the solution with prescribed initial condition, $h_1 : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ is continuous and is a regular perturbation of h_0, that is to say, $h_1(z,t) \simeq h_0(z,t)$ for all limited z and t.

LEMMA B.1. (Short Shadow Lemma) *Let t_0, t_1, z_0 and z_1 be limited such that $t_1 \simeq t_0$ and $z_1 \simeq z_0$. Let $z_0(t) = z^{h_0}(t, t_0, z_0)$ be the solution of system $\dot{z} = h_0(z,t)$ with initial condition $z(t_0) = z_0$. Then any solution $z(t)$ of system $\dot{z} = h_1(z,t)$ with initial condition $z(t_1) = z_1$ is defined as long as t and $z_0(t)$ are limited and we have $z(t) \simeq z_0(t)$.*

PROPOSITION B.1. *Let $f(x,u)$ be limited for all limited x and u. Let u be a limited control. Let x_0 be a limited initial condition. Then $x^f(t, x_0, u(\cdot))$ is limited for all infinitesimal positive t.*

Proof. Let $x(t) = x^f(t, x_0, u(\cdot))$. Assume there exists t_1 infinitesimal such that $x(t_1)$ is not limited. Let $t_0 < t_1$ be such that $\|x(t_0) - x_0\| = 1$ and $\|x(t) - x_0\| \leq 1$ for all $t \in [0, t_0]$. Then $x(t_0) - x_0 = \int_0^{t_0} f(x(s), u(s))ds$ is infinitesimal because we integrate a limited function on an interval of infinitesimal lenght. Thus $x(t_0) \simeq x_0$ which contradicts $\|x(t_0) - x_0\| = 1$. ▮

PROPOSITION B.2. *Let f be standard, t_0 and x_0 be limited and u_0 be a standard control. Assume that $t_1 \simeq t_0$, $x_1 \simeq x_0$ and $u(t) \simeq u_0(t)$ for all $t \geq t_0$. For all $t \geq t_0$ we have $x^f(t, t_0, x_0, u(\cdot)) \simeq x^f(t, t_0, x_0, u_0(\cdot))$ and $x^f(t, t_1, x_1, u_0(\cdot)) \simeq x^f(t, t_0, x_0, u_0(\cdot))$ as long as t and $x^f(t, t_0, x_0, u_0(\cdot))$ are limited.*

Proof. It is a consequence of Lemma B.1. ▮

PROPOSITION B.3. *Let f be standard, x_0 limited and u a control. Assume there exists t^* infinitesimal positive such that $u(t) \simeq 0$ for all $t \geq t^*$ and $x^f(t, x_0, u(\cdot))$ limited for $0 \leq t \leq t^*$. Let $x_0^* = x^f(t^*, x_0, u(\cdot))$. Then for all $t \geq t^*$, we have $x^f(t, x_0, u(\cdot)) \simeq x^f(t, x_0^*, 0)$ as long as t and $x^f(t, x_0^*, 0)$ are limited.*

Proof. . Using Proposition B.2, we have for all $t \geq t^*$:

$$\begin{aligned} x^f(t, x_0, u(\cdot)) &= x^f(t, 0, x_0, u(\cdot)) = x^f(t, t^*, x_0^*, u(\cdot)) \\ &\simeq x^f(t, t^*, x_0^*, 0) \simeq x^f(t, 0, x_0^*, 0) = x^f(t, x_0^*, 0) \end{aligned}$$

as long as t and $x^f(t, x_0^*, 0)$ are limited. ▮

PROPOSITION B.4. *Assume that f is standard and that condition H1 holds. Let u be a control and let $x(t) = x^f(t, x_0, u(\cdot))$. Assume that there* *exists $t_1 \geq 0$ such that $x(t_1) \simeq 0$ and $u(t) \simeq 0$ for $t \geq t_1$. Then we have $x(t) \simeq 0$ for all $t \geq t_1$.*

Proof. Suppose this is not the case, that is, there exists $t > t_1$ such that $x(t) \not\simeq 0$. Let $t_2 > t_1$ be such that $x_2 = x(t_2) \not\simeq 0$, and $x(t)$ is limited for all $t_1 \leq t \leq t_2$. If $T = t_2 - t_1$ was limited then, by Proposition B.2,

$$x_2 = x^f(t_2, x_0, u(\cdot)) = x^f(T, x_1, u(\cdot + t_1)) \simeq x^f(T, x_1, 0) \simeq x^f(T, 0, 0) = 0$$

which contradicts $x_2 \not\simeq 0$. Thus, T is infinitely large. By Proposition B.2, we have, for all limited positive s

$$x(t_2 - s) = x^f(t_2 - s, x_0, u(\cdot)) = x^f(-s, x_2, u(\cdot + t_2)) \simeq x^f(-s, x_2, 0).$$

By Lemma A.2, this property holds for some unlimited s, that can be chosen such that $s \leq T$. Let $t_3 = t_2 - s$ then $t_1 \leq t_3$ and $x_3 = x(t_3)$ is limited. Thus $x_{30} := x^f(-s, x_2, 0)$ is limited also and by Proposition 4.1 we have $x_2 = x^f(s, x_{30}, 0) \simeq 0$ which contradicts $x_2 \not\simeq 0$. ▮

APPENDIX C
IS and UIB : equivalence between standard and nonstandard formulations

PROPOSITION C.1. *Assume that G is standard. The origin of system $\dot{y} = G(y, \gamma)$ is IS when $\gamma \to +\infty$ if and only if it is IS for all positive unlimited γ.*

Proof. To say that the origin of system $\dot{y} = G(y, \gamma)$ is IS for all positive unlimited γ is the same as saying that

$$\forall\gamma\forall y_0\forall t(y_0 \text{ limited } \& \ t \gg 0 \ \& \ \gamma \simeq +\infty \Rightarrow y^G(t, y_0, \gamma) \simeq 0).$$

Using (A.2), this formula becomes

$$\forall\gamma\forall y_0\forall t(\exists^{\mathrm{st}} A \exists^{\mathrm{st}} t_0 > 0 \forall^{\mathrm{st}} \gamma_0 \ P \Rightarrow \forall^{\mathrm{st}} \delta > 0 \ Q)$$

where P is the internal formula $\|y_0\| \leq A \ \& \ t \geq t_0 \ \& \ \gamma > \gamma_0$ and Q is the internal formula $\|y^G(t, y_0, \gamma)\| < \delta$. Here, G is a standard parameter. By (A.1), this is equivalent to

$$\forall A \forall \delta > 0 \forall t_0 > 0 \exists^{\mathrm{fin}} \gamma_0' \forall\gamma\forall y_0\forall t_0(\forall \gamma_0 \in \gamma_0' \ P \Rightarrow Q).$$

For γ_0' a finite set, $\forall\gamma_0 \in \gamma_0'\ \gamma > \gamma_0$ is the same as $\gamma > \gamma_0$ for $\gamma_0 = \max\gamma_0'$ and so our formula is equivalent to

$$\forall A \forall\delta > 0 \forall t_0 > 0 \exists\gamma_0 \forall\gamma\forall y_0 \forall t$$
$$(\|y_0\| \leq A \ \&\ t \geq t_0 \ \&\ \gamma > \gamma_0 \Rightarrow \|y^G(t, y_0, \gamma)\| < \delta).$$

This is the usual definition of being IS when $\gamma \to +\infty$. ▮

Proposition C.2. *Assume that f and G are standard. The system (1) is UIB when $\gamma \to +\infty$ if and only if it is UIB for all positive unlimited γ.*

Proof. To say system (1) is UIB for all positive unlimited γ is the same as saying that

$$\forall\gamma\forall(x_{,0}\, y_0)\forall t \geq 0$$
$$(x_0 \text{ and } y_0 \text{ limited } \&\ t \simeq 0 \ \&\ \gamma \simeq +\infty \Rightarrow x^{(f,G)}(t, (x_0, y_0), \gamma) \text{ limited}).$$

Using (A.2), this formula becomes

$$\forall\gamma\forall(x_0, y_0)\forall t \geq 0(\exists^{\mathrm{st}} A \forall^{\mathrm{st}} t_0 > 0 \forall^{\mathrm{st}}\gamma_0\ P \Rightarrow \exists^{\mathrm{st}} B\ Q)$$

where P is the internal formula $\|x_0\| \leq A \ \&\ \|y_0\| \leq A \ \&\ t \leq t_0 \ \&\ \gamma > \gamma_0$ and Q is the internal formula $\|x^{(f,G)}(t, (x_0, y_0), \gamma)\| \leq B$. Here, f and G are standard parameters. By (A.1), this is equivalent to

$$\forall A \exists^{\mathrm{fin}} B' \exists^{\mathrm{fin}} t_0' \subset]0, +\infty[\exists^{\mathrm{fin}}\gamma_0' \forall\gamma\forall(x_0, y_0)\forall t \geq 0$$
$$(\forall t_0 \in t_0' \forall\gamma_0 \in \gamma_0'\ P \Rightarrow \exists B \in B'\ Q).$$

For γ_0' a finite set, $\forall\gamma_0 \in \gamma_0'\ \gamma > \gamma_0$ is the same as $\gamma > \gamma_0$ for $\gamma_0 = \max\gamma_0'$, for a finite set $t_0' \subset]0, +\infty[$, $\forall t_0 \in t_0'\ t \leq t_0$ is the same as $t \leq t_0$ for $t_0 = \min t_0' > 0$, for B' a finite set, $\exists B \in B'\ Q$ is the same as Q for $B = \max B'$ and so our formula is equivalent to

$$\forall A \exists B \exists t_0 > 0 \exists\gamma_0 \forall\gamma\forall(x_0, y_0)\forall t \geq 0$$
$$(\|x_0\| \leq A \ \&\ \|y_0\| \leq A \ \&\ t \leq t_0 \ \&\ \gamma \geq \gamma_0 \Rightarrow \|x^{(f,G)}(t, (x_0, y_0), \gamma)\| \leq B).$$

This is the usual definition of being UIB when $\gamma \to +\infty$. ▮

Proposition C.3. *Assume that G is standard. Let $\omega(\gamma)$ be a standard positive function. The origin of system $\dot{y} = G(y\gamma)$ is ω-IS when $\gamma \to +\infty$if and only if it is $\omega(\gamma)$-IS for all positive unlimited γ.*

Proof. To say that the origin of system $\dot{y} = G(y, \gamma)$ is $\omega(\gamma)$-IS for all positive unlimited γ is the same as saying that

$$\forall\gamma\forall y_0\forall\tau(y_0 \text{ limited } \& \ \tau \simeq +\infty \ \& \ \gamma \simeq +\infty \Rightarrow y^G(\tau/\omega(\gamma), y_0, \gamma) \simeq 0).$$

Using (A.2), this formula becomes

$$\forall\gamma\forall y_0\forall\tau(\exists^{\rm st} A\forall^{\rm st}\tau_0\forall^{\rm st}\gamma_0 \ P \Rightarrow \forall^{\rm st}\delta > 0 \ Q)$$

where P is the internal formula $\|y_0\| \leq A \ \& \ \tau > \tau_0 \ \& \ \gamma > \gamma_0$ and Q is the internal formula $\|y^G(\tau/\omega(\gamma), y_0, \gamma)\| < \delta$. Here, G and ω are standard parameters. By (A.1), this is equivalent to

$$\forall A\forall\delta > 0\exists^{\rm fin}\tau_0'\exists^{\rm fin}\gamma_0'\forall\gamma\forall y_0\forall\tau(\forall\tau_0 \in \tau_0'\forall\gamma_0 \in \gamma_0' \ P \Rightarrow Q).$$

For γ_0' a finite set, $\forall\gamma_0 \in \gamma_0' \ \gamma > \gamma_0$ is the same as $\gamma > \gamma_0$ for $\gamma_0 = \max\gamma_0'$, for τ_0' a finite set, $\forall\tau_0 \in \tau_0' \ \tau > \tau_0$ is the same as $\tau > \tau_0$ for $\tau_0 = \max\tau_0'$ and so our formula is equivalent to

$$\forall A\forall\delta > 0\exists\tau_0\exists\gamma_0\forall\gamma\forall y_0\forall\tau$$
$$(\|y_0\| \leq A \ \& \ \tau > \tau_0 \ \& \ \gamma > \gamma_0 \Rightarrow \|y^G(\tau/\omega(\gamma), y_0, \gamma)\| < \delta).$$

This is the usual definition of being ω-IS when $\gamma \to +\infty$. ∎

PROPOSITION C.4. *Assume that f and G are standard. Let $\omega(\gamma)$ be a standard positive functions. The system (1) is ω-UIB when $\gamma \to +\infty$ if and only if it is $\omega(\gamma)$-UIB for all positive unlimited γ.*

Proof. To say that system (1) is $\omega(\gamma)$-UIB for all positive unlimited γ is the same as saying that

$$\forall\gamma\forall(x_0, y_0) \ ((x_0, y_0) \text{ limited } \& \ \gamma \simeq +\infty$$
$$\Rightarrow \exists s \simeq +\infty \ \forall t \in [0, s/\omega(\gamma)] \ x^{(f,G)}(t, (x_0, y_0), \gamma) \text{ limited})$$

Using (A.2), this formula becomes

$$\forall\gamma\forall(x_0, y_0)(\exists^{\rm st} A\forall^{\rm st}\gamma_0 \ P \Rightarrow \exists s\forall^{\rm st} s_0 \ s \geq s_0 \ \forall t \in [0, s/\omega(\gamma)]\exists^{\rm st} B \ Q) \quad \text{(C.1)}$$

where P is the internal formula $\|x_0\| \leq A \ \& \ \|y_0\| \leq A \ \& \ \gamma > \gamma_0$ and Q is the internal formula $\|x^{(f,G)}(t, (x_0, y_0), \gamma)\| \leq B$. Here, f, G and ω are standard parameters. We have

$$\forall t \in [0, s/\omega(\gamma)]\exists^{\rm st} B \ Q \Leftrightarrow \exists^{\rm st} B\forall t \in [0, s/\omega(\gamma)] \ Q. \quad \text{(C.2)}$$

Indeed, by the idealization principle [15], the left hand side of (C.2) is equivalent to the formula $\exists^{\mathrm{st\,fin}} B' \forall t \in [0, s/\omega(\gamma)] \exists B \in B'\ Q$, which is itself equivalent to the formula $\exists^{\mathrm{st}} B \forall t \in [0, s/\omega(\gamma)]\ Q$, because for B' a standard finite set, $\exists B \in B'\ Q$ is the same as Q for $B = \max B'$ which is standard. Also, we have

$$\exists s \forall^{\mathrm{st}} s_0\ s \geq s_0 \Leftrightarrow \forall^{\mathrm{st}} s_0 \exists s\ s \geq s_0. \tag{C.3}$$

Indeed, by the idealization principle, the left hand side of (C.3) is equivalent to the formula $\forall^{\mathrm{st\,fin}} s_0' \exists s \forall s \in s_0'\ s \geq s_0$, which is itself equivalent to the formula $\forall^{\mathrm{st}} s_0 \exists s\ s \geq s_0$, because for a s_0' a finite set, $\forall s \in s_0'\ s \geq s_0$ is the same as $s \geq \max s_0'$ which is standard. Using (C.2) and (C.3), we see that formula (C.1) is equivalent to

$$\forall \gamma \forall (x_0, y_0)(\exists^{\mathrm{st}} A \forall^{\mathrm{st}} \gamma_0\ P \Rightarrow \forall^{\mathrm{st}} s_0 \exists^{\mathrm{st}} B \exists s \geq s_0 \forall t \in [0, s/\omega(\gamma)]\ Q)$$

Since we can always take $s = s_0$, this formula is equivalent to

$$\forall \gamma \forall (x_0, y_0)(\exists^{\mathrm{st}} A \forall^{\mathrm{st}} \gamma_0\ P \Rightarrow \forall^{\mathrm{st}} s \exists^{\mathrm{st}} B \forall t \in [0, s/\omega(\gamma)]\ Q)$$

By (A.1), this is equivalent to

$$\forall A \forall s \exists^{\mathrm{fin}} \gamma_0' \exists^{\mathrm{fin}} B' \forall \gamma \forall (x_0, y_0)(\forall \gamma_0 \in \gamma_0'\ P \Rightarrow \exists B \in B' \forall t \in [0, s/\omega(\gamma)]\ Q)$$

For γ_0' a finite set, $\forall \gamma_0 \in \gamma_0'\ \gamma > \gamma_0$ is the same as $\gamma > \gamma_0$ for $\gamma_0 = \max \gamma_0'$, for B' a finite set, $\exists B \in B'\ Q$ is the same as Q for $B = \max B'$ and so our formula is equivalent to

$$\begin{aligned}\forall A \forall s \exists \gamma_0 \exists B \forall \gamma \forall (x_0, y_0)(&\|x_0\| \leq A \ \&\ \|y_0\| \leq A \ \&\ \gamma > \gamma_0 \\ &\Rightarrow \forall t \in [0, s/\omega(\gamma)]\ \|x^{(f,G)}(t, (x_0, y_0), \gamma)\| \leq B)\end{aligned}$$

This is the usual definition of being ω-UIB when $\gamma \to +\infty$. ▮

APPENDIX D
Nonstandard proofs of standard theorems

Proof (of Theorem 2.3). By the transfer principle, we assume that f and G are standard. Let $\gamma \simeq +\infty$. By Theorem 2.2, the origin of system (5) is S-GAS. By Proposition A.1, the origin of (5) seems to be GAS when $\gamma \to +\infty$. ▮

Proof (of Theorem 5.1). By the transfer principle, we assume that f and G are standard functions. Assume that H7 holds. Since the origin of

$\dot{y} = G(y, \gamma)$ is IS when $\gamma \to +\infty$, by Proposition C.1, it is IS for all $\gamma \simeq +\infty$. Since (1) is UIB when $\gamma \to +\infty$, by Proposition C.2, it is UIB for all $\gamma \simeq +\infty$. Hence H5 holds. By Theorem 4.2, the origin of (1) is S-GAS for all $\gamma \simeq +\infty$. By Proposition A.1, the origin of (1) seems to be GAS when $\gamma \to +\infty$. ▮

Proof (of Theorem 5.2). By the transfer principle we assume that f and G are standard functions. Assume that H8 holds. By the transfer principle, there exists a standard positive function ω such that H8 holds. We have $\omega(\gamma) \simeq +\infty$ for all $\gamma \simeq +\infty$. Since the origin of $\dot{y} = G(y, \gamma)$ is ω-IS when $\gamma \to +\infty$, by Proposition C.3, it is $\omega(\gamma)$-IS for all $\gamma \simeq +\infty$. Since (1) is ω-UIB when $\gamma \to +\infty$, by Proposition C.4, it is $\omega(\gamma)$-UIB for $\gamma \simeq +\infty$. Thus H6 holds. By Theorem 4.4, the origin of (1) is S-GAS for all $\gamma \simeq +\infty$. By Proposition A.1, the origin of (1) is seems to be GAS when $\gamma \to +\infty$. ▮

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