Local controllability of families of vector fields

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Abstract

Sufficient conditions involving Lie brackets of arbitrarily high-order are obtained for local controllability of families of vector fields. After providing a general framework for the generation of high-order control variations, a specific method for generating such variations is proposed. The theory is applied to a number of nontrivial examples.

Keywords: local controllability, nonlinear systems.

1 Introduction

This paper introduces high-order conditions for the local controllability of a family of vector fields at a point. Our approach is motivated by the early work of Sussmann on local controllability [9]. As in [9] we consider a set S of analytic vector fields on $\Omega \subset \mathbb{R}^n$ and define an *S-trajectory* to be a continuous curve which is a finite concatenation of integral curves of vector fields in S. A point q is *S-reachable from p* if there exists an S-trajectory $t \to \gamma(t)$ such that $q = \gamma(t)$ for some $t \geq 0$ and *S-reachable from p in time $\leq T$* if $t \leq T$. We say S is *locally controllable* (hereafter abbreviated *l.c.*) if, for every $T > 0$, the set of points S-reachable from p in time $\leq T$ contains p in its interior. In [9] Sussmann defines the set S_p^1 of Lie brackets of order two of vector fields in s. His main result is that S is locally controllable at p if $0 \in \mathrm{int}(\mathrm{co}(S(p) \cup S_p^1(p)))$. The main contribution in this paper is the construction of sets of vector fields $S_p^2, S_p^3 \cdots$ of higher-order Lie brackets of vector fields in S. Our main result is Theorem 3.7 which states that S is locally controllable at p if

$$0 \in \mathrm{int}(\mathrm{co}(S(p) \cup S_p^1(p) \cup \cdots \cup S_p^m(p)))$$

for some $m \geq 1$. Of course, the problem of local controllability, especially for control affine systems, has been studied in detail. We refer particularly to [10, 3, 11, 4, 5, 6, 2, 1].

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The paper is organized as follows. In Section 2 we review Sussmann's results on local controllability and consider an example. In Section 3 we introduce our high-order condition for local controllability. In Section 5 we give some examples illustrating our results.

2 First-order conditions

Suppose that S is a set of vector fields on an open set $\Omega \subset \mathbb{R}^n$ and $0 \in \operatorname{co}(S(p))$ for some $p \in \Omega$, where $\operatorname{co}(S(p))$ is the convex hull in $\mathbb{R}^n \simeq T_p\Omega$ of the set of vectors $S(p) = \{X(p) \mid X \in S\}$. Then, as in [9], we let $L^0(S,p) \subset \mathbb{R}^n$ denote the unique linear subspace of maximal dimension such that

$$0 \in \operatorname{int}_{L^0(S,p)}\big(\operatorname{co}(S(p)) \cap L^0(S,p)\big)$$

and define

$$Z_p^0 = \{X \in S \mid X(p) \in L^0(S,p)\}.$$

Let $\tilde{S}_p^1$ denote the set of second-order Lie Brackets $\tilde{S}_p^1 = \{[X,Y] \mid X, Y \in Z_p^0\}$, where $[X,Y](p) = dY_p X(p) - dX_p Y(p)$. The following sufficient condition was established by Sussmann:

Theorem 2.1 ([9]) *Suppose that S is a finite set of vector fields such that $0 \in \operatorname{int}(\operatorname{co}(S(p) \cup \tilde{S}_p^1(p)))$. Then S is locally controllable at p.*

Remark 2.2 A natural extension of this result would involve $\tilde{S}_p^2$, the set of all triple brackets of elements of Z_p^0. Sussmann points out that the corresponding second-order theorem, that S is locally controllable at p if

$$0 \in \operatorname{int}(\operatorname{co}(S(p) \cup \tilde{S}_p^1(p) \cup \tilde{S}_p^2(p))), \tag{1}$$

is false. One consequence of our results is that this theorem does hold if $\tilde{S}_p^2$ is the set of all triple brackets of elements of Z_p^0 of the form $[X,[X,Y]]$. For example, if in $\mathbb{R}^3$ we take the vector fields

$$W = (1, z, 0), \quad X = (-1, 0, x^2), \quad Y = (0, 1, 0), \quad Z = (0, -1, 0),$$

then (1) holds at $p = (0,0,0)$, but clearly the family is not locally controllable at this point as one can never reach states with negative z coordinate. □

3 Higher-order Lie brackets

In this section we develop our methodology for the generation of control variations involving of arbitrarily high-order brackets of vector fields in S. Our method for doing so begins with some constructions involving what we call complementary sets of vector fields. After these considerations have been discussed in Section 3.1, in Section 3.2 we produce explicit S-trajectories which give us control variations involving certain high-order Lie brackets of vector fields in S. In Section 3.3 we apply these constructions to give a theorem on local controllability of S.

If X is a vector field, we denote its flow by $t \mapsto X_t(p)$. If X, Y are vector fields we let $\mathrm{ad}_X Y$ denote the Lie bracket $[X,Y](x) = dY_x X(x) - dX_x Y(x)$. We shall consider iterated brackets of vector fields from a family of vector fields, and so need the notion of degree of a bracket. For us, this will refer to the number of vector fields involved in the bracket. Thus a plain vector field has degree 1 and $[X,Y]$ is a bracket of degree 2. Of course, to be perfectly clear about this, one should use free Lie algebras. However, the loss of rigor in what we do here does not merit the introduction of the additional terminology.

3.1 Complementary vector fields

A finite subset $\mathcal{X}_p \subset Z_p^0$ is said to be *complementary at* p if

$$0 \in \mathrm{int}_{L^0(S,p)}\big(\mathrm{co}(\mathcal{X}_p(p)) \cap L^0(S,p)\big),$$

or equivalently, if 0 can be written as a linear combination of the $X(p)$, $X \in \mathcal{X}_p$, with strictly positive coefficients. Clearly Z_p^0 is complementary at p. If Z_p^0 is convex then there are many complementary sets. We note that Z_p^0 is convex if S is. Furthermore it is known that S is l.c. if and only if $\mathrm{co}(S)$ is l.c. [7]. While our results do not depend on S being convex to simplify notation *we will assume that S is convex* for the rest of this paper. We will also assume that the family of vector fields has the property that $S(p) \subset T_p\Omega$ is compact.

Proposition 3.1 *Suppose Z_p^0 is convex. Then for every $X \in Z_p^0$ there exists a vector field $Y \in Z_p^0$ such that $\{X, Y\}$ is complementary at p.*

Proof Let $X \in Z_p^0$. From the definition of Z_p^0 there exist $\lambda_i > 0$ and $Y_i \in Z_p^0$ such that $\sum_{i=0}^k \lambda_i = 1$ and $\lambda_0 X(p) + \lambda_1 Y_1(p) + \cdots + \lambda_k Y_k(p) = 0$. Set $\lambda_* = \sum_{i=1}^k \lambda_i$ and $Y = \sum_{i=1}^k (\lambda_i/\lambda_*) Y_i$. Because Z_p^0 is convex $Y \in Z_p^0$. This, together with the fact that $(\lambda_0 X + \lambda_* Y)(p) = 0$ completes the proof. ■

Suppose that $\mathcal{X}_p = \{X^1, \ldots, X^k\} \subset Z_p^0$ is complementary at p. Then $\mathcal{X}_p$ gives rise to vector fields which vanishes at p, namely those which can be expressed as $Z = \lambda_1 X^1 + \cdots + \lambda_k X^k$ for appropriate $\lambda_i > 0$. We define $\mathcal{Z}_p$ be the collection of all such vector fields Z. Since we assume that S is convex we know that $Z \in S$ and thus

$$\mathcal{Z}_p = \{Z \mid Z \in S, Z(p) = 0\}.$$

Part of our approach will be to systematically consider rather general classes of S-trajectories To this end, let π be a permutation of $\{1, \ldots, k\}$. We denote by $\mathcal{X}_t^\pi(p)$ the composition of integral curves of the vector fields in X_p with time rescaled, namely

$$\mathcal{X}_t^\pi(p) = X^{\pi(k)}_{\lambda_{\pi(k)}t} \circ \cdots \circ X^{\pi(1)}_{\lambda_{\pi(1)}t}(p),$$

where $\lambda_i > 0$ and $\sum_{i=0}^{k} \lambda_i X^i(p) = 0$. Note that $\mathcal{X}_t^\pi(p)$ is indeed an S-trajectory, in the sense that all points of the form $\mathcal{X}_t^\pi(p)$ for t sufficiently small are the image of a proper S-trajectory. Let P_k denote the set of sequences of permutations of $\{1, 2, \ldots, k\}$. If $\eta \in P_k$ then $\eta = \pi_\ell \circ \pi_{\ell-1} \circ \cdots \circ \pi_1$ for some $\ell \in \mathbb{N}$, and we define a $\mathcal{X}_p^1$*-trajectory* $\mathcal{X}_t^\eta(p)$ to be the S-trajectory which is the composition of the curves $\mathcal{X}_t^{\pi_i}(p)$. Then the Campbell-Baker-Hausdorff formula asserts that, for t sufficiently small, there exist vector fields $X^{\eta,i}$ and $X^{\pi,i}$ such that

$$\begin{aligned}
\mathcal{X}_t^\pi(p) &= \Big(\sum_{i=1}^{k} \lambda_{\pi(i)} X^{\pi(i)} + X^{\pi,1}t + X^{\pi,2}t^2 + o(t^2)\Big)_t(p) \\
\mathcal{X}_t^\eta(p) &= \mathcal{X}_t^{\pi_s} \circ \cdots \circ \mathcal{X}_t^{\pi_1}(p), \\
&= (X^{\eta,1} + X^{\eta,2}t + o(t))_t(p),
\end{aligned} \tag{2}$$

where $X^{\eta,1}$ is a multiple of $\sum_{i=1}^{k} \lambda_i X^i$ and hence vanishes at p. Note that the Campbell-Baker-Hausdorff formula also provides explicit expressions for these terms in the series. In any event, this leaves as dominant the "second-order" term $X^{\eta,2}(p)$. Sussmann [9] generates a richer class of S-trajectories which allows him to prove his theorem on local controllability (stated as Theorem 2.1 here). However, the local controllability result can be proved using the smaller class of S-trajectories we consider here.

Now we show that a given S-trajectory has associated with it a further collection of S-trajectories. For affine control systems a similar construction is carried out by Bianchini and Stefani [2]. Suppose that $Z \in \mathcal{Z}_p$ so that $Z(p) = 0$. Since $Z \in S$, the integral curve for Z in positive time is an S-trajectory. We denote by α the S-trajectory which results from the concatenation of the integral curve for Z and the S-trajectory $\mathcal{X}_t^\eta(p)$. That is, $\alpha(t, r) = Z_r \circ \mathcal{X}_t^\eta(p)$. Using the fact that p is an equilibrium point for the vector field Z, it may be shown [12, 13] that the S-trajectory α has an absolutely convergent series expansion

$$\begin{aligned}
\alpha(t, r) &= Z_r \circ \mathcal{X}_t^\eta \circ Z_{-r}(p) \\
&= \Big(\sum_{k=0}^{\infty} \frac{r^k}{k!} \mathrm{ad}_Z^k X^{\eta,1} + t \sum_{k=0}^{\infty} \frac{r^k}{k!} \mathrm{ad}_Z^k X^{\eta,2} + o(t)\Big)_t(p).
\end{aligned}$$

Because $X^{\eta,1}$ and Z vanish at p we have $\mathrm{ad}_Z^k X^{\eta,1}(p) = 0$ which leaves as dominant the "second-order" term $\sum_{k=0}^{\infty} \frac{r^k}{k!} \mathrm{ad}_Z^k X^{\eta,2}(p)$. More generally if $Z^1, \ldots, Z^s \in \mathcal{Z}_p$ we set $\boldsymbol{r} = (r_1, \ldots, r_s)$ and consider the S-trajectory

$$\alpha(t, \boldsymbol{r}) = Z_{r_1}^1 \circ Z_{r_2}^2 \circ \cdots \circ Z_{r_s}^s \circ \mathcal{X}_t^\eta(p).$$

Arguing as above we can show that

$$\alpha(t, \boldsymbol{r}) = (Y^{\boldsymbol{r},1} + tY^{\boldsymbol{r},2} + o(t))_t(p) \tag{3}$$

where

$$Y^{\boldsymbol{r},2} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_s=0}^{\infty} \frac{r_1^{k_1}}{k_1!} \cdots \frac{r_s^{k_s}}{k_s!} \mathrm{ad}_{Z^1}^{k_1} \circ \cdots \circ \mathrm{ad}_{Z^s}^{k_s} X^{\eta,2}(p)$$

and $Y^{r,1}(p) = 0$. Thus $Y^{r,2}$ is the "dominant second-order" term in α. We let $S_p^1(\mathcal{X}_p)$ denote the set of all such terms indexed over all $\mathcal{X}_p^1$-trajectories $\mathcal{X}_p^\eta(t)$, $Z^1, \dots, Z^s \in \mathcal{Z}_p$, $s, r_1, \dots, r_s \in \natural$, and let S_p^1 denote the union of the sets $S_p^1(\mathcal{X}_p)$ over all subsets $\mathcal{X}_p$ complementary at p. We note that $Y^{(0,\dots,0),2} = X^{\eta,2}$ so that $X^{\eta,2} \in S_p^1$.

Remark 3.2 In equation (2) we have expressed $\mathcal{X}_t^\eta(p)$ as an integral curve for a "time-dependent" vector field $X(t) = X^{\eta,1} + X^{\eta,2}t + o(t)$. To make this more precise we fix $\tau > 0$ and let $\alpha_\tau(t)$ denote the integral curve of the vector field $X(\tau)$ through p, that is $\alpha_\tau(t) = (X(\tau))_t(p)$ where $\frac{d}{dt}\alpha_\tau(t) = X(\tau)(\alpha_\tau(t))$ and $\alpha_\tau(0) = p$. Then $\mathcal{X}_t^\eta(p)$ denotes the point $\alpha_\tau(t)|_{\tau=t}$. □

Remark 3.3 While our definition for S_p^1 differs slightly from Sussmann's $\tilde{S}_p^1$, we do have $\mathrm{co}(\tilde{S}_p^1) \subset \mathrm{co}(S_p^1)$. To show this we can utilize the limited set of permutations used by Sussmann in his proof of his sufficiency condition for l.c. (Theorem 3 of [9]). □

Before we define S_p^k, we motivate the notion of S-trajectories which approximate integral curves to orders higher than one. Let $X, Y \in S_p^1$. From the definition of S_p^1 there exist S-trajectories $\mathcal{X}_t^\eta(p) = (X^1 + tX + o(t))_t(p)$ and $\mathcal{Y}_t^\eta(p) = (Y^1 + tY + o(t))_t(p)$ such that X^1 and Y^1 are linear combinations of vector fields in S that vanish at p. Now suppose that $(\lambda_1 X + \lambda_2 Y + \lambda_3 Z)(p) = 0$ for some $Z \in S$ and $\lambda_1, \lambda_2, \lambda_3 > 0$. Proceeding as above, while rescaling time to ensure compatibility between the vector fields in S and S_p^1, we construct the S-trajectory

$$\mathcal{X}_t(p) = \mathcal{X}^\eta_{\sqrt{\lambda_1}t} \circ \mathcal{Y}^\eta_{\sqrt{\lambda_2}t} \circ Z_{\lambda_3 t^2}(p).$$

From the Campbell-Baker-Hausdorff formula we obtain

$$\begin{aligned}
\mathcal{X}_t(p) &= \left(X^1 + X\sqrt{\lambda_1}t + o(t)\right)_{\sqrt{\lambda_1}t} \\
&\quad \circ \left(Y^1 + Y\sqrt{\lambda_2}t + o(t)\right)_{\sqrt{\lambda_2}t} \circ Z_{\lambda_3 t^2}(p) \\
&= \left(t(\sqrt{\lambda_1}X^1 + \sqrt{\lambda_2}Y^1) + t^2(\lambda_1 X + \lambda_2 Y + \lambda_3 Z) + \right. \\
&\quad \left. t^3 W + o(t^3)\right)_1(p) \\
&= (tX^{\eta,1} + t^2 X^{\eta,2} + t^3 X^{\eta,3})_1(p)
\end{aligned}$$

for vector fields $X^{\eta,1}, X^{\eta,2}, X^{\eta,3}$ with the following properties:

1. $X^{\eta,1}$ is a linear combination of vector fields from S;
2. $X^{\eta,2}$ is a linear combination of degree 1 and 2 brackets of vector fields in S;
3. $X^{\eta,3}$ is a linear combination of degree 2 and 3 brackets of vector fields in S;
4. $X^{\eta,1}$ and $X^{\eta,2}$ vanish at p.

Since the coefficients of t and t^2 vanish at p we have produced an S-trajectory which approximates, to the "third-order" in t, the integral curve of $X^{\eta,3}$. If $Z^1,\ldots,Z^\ell \in \mathcal{Z}_p$, then, as above, we set

$$\alpha(t,\boldsymbol{w}) = Z^1_{w_1} \circ Z^2_{w_2} \circ \cdots \circ Z^\ell_{w_\ell} \circ \mathcal{X}_t(p),$$

with $w_i \geq 0$, $i = 1,\ldots,\ell$. Using the Campbell-Baker-Hausdorff formula we can show that this S-trajectory has the property that

$$\alpha(t,\boldsymbol{w}) = (Y^{\boldsymbol{w},1} + tY^{\boldsymbol{w},2} + t^2 Y^{\boldsymbol{w},3} + o(t^2))_t(p)$$

where

$$Y^{\boldsymbol{w},3} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_\ell=0}^{\infty} \frac{w_1^{k_1}}{k_1!} \cdots \frac{w_\ell^{k_\ell}}{k_\ell!} \mathrm{ad}^{k_1}_{Z^1} \circ \cdots \circ \mathrm{ad}^{k_\ell}_{Z^\ell} X^{\eta,3}(p)$$

and $Y^{\boldsymbol{w},1}(p) = Y^{\boldsymbol{w},2}(p) = 0$. This formula may be derived in a manner similar to formula (3). In any event, $Y^{\boldsymbol{w},3}$ is the "dominant third-order" term in α. We let $S^2_p(\mathcal{X}_p)$ denote the set of all such terms and S^2_p the union of the sets $S^2_p(\mathcal{X}_p)$ over all subsets $\mathcal{X}_p$ complementary at p.

3.2 Higher-order variations

We now define S^i_p for $i > 1$ inductively. Suppose that we have defined sets of vector fields $S^1_p,\ldots,S^m_p$ with the following property: for any $X \in S^j_p$ there exists an S-trajectory of the form $\mathcal{X}^\eta_t(p) = (X^\eta(t))_t(p) = (tX^\eta(t))_1(p)$, with $X^\eta(t)$ a time varying vector field so that for t sufficiently small $tX^\eta(t)$ can be represented by the convergent power series

$$tX^\eta(t) = tX^{\eta,1} + t^2 X^{\eta,2} + \cdots + t^{\sigma_j - 1} X^{\eta,\sigma_j - 1} + t^{\sigma_j} X^{\eta,\sigma_j} + o(t^{\sigma_j})$$

where the vector fields $X^{\eta,1},\ldots,X^{\eta,\sigma_j-1}$ vanish at p; $X = X^{\eta,\sigma_j}$; and σ_j is defined inductively by $\sigma_1 = 2$,

$$\sigma_{k+1} = \frac{(\sigma_k + 1)\mathrm{lcm}\{\sigma_1,\ldots,\sigma_k\}}{\sigma_k}. \tag{4}$$

We note that one consequence of the above definition is that $\sigma_{m+1} > \sigma_m > \cdots > \sigma_1$. The reason for this definition becomes apparent in the proof of Lemma 3.4. Let $L^m(S,p)$ denote the unique linear subspace of maximal dimension such that

$$0 \in \mathrm{int}_{L^m(S,p)}[\mathrm{co}(S(p) \cup S^1_p(p) \cup \cdots \cup S^m_p(p)) \cap L^m(S,p)]$$

and set

$$Z^m_p = \{X \in S \cup S^1_p \cup \cdots \cup S^m_p \mid X(p) \in L^m(S,p)\}.$$

A finite subset $\mathcal{X}_p \subset Z^m_p$ is said to be *complementary at p* if

$$0 \in \mathrm{int}_{L^m(S,p)}[\mathrm{co}(\mathcal{X}_p(p) \cap L^m(S,p))],$$

or equivalently, if 0 can be written as a linear combination of the vectors $X(p), X \in \mathcal{X}_p$, with strictly positive coefficients. Suppose that $\mathcal{X}_p = \{X^1, \dots, X^k\}$ is a subset of Z_p^m complementary at p, so that $\sum_{i=0}^k \lambda_i X^i(p) = 0$ for some $\lambda_i > 0$. Let π be a permutation of $\{1, 2, \dots, k\}$. Then $X^i \in S$ or $X^i \in S_p^{m_i}$ where $m_i \in \{1, \dots, m\}$. If $X^i \in S_p^{m_i}$ then, by our induction hypothesis, there exists an S-trajectory of the form $\mathcal{X}_t^{\eta_i}(p) = (X^{\eta_i}(t))_t(p)$ where the time-varying vector field $X^{\eta_i}(t)$ has the power series expansion

$$tX^{\eta_i}(t) = tX^{\eta_i,1} + \cdots + t^{\sigma_{m_i}-1}X^{\eta_i,\sigma_{m_i-1}} + t^{\sigma_{m_i}}X + o(t^{\sigma_{m_i}}) \tag{5}$$

and such that $X^{\eta_i,1}, \dots, X^{\eta_i,\sigma_{m_i}-1}$ vanish at p. We rescale time by $t \mapsto \alpha_i t^{\gamma_i}$ where $\alpha_i = \lambda_i^{1/\sigma_{m_i}}$, and $\gamma_i = \operatorname{lcm}\{\sigma_1, \dots, \sigma_m\}/\sigma_{m_i}$. If $X^i \in S$ we rescale time by $t \mapsto \alpha_i t^{\gamma_i}$ where $\gamma_i = \operatorname{lcm}\{\sigma_1, \dots, \sigma_m\}$—in effect we define $\sigma_0 = 1$. We denote by $\mathcal{X}_t^\pi(p)$ the S-trajectory

$$\mathcal{X}_t^\pi(p) = \mathcal{X}^{\eta_{\pi(k)}}_{\alpha_{\pi(k)}t^{\gamma_{\pi(k)}}} \circ \cdots \circ \mathcal{X}^{\eta_{\pi(1)}}_{\alpha_{\pi(1)}t^{\gamma_{\pi(1)}}}(p). \tag{6}$$

This rescaling is needed because, if $X \in S_p^k$, then, using a suitable "control variation," we can generate an S-trajectory which achieves motion in the "X direction" to order σ_k in t. Finally, if $\eta \in P_k$, so that $\eta = \pi_{s-1} \circ \pi_{k_{s-1}} \circ \cdots \circ \pi_1$, we define $\mathcal{X}_t^\eta(p)$ to be the composition of the curves $\mathcal{X}_t^{\pi_i}(p)$ and say that $\mathcal{X}_t^\eta(p)$ is an $\mathcal{X}_p^{m+1}$*-trajectory*. Then

$$\mathcal{X}_t^\eta(p) = \mathcal{X}_t^{\pi_s} \circ \cdots \circ \mathcal{X}_t^{\pi_1}(p) = (X^\eta(t))_t(p).$$

For t sufficiently small the Campbell-Baker-Hausdorff formula yields

$$tX^\eta(t) = tX^{\eta,1} + \cdots + t^{\sigma_{m+1}-1}X^{\eta,\sigma_{m+1}-1} + t^{\sigma_{m+1}}X^{\eta,\sigma_{m+1}} + o(t^{\sigma_{m+1}}), \tag{7}$$

for vector fields $X^{\eta,i}$.

The following lemma makes clear why the inductive definition of the σ_k's are as in (4). The idea essentially is that one needs to define time rescalings along vector fields in an S-trajectory to ensure that the desired term is the first nonzero term in the series expansion. This makes sense of our inductive definition of $S_p^m(p)$.

Lemma 3.4 *The vector fields $X^{\eta,1}, \dots, X^{\eta,\sigma_{m+1}-1}$ that appear in equation* (7) *vanish at p.*

Proof Let $X \in S_p^i, Y \in S_p^j$ where $i, j \in \{0, \dots, m\}$ and $S_p^0 = S$. By our induction hypotheses there exist time-varying vector fields $X^{\eta_i}(t), X^{\eta_j}(t)$ where

$$tX^{\eta_i}(t) = (tX^{\eta_i,1} + \cdots + t^{\sigma_i-1}X^{\eta_i,\sigma_i-1} + t^{\sigma_i}X + o(t^{\sigma_i})$$
$$tX^{\eta_j}(t) = (tX^{\eta_j,1} + \cdots + t^{\sigma_j-1}X^{\eta_j,\sigma_j-1} + t^{\sigma_j}Y + o(t^{\sigma_j})$$

with $X^{\eta_j,k}, X^{\eta_i,\ell}$ vanishing at p as in (5) above. The corresponding S-trajectories are $\mathcal{X}_t^{\eta_j}(p) = (X^{\eta_j}(t))_t(p) = (tX^{\eta_j}(t))_1(p)$ and $\mathcal{X}_t^{\eta_i}(p) =$

$(X^{\eta_i}(t))_t(p) = (tX^{\eta_i}(t))_1(p)$. Rescaling time as above and concatenating these curves yields the S-trajectory

$$\begin{aligned}
\beta(t) &= \mathcal{X}^{\eta_j}_{\alpha_j t^{\gamma_j}} \circ \mathcal{X}^{\eta_i}_{\alpha_i t^{\gamma_i}} \\
&= (X^{\eta_j}(\alpha_j t^{\gamma_j}))_{\alpha_j t^{\gamma_j}} \circ (X^{\eta_i}(\alpha_i t^{\gamma_i}))_{\alpha_i t^{\gamma_i}}(p) \\
&= (\alpha_j t^{\gamma_j} X^{\eta_j}(\alpha_j t^{\gamma_j}))_1 \circ (\alpha_i t^{\gamma_i} X^{\eta_i}(\alpha_i t^{\gamma_i}))_1(p) \\
&= \Big(\sum_{k=1}^{\sigma_j+1} (\alpha_j t^{\gamma_j})^k X^{\eta_j,k} + o(t^{\gamma_j(\sigma_j+1)}) \Big)_1 \\
&\quad \circ \Big(\sum_{\ell=1}^{\sigma_i+1} (\alpha_i t^{\gamma_i})^\ell X^{\eta_i,\ell} + o(t^{\gamma_i(\sigma_i+1)}) \Big)_1 (p),
\end{aligned}$$

where $X = X^{\eta_i,\sigma_i}$, $Y = X^{\eta_j,\sigma_j}$, and $X^{\eta_i,\ell}$ and $X^{\eta_j,k}$ vanish at p for $k < \sigma_j, \ell < \sigma_i$. For t sufficiently small, the Campbell-Baker-Hausdorff formula gives the coefficients of t in the power series expansion for $\beta(t)$. In particular $\beta(t)$ can be written as a convergent power series whose terms are expressible as linear combinations of Lie brackets of the vector fields $X^{\eta_i,\ell}$ and $X^{\eta_j,k}$ and Lie brackets of these vector fields of all orders. Our induction hypothesis implies that $X^{\eta_i,\ell}$ and $X^{\eta_j,k}$ vanish at p if $k < \sigma_j$ and $\ell < \sigma_i$. Hence Lie brackets of these vector fields also vanish at p. Thus the lowest order term with respect to t in the power series expansion for $\beta(t)$ which does not necessarily vanish at p will be

$$(\alpha_j t^{\gamma_j})^{\sigma_j} X^{\eta_j,\sigma_j} + (\alpha_i t^{\gamma_i})^{\sigma_i} X^{\eta_i,\sigma_i}.$$

From the above definitions

$$(\alpha_j t^{\gamma_j})^{\sigma_j} X^{\eta_j,\sigma_j} = (\lambda_j^{1/\sigma_j})^{\sigma_j}) t^{\gamma_j \sigma_j} X^{\eta_j,\sigma_j}$$

and

$$(\alpha_i t^{\gamma_i})^{\sigma_i} X^{\eta_i,\sigma_i} = (\lambda_i^{1/\sigma_i})^{\sigma_i}) t^{\gamma_i \sigma_i} X^{\eta_i,\sigma_i}.$$

Thus

$$(\alpha_j t^{\gamma_j})^{\sigma_j} X^{\eta_j,\sigma_j} + (\alpha_i t^{\gamma_i})^{\sigma_i} X^{\eta_i,\sigma_i} = t^{\operatorname{lcm}\{\sigma_1,\ldots,\sigma_m\}} (\lambda_j X^{\eta_i,\sigma_i} + \lambda_i X^{\eta_i,\sigma_i}).$$

The next (higher) power of t which appears in the power series for $\beta(t)$ is t^r which has as coefficient the linear combination of vector fields

$$\begin{aligned}
&(\alpha_j t^{\gamma_j})^{\sigma_j+1} X^{\eta_j,\sigma_j+1} + (\alpha_i t^{\gamma_i})^{\sigma_i+1} X^{\eta_i,\sigma_i+1} = \\
&\qquad\qquad \alpha_j^{\sigma_j+1} t^{\gamma_j(\sigma_j+1)} X^{\eta_j,\sigma_j+1} + \alpha_i^{\sigma_i+1} t^{\gamma_i(\sigma_i+1)} X^{\eta_i,\sigma_i+1}.
\end{aligned}$$

We now show that $r \geq \sigma_{m+1}$. Since $\gamma_j(\sigma_j+1) = (\frac{\sigma_j+1}{\sigma_j})\operatorname{lcm}\{\sigma_1,\ldots,\sigma_m\}$, the sequence $\{\sigma_j\}$ is, by definition, monotone increasing, and $\sigma_{m+1} = (\frac{\sigma_m+1}{\sigma_m})\operatorname{lcm}\{\sigma_1,\ldots,\sigma_m\}$ we see that $\gamma_j(\sigma_j+1) > \sigma_{m+1}$ for $j < m$ and $\gamma_j(\sigma_j+1) = \sigma_{m+1}$ if $j = m$. Among the Lie brackets of order 2 in the power series expansion of $\beta(t)$ which do not vanish at p, the terms with the lowest power of t will have the form

$$[\alpha_j t^{\gamma_j} X^{\eta_j,1}, \alpha_i t^{\gamma_i} X^{\eta_i,\sigma_i}] = \alpha_j \alpha_j t^{\gamma_j+\gamma_i\sigma_i} [X^{\eta_j,1}, X^{\eta_i,\sigma_i}].$$

Here we have t to the power $\gamma_j + \gamma_i\sigma_i$ and

$$\begin{aligned}\gamma_j + \gamma_i\sigma_i &= \frac{\operatorname{lcm}\{\sigma_1,\ldots,\sigma_m\}}{\sigma_j} + \operatorname{lcm}\{\sigma_1,\ldots,\sigma_m\} \\ &= \Big(\frac{\sigma_j+1}{\sigma_j}\Big)\operatorname{lcm}\{\sigma_1,\ldots,\sigma_m\} \\ &\geq \sigma_{m+1},\end{aligned}$$

with equality holding if and only if $j = m$. Lie brackets of order greater than 2 which are "coefficients" of t^s with $s \leq \sigma_{m+1}$ clearly must vanish at p. Thus if $\ell = \operatorname{lcm}\{\sigma_1,\ldots,\sigma_m\}$ then the power series expansion for $\mathcal{X}_t^{\pi}(p)$ defined by (6) is of the form $tZ^1 + \cdots + t^{\ell-1}Z^{\ell-1} + t^{\ell}Z^{\ell} + t^r Z^r + o(t^r))_1(p)$ where $Z^1,\ldots,Z^{\ell-1}$ vanish at p, $Z^{\ell} = \sum_{i=1}^m \lambda_i X^i$, hence by our choice of the λ_i's we have $Z^{\ell}(p) = 0$, and $r \geq \sigma_{m+1}$ with $r = \sigma_{m+1}$ if and only if one of the vector fields $X^i \in S_p^m$. Extending this argument to $\mathcal{X}_t^{\eta}(p)$ completes the proof. ∎

This Lemma implies that $\mathcal{X}_t^{\eta}(p)$ is an S-trajectory which approximates, to order $t^{\sigma_{m+1}}$, the integral curve of $X^{\eta,\sigma_{m+1}}$ with time rescaled to $t^{\sigma_{m+1}}$. If $Z^1,\ldots,Z^{\ell} \in \mathcal{Z}_p$ we set

$$\alpha(t,\boldsymbol{w}) = Z^1_{w_1} \circ Z^2_{w_2} \circ \cdots \circ Z^{\ell}_{w_\ell} \circ \mathcal{X}_t(p).$$

As above the Campbell-Baker-Hausdorff formula shows that this S-trajectory has the form

$$\alpha(t,\boldsymbol{w}) = (Y^{\boldsymbol{w},1} + \cdots + t^{\sigma_{m+1}-2}Y^{\boldsymbol{w},\sigma_{m+1}-1} + \\ t^{\sigma_{m+1}-1}Y^{\boldsymbol{w},\sigma_{m+1}} + o(t^{\sigma_{m+1}-1}))_t(p)$$

where

$$Y^{\boldsymbol{w},\sigma_{m+1}} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_\ell=0}^{\infty} \frac{w_1^{k_1}}{k_1!} \cdots \frac{w_\ell^{k_\ell}}{k_\ell!} \operatorname{ad}^{k_1}_{Z^1} \circ \cdots \circ \operatorname{ad}^{k_\ell}_{Z^\ell} X^{\eta,\sigma_{m+1}}(p) \qquad (8)$$

and $Y^{\boldsymbol{w},1}(p) = \cdots = Y^{\boldsymbol{w},\sigma_{m+1}-1}(p) = 0$. Thus $Y^{\boldsymbol{w},\sigma_{m+1}}$ is the "dominant σ_{m+1}th-order" term in α. We let $S_p^{m+1}(\mathcal{X}_p)$ denote the set of all such terms indexed over all $\mathcal{X}_p^{m+1}$-trajectories $\mathcal{X}_t^{\eta}(p)$, all $Z^1,\ldots,Z^{\ell} \in \mathcal{Z}_p$, and all $\ell, w_1,\ldots,w_\ell \geq 0$.

Definition 3.5 *S_p^{m+1} is defined to be the union of the sets $S_p^{m+1}(\mathcal{X}_p)$ over all subsets $\mathcal{X}_p$ complementary at p.*

We note that vector fields in S_p^m will be linear combinations of brackets of degree at most $m+1$ of vector fields in S. We also note that $X^{\eta,\sigma_{m+1}} \in S_p^{m+1}$ as $Y^{\mathbf{0},\sigma_{m+1}} = X^{\eta,\sigma_{m+1}}$.

If X and Y are vector fields and $s \geq 0$ we set

$$\exp(s\mathrm{ad}_X)Y = \sum_{k=0}^{\infty} \frac{s^k}{k!}\mathrm{ad}_X^k Y.$$

The following is a consequence of the above discussion:

Proposition 3.6 *Suppose that $X \in S_p^m$. Then for any $Z^1, \ldots, Z^\ell \in \mathcal{Z}_p$ we have:*

1. $\exp(s\mathrm{ad}_{Z^i})X \in S_p^m$ *for all $s \geq 0, 1 \leq i \leq \ell$.*
2. $\pm\mathrm{ad}_{Z^1} \circ \mathrm{ad}_{Z^2} \circ \cdots \circ \mathrm{ad}_{Z^\ell} X(p) \in S_p^m(p)$ *for any vector fields Y satisfying $Y(p) \in S_p^m(p)$ and $Y(p) + X(p) = 0$.*
3. *for t sufficiently small, there exists an S-trajectory $\mathcal{X}_t^\eta(p)$ of the form*

$$\mathcal{X}_t^\eta(p) = (X^{\eta,1} + \cdots + t^{\sigma_m - 1}X^{\eta,\sigma_m} + t^{\sigma_m}X^{\eta,\sigma_m+1} + o(t^{\sigma_m}))_t(p),$$

where $X = X^{\eta,\sigma_m}$ and the vector fields $X^{\eta,k}$ vanish at p for $k = 1, \ldots, \sigma_m - 1$. If $X(p) = 0$ then $X^{\eta,\sigma_m+1} \in S_p^{m+1}$.

Proof Assertion 1 follows from (8) where $Z^1 = \cdots = Z^\ell = Z^i$. To prove 2 we let W_p denote the subspace spanned by vectors of the form

$$v(\boldsymbol{w}, \boldsymbol{k}) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_\ell=0}^{\infty} \frac{w_1^{k_1}}{k_1!} \cdots \frac{w_\ell^{k_\ell}}{k_\ell!}\mathrm{ad}_{Z^1}^{k_1} \circ \cdots \circ \mathrm{ad}_{Z^\ell}^{k_\ell} X(p)$$

for $w_i \in \mathbb{R}$, $k_1 + \cdots + k_\ell \geq 0$. Since $X(p) \in S_p^m(p)$ we have $X = \bar{X} + Z^X$ where $Z^X(p) = 0$ and $\bar{X}(p) = X(p)$. It follows from (8) that $+v_X(\boldsymbol{w}, \boldsymbol{k})(p) \in S_p^m(p)$ for $w_i \geq 0$. But $Y(p) \in S_p^m(p)$ and $Y(p) = -X(p)$. Furthermore, $\mathrm{ad}_{Z^i}^j Y(p) = -\mathrm{ad}_{Z^i}^j X(p)$ as $Z^i(p) = 0$ and $Y = \bar{Y} + Z^Y$ where $Z^Y(p) = 0, \bar{Y} \in S_p^m$. Thus $v_Y(\boldsymbol{w}, \boldsymbol{k})(p) \in S_p^m(p)$ which implies that $-v_X(\boldsymbol{w}, \boldsymbol{k})(p) \in S_p^m(p)$. In particular $W_p \subset S_p^m(p)$. Now consider the function from a neighborhood of $0 \in \mathbb{R}^\ell$ to a neighborhood of $0 \in W_p$ given by $\boldsymbol{w} \mapsto v(\boldsymbol{w}, \boldsymbol{k})$. By taking successive derivatives of this map at $\boldsymbol{w} = 0$ we see that each of the vectors

$$\mathrm{ad}_{Z^1}^{k_1} \circ \cdots \circ \mathrm{ad}_{Z^\ell}^{k_\ell} X(p)$$

must lie in W_p, and therefore in $S_p^m(p)$.

Assertion 3 follows from our definition of S_p^m. In particular the fact that, for t sufficiently small, there exists an S-trajectory $\mathcal{X}_t^\eta(p)$ of the form

$$\mathcal{X}_t^\eta(p) = (X^{\eta,1} + \cdots + t^{\sigma_m - 1}X^{\eta,\sigma_m} + t^{\sigma_m}X^{\eta,\sigma_m+1} + o(t^{\sigma_m}))_t(p),$$

where $X = X^{\eta,\sigma_m}$ and the vector fields $X^{\eta,k}$ vanish at p for $k = 1, \ldots, \sigma_m - 1$ follows from the definition of S_p^m and Lemma 3.4. Suppose that X also vanishes at p. Then, by definition, $\mathcal{X}_p = \{X\} \subset Z_p^m$ is a set of vector fields complementary at p and hence the $\mathcal{X}_p^m$-trajectory $\mathcal{X}_t^\eta(p)$ is also a $\mathcal{X}_p^{m+1}$-trajectory and then $X^{\eta,\sigma_m+1} \in S_p^{m+1}$ by definition. ■

3.3 A theorem on local controllability

The main results in this section is the following "high-order" sufficient condition for local controllability.

Theorem 3.7 *Suppose that S is a set of vector fields on $\Omega \subset \mathbb{R}^n$ such that*

$$0 \in \mathrm{int}\big(\mathrm{co}(S(p) \cup S_p^1(p) \cup \cdots \cup S_p^m(p))\big)$$

for some $m \geq 1$. Then S is locally controllable at p.

Before we present the proof we establish the following technical lemma:

Lemma 3.8 *Suppose that $X \in S_p^m$. Then there exists an S-trajectory $\mathcal{X}_t^\eta(p)$ with the property that*

$$\mathcal{X}_t^\eta(p) = p + \frac{t^{\sigma_m}}{\sigma_m} X(p) + o(t^{\sigma_m}).$$

Proof Here $X \in S_p^m$ and from the construction of S_p^m we know that

$$\mathcal{X}_t^\eta(p) = (X^{\eta,1} + \cdots + t^{\sigma_m - 2} X^{\eta,\sigma_m - 1} + t^{\sigma_m - 1} X + o(t^{\sigma_m - 1}))_t(p)$$

where the vector fields $X^{\eta,k}$ vanish at p for $k = 1, \ldots, \sigma_m - 1$. Set

$$X(t) = X^{\eta,1} + \cdots + t^{\sigma_m - 2} X^{\eta,\sigma_m - 1} + t^{\sigma_m - 1} X + o(t^{\sigma_m - 1}),$$

a time-dependent vector field and let $\alpha(t) = (X(t))_t(p)$, the "approximate integral curve" as in Remark 3.2. Given $x \in \Omega$ we set

$$\begin{aligned}
X(t)(x) &= X^{\eta,1}(x) + \cdots + t^{\sigma_m - 2} X^{\eta,\sigma_m - 1}(x) + t^{\sigma_m - 1} X(x) + o(t^{\sigma_m - 1}) \\
dX_x(t) &= dX_x^{\eta,1} + \cdots + t^{\sigma_m - 2} dX_x^{\eta,\sigma_m - 1} + t^{\sigma_m - 1} dX_x + o(t^{\sigma_m - 1}) \\
\dot{X}(t) &= X^{\eta,2} + \cdots + \sigma_m - 2t^{\sigma_m - 3} X^{\eta,\sigma_m - 1} + (\sigma_m - 1) t^{\sigma_m - 2} X + o(t^{\sigma_m - 2}).
\end{aligned}$$

Then the derivatives of $\alpha(t) \in \mathbb{R}^n$ with respect to t take the form $\dot{\alpha}(t) = X(t)(\alpha(t))$, and $\ddot{\alpha}(t) = dX_{\alpha(t)}(t)\dot{\alpha}(t) + \dot{X}(t)(\alpha(t))$. Thus $\alpha(0) = p$, $\dot{\alpha}(0) = X(0)(p) = X^{\eta,1}(p)$, and $\ddot{\alpha}(0) = dX_p^{\eta,1} X^{\eta,1}(p) + X^{\eta,2}(p)$. For $\sigma_m > 2$ we have $X^{\eta,1}(p) = X^{\eta,2}(p) = 0$ and hence $\dot{\alpha}(0) = \ddot{\alpha}(0) = 0$. It is straightforward to show that $\alpha^{(k)}(0) = (k-1)! X^{\eta,k}(p) = 0$ for $1 \leq k \leq \sigma_m - 1$ and $\alpha^{(\sigma_m)}(0) = (\sigma_m - 1)! X(p)$. In particular we have the Taylor series expansion

$$\begin{aligned}
\alpha(t) &= \alpha(0) + \alpha^{(1)}(0) t + \cdots + \frac{1}{\sigma_m!} \alpha^{(\sigma_m)}(0) t^{\sigma_m} + o(t^{\sigma_m}) \\
&= p + \frac{t^{\sigma_m}}{\sigma_m} X(p) + o(t^{\sigma_m}).
\end{aligned}$$

The observation that $\mathcal{X}_t^\eta(p) = \alpha(t)$ completes the proof. ■

Proof of Theorem 3.7 By assumption there exist vector fields $X_1^i, \dots, X_{k_i}^i \in S_p^i$ for $0 \leq i \leq m$ such that 0 is contained in the absolute interior of the convex hull of $\{X_j^i(p) \mid 0 \leq i \leq m,\ 1 \leq j \leq k_i\}$. Here we set $S_p^0 = S$. In light of Lemma 3.8 we can find corresponding S-trajectories

$$\mathcal{X}_t^{\eta_{i,j}}(p) = p + \frac{t^{\sigma_i}}{\sigma_i} X_j^i(p) + o(t^{\sigma_i}).$$

Rescaling time by $t^{\sigma_i} = \sigma_i s_{i,j}$ for $s_{i,j} > 0$ we have $\tilde{\mathcal{X}}_t^{\eta_{i,j}}(p) = p + s_{i,j} X_j^i(p) + o(s_{i,j})$. The composition of such S-trajectories yields

$$\alpha(s_{1,1}, s_{1,2}, \dots, s_{m,k_m}) = \\ p + \sum_{i=0}^{m} \sum_{j=1}^{k_i} s_{i,j} X_j^i(p) + o(s_{1,1} + s_{1,2} + \cdots + s_{m,k_m}).$$

This is the form of the S-trajectories used in the proof of Theorem 3 in [9]. We can then apply Lemma 4 of [9] to conclude that S is l.c. at p. ■

4 A concrete class of higher-order variations

While Theorem 3.7 is interesting, it is not so easy to apply as we have not been very concrete about describing tangent vectors in $S_p^m(p)$. In this section we provide a description of some such tangent vectors. Our description arises from developing S-trajectories associated with sequences of permutations. One of the consequences of our development is the identification of terms in the series expansion for the S-trajectories that are independent of permutation. These are obstructions to local controllability in our setup. In the parlance of Sussmann [11], these are fixed points of a group action in a free Lie algebra.

4.1 Variations associated with sequences of permutations

Suppose that $X, Y \in S$. Then, for t sufficiently small,

$$Y_t \circ X_t(p) = \big(A^0(X,Y) + A^1(X,Y)t + A^2(X,Y)t^2 + \\ A^3(X,Y)t^3 + \cdots\big)_t(p)$$

where, from the Campbell-Baker-Hausdorff formula [12],

$$\begin{aligned}
A^0(X,Y) &= X+Y \\
A^1(X,Y) &= \frac{1}{2}\mathrm{ad}_X Y \\
A^2(X,Y) &= \frac{1}{12}(\mathrm{ad}_Y^2 X + \mathrm{ad}_X^2 Y) \\
A^3(X,Y) &= -\frac{1}{24}\mathrm{ad}_Y \mathrm{ad}_X^2 Y \\
A^4(X,Y) &= -\frac{1}{180}\mathrm{ad}_Y \mathrm{ad}_X^3 Y - \frac{1}{120}\left[\mathrm{ad}_X Y, \mathrm{ad}_X^2 Y\right] + \frac{1}{180}\,\mathrm{ad}_Y^2 \mathrm{ad}_X^2 Y + \\
&\qquad \frac{1}{360}\left[\mathrm{ad}_X Y, \mathrm{ad}_Y^2 X\right] - \frac{1}{720}\,\mathrm{ad}_X^4 Y - \frac{1}{720}\,\mathrm{ad}_Y^4 X
\end{aligned} \tag{9}$$

and $A^k(X,Y)$ can, in principal, be expressed explicitly as a function of X,Y for all $k>0$. Let $\mathbb{N}$ denote the positive integers. If $X^i, Y^i \in S$, $\boldsymbol{r} = (r_1,\ldots,r_k) \in \mathbb{N}^k$, then for $\sigma \in P_k^0$, the group of permutations of $\{1,2,\ldots,k\}$, we form the S-trajectory

$$\begin{aligned}
\mathcal{X}_t^\sigma(p) &= Y^{\sigma(1)}_{t^{r_{\sigma(1)}}} \circ X^{\sigma(1)}_{t^{r_{\sigma(1)}}} \circ Y^{\sigma(2)}_{t^{r_{\sigma(2)}}} \circ X^{\sigma(2)}_{t^{r_{\sigma(2)}}} \circ \cdots \circ Y^{\sigma(k)}_{t^{r_{\sigma(k)}}} \circ X^{\sigma(k)}_{t^{r_{\sigma(k)}}}(p) \\
&= (Q_\sigma^0 + Q_\sigma^1 t + Q_\sigma^2 t^2 + \cdots)_t(p)
\end{aligned} \tag{10}$$

where the vector fields $Q_\sigma^\ell = Q_\sigma^\ell(X^1,Y^1,\ldots,X^k,Y^k,\boldsymbol{r})$ are linear combinations of the vector fields $A^j(X^i,Y^i)$ and their Lie brackets. For example for $\boldsymbol{r} = (1,\ldots,1)$ we have

$$\begin{aligned}
Q_\sigma^0 &= A^0(X^1,Y^1) + \cdots + A^0(X^k,Y^k), \\
Q_\sigma^1 &= \sum_{i=1}^k A^1(X^i,Y^i) + \frac{1}{2}\sum_{1\le i<j\le k}[A^0(X^{\sigma(i)},Y^{\sigma(i)}), A^0(X^{\sigma(j)},Y^{\sigma(j)})].
\end{aligned}$$

The order of the group P_k^0 is $k!$ and we define P_k^1 to be the elements of the $k!$-fold direct product of P_k^0 with itself, $\Pi_{i=1}^{k!} P_k^0$, of the form $\sigma = (\sigma_1,\ldots,\sigma_{k!})$ where $\sigma_i \in P_k^0$ are *distinct*. We note that P_k^1 is a set with $\Gamma = k!!$ elements. If $\sigma = (\sigma_1,\ldots,\sigma_{k!}) \in P_k^1$ we define a corresponding S-trajectory

$$\mathcal{X}_t^\sigma(p) = \mathcal{X}_t^{\sigma_1} \circ \cdots \circ \mathcal{X}_t^{\sigma_{k!}}(p) = (Q_\sigma^0 + Q_\sigma^1 t + Q_\sigma^2 t^2 + \cdots)_t(p)$$

where, as above, $Q_\sigma^\ell = Q_\sigma^\ell(X^1,Y^1,\ldots,X^k,Y^k,\boldsymbol{r})$ is a linear combination of the vector fields $A^j(X^i,Y^i)$ and their Lie brackets. Similarly $\sigma \in P_k^2$ if $\sigma = (\sigma_1,\ldots,\sigma_\gamma)$ where $\sigma_i \in P_k^1$ and

$$\mathcal{X}_t^\sigma(p) = \mathcal{X}_t^{\sigma_1} \circ \cdots \circ \mathcal{X}_t^{\sigma_\Gamma}(p) = (Q_\sigma^0 + Q_\sigma^1 t + Q_\sigma^2 t^2 + \cdots)_t(p).$$

In this way we can inductively define subsets of permutations P_k^ℓ. It will be convenient to use the notation $\Gamma(k,\ell)$ to denote the cardinality of P_k^ℓ. Thus

$\Gamma(k,0) = k!$ and $\Gamma(k,\ell+1) = \Gamma(k,\ell)!$. Note that if $\sigma = (\sigma_1, \ldots, \sigma_{\Gamma(k,\ell)}) \in P_k^{\ell+1}$ where $\sigma_i \in P_k^\ell$, then $\mathcal{X}_t^\sigma(p)$ denotes the S-trajectory

$$\begin{aligned} \mathcal{X}_t^\sigma(p) &= \mathcal{X}_t^{\sigma_1} \circ \cdots \circ \mathcal{X}_t^{\sigma_{\Gamma(k,\ell)}}(p) \\ &= (Q_\sigma^0 + Q_\sigma^1 t + \cdots)_t(p). \end{aligned} \tag{11}$$

For example, $P_2^0 = \{\sigma_1, \sigma_2\}$ with

$$\sigma_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned} P_2^1 &= \{\sigma_1 \circ \sigma_2,\ \ \sigma_2 \circ \sigma_1\} \\ P_2^2 &= \{\sigma_1 \circ \sigma_2 \circ \sigma_2 \circ \sigma_1,\ \ \sigma_2 \circ \sigma_1 \circ \sigma_1 \circ \sigma_2\}. \end{aligned}$$

For $\sigma = \sigma_1 \circ \sigma_2 \in P_2^1$, then

$$\mathcal{X}_t^\sigma(p) = Y_{t^{r_1}}^1 \circ X_{t^{r_1}}^1 \circ Y_{t^{r_2}}^2 \circ X_{t^{r_2}}^2 \circ Y_{t^{r_2}}^2 \circ X_{t^{r_2}}^2 \circ Y_{t^{r_1}}^1 \circ X_{t^{r_1}}^1$$

and if $\sigma = \sigma_2 \circ \sigma_1 \in P_2^1$, then

$$\mathcal{X}_t^\sigma(p) = Y_{t^{r_2}}^2 \circ X_{t^{r_2}}^2 \circ Y_{t^{r_1}}^1 \circ X_{t^{r_1}}^1 \circ Y_{t^{r_1}}^1 \circ X_{t^{r_2}}^1 \circ Y_{t^{r_2}}^2 \circ X_{t^{r_2}}^2.$$

Similar expressions then hold for the elements of P_2^2. In essence these are analogous to the time reversal permutations considered by Sussmann [11].

4.2 Permutation-invariant elements

Next we turn to a more detailed investigation of the terms in the power series expansion for the S-trajectories of the preceding section. In particular, we show that such power series expansions possess terms that are independent of the sequence of permutations. In essence, these are terms in the series which cannot be modified by changing the sequence, and so may be thought of as obstructions to local controllability.

The first result exposes the pattern in which invariant terms arise in the series expansion (11) under sequences of permutations of a given length. If $\boldsymbol{r} = (r_1, \ldots, r_k) \in \mathbb{N}^k$ we set

$$m(\boldsymbol{r}) = \min\{r_i|\ \ 1 \le i \le k\}$$

and define $m_i(\boldsymbol{r})$ inductively by

$$m_0(\boldsymbol{r}) + 2 = \min\{r_i + r_j|\ \ 1 \le i, j \le k,\ \ i \ne j\}$$

and

$$m_\ell(\boldsymbol{r}) = m_0(\boldsymbol{r}) + \ell m(\boldsymbol{r}).$$

Lemma 4.1 *Let $\sigma \in P_k^\ell$ and let $\mathcal{X}_t^\sigma(p)$ be the S-trajectory*

$$\mathcal{X}_t^\sigma(p) = (Q_\sigma^0 + Q_\sigma^1 t + \cdots)_t(p).$$

defined by (11). *Then $Q_\sigma^0, \ldots, Q_\sigma^{m_\ell(\boldsymbol{r})}$ are independent of σ and $Q_\sigma^0, \ldots, Q_\sigma^{m(\boldsymbol{r})-2}$ vanish identically.*

Proof We begin by considering the case $\sigma \in P_k^0$. To help with notation we set

$$X_j(t) = \sum_{i=0}^{\infty} A^i(X^{\sigma(j)}, Y^{\sigma(j)}) t^{(i+1)r_{\sigma(j)}}$$

so that the S-trajectory $\mathcal{X}_t^\sigma(p)$ defined by (10) is the composition of integral curves of the vector fields X_j followed for one unit of time. Thus $\mathcal{X}_t^\sigma(p) = (X_1(t))_1 \circ \cdots \circ (X_k(t))_1(p)$, and, using the Campbell-Baker-Hausdorff formula, we have

$$\mathcal{X}_t^\sigma(p) = (\sum_{j=1}^{k} X_j(t) + \sum_{1 \le i < j \le k} \frac{1}{2}[X_i(t), Y_i(t)] + \cdots)_1(p), \tag{12}$$

where the additional terms are iterated brackets of the vector fields $X_i(t)$ of degree greater than two. We note that $\sum_{j=1}^k X_j(t)$ is independent of our choice of $\sigma \in P_k^0$. Writing the above vector field explicitly as a power series in t,

$$\mathcal{X}_t^\sigma(p) = (Q_\sigma^0 t + Q_\sigma^1 t^2 + \cdots)_t(p),$$

we see that, from the definition of $X_j(t)$, the lowest power of t with a nonzero coefficient will be t^m where $m(\boldsymbol{r}) = \min\{r_i \mid 1 \le i \le k\}$ as above. In particular $Q_\sigma^0, \ldots, Q_\sigma^{m(\boldsymbol{r})-2}$ are identically zero. Similarly the lowest power of t with a nonzero coefficient in $\sum_{1 \le i < j \le k} \frac{1}{2}[X_i(t), Y_j(t)]$ will be $t^{m_0(\boldsymbol{r})+2}$ so that $Q_\sigma^{m_0(r)+1}$ is the coefficient of the lowest power of t which could vary with $\sigma \in P_k^0$. From our definition of $m_0(\boldsymbol{r})$ we have $m(\boldsymbol{r}) < m_0(\boldsymbol{r})$ and hence

$$\mathcal{X}_t^\sigma(p) = (Q_\sigma^{m(\boldsymbol{r})-1} t^{m(\boldsymbol{r})-1} + \cdots + Q_\sigma^{m_0(\boldsymbol{r})} t^{m_0(\boldsymbol{r})} + Q_\sigma^{m_0(\boldsymbol{r})+1} t^{m_0(\boldsymbol{r})+1} + \cdots)_t(p),$$

where $Q_\sigma^{m(\boldsymbol{r})-1}, \ldots, Q_\sigma^{m_0(\boldsymbol{r})}$ are invariant with respect to $\sigma \in P_k^0$. This proves the lemma in the case $\ell = 0$. Now suppose that the lemma holds for $\sigma \in P_k^\ell$. Let $\sigma = (\sigma_1, \ldots \sigma_{\Gamma(k,\ell)}) \in P_k^{\ell+1}$ where $\sigma_i \in P_k^\ell$ and set

$$\mathcal{X}_t^\sigma(p) = \mathcal{X}_t^{\sigma_1} \circ \cdots \circ \mathcal{X}_t^{\sigma_{\Gamma(k,\ell)}}(p).$$

By assumption

$$\mathcal{X}_t^{\sigma_i}(p) = (Q_{\sigma_i}^{m(\boldsymbol{r})-1} + Q_{\sigma_i}^{m(\boldsymbol{r})} t + \cdots)_t(p)$$

where $Q_{\sigma_i}^{m(\boldsymbol{r})-1}, \ldots, Q_{\sigma_i}^{m_\ell(\boldsymbol{r})}$ are independent of σ_i. Setting $Q^j = Q_{\sigma_i}^j$ for $j = m(\boldsymbol{r}) - 1, \ldots, m_\ell(\boldsymbol{r})$ it follows that $\mathcal{X}_t^{\sigma_i}(p) = (\bar{X}_i(t))_t(p)$ where

$$\bar{X}_i(t) = Q^{m(\boldsymbol{r})-1} t^{m(\boldsymbol{r})-1} + Q^m(\boldsymbol{r}) t^m(\boldsymbol{r}) + \cdots + Q^{m_\ell(\boldsymbol{r})} t^{m_\ell(\boldsymbol{r})} + Q_{\sigma_i}^{m_\ell(\boldsymbol{r})+1} t^{m_\ell(\boldsymbol{r})+1} + \cdots .$$

As in (12), the Campbell-Baker-Hausdorff formula yields an expression for $\mathcal{X}_t^\sigma$ with $\bar{X}_i(t)$ replacing $X_i(t)$. Arguing as in the case $\ell = 0$ above we can conclude that

$$\begin{aligned}\mathcal{X}_t^\sigma(p) =& \big(\Gamma(k,\ell)Q^{m(\boldsymbol{r})-1}t^{m(\boldsymbol{r})-1} + \cdots + \Gamma(k,\ell)Q^{m_\ell(\boldsymbol{r})}t^{m_\ell(\boldsymbol{r})} + \\ &(Q_{\sigma_1}^{m_\ell(\boldsymbol{r})+1} + \cdots + Q_{\sigma_{\Gamma(k,\ell)}}^{m_\ell(\boldsymbol{r})+1})t^{m_\ell(\boldsymbol{r})+1} + \cdots (Q_{\sigma_1}^{m_\ell(\boldsymbol{r})+m} + \cdots + \\ &\qquad Q_{\sigma_{\Gamma(k,\ell)}}^{m_\ell(\boldsymbol{r})+m})t^{m_\ell(\boldsymbol{r})+m} + Q_\sigma^{m_\ell(\boldsymbol{r})+m+1}t^{m_\ell(\boldsymbol{r})+m+1} + \cdots\big)_t(p).\end{aligned}$$

Since $m_\ell(\boldsymbol{r}) + m = m_{\ell+1}(\boldsymbol{r})$ and in the above equation the coefficients of t^i with $i \le m_{\ell+1}(\boldsymbol{r})$ are σ-invariant the induction is complete. ■

Let $\sigma \in P_k^\ell$. In light of Lemma 4.1 we set

$$Q_{\mathrm{inv}}^i = Q_\sigma^i, \quad m_{\ell-1}(\boldsymbol{r}) < i \le m_\ell(\boldsymbol{r}).$$

where $Q_{\mathrm{inv}}^i = Q_{\mathrm{inv}}^i(X^1, Y^1, \ldots, X^k, Y^k, \boldsymbol{r})$ depends on X^i, Y^i and $\boldsymbol{r}$ but is independent of σ. For $\ell = 0$ we set

$$Q_{\mathrm{inv}}^i = Q_\sigma^i, \quad i \in \{0, 1, \ldots, m_0(\boldsymbol{r})\}.$$

In the case $\boldsymbol{r} = (1, \ldots, 1)$ it is straightforward to show that $m_\ell(\boldsymbol{r}) = \ell$ and

$$\begin{aligned}Q_{\mathrm{inv}}^0 &= A^0(X^1, Y^1) + \cdots + A^0(X^k, Y^k),\\ Q_{\mathrm{inv}}^1 &= A^1(X^1, Y^1) + \cdots + A^1(X^k, Y^k),\\ Q_{\mathrm{inv}}^2 &= A^2(X^1, Y^1) + \cdots + A^2(X^k, Y^k) + B,\end{aligned}$$

where B is a linear combination of degree 3 brackets of the vector fields $A^0(X^i, Y^i)$. For our application, the pairs $\{X^i, Y^i\}$ above will be complementary at p so that $A^0(X^i, Y^i)$ and hence B vanish at p.

The following Proposition relates the definition of Q_{inv}^i to the S-trajectory corresponding to $\sigma \in P_k^\ell$ where $i \le m_\ell(\boldsymbol{r})$.

Proposition 4.2 *For each $\ell \ge 0$ and $\sigma \in P_k^\ell$ there corresponds an S-trajectory of the form*

$$\mathcal{X}_t^\sigma(p) = \big(\alpha_0 Q_{\mathrm{inv}}^0 + \cdots + \alpha_{m_\ell(\boldsymbol{r})} Q_{\mathrm{inv}}^{m_\ell(\boldsymbol{r})} t^{m_\ell(\boldsymbol{r})} + Q_\sigma^{m_\ell(\boldsymbol{r})+1} t^{m_\ell(\boldsymbol{r})+1} + \cdots\big)_t(p),$$

where $\alpha_i > 0$, $i \in \{0, 1, \ldots, m_\ell(\boldsymbol{r})\}$.

Proof The proof of Lemma 4.1 contains this result with a slight change of notation using the subscript inv to keep track of the vector fields invariant with respect to the appropriate collection of permutations. ■

Remark 4.3 In the case of an single-input affine system $\dot{x} = f(x) + ug(x)$, consider the sets $\{X_1 = f + g, Y_1 = f - g\}$ and $\{X_2 = f - g, Y_2 = f + g\}$, and take $r_1 = r_2 = 1$ to compute

$$\begin{aligned}
Q^0_{\mathrm{inv}} &= 8f, \\
Q^2_{\mathrm{inv}} &= \frac{16}{3}\mathrm{ad}^2_g f, \\
Q^4_{\mathrm{inv}} &= \frac{16}{15}\mathrm{ad}^4_g f + \frac{112}{45}\mathrm{ad}_g \mathrm{ad}^3_f g - \frac{992}{45}[\mathrm{ad}_f g, \mathrm{ad}^2_f g], \\
Q^6_{\mathrm{inv}} &= \frac{2272}{315}[[f,g], \mathrm{ad}^4_f g] - \frac{239824}{945}[\mathrm{ad}^2_f g, \mathrm{ad}^3_f g] + \frac{64}{105}\mathrm{ad}^3_g \mathrm{ad}^3_f g \\
&\quad - \frac{2048}{315}[\mathrm{ad}_f g, \mathrm{ad}^3_g \mathrm{ad}^2_f g] + \frac{6752}{315}[\mathrm{ad}_f g, [\mathrm{ad}_f g, \mathrm{ad}^2_g f]] - \frac{288}{35}[\mathrm{ad}^2_f g, \mathrm{ad}^3_g f]] \\
&\quad + \frac{32}{315}\mathrm{ad}^6_g f + \frac{352}{945}[\mathrm{ad}^2_g f, [g, \mathrm{ad}^2_f g]] - \frac{352}{945}[g, \mathrm{ad}^5_f g].
\end{aligned}$$

These are linear combinations of *bad* vector fields as per [11]. We show in Corollary 4.8 that for two pairs of complementary vector fields, $Q^\ell_{\mathrm{inv}} = 0$ for ℓ odd. We also remark that the eccentric character of the coefficients in the expressions for the permutation invariant brackets is a consequence of our use of the Campbell-Baker-Hausdorff formula. □

Remark 4.4 In a given example one may have many more permutation-invariant vector fields than the Q^i_{inv}, which are invariant on essentially the free Lie algebra level. □

4.3 Applications to local controllability

In this section we summarize the above developments as they apply to conditions for local controllability. The following result relates the permutation dependent constructions to the more general constructions of Section 3.2.

Theorem 4.5 *Suppose that* $\{X^i, Y^i\} \subset S$ *for* $i = 1, \ldots, k$*,* $\boldsymbol{r} \in \mathbb{N}^k$*, and* $Q^0_{\mathrm{inv}}(p) = Q^1_{\mathrm{inv}}(p) = \cdots = Q^{m_\ell(\boldsymbol{r})}_{\mathrm{inv}}(p) = 0$*. Then*

1. $Q^{m_\ell(\boldsymbol{r})+1}_\sigma \in S^{\ell+1}_p$ *for all* $\sigma \in P^\ell_k$*,*
2. $Q^{m_\ell(\boldsymbol{r})+1}_{\mathrm{inv}} \in S^{\ell+1}_p$*.*

The next three corollaries specialize the theorem to interesting cases. The first deals with the case when all time rescalings are equal to 1. In practice, this will often be the case, but in Remark 4.10 we provide a situation where it is beneficial to allow the more general class of rescalings.

Corollary 4.6 *Suppose that* $\{X^i, Y^i\} \subset S$ *for* $i = 1, \ldots, k$ *and* $\boldsymbol{r} = (1, \ldots, 1)$*. Then*

1. *if* $Q^0_{\mathrm{inv}}(p) = Q^1_{\mathrm{inv}}(p) = \cdots = Q^\ell_{\mathrm{inv}}(p) = 0$ *then* $Q^{\ell+1}_\sigma \in S^{\ell+1}_p$ *for all* $\sigma \in P^\ell_k$*,*
2. *if* $\{X^i, Y^i\}$ *are complementary at* p *for* $i = 1, \ldots, k$ *then*

(a) $\pm\mathrm{ad}_{X^i}Y^i \in S_p^1$ and $\pm\mathrm{ad}_{(X^{i_1}+Y^{i_1})} \circ \cdots \circ \mathrm{ad}_{(X^{i_s}+Y^{i_s})}\mathrm{ad}_{X^i}Y^i(p) \in S_p^1(p)$ where $i_1, \ldots, i_s \in \{1, \ldots, k\}$,

(b) $2\mathrm{ad}^2_{X^i}Y^i - \mathrm{ad}^2_{Y^i}X^i \in S_p^2$ and $\mathrm{ad}^2_{X^i}Y^i \in S_p^2$,

(c) $\pm\mathrm{ad}_{Y^i}\mathrm{ad}^2_{X^i}Y^i \in S_p^3$ if $\sum_{i=1}^k(\mathrm{ad}^2_{X^i}Y^i + \mathrm{ad}^2_{Y^i}X^i)(p) = 0$.

Our next result specializes Theorem 4.5 to two pairs of vector fields.

Corollary 4.7 *Suppose that* $\{X^1, Y^1\}, \{X^2, Y^2\} \subset S$, $\boldsymbol{r} = (1, 1)$, *and* $Q^0_{\mathrm{inv}}(p) = Q^1_{\mathrm{inv}}(p) = \cdots = Q^\ell_{\mathrm{inv}}(p) = 0$ *for* ℓ *even. Then*

1. $\pm Q^{\ell+1}_\sigma \in S_p^{\ell+1}$ *for all* $\sigma \in P_2^\ell$,

2. $Q^{\ell+2}_\sigma \in S_p^{\ell+2}$ *for all* $\sigma \in P_2^{\ell+1}$.

Finally, we consider the case of a single pair of vector fields. In practice, this simple result is often the most useful, as we shall see in Section 5.

Corollary 4.8 *Suppose* $\{X, Y\} \subset S$.

1. *if* $\boldsymbol{r} = (1, 1)$ *then for the pairs* $\{X^1, Y^1\} = \{X, Y\}$ *and* $\{X^2, Y^2\} = \{Y, X\}$ *we have* $Q^\ell_{\mathrm{inv}} = 0$ *for* ℓ *odd.*

2. *if* $\boldsymbol{r} = (1)$ *then* $Q^\ell_{\mathrm{inv}} = A^\ell(X, Y)$. *In particular,* $A^k(X, Y)(p) = 0$, $k \in \{0, 1, \ldots, \ell\}$ *implies* $A^{\ell+1}(X, Y) \in S_p^{\ell+1}$.

Remark 4.9 We can replace one or more of the pairs $\{X^i, Y^i\}$ in Theorem 4.5 with $\{Y^i, X^i\}$ to generate additional vector fields in $S_p^{\ell+1}$. $\square$

Remark 4.10 In Theorem 4.5 the vanishing of the vector fields Q^i_{inv} at p can be replaced by conditions for neutralization resembling those in the existing literature (e.g., [11, 7]). That is, we may ask not that $Q^0_{\mathrm{inv}}(p) = \cdots = Q^\ell_{\mathrm{inv}}(p) = 0$ but that $Q^0_{\mathrm{inv}}(p) = \cdots = Q^{\ell-1}_{\mathrm{inv}}(p) = 0$ and $0 \in \mathrm{co}\{Q^0_{\mathrm{inv}}(p), Q^1_{\mathrm{inv}}(p), \ldots, Q^\ell_{\mathrm{inv}}(p)\}$. More generally, suppose that for $i, j \in \mathbb{N}$ we denote

$$Q^i_{\mathrm{inv}} = Q^i_{\mathrm{inv}}(X^1, Y^1, \ldots, X^k, Y^k, \boldsymbol{r}), \quad \tilde{Q}^j_{\mathrm{inv}} = \tilde{Q}^j_{\mathrm{inv}}(\tilde{X}^1, \tilde{Y}^1, \ldots, \tilde{X}^{\tilde{k}}, \tilde{Y}^{\tilde{k}}, \tilde{\boldsymbol{r}}),$$

and that for a specific $\ell, m \in \mathbb{N}$ we have $Q^\ell_{\mathrm{inv}}(p) + \tilde{Q}^m_{\mathrm{inv}}(p) = 0$. Then we consider the augmented collection of pairs of vector fields

$$\{X^1, Y^1\}, \ldots, \{X^k, Y^k\}, \{\tilde{X}^1, \tilde{Y}^1\}, \ldots, \{\tilde{X}^{\tilde{k}}, \tilde{Y}^{\tilde{k}}\}$$

and choose $\hat{\boldsymbol{r}} = (mr_1, \ldots, mr_k, \ell\tilde{r}_1, \ldots, \ell\tilde{r}_{\tilde{k}})$. The resulting set of invariant vector fields $\hat{Q}^i_{\mathrm{inv}}$ will have $\hat{Q}^{m\ell-1}_{\mathrm{inv}}$ a positive multiple of $Q^\ell_{\mathrm{inv}} + \tilde{Q}^m_{\mathrm{inv}}$ and for $j < ml - 1$ the vector fields $\hat{Q}^j_{\mathrm{inv}}$ will be linear combinations of $Q^0_{\mathrm{inv}}, \ldots, Q^{\ell-1}_{\mathrm{inv}}, \tilde{Q}^0_{\mathrm{inv}}, \ldots, \tilde{Q}^{m-1}_{\mathrm{inv}}$, and their Lie brackets. The notion of rescaling time to generate new "higher-order" S-trajectories is inherent in the definition of the sets S_p^k. Examples 5.3 and 5.4 illustrates this point. $\square$

Remark 4.11 Stefani's example [8]

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= x\\ \dot{z} &= x^3 y\end{aligned}$$

in $\mathbb{R}^3$ fits the framework of Corollary 4.7. As noted in Sussmann's paper [11], the Lie brackets in $f = (0, y, x^3 y)$ and $g = (1, 0, 0)$ of degree 3, 4, and 5 vanish at $p = (0, 0, 0)$ and thus Corollary 4.7–1 implies $\pm \mathrm{ad}_g^3 \mathrm{ad}_f^2 g$ is in S_p^5. Theorem 3.7 implies local controllability. □

Proof of Theorem 4.5 Choose $\sigma_i \in P_k^\ell$. Then Proposition 4.2 asserts that there exists an S-trajectory

$$\mathcal{X}_t^\sigma(p) = \big(\alpha_0^i Q_{\mathrm{inv}}^0 + \cdots + \alpha_{m_\ell(\boldsymbol{r})}^i Q_{\mathrm{inv}}^{m_\ell(\boldsymbol{r})} t^{m_\ell(\boldsymbol{r})} + Q_{\sigma_i}^{m_\ell(\boldsymbol{r})+1} t^{m_\ell(\boldsymbol{r})+1} + \cdots\big)_t(p).$$

Since $Q_{\mathrm{inv}}^i(p)$ vanishes for $0 \le i \le m_\ell(\boldsymbol{r})$ it follows that $Q_{\sigma_i}^{m_\ell(\boldsymbol{r})+1} \in S_p^a$ for some $a \in \mathbb{N}$. Here $Q_{\sigma_i}^{m_{\ell+1}(\boldsymbol{r})}$ is the coefficient Q_{σ_i} of the lowest power of t with the property that Q_{σ_i} could vary with $\sigma_i \in P_k^\ell$. A straightforward computation based on the proof of Lemma 4.1 and the definitions of $m(\boldsymbol{r})$ and $m_\ell(\boldsymbol{r})$ shows that $a = \ell + 1$. Finally, if $P_k^{\ell+1} = \{\sigma_1, \ldots, \sigma_{\Gamma(k,\ell)}\}$ then we form the S-trajectory

$$\mathcal{X}_t^\sigma(p) = \big(\alpha_0 Q_{\mathrm{inv}}^0 + \cdots + \alpha_{m_\ell(\boldsymbol{r})} Q_{\mathrm{inv}}^{m_\ell(\boldsymbol{r})} t^{m_\ell(\boldsymbol{r})} + \sum_{i=1}^{\Gamma(k,\ell)} Q_{\sigma_i}^{m_\ell(\boldsymbol{r})+1} t^{m_\ell(\boldsymbol{r})+1} + \cdots\big)_t(p),$$

where $\alpha_i > 0$. Since

$$Q_{\mathrm{inv}}^{m_\ell(\boldsymbol{r})+1} = \sum_{i=1}^{\Gamma(k,\ell)} Q_{\sigma_i}^{m_\ell(\boldsymbol{r})+1},$$

it follows that $Q_{\mathrm{inv}}^{m_\ell(\boldsymbol{r})+1} \in S_p^{\ell+1}$. ■

Before proving the Corollaries to Theorem 4.5 we establish some technical lemmas.

Lemma 4.12 *Suppose that P, Q are vector fields on $\Omega \subset \mathbb{R}^n$. Then, for t sufficiently small, the integral curve $Q_t \circ P_t(p) = (\sum_{\ell=0}^{\infty} M^\ell(P,Q))_t(p)$ for vector fields $M^\ell(P,Q)$ with the following properties:*

1. *$M^\ell(P,Q) = (-1)^\ell M^\ell(Q,P)$,*
2. *if $P = \sum_{i=0}^{\infty} A_1^i t^i$ and $Q = \sum_{i=0}^{\infty} A_2^i t^i$ then $M^\ell(P,Q)$ has a power series expansion in t whose coefficients are Lie brackets of the vector fields A_j^i of degree $\ell + 1$.*

Proof The existence of the vector fields $M^\ell(Q,P)$ follows from the Campbell-Baker-Hausdorff formula [12] and we have $M^0(P,Q) = P+Q = M^0(Q,P)$ and $M^1(P,Q) = \frac{1}{2}\mathrm{ad}_P Q = -\frac{1}{2}\mathrm{ad}_Q P = -M^1(Q,P)$. Thus $P+Q$ is a linear combination of the vector fields A^i_j, which we call Lie brackets of degree 1 while $M^1(P,Q)$ is a linear combination of the vector fields of the form $[A^i_1, A^j_2]$ which are Lie brackets of degree 2. Thus 1 and 2 hold for $\ell = 0,1$. We now establish 1. Set $M^k = M^k(P,Q)$ and $\bar{M}^k = M^k(Q,P)$. Suppose that $M^j = (-1)^j \bar{M}^j$ for $j<\ell$. For $j=\ell$ the Campbell-Baker-Hausdorff formula [12] asserts that

$$(\ell+1)M^\ell = \frac{1}{2}[P-Q, M^{\ell-1}] + \sum_{\substack{p\geq 1\\ 2p\leq \ell}} K_{2p} V_p(P,Q)$$

with

$$V_p(P,Q) = \sum_{\substack{k_1,\ldots,k_{2p}>0\\ k_1+\cdots+k_{2p}=\ell}} [M^{k_1-1},[M^{k_2-1},\ldots,[M^{k_{2p}-1},P+Q]\ldots]$$

Hence

$$(\ell+1)\bar{M}^\ell = \frac{1}{2}[Q-P, \bar{M}^{\ell-1}] + \sum_{\substack{p\geq 1\\ 2p\leq \ell}} K_{2p} \bar{V}_p(P,Q)$$

with

$$\bar{V}_p(P,Q) = \sum_{\substack{k_1,\ldots,k_{2p}>0\\ k_1+\cdots+k_{2p}=\ell}} [\bar{M}^{k_1-1},[\bar{M}^{k_2-1},\ldots,[\bar{M}^{k_{2p}-1},P+Q]\cdots].$$

By our induction hypothesis we know that $M^j = (-1)^j\bar{M}^j$ for $j<\ell$ thus $\frac{1}{2}[Q-P,\bar{M}^{\ell-1}] = (-1)^\ell\frac{1}{2}[P-Q,M^{\ell-1}]$ and

$$\begin{aligned}
\bar{V}_p(P,Q) &= \sum_{\substack{k_1,\ldots,k_{2p}>0\\ k_1+\cdots+k_{2p}=\ell}} [(-1)^{k_1-1}M^{k_1-1},[\cdots,[(-1)^{k_{2p}-1}M^{k_{2p}-1},P+Q]\ldots]\\
&= \sum_{\substack{k_1,\ldots,k_{2p}>0\\ k_1+\cdots+k_{2p}=\ell}} (-1)^{k_1+\cdots+k_{2p}-2p}[M^{k_1-1},[\cdots,[M^{k_{2p}-1},P+Q]\ldots]\\
&= (-1)^\ell V_p(P,Q)
\end{aligned}$$

since $(-1)^{k_1+\cdots+k_{2p}-2p} = (-1)^\ell(-1)^{2p} = (-1)^\ell$. This implies $M^\ell(P,Q) = (-1)^\ell M^\ell(Q,P)$.

To establish 2 we note that 2 holds for $\ell = 0,1$. Suppose that $M^j = (-1)^j\bar{M}^j$ for $j<\ell$. Thus $M^{\ell-1}(P,Q)$ has a power series expansion in t whose coefficients are Lie brackets of the vector fields A^i_j of degree ℓ. Now $P-Q$ has a power series expansion in t whose coefficients are Lie brackets of the vector fields A^i_j of degree 1 hence, in the above formula for $M^\ell(P,Q)$, the term $[P-Q,M^{\ell-1}]$ is a combination of Lie brackets of the vector fields A^i_j of degree

$\ell+1$. The remaining terms in $M^\ell(P,Q)$ involve the vector fields $V_p(P,Q)$. By our induction hypothesis the vector fields M^{k_i-1} in $V_p(P,Q)$ involve Lie brackets of the vector fields A_j^i of degree k_i. Since $P+Q$ involve Lie brackets of the vector fields A_j^i of degree 1 it follows that $[M^{k_1-1},[M^{k_2-1},\ldots,[M^{k_{2p}-1},P+Q]\ldots]$ has a power series expansion in t whose coefficients are Lie brackets of the vector fields A_j^i of degree $k_1+\cdots+k_{2p}+1=\ell+1$. This completes the induction. ∎

Lemma 4.13 *Suppose that* $\{X^1,Y^1\},\{X^2,Y^2\}\subset S$ *and* $\boldsymbol{r}=(1,1)$. *Then* $Q^\ell_{\rm inv}$ *is a power series in* t *whose coefficients are Lie brackets of odd degree of the vector fields* A_j^i *for all* $\ell\geq 0$.

In particular, $Q^\ell_{\rm inv}(X^1,Y^1,X^2,Y^2,\boldsymbol{r})=(-1)^\ell Q^\ell_{\rm inv}(Y^1,X^1,Y^2,X^2,\boldsymbol{r})$.

Proof We begin by examining the S-trajectories which correspond to permutation in S_2^0. By definition $S_2^0=\{\sigma_1,\sigma_2\}$ where $\sigma_1(1)=1,\sigma_1(2)=2$ and $\sigma_2(1)=2,\sigma_2(2)=1$. Then

$$\mathcal{X}_t^{\sigma_1}(p)=X_t^1\circ Y_t^1\circ X_t^2\circ Y_t^2(p)=\Big(\sum_{i=0}^\infty A_1^it^i)_t\circ(\sum_{i=0}^\infty A_2^it^i\Big)_t(p)$$

where $A_1^i=A^i(X^1,Y^1)$ and $A_2^i=A^i(X^2,Y^2)$. Set $P=\sum_{i=0}^\infty A_1^it^i, Q\sum_{i=0}^\infty A_2^it^i$ so P and Q are power series in t whose coefficients are Lie brackets of the vector fields A_j^i of degree 1 (odd). Thus $\mathcal{X}_t^{\sigma_1}(p)=P_t\circ Q_t(p)$ and, in light of Lemma 4.12, there exist vector fields $M^i(P,Q)$ such that $\mathcal{X}_t^{\sigma_1}(p)=(\sum_{i=0}^\infty M^i(P,Q))_t(p)$. Since $\mathcal{X}_t^{\sigma_2}(p)=Q_t\circ P_t(p)$ we have

$$\mathcal{X}_t^{\sigma_2}(p)=\Big(\sum_{i=0}^\infty M^i(Q,P)\Big)_t(p)=\Big(\sum_{i=0}^\infty(-1)^iM^i(P,Q)\Big)_t(p)$$

where $M^\ell(P,Q)$ is a power series in t whose coefficients are Lie brackets of the vector fields A_j^i of degree $\ell+1$. We let $M^{\rm odd}(P,Q)=\sum_{i=0}^\infty M^{2i}(P,Q)$ and $M^{\rm even}(P,Q)=\sum_{i=0}^\infty M^{2i+1}(P,Q)$ so that $M^{\rm odd}(P,Q)$ $(M^{\rm even}(P,Q))$ is a power series in t whose coefficients are Lie brackets of the vector fields A_j^i of odd (even) degree. Furthermore $M^{\rm odd}(P,Q)=M^{\rm odd}(Q,P)$ while $M^{\rm even}(P,Q)=-M^{\rm even}(Q,P)$. We now explore the same issues for $S_2^1=\{\hat\sigma_1,\hat\sigma_2\}$ where $\hat\sigma_1=(\sigma_1,\sigma_2)$ and $\hat\sigma_2=(\sigma_2,\sigma_1)$ for the permutations $\sigma_1,\sigma_2\in S_2^0$ defined above. Setting $\hat P=\sum_{i=0}^\infty M^i(P,Q)$ and $\hat Q=\sum_{i=0}^\infty M^i(Q,P)$ we have, as above,

$$\mathcal{X}_t^{\hat\sigma_1}(p)=\mathcal{X}_t^{\sigma_1}\circ\mathcal{X}_t^{\sigma_2}(p)=P_t\circ Q_t(p).$$

From Lemma 4.12 there exist vector fields $\hat M^\ell(\hat P,\hat Q)$ such that $\hat M^\ell(\hat Q,\hat P)=(-1)^\ell\hat M^\ell(\hat P,\hat Q)$ hence

$$\mathcal{X}_t^{\hat\sigma_1}(p)=\Big(\sum_{\ell=0}^\infty\hat M^\ell(\hat P,\hat Q)\Big)_t(p),\quad \mathcal{X}_t^{\hat\sigma_2}(p)=\Big(\sum_{\ell=0}^\infty(-1)^\ell\hat M^\ell(\hat P,\hat Q)\Big)_t(p).$$

We now establish that the vector fields $\hat M^\ell(\hat P,\hat Q)$ are power series in t whose coefficients are Lie brackets of the vector fields A_j^i of odd degree. We showed above that we have $\hat{P} = \sum_{i=0}^{\infty} \hat{M}^i(P,Q) = M^{\text{odd}}(P,Q) + M^{\text{even}}(P,Q)$ while $\hat{Q} = \sum_{i=0}^{\infty} \hat{M}^i(Q,P) = M^{\text{odd}}(P,Q) - M^{\text{even}}(P,Q)$. From the Campbell-Baker-Hausdorff formula we know that $\hat{M}^0 = \hat{P} + \hat{Q} = 2M^{\text{odd}}(P,Q)$, a power series in t whose coefficients are Lie brackets of the vector fields A^i_j of odd degree. Also

$$\hat{M}^1 = 1/2[\hat{P}, \hat{Q}] = [M^{\text{even}}(P,Q), M^{\text{odd}}(P,Q)].$$

Since $M^{\text{even}}(P,Q)$ is composed of Lie brackets of A^i_j of even degree and $M^{\text{odd}}(P,Q)$ is composed of Lie brackets of A^i_j of odd degree it follows that $\hat{M}^1$ is composed of Lie brackets of A^i_j of odd degree. Now suppose that this holds for $\hat{M}^2, \hat{M}^3, \ldots, \hat{M}^{\ell-1}$. From the Campbell-Baker-Hausdorff formula

$$(\ell+1)\hat{M}^\ell = \frac{1}{2}[2M^{\text{even}}(P,Q), M^{\ell-1}] + \sum_{\substack{p \geq 1 \\ 2p \leq \ell}} K_{2p} V_p(P,Q)$$

with

$$V_p(P,Q) = \sum_{\substack{k_1,\ldots,k_{2p}>0 \\ k_1+\cdots+k_{2p}=\ell}} [\hat{M}^{k_1-1}, [\hat{M}^{k_2-1}, \ldots, [\hat{M}^{k_{2p}-1}, 2M^{\text{odd}}(P,Q)] \cdots].$$

By our induction hypothesis and the fact that $M^{\text{even}}(P,Q)$ is composed of Lie brackets of A^i_j of even degree we see that $[2M^{\text{even}}(P,Q), M^{\ell-1}]$ is is composed of Lie brackets of A^i_j of odd degree. Looking at the terms in $V_p(P,Q)$ we note that

$$[\hat{M}^{k_2-1}, \ldots, [\hat{M}^{k_{2p}-1}, 2M^{\text{odd}}(P,Q)] \cdots]$$

has an even number of terms M^{k_i-1} and $M^{\text{odd}}(P,Q)$ is composed of Lie brackets of A^i_j of odd degree it follows that $\hat{M}^\ell$ is composed of Lie brackets of A^i_j of odd degree. We can now repeat the initial argument to show that if $S_2^2 = \{\sigma_1, \sigma_2\}$ then there exist vector fields $M^{\text{even}}(P,Q)$ composed of Lie brackets of A^i_j of even degree and $M^{\text{odd}}(P,Q)$ composed of Lie brackets of A^i_j of odd degree such that

$$\mathcal{X}_t^{\sigma_1}(p) = \Big(\sum_{i=0}^{\infty} M^i(P,Q))_t(p) = (M^{\text{odd}}(P,Q) + M^{\text{even}}(P,Q)\Big)_t(p)$$

and

$$\begin{aligned}\mathcal{X}_t^{\sigma_2}(p) &= \Big(\sum_{i=0}^{\infty} M^i(Q,P)\Big)_t(p) = \Big(\sum_{i=0}^{\infty}(-1)^i M^i(P,Q)\Big)_t(p) \\ &= (M^{\text{odd}}(P,Q) - M^{\text{even}}(P,Q))_t(p).\end{aligned}$$

We simply repeat the above steps for $S_2^3, S_2^4, \ldots$ to conclude that the vector fields $\hat{M}^\ell(\hat{P}, \hat{Q})$ are power series in t whose coefficients are Lie brackets of the vector fields A^i_j of odd degree.

We now are in a position to verify that $Q^\ell_{\rm inv}$ is a power series in t whose coefficients are Lie brackets of the vector fields A^i_j of odd degree. We begin by choosing any $\sigma \in S^k_2$. Then we know from above that the corresponding S-trajectory $\mathcal{X}^\sigma_t(p) = (M^{\rm odd}(P,Q) + M^{\rm even}(P,Q))_t(p)$ where $M^{\rm even}(P,Q)$ composed of Lie brackets of A^i_j of even degree and $M^{\rm odd}(P,Q)$ composed of Lie brackets of A^i_j of odd degree. In the case where k is odd we showed that $M^{\rm even}(P,Q) = 0$. Suppose $k = 1$. Then $\mathcal{X}^\sigma_t(p) = (M^{\rm odd}(P,Q))_t(p)$ where $M^{\rm odd}(P,Q)$ is a power series in t whose coefficients are Lie brackets of the vector fields A^i_j of odd degree. Thus there exist vector fields $Q^{\rm odd}_{0,1}, Q^{\rm odd}_{1,1}, \ldots,$ which are linear combinations of Lie brackets of the vector fields A^i_j of odd degree, such that

$$\mathcal{X}^\sigma_t(p) = (Q^{\rm odd}_{0,1} + Q^{\rm odd}_{1,1}t + Q^{\rm odd}_{2,1}t^2 + \cdots)_t(p).$$

But from Proposition 4.2 we have

$$\mathcal{X}^\sigma_t(p) = (Q^0_{\rm inv} + Q^1_\sigma t + Q^2_\sigma t^2 + \cdots)_t(p).$$

This means that $Q^0_{\rm inv} = Q^{\rm odd}_{0,1}$, a linear combinations of Lie brackets of the vector fields A^i_j of odd degree. Next we consider the case where $\sigma \in S^2_2 = \{\sigma_1, \sigma_2\}$. Then

$$\mathcal{X}^{\sigma_1}_t(p) = (M^{\rm odd}(P,Q) + M^{\rm even}(P,Q))_t(p)$$

where $M^{\rm odd}(P,Q)$ (resp. $M^{\rm even}(P,Q)$) is a power series in t whose coefficients are Lie brackets of the vector fields A^i_j of odd (resp. even) degree. Thus there exist vector fields $Q^{\rm odd}_{0,1}, Q^{\rm odd}_{1,1}, \ldots,$ and $Q^{\rm even}_{0,1}, Q^{\rm even}_{1,1}, \ldots,$ which are linear combinations of Lie brackets of the vector fields A^i_j of odd and even degrees respectively, such that

$$\mathcal{X}^{\sigma_1}_t(p) = ((Q^{\rm odd}_{0,1} + Q^{\rm even}_{0,1}) + (Q^{\rm odd}_{1,1} + Q^{\rm even}_{1,1})t + (Q^{\rm odd}_{2,1} + Q^{\rm even}_{2,1})t^2 + \cdots)_t(p).$$

But from Proposition 4.2 we have

$$\mathcal{X}^{\sigma_1}_t(p) = (\alpha_0 Q^0_{\rm inv} + \alpha_1 Q^1_{\rm inv} t + Q^2_{\sigma_1} t^2 + \cdots)_t(p).$$

Similarly

$$\begin{aligned}\mathcal{X}^{\sigma_2}_t(p) &= (\alpha_0 Q^0_{\rm inv} + \alpha_1 Q^1_{\rm inv} t + Q^2_{\sigma_2} t^2 + \cdots)_t(p)\\ &= ((Q^{\rm odd}_{0,1} - Q^{\rm even}_{0,1}) + (Q^{\rm odd}_{1,1} - Q^{\rm even}_{1,1})t + (Q^{\rm odd}_{2,1} - Q^{\rm even}_{2,1})t^2 + \cdots)_t(p).\end{aligned}$$

Since $Q^\ell_{\rm inv}$ is invariant with respect to our choice of permutation in $S^{\ell+1}_2$, we can conclude that $Q^{\rm even}_{0,1} = Q^{\rm even}_{1,1} = 0$. This in turn implies that $Q^0_{\rm inv}$ and $Q^1_{\rm inv}$ are linear combinations of Lie brackets of the vector fields A^i_j of odd degree. It is straightforward to show by induction that this is the case for all $Q^\ell_{\rm inv}$.

Finally we show that

$$Q^\ell_{\rm inv}(X^1, Y^1, X^2, Y^2) = (-1)^\ell Q^\ell_{\rm inv}(Y^1, X^1, Y^2, X^2).$$

Using Lemma 4.12 with $P = X, Q = Y$ we conclude that $A^i(X,Y) = (-1)^i A^i(Y,X)$. The vector fields A^i_j enter into our S-trajectory in the power series $\sum_{i=0}^{\infty} A^i(X^j,Y^j)t^i$. Since $Q^\ell_{\rm inv}$ is the coefficient of t^ℓ in a power series expansion of a similar S-trajectory we can conclude that $Q^\ell_{\rm inv}$ is a linear combination of iterated Lie brackets

$$B = [A^{i_1}_{j_1}, [[A^{i_2}_{j_2}, \dots, [A^{i_{2k}}_{j_{2k}}, A^{i_{2k+1}}_{j_{2k+1}}] \dots]$$

of an odd number of A^i_j's where $j_m \in \{1,2\}$ and $i_1 + \cdots + i_{2k+1} = \ell$. In light of Lemma 4.12 with $Q = Y, P = X$ we know that if i_m is even then $A^{i_m}_j(Y^j,X^j) = A^{i_m}_j(X^j,Y^j)$ and if i_m is odd then $A^{i_m}_j(Y^j,X^j) = -A^{i_m}_j(X^j,Y^j)$ for $j \in \{1,2\}$. If ℓ is even then there must be an even number of integers in $\{i_1,\dots,i_{2k+1}\}$ which are odd, and hence B does not change sign when X^i and Y^i are interchanged and this completes the proof. ∎

Proof of Corollary 4.6 Suppose that $\boldsymbol{r} = (1,\dots,1)$. Then 1 follows from Theorem 4.5 and the observation that in the case $\boldsymbol{r} = (1,\dots,1)$ we have $m_i(\boldsymbol{r}) = i$. Suppose that the subsets $\{X^i,Y^i\} \subset S$ are complementary at p for $i = 1,\dots,k$. From Remark 3.3 (or from the definition of $A^1(X^i,Y^i)$) we know that $\mathrm{ad}_{X^i}Y^i \in S^1_p$. Also $\{X^i,Y^i\}$ complementary at p implies $\{Y^i,X^i\}$ complementary at p hence $-\mathrm{ad}_{X^i}Y^i \in S^1_p$. This fact lets us use Proposition 3.6–2 to establish 2 of the Corollary. To establish 2 we can use Lemma 4.13 with the choices $X^1 = X^i, Y^1 = Y^i, X^2 = Y^i, Y^2 = X^i$ to conclude $Q^1_{\rm inv}(X^1,Y^1,X^2,Y^2,\boldsymbol{r}) = -Q^1_{\rm inv}(X^2,Y^2,X^1,Y^1,\boldsymbol{r})$. Since $Q^1_{\rm inv}$ is invariant with respect to permutations of $\{1,2\}$ we conclude that $Q^1_{\rm inv}(X^1,Y^1,X^2,Y^2,\boldsymbol{r}) = 0$. As a result of Theorem 4.5, we have $Q^2_\sigma \in S^2_p$ for all $\sigma \in P^2_2$. One can easily check from the definition that $Q^2_{\rm inv} = (\mathrm{ad}^2_{X^i}Y^i + \mathrm{ad}^2_{Y^i}X^i)/6$ while $Q^2_\sigma = 2\mathrm{ad}^2_{X^i}Y^i - \mathrm{ad}^2_{Y^i}X^i$ for all $\sigma \in P^2_2$. Finally, if we reverse X^i and Y^i we get $2\mathrm{ad}^2_{Y^i}X^i - \mathrm{ad}^2_{X^i}Y^i$ and $\mathrm{co}\{2\mathrm{ad}^2_{Y^i}X^i - \mathrm{ad}^2_{X^i}Y^i, 2\mathrm{ad}^2_{X^i}Y^i - \mathrm{ad}^2_{Y^i}X^i\}$ contains a positive multiple of $\mathrm{ad}^2_{X^i}Y^i$ hence $\mathrm{ad}^2_{X^i}Y^i \in S^2_p$. To complete the proof we must show that 2 holds. Here we simply augment our set of complementary vector fields by adding in k additional pairs, namely those of the form (Y^i,X^i). Arguing as in the proof of 2 above, we find that $Q^1_{\rm inv} = 0$, $Q^2_{\rm inv} = \sum_{i=1}^k(\mathrm{ad}^2_{X^i}Y^i + \mathrm{ad}^2_{Y^i}X^i)$, hence $Q^2_{\rm inv}(p) = 0$, and $Q^3_{\rm inv} = \sum_{i=1}^k \mathrm{ad}_{Y^i}\mathrm{ad}^2_{X^i}Y^i$. Thus Theorem 4.5 implies $\sum_{i=1}^k \mathrm{ad}_{Y^i}\mathrm{ad}^2_{X^i}Y^i \in S^3_p$. Now we note that reversing X^i and Y^i in $\mathrm{ad}_{Y^i}\mathrm{ad}^2_{X^i}Y^i$ gives the negative of this vector field. In this we can isolate each term in the above sum and conclude that $\pm\mathrm{ad}_{Y^i}\mathrm{ad}^2_{X^i}Y^i \in S^3_p$. ∎

Proof of Corollary 4.7 Suppose that $\{X^1,Y^1\},\{X^2,Y^2\} \subset S$, $\boldsymbol{r} = (1,1)$, and $Q^0_{\rm inv}(p) = Q^1_{\rm inv}(p) = \cdots = Q^\ell_{\rm inv}(p) = 0$ for ℓ even. Then $Q^{\ell+1}_\sigma \in S^{\ell+1}_p$ by Corollary 4.6. Here $\ell+1$ is odd and in the proof of Lemma 4.13 it is shown that there exists an S-trajectory $\mathcal{X}^\sigma_t(p)$ corresponding to $Q^{\ell+1}_\sigma$ with $\mathcal{X}^\sigma_t(p) = (M^{\rm odd}(P,Q))_t(p)$ where $M^{\rm odd}(P,Q)$ is a power series in t whose coefficients are Lie brackets of the vector fields $A^i_j(X^j,Y^j)$ of odd degree. In light of (7) $Q^{\ell+1}_\sigma$ is the coefficient of $t^{\sigma_\ell+1}$ in this expansion hence is a linear combination of iterated Lie brackets

$$B = [A^{i_1}_{j_1}, [[A^{i_2}_{j_2}, \dots, [A^{i_{2k}}_{j_{2k}}, A^{i_{2k+1}}_{j_{2k+1}}] \dots]$$

of an odd number of A_j^i's where $j_m \in \{1,2\}$ and $i_1 + \cdots + i_{2k+1} = \ell + 1$. Since $\ell+1$ is odd there must be an odd number of integers in $\{i_1, \ldots, i_{2k+1}\}$ which are odd, hence $Q_\sigma^{\ell+1}$ changes sign when X^i and Y^i are interchanged. Thus using the complementary sets $\{Y^1, X^1\}, \{Y^2, X^2\}$ in place of $\{X^1, Y^1\}, \{X^2, Y^2\}$ results in

$$Q_\sigma^{\ell+1}(Y^1, X^1, Y^2, X^2, \boldsymbol{r}) = -Q_\sigma^{\ell+1}(X^1, Y^1, X^2, Y^2, \boldsymbol{r})$$

which establishes 1. Since $Q_{\text{inv}}^{\ell+1}$ is a linear combination of the vector fields $Q_\sigma^{\ell+1}$ for $\sigma \in P_2^{\ell+1}$ we have $Q_{\text{inv}}^{\ell+1}(Y^1, X^1, Y^2, X^2, \boldsymbol{r}) = -Q_{\text{inv}}^{\ell+1}(X^1, Y^1, X^2, Y^2, \boldsymbol{r})$. Proceeding as above and using the fact that $\ell+2$ is even we see that

$$Q_\sigma^{\ell+2}(Y^1, X^1, Y^2, X^2, \boldsymbol{r}) = Q_\sigma^{\ell+2}(X^1, Y^1, X^2, Y^2, \boldsymbol{r})$$

for $\sigma \in P_2^{\ell+1}$. Set

$$\tilde{Q}_\sigma^i = Q_\sigma^i(Y^1, X^1, Y^2, X^2, \boldsymbol{r}), \quad Q_\sigma^i = Q_\sigma^i(X^1, Y^1, X^2, Y^2, \boldsymbol{r})$$

so that $\tilde{Q}_\sigma^{\ell+2} = Q_\sigma^{\ell+2}$. From Proposition 4.2 there exists an S-trajectory

$$\tilde{\mathcal{X}}_t(p) = \left(\tilde{\alpha}_0 \tilde{Q}_{\text{inv}}^0 + \cdots + \tilde{\alpha}_{\ell+1} \tilde{Q}_{\text{inv}}^{\ell+1} t^{\sigma_{\ell+1}} + Q_\sigma^{\ell+2} t^{\sigma_{\ell+2}} + \cdots\right)_t(p).$$

By assumption $Q_{\text{inv}}^0, \ldots, Q_{\text{inv}}^\ell$ vanish at p, hence $\tilde{Q}_{\text{inv}}^0, \ldots, \tilde{Q}_{\text{inv}}^\ell$ also vanish at p. By the Campbell-Baker-Hausdorff formula the S-trajectory

$$\tilde{\mathcal{X}}_t \circ \mathcal{X}_t \circ \mathcal{X}_t \circ \tilde{\mathcal{X}}_t(p) = (Z^0 + Z^1 t + \cdots + Z^{\ell+1} t^{\sigma_{\ell+1}} + Q_\sigma^{\ell+2} t^{\sigma_{\ell+2}} + \cdots)_t(p)$$

hence $Q_\sigma^{\ell+2} \in S_p^{\ell+1}$ for $\sigma \in P_2^{\ell+2}$. ■

Proof of Corollary 4.8 Lemma 4.13 together with the fact that reversing the roles of $\{X^1, Y^1\}$ and $\{X^2, Y^2\}$ in this case is the same as interchanging X^i and Y^i. Thus permutation invariance means that we must have Q_{inv}^ℓ vanish for ℓ odd, establishing 1. If $\boldsymbol{r} = (1)$ then P_1^ℓ consists of the single permutation $1 \mapsto 1$ so $Q_{\text{inv}}^\ell = A^\ell(X, Y)$. Then 2 follows from Corollary 4.6. ■

5 Examples

Example 5.1 As in [9] we consider the system $S = \{X, Y, Z\}$ in the plane where, in local coordinates (x, y),

$$X = (1, 0), \quad Y = (-1, x^2), \quad Z = (0, -1).$$

Then

$$[X, Y] = (0, 2x), \quad [X, [X, Y]] = (0, 2),$$
$$[Y, [X, Y]] = (0, -2), \quad [Z, Y] = [Z, X] = (0, 0).$$

For $p = (0, 0)$ we have $L^0(S, p) = \mathbb{R} \times \{0\}$, and hence $Z_p^0 = \{X, Y\}$ and $S_p^1 = \{[X, Y], [Y, X]\}$. This implies that $S(p) = \{(1, 0), (-1, 0), (0, -1)\}$, $S_p^1(p) =$

$\{(0,0)\}$, and it follows that $L^1(S,p) = L^0(S,p)$ and $Z_p^1 = \{X, Y, [X,Y], [Y,X]\}$. Since $X(p)+Y(p) = (0,0)$ the subset $\{X,Y\} \subset S$ is complementary at p. From Corollary 4.6–2 we have $\mathrm{ad}_X^2 Y = (0,2) \in S_p^2$. Thus

$$\mathrm{co}(\{(1,0),(-1,0),(0,-1),(0,2)\}) \subset \mathrm{co}(S(p) \cup S_p^1(p) \cup S_p^2(p))$$

and $0 \in \mathrm{int}(\mathrm{co}(S(p) \cup S_p^1(p) \cup S_p^2(p)))$. Local controllability at p then follows from Theorem 3.7. □

Example 5.2 Consider the affine system

$$\begin{aligned}\dot{x} &= u_1\\ \dot{y} &= u_2\\ \dot{z} &= x^2 - y^4,\end{aligned} \tag{13}$$

with $|u_a| \leq 1$ for $a = 1,2$ and $p = (x(0), y(0), z(0)) = (0,0,0)$. Here the system model is of the form

$$\dot{x}(t) = f_0(x(t)) + u_1(t) f_1(x(t)) + u_2(t) f_2(x(t))$$

where f_0, f_1, f_2 are smooth vector fields on $\mathbb{R}^3$ which, in local coordinates, are defined by $f_0 = (0,0,x^2 - y^4)$, $f_1 = (1,0,0)$, and $f_2 = (0,1,0)$. The nonzero Lie brackets are

$$\begin{aligned}
&\mathrm{ad}_{f_1} f_0 = (0,0,2x) && \mathrm{ad}_{f_1}^2 f_0 = (0,0,2)\\
&\mathrm{ad}_{f_2} f_0 = (0,0,-4y^3) && \mathrm{ad}_{f_2}^2 f_0 = (0,0,-12y^2)\\
&\mathrm{ad}_{f_2}^3 f_0 = (0,0,-24y) && \mathrm{ad}_{f_2}^4 f_0 = (0,0,-24)
\end{aligned}$$

while $\mathrm{ad}_{f_1}^3 f_0 = \mathrm{ad}_{f_0}^k \mathrm{ad}_{f_1}^j f_0 = (0,0,0)$ for $j,k \geq 1$ and $\mathrm{ad}_{f_2}^5 f_0 = \mathrm{ad}_{f_0}^k \mathrm{ad}_{f_2}^j f_0 = (0,0,0)$ for $j,k \geq 1$. The tangent space to $\mathbb{R}^3$ at p is spanned by $f_1(p)$,$f_2(p)$, and $[f_1,[f_1,f_0]](p)$ hence the first-order sufficient condition Theorem 2.1 cannot be employed. The generalization of Hermes' condition, Theorem 7.3 of [11], does not apply because the "bad" bracket $\mathrm{ad}_{f_1}^2 f_0$ is not expressible in terms of "good" and "bad" brackets of the required orders. On the other hand, the drift vector field f_0 vanishes at p so that $\{X^1, Y^1\} = \{f_0 + f_1, f_0 - f_1\}$ is complementary at p, as is $\{X^2, Y^2\} = \{f_0 + f_2, f_0 - f_2\}$. In light of equation (9) we have $A^0(X^1,Y^1)(p) = A^1(X^1,Y^1)(p) = (0,0,0)$ while $A^2(X^1,Y^1)(p)$ is a positive multiple of $(0,0,1)$. Corollary 4.8 lets us conclude that $(0,0,1) \in S_p^2(p)$. Similarly $A^i(X^2,Y^2)(p) = (0,0,0)$ for $i = 0,1,2,3$ and $A^4(X^2,Y^2)(p)$ is a positive multiple of $(0,0,-1)$ so that $(0,0,-1) \in S_p^4(p)$ as a consequence of Corollary 4.8. Finally, we note that $f_0(p) \pm f_1(p) = (\pm 1, 0, 0) \in S(p)$ and $f_0(p) \pm f_2(p) = (0, \pm 1, 0) \in S(p)$ hence $0 \in \mathrm{int}(\mathrm{co}(S(p) \cup S_p^1(p) \cup \cdots \cup S_p^4(p)))$. Thus the system (13) is l.c. as a consequence of Theorem 3.7. □

The next example illustrates the weakening of the hypotheses of Theorem 4.5 described in Remark 4.10.

Example 5.3 Consider the system $S = \{W, X, Y\}$ in $\mathbb{R}^3$ where, in local coordinates (x, y, z),

$$W = (0, 0, -1), \quad X = (1, z, 0), \quad Y = (-1, 0, x^2).$$

Then

$$[X, Y] = (0, -x^2, 2x), \quad [X, [X, Y]] = (0, -4x, 2),$$
$$[Y, [Y, X]] = (0, -2x, 2), \quad [Y, [X, [X, Y]]] = (0, 4, 4x).$$

We take $p = (0, 0, 0)$. Since $(X + Y)(p) = 0$ we have $\{X, Y\}$ complementary at p. In light of (9) and Corollary 4.8, Q^i_{inv} is a positive multiple of $A^i(X, Y)$ and we have the S-trajectory

$$\begin{aligned} X_t \circ Y_t(p) &= \left(A^0(X, Y) + A^1(X, Y)t + A^2(X, Y)t^2 + A^3(X, Y)t^3 + \cdots\right)_t(p) \\ &= \Big((X + Y) + \frac{1}{2}\mathrm{ad}_X Y t + \frac{1}{12}(\mathrm{ad}^2_Y X + \mathrm{ad}^2_X Y)t^2 \\ &\quad - \frac{1}{24}\mathrm{ad}_Y \mathrm{ad}^2_X Y t^3 + \cdots\Big)_t(p). \end{aligned}$$

Here $A^0(X, Y)(p) = (X + Y)(p) = 0$ and $A^1(X, Y)(p) = \frac{1}{2}\mathrm{ad}_X Y(p) = 0$. Thus $A^2(X, Y) = \frac{1}{12}(\mathrm{ad}^2_Y X + \mathrm{ad}^2_X Y) \in S^2_p$ by Corollary 4.8. We note that

$$W \in S, \quad A^2 = \frac{1}{12}(\mathrm{ad}^2_Y X + \mathrm{ad}^2_X Y) \in S^2_p, \quad \Big(\frac{1}{3}W + A^2\Big)(p) = (0, 0, 0),$$

where $A^i = A^i(X, Y)$. As in Remark 4.10 we consider the pairs $\{\frac{1}{6}W, \frac{1}{6}W\}, \{X, Y\} \subset S$ and take $\boldsymbol{r} = (3, 1)$. Here $k = 2$ thus $P^0_2 = \{\sigma_1, \sigma_2\}$ where $\sigma_1(1) = 1, \sigma_1(2) = 2, \sigma_2(1) = 2, \sigma_2(2) = 1$. The S-trajectories (10) corresponding to σ_1, σ_2 are

$$\begin{aligned} \mathcal{X}^{\sigma_1}_t(p) &= \Big(\frac{1}{6}W\Big)_{t^3} \circ \Big(\frac{1}{6}W\Big)_{t^3} \circ X_t \circ Y_t(p) \\ &= \Big(\frac{1}{3}W\Big)_{t^3} \circ (A^0 + A^1 t + A^2 t^2 + \cdots)_t(p) \\ &= \Big(A^0 + A^1 t + \Big(A^2 + \frac{1}{3}W\Big)t^2 + \Big(A^3 - \frac{1}{6}[A^0, W]\Big)t^3 + \cdots\Big)_t(p) \\ &= (Q^0_{\sigma_1} + Q^1_{\sigma_1} t + Q^2_{\sigma_1} t^2 + \cdots)_t(p) \end{aligned}$$

and

$$\begin{aligned} \mathcal{X}^{\sigma_2}_t(p) &= \Big(A^0 + A^1 t + (A^2 + \frac{1}{3}W\Big)t^2 + \Big(A^3 + \frac{1}{6}[A^0, W]\Big)t^3 + \cdots\Big)_t(p) \\ &= (Q^0_{\sigma_2} + Q^1_{\sigma_2} t + Q^2_{\sigma_2} t^2 + \cdots)_t(p). \end{aligned}$$

Here $m(\boldsymbol{r}) = 1, m_0(\boldsymbol{r}) = 2, m_1(\boldsymbol{r}) = 3, m_2(\boldsymbol{r}) = 4$ and Lemma 4.1 implies that $Q^0_\sigma, Q^1_\sigma, Q^2_\sigma$ are constant functions of $\sigma \in P^0_2$, which is shown explicitly above.

From our definition it follows that $Q^0_{\rm inv} = A^0$, $Q^1_{\rm inv} = A^1$, and $Q^2_{\rm inv} = A^2 + \frac{1}{3}W$. Similarly if $\sigma = \sigma_1 \circ \sigma_2 \in P^1_2$ we have

$$\begin{aligned}\mathcal{X}^\sigma_t(p) &= \mathcal{X}^{\sigma_1}_t \circ \mathcal{X}^{\sigma_2}_t(p)\\ &= \Big(2A^0 + 2A^1 t + 2\Big(A^2 + \frac{1}{3}W\Big)t^2 + 2A^3t^3 + \cdots\Big)_t(p)\end{aligned}$$

hence $Q^3_{\rm inv} = 2A^3 = -\frac{1}{12}\mathrm{ad}_Y\mathrm{ad}^2_X Y$. Since $Q^0_{\rm inv}$, $Q^1_{\rm inv}$, and $Q^2_{\rm inv}$ each vanish at p, Theorem 4.5 implies that

$$Q^3_{\rm inv} = -\frac{1}{12}\mathrm{ad}_Y\mathrm{ad}^2_X Y = (0, -\frac{1}{3}, -\frac{1}{3}x) \in S^3_p.$$

Interchanging X and Y and repeating the previous steps, we can conclude that $-\frac{1}{12}\mathrm{ad}_X\mathrm{ad}^2_Y X = (0, \frac{1}{3}, \frac{1}{3}x) \in S^3_p$. In summary $S(p)$ contains the vectors

$$W(p) = (0,0,-1), \quad X(p) = (1,0,0), \quad Y(p) = (-1,0,0),$$

$S^2_p(p)$ contains the vector $\frac{1}{12}(\mathrm{ad}^2_Y X + \mathrm{ad}^2_X Y)(p) = (0,0,\frac{1}{3})$, and $S^3_p(p)$ contains the vectors

$$-\frac{1}{12}\mathrm{ad}_Y\mathrm{ad}^2_X Y(p) = (0, -\frac{1}{3}, 0), \quad -\frac{1}{12}\mathrm{ad}_X\mathrm{ad}^2_Y X(p) = (0, \frac{1}{3}, 0).$$

Thus

$$\begin{aligned}&\mathrm{co}(\{(\pm 1,0,0), (0,\pm\tfrac{1}{3},0), (0,0,\tfrac{1}{3}), (0,0,-1)\}) \subset\\ &\qquad\qquad \mathrm{co}(S(p) \cup S^1_p(p) \cup S^2_p(p) \cup S^3_p(p))\end{aligned}$$

and $0 \in \mathrm{int}(\mathrm{co}(S(p) \cup S^1_p(p) \cup S^2_p(p) \cup S^3_p(p)))$. Local controllability at p follows from Theorem 3.7. This example illustrates the use of time rescaling to generate new "higher-order" S-trajectories (see Remark 4.10). $\square$

Example 5.4 Here is a control affine system which has a "bad" bracket that can be neutralized as in [2]:

$$\begin{aligned}\dot{x} &= yz + u_1\\ \dot{y} &= -xz + u_2\\ \dot{z} &= -u_2,\end{aligned}$$

with $|u_i| \le 1$, $i = 1, 2$, and $p = (0,0,0)$. The brackets are

$$\begin{gathered}[f, g_1] = (0, z, 0), \quad [f, g_2] = (y - z, -x, 0),\\ [g_1, [f, g_1]] = (0,0,0), \quad [g_2, [f, g_2]] = (2,0,0), \quad [g_1, [f, g_2]] = (0,-1,0).\end{gathered}$$

Motivated by Remark 4.10 we will show that the *bad* bracket $[g_2, [f, g_2]]$ can be neutralized. To this end we set

$$\begin{aligned}&S = \{f + ag_1 + bg_2 \mid \ -1 \le a, b \le 1\},\\ &W = f + g_1, \quad X = f + g_2, \quad Y = f - g_2,\end{aligned}$$

and consider the pairs $\{\frac{1}{3}W, \frac{1}{3}W\}, \{X, Y\} \subset S$ and $\boldsymbol{r} = (3,1)$. With $P_2^0 = \{\sigma_1, \sigma_2\}$ and $A^i = A^i(X,Y)$ as defined in Example 5.3, the S-trajectory (10) corresponding to σ_1 is

$$\begin{aligned}
\mathcal{X}_t^{\sigma_1}(p) &= \Big(\frac{1}{3}W\Big)_{t^3} \circ \Big(\frac{1}{3}W\Big)_{t^3} \circ X_t \circ Y_t(p) \\
&= \Big(\frac{2}{3}W\Big)_{t^3} \circ (A^0 + A^1 t + A^2 t^2 + \cdots)_t(p) \\
&= \Big(A^0 + A^1 t + \Big(A^2 + \frac{2}{3}W\Big)t^2 + \Big(A^3 - \frac{1}{3}[A^0, W]\Big)t^3 + \\
&\quad \Big(A^4 - \frac{1}{3}[A^1, W] + \frac{1}{18}[A^0, [A^0, W]]\Big)t^4 + \cdots\Big)_t(p) \\
&= (Q_{\sigma_1}^0 + Q_{\sigma_1}^1 t + Q_{\sigma_1}^2 t^2 + \cdots)_t(p).
\end{aligned}$$

Here $A^2 = \frac{1}{3}\mathrm{ad}_{g_2}^2 f$ does not vanish at p but is neutralized in the above S-trajectory as $A^2 + \frac{2}{3}W$ does vanish at p. It is straightforward to check that $Q_{\sigma_1}^0, \ldots, Q_{\sigma_1}^3$ vanish at p while

$$Q_{\sigma_1}^4(p) = -\frac{1}{3}[A^1, W](p) = -\frac{1}{3}[g_1, [f, g_2]](p) = (0, \tfrac{1}{3}, 0).$$

From our definition of S_p^m (or from Proposition 3.6–3) it follows that $(0, \frac{1}{3}, 0) \in S_p^4(p)$. Now we can repeat the above construction with X and Y interchanged to conclude that $(0, \frac{1}{3}, 0) \in S_p^4(p)$. Since $f \pm g_1, f \pm g_2 \in S$ we have

$$\begin{aligned}
&\mathrm{co}(\{\pm(1,0,0), \pm(0,1,-1), \pm(0, \tfrac{1}{3}, 0)\}) \subset \\
&\qquad\qquad \mathrm{co}(S(p) \cup S_p^1(p) \cup S_p^2(p) \cup S_p^3(p) \cup S_p^4(p))
\end{aligned}$$

and $0 \in \mathrm{int}(\mathrm{co}(S(p) \cup \cdots \cup S_p^4(p)))$. Local controllability at p follows from Theorem 3.7. $\square$

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