Reachable Sets and the Maximum Principle of Optimal Control[1]

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1 Introduction

In this lecture we give a selfcontained introduction to the fundamental result on necessary conditions for optimality in a finite-dimensional optimal control problem, the Maximum Principle of optimal control theory or otherwise known as the Pontryagin Maximum Principle. We will present the classical construction as it was developed by Pontryagin, Boltyansky, Gamkrelidze and Mishchenko [47]. As such this result is given in numerous textbooks on the subject from the sixties and seventies like for instance in the books by Hestenes (1966, [29]), Lee and Markus (1967, [42]), Young (1969, [76]), Berkovitz (1974, [3]), Fleming and Rishel (1975, [22]), or Cesari (1980, [17]). Although we follow the original construction in the proof, our presentation, however, differs significantly from those found in these textbooks in a consistent use of geometric language.

In the seventies significant progress has been made in the analysis of nonlinear control systems and it has been realised that much like the methods of linear algebra are the correct framework to describe and analyze linear systems, the language of differential geometry provides such a framework for nonlinear systems. Much of this early work was focused on problems of nonlinear controllability initiated with pathbraking papers of Lobry (1970, [43]), Brockett (1972, [14]), Sussmann and Jurdjevic (1972, [74]), Sussmann (1973, [63]), Stefan (1974, [61]) and many to follow. The paper by Krener on the high-order Maximum Principle (1977, [38]) provides a first glimpse of applications of these modern tools to problems in optimal control. These methods were accepted rather quickly in the literature on control engineering with significant textbooks being published in the eighties by Isidori (1985, 1989, [31]) and Nijmeijer and van der Schaft (1990, [44]). Sontag's book (1990, [59]) gives a rigorous introduction to the topics of mathematical control theory for finite-dimensional systems, but still with an emphasis on control-engineering aspects and no applications of these techniques to the optimal control problem. It was not until Jurdjevic's book (1997, [34]) and the second edition of Sontag's book (1998, [60]) that a mathematical treatise of this subject appeared which includes optimal control appeared on the level of a textbook. Yet even these textbooks do not give a proof of the Maximum Principle in modern geometric language as it

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will be done in these lecture notes. Precisely this language provides the natural framework to present a streamlined, clear and concise proof of the Pontryagin Maximum Principle. In these notes we attempt to give such a proof for what is called the *smooth* version of the Maximum Principle. We deliberately make simplifying smoothness assumptions (but no differentiability in the control variable will be required) and we only consider time-invariant systems to avoid technical arguments so that the geometric contents of the constructions will be brought out clearly.

Roughly speaking[2], for these lecture notes a *control system* Σ is described by a differential equation $\dot{x} = f(x, u)$ where the state variable x takes values in a finite-dimensional manifold M, the *state-space*, and the control variable u takes values in a *control space* U which we will always take as a subset of $\mathbb{R}^m$. For control functions $\eta : I \mapsto U$ defined on some compact interval in $\mathbb{R}$ in some class $\mathcal{U}$ of *admissible controls*, a time-varying ordinary differential equation $\dot{x} = f(x, \eta(t))$ is obtained whose solution curves ξ are called the *trajectories* of the system. We say a point $\bar{q}$ is *reachable* from an initial state ξ_0 if there exists a trajectory which starts at ξ_0 and passes through $\bar{q}$ at some later time. The set of all reachable points from ξ_0 is the *reachable set* from ξ_0 and will be denoted by $Reach_\Sigma(\xi_0)$. Our object of interest is to study both the collection of all trajectories and to characterize special trajectories which would minimize some objective over a subset of trajectories. The latter is the typical situation considered in the *optimal control problem*[3]: Given subsets N_0 and N_1 of M of initial and terminal conditions, respectively, among all trajectories which start in N_0 and end in N_1 find the ones which minimize some cost functional J. As we shall see, the necessary conditions for optimality of a reference trajectory Γ given in the Maximum Principle are a direct corollary of seperation results. Specifically, if Γ is optimal, then the set of all terminal conditions which would give a cost better than Γ can be seperated from a suitably constructed convex approximating cone to the set of all points reachable in N_1 from initial conditions in N_0. We shall see that the Maximum Principle can easily be reduced to the study of (suitably defined) approximating cones for the reachable sets. Necessary conditions for optimality are recast as geometric conditions on reachable sets and the geometric contents of the Maximum Principle is one of a seperation result.

This will be our approach in these lecture notes. Although *modern language* and concepts will be used, this is a classical approach in the sense of following the original construction of the group of mathematicians around Pontryagin. The needle-variations employed in the proof of the Maximum Principle probably have their historic origin in variations made by Weierstrass in the proof of his side-condition. In a recent article [75] commemorating the 300th anniversary of the brachystochrone problem Sussmann and Willems view this problem as the birth of optimal control. Indeed, the brachystochrone problem is the first problem to deal with dynamic behavior asking for an optimal selection of a curve. Sussmann and

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[2] A precise description including technical assumptions will be given in section 3.2

[3] In these notes we only consider control constraints and constraints on the terminal points, but will not allow state-space or mixed control-state constraints. Also, we do not address other important aspects such as existence results. A well-developed theory exists for existence of optimal solutions and we refer the reader to the books by Berkovitz [3] or Cesari [17]. The tools for these results come from functional analysis typically combining lower semicontinuity of the objective with results which guarantee that the set of admissible controls is compact in a suitable weak topology. There also exist functional analytic approaches to necessary conditions for optimality which are based on methods and techniques from optimization theory. Again, this is not the topic of these notes and we only mention the Dubovitskii-Milyutin method [21, 25] or the smooth-convex approach of Ioffe and Tikhomirov [30] as examples.

Willems make a compelling argument for "the superiority of the optimal control method for the brachystochrone problem" [75, p. 44], which in modern optimal control would be considered a minimum time problem. Viewing the history of the calculus of variations as a "search for the simplest and most general statement of the necessary conditions for optimality, ... this statement is provided by the maximum principle of optimal control" [75, p. 35]. Optimal control theory is deeply rooted in the classical calculus of variations in its problems and results, but it provides superior techniques, which, contrary to a still widespread misconception, go well beyond just incorporating inequality constraints into the classical framework.

The topic of these lecture notes is to give a proof of the Maximum Principle emphasizing the geometric contents of the result and to illustrate how the conditions of the Maximum Principle can be used to get both qualitative and quantitative information about the structure of reachable sets. In section 2 we give a brief introduction to optimal control problems and the Maximum Principle. We give a formulation of the result and include the connections with the simplest problem in the calculus of variations. The presentation in this section is elementary and like the classical textbooks we only formulate the results for open subsets of $\mathbb{R}^n$ as state-space. While this may be perfectly adequate to solve specific problems, differential geometric concepts provide deeper insights[4]. Therefore the proof of the Maximum Principle in section 3 will be given for more general control systems on manifolds. The point of view of a control system as a parametrized family of vector fields on a finite-dimensional manifold is taken and technical assumptions are made which lead to what is called the "smooth" Maximum Principle. Our assumptions will deliberately be kept more restrictive than needed even for the smooth version in order to keep the technical aspects of the construction at a minimum. For instance, we will only consider time-invariant systems to avoid technicalities regarding the dependence of the dynamics on the time variable t. A trivial extension of the result presented here to time-varying systems can be obtained by adding time as extra variable in the dynamics, $\dot{t} = 1$. However, such a procedure requires that the dependence in t is as smooth as it is required for the state variable x. Such a regularity is by no means necessary. This is one of the aspects of generalizing the Maximum Principle of optimal control considered in Sussmann's paper in this volume. Also we only consider the case that the terminal constraint is given by an imbedded submanifold of the state-space. This makes for a particularly simple and appealing transversality condition. Arbitrary terminal constraints could be allowed, but at the expense of less geometric transversality statements. Our assumptions are easily verifiable, yet broad enough to cover many, if not most, important and typical systems, while at the same time keeping the technical aspects of the approximation to the reachable set at a minimum. The spirit of the presentation is tutorial emphasizing the geometric contents of the result. There exist numerous generalizations of the Maximum Principle under weaker assumptions giving so-called "non-smooth" formulations of the results. But these are not the topic of this lecture. A comprehensive generalization of the Maximum Principle which subsumes both the classical constructions presented here and its nonsmooth generalizations has recently been proved by Sussmann [73] and is the topic of Sussmann's article in this volume. The classical construction presented in this lecture

[4] Gaussian elimination is all we need to know to actually solve systems of linear equations. But if we are interested in understanding questions about solvability or the overall structure of the solutions, then concepts like linear subspace, rank or determinants matter.

is still the basis for Sussmann's generalization, but much more powerful technical tools including a specifically for this problem developed differential calculus [72] are being used.

In section 4 we illustrate how the conditions of the Maximum Principle can be used to analyze systems. However, rather than considering specific problems, we pursue a more geometric approach and use modern techniques to obtain qualitative and quantitative information about reachable sets for single-input nonlinear systems with bounded control in low dimensions. The results give explicit and precise descriptions of the structure of the reachable sets in small times depending on linear independence assumptions on the vector fields of the system and their Lie-brackets at the reference point. These results have been reported in the literature before and are used here to illustrate the geometric approach to optimal control. The techniques heavily depend on a Lie-algebraic formalism to perform explicit calculations. We include an exposition of the required calculus deriving all formulas which will be needed. From the structures of the (small-time) reachable sets constructed in this way (local) optimal solutions for corresponding time-optimal control problems can directly be derived. But for these results we refer the reader to the literature. Our constructions bridge a gap between the necessary conditions for optimality in the Maximum Principle and the sufficient conditions required in a synthesis [6].

In the spirit of lecture notes and in an effort to keep these notes as much self-contained as possible complete proofs (even of elementary facts) are given. Many footnotes provide background information about definitions and results which are being used, but are not central to the topic.

2 An Introduction to Optimal Control Problems

We state the Maximum Principle of optimal control and discuss some examples, notably the connection with problems of the calculus of variations. The level of presentation will be elementary and we will only consider systems whose state space is an open subset of $\mathbb{R}^n$. But in this section we formulate the results for time-varying systems. We start in section 2.1 with the probably most important example for control engineering applications, the linear-quadratic regulator. For this problem we give a completely elementary and selfcontained derivation of the optimal solution based on a classical argument of Legendre from the calculus of variations. Then in section 2.2 we formulate the Maximum Principle. Special cases of optimal control problems are given by the problems of the calculus of variations and we briefly highlight this connection in Section 2.3. We also include a derivation of those necessary conditions for optimality which can be derived from variational arguments. Although quite instructive, these simple derivations can only be made rigorous if the control set is open. This, however, is a much too stringent restriction.

2.1 An Example: Linear-Quadratic Optimal Control. We start with an example which can be solved completely by elementary means without having to use any theory, yet probably is the most important application of optimal control in control engineering. We consider a fixed time horizon $[0,T]$ and let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, Q \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times m}$ be matrices whose entries are continuous functions defined on $[0,T]$. We assume Q and R are symmetric, Q is positive semi-definite and R is positive definite. Also let $S_T \in \mathbb{R}^{n\times n}$ be a symmetric, positive semi-definite matrix. The *linear-quadratic optimal control problem* is to

(LQ) find a (continuous) control $\eta : [0,T] \to \mathbb{R}^m$ which minimizes the quadratic objective

$$J(\eta) = \frac{1}{2} \int_0^T \left(\xi^T Q \xi + \eta^T R \eta \right) dt + \frac{1}{2} \xi^T(T) S_T \xi(T) \tag{2.1}$$

subject to the dynamics

$$\dot{\xi} = A(t)\xi + B(t)\eta, \qquad \xi(0) = \xi_0 \tag{2.2}$$

This problem can be solved completely by a simple "completing the square argument", a classical argument of Legendre from the calculus of variations. Given any continuous function $\eta : [0,T] \to \mathbb{R}^m$ there exists a unique solution ξ to the initial value problem (2.2) defined over the full interval $[0,T]$. For any differentiable matrix function $S \in \mathbb{R}^{n \times n}$ we then have that

$$\frac{d}{dt} \left(\xi^T S \xi \right) = (A\xi + B\eta)^T S \xi + \xi^T \dot{S} \xi + \xi^T S (A\xi + B\eta)$$

and thus we can express the cost equivalently as

$$\begin{aligned} J(\eta) \quad &= \quad \frac{1}{2} \int_0^T \xi^T (Q + A^T S + \dot{S} + SA) \xi + \xi^T S B \eta + \eta^T B^T S^T \xi + \eta^T R \eta \, dt \\ &\qquad + \frac{1}{2} \xi^T(T)[S_T - S(T)]\xi(T) + \frac{1}{2} \xi_0^T S(0) \xi_0. \end{aligned}$$

Take S as a symmetric matrix and complete the square to get

$$\begin{aligned} J(\eta) \quad &= \quad \frac{1}{2} \int_0^T \xi^T (\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q) \xi \\ &\qquad + (\eta + R^{-1} B^T S \xi)^T R (\eta + R^{-1} B^T S \xi) dt \\ &\qquad + \frac{1}{2} \xi^T(T)[S_T - S(T)]\xi(T) + \frac{1}{2} \xi_0^T S(0) \xi_0. \end{aligned}$$

For the moment let us assume that there exists a solution S to the matrix Riccati equation

$$\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q \equiv 0, \qquad S(T) = S_T, \tag{2.3}$$

on the full interval $[0,T]$. Then the objective simplifies to

$$J(\eta) = \frac{1}{2} \int_0^T (\eta + R^{-1} B^T S \xi)^T R (\eta + R^{-1} B^T S \xi) dt + \frac{1}{2} \xi_0^T S(0) \xi_0. \tag{2.4}$$

Hence

$$J(\eta) \geq \frac{1}{2} \xi_0^T S(0) \xi_0$$

with equality if and only if

$$\eta_*(t) = -R^{-1}(t) B^T(t) S(t) \xi(t).$$

This generates the optimal solution to the linear-quadratic control problem as a linear *feedback* function, i.e., a function $u_* : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ defined in the time-state space given by

$$u_*(t,x) = -R^{-1}(t)B^T(t)S(t)x.$$

It remains to argue that the solution S does indeed exist on all of $[0,T]$. This follows, for instance, from results about matrix Riccati equations [37]. But we can also give a direct and independent proof using optimal control arguments. It is clear by the theorem about the existence of local solutions to ODE's that the solution will exist on some maximal interval $(\tau, T]$. Let us now consider the linear-quadratic optimal control problem only on a interval $[t_0, T]$ with $\tau < t_0 < T$. For this problem the optimal solution is then given by the control η_* and the optimal value is $J(\eta_*) = \frac{1}{2}\xi_0^T S(t_0)\xi_0$. This holds for arbitrary initial conditions ξ_0. Since the objective is non-negative, it follows that $S(t_0)$ is positive semi-definite. Furthermore, S is trivially bounded above: Let $\Phi(t,s)$ denote the fundamental matrix for the linear ODE $\dot{X}(t) = A(t)X(t)$, $X(s) = Id$. Then the trajectory corresponding to $\eta \equiv 0$ is given by $\xi(t) = \Phi(t,t_0)\xi_0$ and thus

$$0 \le \frac{1}{2}\xi_0^T S(t_0)\xi_0 \le \frac{1}{2}\xi_0^T \left(\int_{t_0}^T \Phi(t,t_0)^T Q(t)\Phi(t,t_0)dt + \Phi(T,t_0)^T S_T \Phi(T,t_0) \right) \xi_0$$

for every $\xi_0 \in \mathbb{R}^n$. Taking the limit $t_0 \to \tau$, $t_0 > \tau$ it follows that S cannot blow up as $t_0 \to \tau$. Hence there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \to \tau$, $\tau_n > \tau$, and $S(\tau_n)$ converges to some finite matrix $\tilde{S}$. But then the solution S to (2.3) can be continued beyond $t = \tau$ contradicting the maximality of the interval $(\tau, T]$. Consequently the solution S of (2.3) exists on all of $[0,T]$. Thus we have shown

Theorem 2.1 *The solution to the linear-quadratic optimal control problem (LQ) is given by the linear feedback control*

$$u_*(t,x) = -R(t)^{-1}B(t)^T S(t)x \tag{2.5}$$

where S is the solution to the Riccati terminal value problem

$$\dot{S} + SA + A^T S - SBR^{-1}B^T S + Q \equiv 0, \qquad S(T) = S_T. \tag{2.6}$$

This solution is positive semi-definite and exists on all of $[0,T]$. The minimal value of the objective is given by $\frac{1}{2}\xi_0^T S(0)\xi_0$.

2.2 Statement of the Maximum Principle. We describe a simple model for an optimal control problem on $\mathbb{R}^n$: The *state space* is an open (and connected) subset O of $\mathbb{R}^n$ and the *control set* is an arbitrary subset U of $\mathbb{R}^m$. The *dynamics* assigns to every point $(t,x,u) \in \mathbb{R} \times O \times U$ a tangent vector $f(t,x,u) \in \mathbb{R}^n$. We **assume** that the time-varying vector fields

$$f : \mathbb{R} \times O \times U \to \mathbb{R}^n, \quad (t,x,u) \mapsto f(t,x,u), \tag{2.7}$$

are continuous in (t,x,u), differentiable in x for fixed $(t,u) \in \mathbb{R} \times U$ and that the derivative $\frac{\partial f}{\partial x}$ is continuous as a function of all variables. The class $\mathcal{U}$ of *admissible* *controls* is given by all locally bounded Lebesgue measurable functions η which take values in the control set U, i.e., given a compact interval $I \subset \mathbb{R}$ there exists a compact subset V of U such that η takes values in V almost everywhere on I. In particular, if the control set U is compact, then admissible controls are simply Lebesgue-measurable functions with values in U.

Given an admissible control η defined on some open interval J and an initial condition $(t_0, \xi_0) \in J \times O$, we want that there exists a unique solution $\xi(\cdot)$, defined on a maximal open subinterval I of J, such that $\dot{\xi}(t) = f(t, \xi(t), \eta(t))$ holds almost everywhere on I, $\xi(t_0) = \xi_0$, and $\xi(t)$ lies in O for $t \in I$. For the case when the right hand side of the differential equation is continuous in t this is textbook material. For our model which only has a Lebesgue measurable right-hand side we need to refer to (equally classical) results about the so-called C^1-Caratheodory conditions. For sake of completeness, we give a definition of solution in this case and briefly recall the main result about the Caratheodory conditions. We also refer the reader to the corresponding discussion in Grasse's article in this volume.

Let F be a vector function defined on some open set $J \times O \subset \mathbb{R} \times \mathbb{R}^n$. An absolutely continuous function ξ defined on an interval $I \subset J$, $\xi : I \to \mathbb{R}^n, t \mapsto \xi(t)$ is called a solution to the initial value problem

$$\dot{\xi} = F(t, \xi), \quad \xi(t_0) = \xi_0 \tag{2.8}$$

over the interval I if $\dot{\xi}(t) = F(t, \xi(t))$ holds almost everywhere in I, if $\xi(t_0) = \xi_0$, and if $\xi(t) \in O$ for all $t \in I$.

Definition 2.1 **(C^1-Caratheodory conditions)** *F satisfies the C^1-Caratheodory conditions on $J \times O$ if*

- *for every $t \in J$ the map $x \mapsto F(t, x)$ is differentiable in x on O*
- *for every $x \in O$ the maps $t \mapsto F(t, x)$ and $t \mapsto \frac{\partial F}{\partial x}(t, x)$ are Lebesgue measurable*
- *for every compact subset $K \subset O$ and every compact subinterval $I \subset J$ there exists an integrable function $a \in L^1(I)$ such that for all $t \in I$ and all $x \in K$ we have*

$$\|F(t, x)\| + \|\frac{\partial F}{\partial x}(t, x)\| \leq a(t). \tag{2.9}$$

Theorem 2.2 **[45]** *If η satisfies the C^1-Caratheodory conditions, then for any $(t_0, \xi_0) \in J \times O$ the initial value problem (2.8) has a unique local solution $\xi(t; t_0, \xi_0)$. This solution is defined on a maximal open interval $(\tau_-(t_0, \xi_0), \tau_+(t_0, \xi_0)) \subset J$ and $\xi(t; t_0, \xi_0)$ is a continuous function of all its variables on its domain*

$$D = \{(t; t_0, \xi_0) : (t_0, \xi_0) \in J \times O, \quad t \in (\tau_-(t_0, \xi_0), \tau_+(t_0, \xi_0)\}.$$

Furthermore, $\xi(t; t_0, \xi_0)$ is continuously differentiable in the variables (t_0, ξ_0) and the partial derivatives $\frac{\partial \xi}{\partial t_0}$ and $\frac{\partial \xi}{\partial x_0}$ of $\xi(t; t_0, \xi_0)$ can be calculated as the solutions of the variational equation

$$\dot{z} = \frac{\partial F}{\partial x}(t, \xi(t; t_0, \xi_0))z \tag{2.10}$$

with initial conditions $z(t_0) = -F(t_0, \xi_0)$ respectively $z(t_0) = Id$ where Id denotes the $n \times n$ identity matrix.

Note that (2.10) is a linear time-varying differential equation along the reference solution $\xi(\cdot; t_0, \xi_0)$ which can be obtained by formally differentiating the equation $\dot{\xi}(t; t_0, \xi_0) = F(t, \xi(t; t_0, \xi_0))$ and the respective initial condition $\xi(t_0; t_0, \xi_0) = \xi_0$ with respect to t_0 and ξ_0.

It is clear that the C^1-Caratheodory conditions are satisfied under our assumptions: given an admissible control η defined on a compact interval I, the function $(t, x) \mapsto F(t, x) = f(t, x, \eta(t))$ satisfies the desired measurability and differentiability conditions and since η takes values in a compact subset V of U, (without loss of generality we may assume this holds everywhere on I) it follows that both F and $\frac{\partial F}{\partial x}$ are bounded over a compact subset K of O since $f(t, x, u)$ and $\frac{\partial f}{\partial x}(t, x, u)$ are continuous on the compact set $I \times K \times V$. Thus, given any admissible control η defined on some interval J and an initial condition $(t_0, \xi_0) \in J \times O$, there exists a unique solution $\xi(\cdot)$ of the initial value problem $\dot{\xi} = f(t, \xi, \eta(t))$, $\xi(t_0) = \xi_0$. We call this curve the corresponding *trajectory*.

Returning to the model for the optimal control problem, we assume that the *terminal constraint* is modelled by a k-dimensional embedded C^1-submanifold N of $\mathbb{R} \times O$, the terminal manifold. Thus near every point $q \in N$ there exist an open set $\Omega \subset \mathbb{R} \times O$ containing q and a C^1-map $\Theta : \Omega \to \mathbb{R} \times \mathbb{R}^{n-k+1}$ with components ϑ_i, $i = 0, \ldots, n-k$, which have linearly independent gradients $\nabla \vartheta_i$ and satisfy $N \cap \Omega = \{(t, x) \in \Omega : \vartheta_i(t, x) = 0, i = 0, \ldots, n-k\}$.

Also assume that the *objective* is given in Bolza form as a Lagrangian plus a penalty term. Suppose

$$L : \mathbb{R} \times O \times U \to \mathbb{R}, \quad (t, x, u) \mapsto L(t, x, u), \tag{2.11}$$

satisfies the same smoothness requirements as f, i.e., L is continuous in (t, x, u), differentiable in x for fixed $(t, u) \in \mathbb{R} \times U$ and the derivative $\frac{\partial L}{\partial x}$ is continuous as a function of all variables. Furthermore let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuously differentiable function. Then the objective or cost-functional is defined as

$$\mathcal{J}(\eta) = \int_{t_0}^{T} L(s, \xi(s), \eta(s)) ds + \varphi(T, \xi(T)) \tag{2.12}$$

The terminal time T is free. A fixed terminal time would be modelled as a constraint in N. The initial time t_0 and initial condition ξ_0 are fixed, but arbitrary. Then the optimal control problem is to

(OC) minimize the objective $\mathcal{J}(\eta)$ over all admissible controls $\eta \in \mathcal{U}$ for which the corresponding trajectory ξ, $\dot{\xi} = f(t, \xi, \eta(t))$, satisfies the initial condition $\xi(0) = \xi_0$ and terminal condition $(T, \xi(T)) \in N$.

The Maximum Principle gives necessary conditions for a controlled trajectory (ξ, η) to be optimal. In our notation we distinguish between tangent vectors which we write as column vectors (such as x, $f(t, x, u)$ etc.) and cotangent vectors which we write as row vectors (like the multiplier ψ in the statement of the Maximum Principle below). We denote the space of n-dimensional row vectors by $(\mathbb{R}^n)^*$. Define the Hamiltonian function H,

$$H : \mathbb{R} \times [0, \infty) \times (\mathbb{R}^n)^* \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$$

as

$$H(t, p_0, p, x, u) = pf(t, x, u) - p_0 L(t, x, u) \tag{2.13}$$

Theorem 2.3 (Pontryagin Maximum Principle) [47] *Suppose the controlled trajectory (ξ, η) defined over the interval $[t_0, T]$ is optimal. Then there exist a constant $\psi_0 \geq 0$ and an absolutely continuous function $\psi : [t_0, T] \to (\mathbb{R}^n)^*$, the* adjoint variable*, such that $(\psi_0, \psi(t)) \neq 0$ for all $t \in [t_0, T]$ and the following conditions are satisfied:*

1. Adjoint equation: *almost everywhere in* $[t_0, T]$
$$\dot{\psi}(t) = \psi_0 L_x(t, \xi(t), \eta(t)) - \psi(t) f_x(t, \xi(t), \eta(t)), \tag{2.14}$$

2. Maximum condition: *almost everywhere in* $[t_0, T]$
$$H(t, \psi_0, \psi(t), \xi(t), \eta(t)) = \max_{u \in U} H(t, \psi_0, \psi(t), \xi(t), u) \tag{2.15}$$

3. Transversality condition: *the vector $(-H + \psi_0 \varphi_t, \psi + \psi_0 \varphi_x)$ is orthogonal to the terminal constraint in the endpoint, i.e., there exists a covector $\nu \in (\mathbb{R}^{n+1-k})^*$ such that at the terminal time T*
$$-H + \psi_0 \varphi_t = \nu D_t \Theta, \qquad \psi + \psi_0 \varphi_x = \nu D_x \Theta. \tag{2.16}$$

Controlled trajectories (ξ, η) for which there exist multipliers ψ_0 and ψ such that the conditions of the Maximum Principle are satisfied are called *extremals*. Note that the conditions are linear in the multipliers (ψ_0, ψ) and thus it is possible to normalize this vector. In particular, if $\psi_0 > 0$, then we can divide by ψ_0 and thus assume $\psi_0 = 1$. These extremals are called *normal* while extremals with $\psi_0 = 0$ are called *abnormal*. The existence of abnormal extremals can in general not be ruled out and there are simple examples which have optimal abnormal solutions[5].

2.3 Example: The Simplest Problem in the Calculus of Variations. Problems in the calculus of variations are a special case of optimal control problems. We highlight this connection here, but we only consider the so-called *simplest problem* in the calculus of variations and only give the basic results. We refer the reader to the literature, especially to [75], for more insights into these connections.

Let $L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(t, x, u) \mapsto L(t, x, u)$ be a continuous function which for fixed $t \in [a, b]$ is differentiable in (x, u) with partial derivatives $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial u}$ continuous in all variables. Also let A and B be two given points in $\mathbb{R}^n$.

(CV) Among all continuously differentiable curves $\xi : [a, b] \mapsto \mathbb{R}^n$ which satisfy the boundary conditions $\xi(a) = A$ and $\xi(b) = B$ find the curve which minimizes the functional
$$J(\xi) = \int_a^b L(t, \xi(t), \dot{\xi}(t)) dt \tag{2.17}$$

In this case the state space is given by $O = \mathbb{R}^n$, the control set U is all of $\mathbb{R}^n$, and the class $\mathcal{U}$ is therefore given by all locally bounded Lebesgue measurable

[5]These examples are by no means pathological. The optimal synthesis for the time-optimal control problem for the harmonic oscillator with bounded controls, a standard text-book example, contains abnormal extremals [34].

functions, $\mathcal{U} = L^\infty([a,b];\mathbb{R}^n)$. The terminal manifold is the point B while the dynamics takes the trivial form $\dot{x} = u$. If $\xi_* : [a,b] \mapsto \mathbb{R}^n$ is an optimal solution, then with $\eta_*(t) = \dot{\xi}_*(t)$ the conditions of the Maximum Principle state that there exist a constant ψ_0 and an absolutely continuous curve $\psi : [a,b] \to (\mathbb{R}^n)^*$ satisfying

$$\dot{\psi}(t) = \psi_0 \frac{\partial L}{\partial x}(t, \xi_*(t), \eta_*(t)) \tag{2.18}$$

such that $(\psi_0, \psi(t)) \neq 0$ for all $t \in [a,b]$ and

$$-\psi_0 L(t, \xi_*(t), \eta_*(t)) + \psi(t)\eta_*(t) = \max_{u \in \mathbb{R}^n} \left\{ -\psi_0 L(t, \xi_*(t), u) + \psi(t) u \right\}. \tag{2.19}$$

Extremals for this problem are normal: For, if $\psi_0 = 0$, then the minimum condition (2.19) implies that also $\psi(t) \equiv 0$ contradicting the nontriviality of the multipliers. Hence, without loss of generality we may normalize $\psi_0 = 1$. The necessary conditions for optimality of $-L(t, \xi_*(t), u) + \psi(t)u$ in u over $\mathbb{R}^n$ then imply that

$$-\frac{\partial L}{\partial u}(t, \xi_*(t), \eta_*(t)) + \psi(t) = 0 \tag{2.20}$$

Combining (2.20) with the adjoint equation (2.18) and changing the notation from u to $\dot{x}$ as is customary in the calculus of variations gives the *Euler-Lagrange equation*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, \xi_*(t), \dot{\xi}_*(t)) = \frac{\partial L}{\partial x}(t, \xi_*(t), \dot{\xi}_*(t)). \tag{2.21}$$

The minimum condition (2.19) is the *Weierstrass condition*: if we define the excess-function as

$$E(t,x,y,u) = L(t,x,u) - L(t,x,y) - \frac{\partial L}{\partial \dot{x}}(t,x,y)(u-y), \tag{2.22}$$

then condition (2.19) states that

$$E(t, \xi_*(t), \dot{\xi}_*(t), u) \geq 0 \qquad \text{for all } u \in \mathbb{R}^n. \tag{2.23}$$

If L is twice continuously differentiable, additional regularity statements like the Legendre condition or the Hilbert differentiability theorem about regularity of the extremals can easily be derived from the Maximum Principle. In this case the minimum condition (2.19) implies that the matrix $\frac{\partial^2 L}{\partial \dot{x}^2}(t, \xi_*(t), \dot{\xi}_*(t))$ is positive semi-definite for $t \in [a,b]$, i.e., the *Legendre condition* is satisfied. If the strengthened Legendre condition holds, i.e., if the matrix $\frac{\partial^2 L}{\partial \dot{x}^2}(t, \xi_*(t), \dot{\xi}_*(t))$ is positive definite for $t \in [a,b]$, then the solution ξ_* is also twice continuously differentiable. This is the classical calculus of variations argument: for a constant c the equation

$$F(t,u) := \frac{\partial L}{\partial \dot{x}}(t, \xi_*(t), u) - \int_a^t \frac{\partial L}{\partial x}(t, \xi_*(s), \dot{x}_*(s))ds - c = 0$$

has the solution $u = \dot{\xi}_*(t)$. But by the implicit function theorem this solution is continuously differentiable if the strengthened Legendre condition holds. Hence ξ_* is twice continuously differentiable at points where the strengthened Legendre condition holds.

These connections with problems in the calculus of variations can be carried further including generalizations of the Jacobi-condition and field theory. But these go beyond the Maximum Principle and the scope of these notes. Here we only want to highlight the relations.

2.4 A Variational Motivation for the Maximum Principle. Conversely, some form of the Maximum Principle can be derived using variational arguments. These arguments essentially require that the control set U be open which is too restrictive, but the calculations are quite instructive and convincing. We therefore include a brief motivation of the Maximum Principle from the classical variational point of view. However, we again only consider what would be the direct generalization of the simplest problem in the calculus of variations to the optimal control case.

Consider the fixed end-time problem to minimize the functional

$$J(\eta) = \int_0^T L(t, \xi(t), \eta(t))dt + \varphi(\xi(T)) \tag{2.24}$$

subject to the dynamics $\dot{\xi} = f(t, \xi, \eta(t))$ with initial condition $\xi(0) = \xi_0$. No control constraints nor terminal constraints are imposed. Adjoin the dynamics to the objective with a multiplier $\psi = \psi(t)$ (which, consistent with our notation, we write as a row vector) to form the so-called augmented objective

$$\begin{aligned} J(\eta) &= \int_0^T [L(t, \xi, \eta) + \psi(\dot{\xi} - f(t, \xi, \eta))]dt + \varphi(\xi(T)) \\ &= \int_0^T \left(-H(t, \psi, \xi, \eta) + \psi\dot{\xi}\right) dt + \varphi(\xi(T)) \end{aligned}$$

where

$$H(t, p, x, u) = -L(t, x, u) + pf(t, x, u).$$

Integrating $\psi\dot{\xi}$ by parts gives

$$J(\eta) = \int_0^T \left(-H(t, \psi, \xi, \eta) - \dot{\psi}\xi\right) dt + \psi(T)\xi(T) - \psi(0)\xi(0) + \varphi(\xi(T)).$$

Now suppose η_* is an optimal control and let ξ_* be the corresponding trajectory. Since U is open for any function $\delta\eta$ and sufficiently small ϵ the control $\eta_* + \epsilon\delta\eta$ also takes values in the control set and thus is admissible. Hence $J(\eta_*) \leq J(\eta_* + \epsilon\delta\eta)$ and the first variation

$$\delta J = \frac{d}{d\epsilon}_{|\epsilon=0} J(\eta_* + \epsilon\delta\eta) \tag{2.25}$$

vanishes while the second variation

$$\delta^2 J = \frac{d^2}{d\epsilon^2}_{|\epsilon=0} J(\eta_* + \epsilon\delta\eta) \tag{2.26}$$

is positive semi-definite. Write the trajectory corresponding to the control $\eta_* + \epsilon\delta\eta$ and the same initial condition ξ_0 in the form $\xi_* + \epsilon\delta\xi$. Thus this equation defines $\delta\xi$. Then we have

$$\begin{aligned} J(\eta_* + \epsilon\delta\eta) &= \int_0^T \left(-H(t, \psi, \xi_* + \epsilon\delta\xi, \eta_* + \epsilon\delta\eta) - \dot{\psi}(\xi_* + \epsilon\delta\xi)\right) dt \\ &\quad +\psi(T)(\xi_*(T) + \epsilon\delta\xi(T)) - \psi(0)\xi(0) + \varphi(\xi_*(T) + \epsilon\delta\xi(T)) \end{aligned}$$

and hence

$$\delta J(u_*) = -\int_0^T \left\{ \frac{\partial H}{\partial x}(t,\psi,\xi_*,\eta_*) + \dot{\psi} \right\} \delta\xi(t) + \frac{\partial H}{\partial u}(t,\psi,\xi_*,\eta_*)\delta\eta(t)dt$$
$$+[\nabla\varphi(\xi_*(T)) + \psi(T)]\delta\xi(T)$$

By *choosing* ψ as the solution of the adjoint equation

$$\begin{aligned} \dot{\psi}(t) &= -H_x(t,\psi(t),\xi_*(t),\eta_*(t)) \\ &= L_x(t,\xi_*(t),\eta_*(t)) - \psi(t)f_x(t,\xi_*(t),\eta_*(t)), \end{aligned} \tag{2.27}$$

with terminal condition

$$\psi(T) = -\nabla\varphi(\xi_*(T)), \tag{2.28}$$

the condition that the first variation vanishes reduces to

$$\int_0^T \frac{\partial H}{\partial u}(t,\psi(t),\xi_*(t),\eta_*(t))\delta\eta(t))dt = 0 \tag{2.29}$$

for all locally bounded Lebesgue measurable functions. Clearly this is equivalent to

$$\frac{\partial H}{\partial u}(t,\psi(t),\xi_*(t),\eta_*(t)) \equiv 0, \tag{2.30}$$

the stationary point condition. Similarly it can easily be shown that the condition that the second variation is positive semidefinite implies the Legendre condition in the form that the matrix H_{uu} is positive semi-definite along (t,ψ,ξ_*,η_*).

Thus a form of the Maximum Principle which gives the basic equivalents to the Euler-Lagrange equation and the Legendre condition for a set-up like in the simplest problem in the calculus of variations can easily be derived. However, these arguments only prove that the function $u \mapsto H(t,\psi(t),\xi_*(t),u)$ satisfies the first- and second-order necessary conditions for having a maximum at $u = \eta_*(t)$, the so-called *weak* version of the Maximum Principle. The full result, sometimes also called the *strong* version, establishes that this function indeed takes on its maximum at $u = \eta_*(t)$.

The conditions of the weak version are of course sufficient if the Hamiltonian is convex in the control. For instance, we can use them to derive the form of the optimal control in the linear-quadratic optimal control problem (LQ) considered in section 2.1. In this case the Hamiltonian function takes the form

$$H = -\frac{1}{2}x^T Q(t)x - \frac{1}{2}u^T R(t)u + p(A(t)x + B(t)u)$$

and the stationarity condition $H_u \equiv 0$ therefore implies

$$u = -R^{-1}(t)B^T(t)p^T.$$

Substituting this formula in the system and adjoint equation gives the following linear two-point boundary value problem for x and $\lambda = \psi^T$:

$$\begin{pmatrix} \dot{\xi} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} \xi \\ \lambda \end{pmatrix}, \qquad \begin{pmatrix} \xi(0) \\ \lambda(T) \end{pmatrix} = \begin{pmatrix} \xi_0 \\ S_T\xi(T) \end{pmatrix}. \tag{2.31}$$

It is well-known that the solution is of the form $\lambda(t) = S(t)\xi(t)$ where S is the solution of the Riccati terminal value problem (2.3) [15].

The full power of the Maximum Principle, however, is the fact that it applies to general control sets $U \subset \mathbb{R}^m$. The key idea for this generalization can already be found in Weierstrass's argument for the proof of his necessary condition for optimality in calculus of variations where he constructs variations of a very different type than the ones considered here but similar to the needle-variations which will be constructed in the proof of the Maximum Principle (see, for instance [29]). In the next chapter these constructions will be given.

3 Approximations of the Reachable Set

The reachable set of a control system from a point ξ_0 consists of all points which lie on admissible trajectories forward in time starting at ξ_0. In this chapter we present a framework to construct approximating directions to the reachable set at a point $\bar{q}$. Essential ingredients of this framework are an imbedding of a reference trajectory which steers ξ_0 into $\bar{q}$ and the definition of the so-called needle variations given in section 3.2. These needle-variations will be used to generate a cone of variational directions at a point on the reference trajectory and the imbedding provides the mechanism to move these vectors along the flow of the reference trajectory to the terminal point $\bar{q}$. We also need an adequate theory which allows to combine single variational vectors which are constructed by one-parameter families of variations into a meaningful tool for approximation. Here our guideline is that we want to have a result which states that if arbitrary vectors are approximating directions to the reachable set at $\bar{q}$, then $\bar{q}$ is an interior point of the reachable set. Hence by negation we get necessary conditions for a point to lie in the boundary of the reachable set. The necessary conditions for optimality of the Maximum Principle directly follow from these results using classical seperation theorems about convex sets. In its purest form *the Maximum Principle* indeed *gives necessary conditions for trajectories to lie in the boundary of the reachable set from a point.*

The notion of an approximating cone C introduced in section 3.1 is essentially due to Boltyansky [47, 7]. There exist more general approaches, but we prefer the direct geometric nature of Boltyansky's definition and this concept is strong enough to handle the construction within the model assumptions which will be made in section 3.2. The technical assumptions which we make are by no means the most general conditions possible (and for these we refer to Sussmann's lecture notes), but are chosen to make the presentation easy. This will allow us to concentrate on the main ideas of the construction, not on technical aspects. Whenever reasonable we chose conditions which avoid technicalities. Yet our assumptions are sufficiently general to cover typical optimal control models as they arise in a wide range of applications. Furthermore, for specific problems these conditions are easily verifiable. In section 3.3 we then define point-variations. This class of variations is modelled after the classical needle-variations of Pontryagin, Boltyansky, Gamkrelidze and Mishchenko [47] and contains them as special case. Making these point variations we will generate approximating directions to the reachable set at $\bar{q}$. After a brief digression on the topic of variational and covariational vectorfield which is necessary to discuss how vectors and covectors are being moved along the flow of a reference trajectory, in section 3.5 we then describe how approximating directions can be combined into an approximating map finishing our construction of an approximating cone. This cone will then be used to give necessary conditions for a trajectory to lie in the boundary of the reachable set in section 3.6 and to derive the necessary conditions for optimality of the Maximum Principle in section 3.7.

3.1 Approximating Cones. Recall that a *cone* in an $\mathbb{R}$-vector space Z is a subset C which has the property that $rC \subset C$ for all $r > 0$. A cone C is convex if and only if $rC + sC \subset C$ for all $r, s > 0$. Our goal is to approximate the reachable set at a point $\bar{q}$ by a convex cone C in such a way that if the full tangent space at $\bar{q}$ is an approximating cone, then $\bar{q}$ is an interior point to the reachable set. A first idea might be to take for C the cone of tangent vectors[6] to the reachable set at $\bar{q}$: a vector v is tangent to a set S at a point $\bar{q}$ if there exists an $\epsilon > 0$ and a curve $\gamma : [0, \epsilon] \to S$, $\gamma(0) = \bar{q}$, which is differentiable from the right at $t = 0$ with tangent vector $\dot{\gamma}(0) = v$. However, it is easily seen that this concept is too general.

Example: Let $A = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ or } y \leq 0 \text{ or } y \geq x^2\}$. Here for any line ℓ passing through 0 there exists a line segment which has $(0, 0)$ in its relative interior which entirely lies in A. Thus the set of tangent vectors to A at $(0, 0)$ is $\mathbb{R}^2$, but $(0, 0)$ is not an interior point of A.

What fails in the example is that the approximation of the set by the tangent line tv_ϵ becomes increasingly worse in t for vectors $v_\epsilon = (1, \epsilon)$ as $\epsilon \to 0$ and it is not possible to approximate the set uniformly for all tangent lines. The following somewhat technical condition which in its basic idea goes back to Boltyansky makes such a notion of uniform approximation precise. We first establish some notation: $B^n(r) = \{x \in \mathbb{R}^n : ||x|| \leq r\}$, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i = 1, \dots, n\}$, and $Q^n(r) = \mathbb{R}^n_+ \cap B^n(r)$.

Definition 3.1 (approximating cone) *Let M be a finite-dimensional differentiable manifold, let A be a subset of M and let $\bar{q}$ be a point in the closure of A in M. We call a cone $C \subset T_{\bar{q}}M$ an approximating cone for A at $\bar{q}$ if for any finite set of vectors $v_0, \dots, v_n$ in C and for any $\epsilon > 0$ there exist vectors $v'_0, \dots, v'_n$ in $T_{\bar{q}}M$ which satisfy $||v_i - v'_i|| < \epsilon$ for $i = 0, \dots, n$, and a continuous map Ψ defined on some sufficiently small set $Q^{n+1}(\delta)$, $\Psi : Q^{n+1}(\delta) \to A \cup \{q\}$, with the property that*

$$\Psi(z_0, \dots, z_n) = \bar{q} + z_0 v'_0 + \cdots z_n v'_n + o(||z||). \tag{3.1}$$

Several COMMENTS are in order: Clearly this definition only makes sense once coordinates have been chosen at $\bar{q}$, but it is easy to see that the definition is independent of the coordinates. Also, it is a simple exercise to show that a convex combination of approximating maps for different sets of vectors defines an approximating map for the convex combination of the vectors. It therefore follows that, if C is an approximating cone to A at $\bar{q}$, then so is the convex hull of C. Without loss of generality we may therefore assume that approximating cones are convex. Furthermore, postulating that the map Ψ only needs to approximate closeby vectors v'_i, although clearly being more general, is completely unintuitive. We shall see in section 3.4 that the reason for this is purely technical. It helps to combine approximating directions constructed with needle variations at the same time into an approximating map. This avoids serious problems one encounters when trying to combine variations made at the same time (see, for example [62]). Geometrically

[6] in the sense of differential geometry

little generality is lost if the reader prefers to think of v_i' as v_i.

We now show that this definition serves our purpose.

Theorem 3.1 *If $T_{\bar{q}}M$ is an approximating cone for a set A at $\bar{q}$, then $\bar{q}$ is an interior point of $A \cup \{\bar{q}\}$.*

Proof: By choosing coordinates we may assume that $M = \mathbb{R}^n$ and $\bar{q} = 0$. Let $e_1, \dots, e_n$ be a basis for $\mathbb{R}^n = T_{\bar{q}}M$. Set $e_0 = -(e_1 + \cdots + e_n)$ and let Σ be the convex hull of $e_0, e_1, \dots, e_n$, i.e., the n-dimensional unit simplex in $\mathbb{R}^{n+1}$. Every element $\sigma \in \Sigma$ can be written in a unique way as $\sigma = z_0 e_0 + \cdots + z_n e_n$ with $z_0 + \cdots + z_n = 1$. Let $\vec{z} = (z_0, \dots, z_n)$ denote this vector of so-called barycentric coordinates. Since by assumption $\mathbb{R}^n$ is an approximating cone for A at 0, given $\epsilon > 0$ there exist vectors $e_0', \dots, e_n'$ such that $||e_i - e_i'|| < \epsilon$ for all $i = 0, \dots, n$, and a continuous map $\Psi : Q^{n+1}(\delta) \to A \cup \{0\}$ such that $\Psi(\vec{z}) = z_0 e_0' + \cdots z_n e_n' + o(||z||)$ as $z \to 0$. Choose an $\alpha > 0$ so that the ball $B^n(\alpha)$ lies in Σ and for sufficiently small r define a map $\phi_r : \Sigma \to \mathbb{R}^n$ by

$$\phi_r(\sigma) = \frac{1}{r}\Psi(r\vec{z})$$

It then follows that for ϵ and r small enough we have for all $\sigma \in \Sigma$ that

$$||\phi_r(\sigma) - \sigma|| = ||\sum_{i=0}^{n} z_i(e_i' - e_i) + \frac{o(r)}{r}|| \le \epsilon + \frac{o(r)}{r} \le \frac{\alpha}{2}.$$

Therefore for a fixed $q \in B^n(\frac{\alpha}{2})$ the continuous map

$$\begin{array}{rcl} F : \Sigma \times B^n(\frac{\alpha}{2}) & \to & \mathbb{R}^n \\ (\sigma, q) & \mapsto & F(\sigma, q) = q - \phi_r(\sigma) + \sigma \end{array}$$

actually maps Σ into $B^n(\alpha)$. Hence by the Brouwer fixed point theorem[7] it has a fixed point $\mu = \mu(q)$ in $B^n(\alpha) \subset \Sigma$. But $F(\mu, q) = \mu$ is equivalent to $\phi_r(\mu) = q$ and thus the neigborhood $B^n(\frac{\alpha}{2})$ lies in the image of $\phi_r(\Sigma)$. Hence $A \cup \{0\}$ contains the neighborhood $rB^n(\frac{\alpha}{2})$ of zero. *QED*

The next theorem allows to translate the geometric conditions about reachable sets into necessary conditions for optimality in the optimal control problem. We first introduce some useful terminology due to Sussmann [72].

Definition 3.2 (transversal cones) *Let C_1 and C_2 be convex cones in a (finite-dimensional) vector space Z. We call C_1 and C_2 transversal if $C_1 - C_2 = \mathbb{R}^n$ where $C_1 - C_2$ denotes the set $\{x - y : x \in C_1, y \in C_2\}$. We call C_1 and C_2 strongly transversal if C_1 and C_2 are transversal and $C_1 \cap C_2 \neq 0$, i.e., C_1 and C_2 intersect at least along a half-line.*

Theorem 3.2 *Let A_1 and A_2 be subsets of a a finite-dimensional differentiable manifold M and suppose $\bar{q}$ is a point in the intersection of the closures of these sets. Let C_1 and C_2 be (convex) approximating cones at $\bar{q}$ for A_1 and A_2 respectively which are strongly transversal. Then $\bar{q}$ is a limit point of $A_1 \cap A_2$, i.e., every neighborhood of $\bar{q}$ contains a point $q \in A_1 \cap A_2$, $q \neq \bar{q}$.*

[7] Brouwer fixed point theorem [20]: Let $D \subset \mathbb{R}^n$ be a nonempty, compact, and convex set and let $f : D \to D$ be a continuous map which maps D into itself. Then f has a fixed point in D.

Proof: Without loss of generality we may again assume that $M = \mathbb{R}^n$, $\bar{q} = 0$ and that the cones C_1 and C_2 are convex. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n = T_{\bar{q}}M$. Since C_1 and C_2 are transversal, we can write $e_i = f_i - g_i$, $f_i \in C_1$, $g_i \in C_2$, and since there exists a non-zero vector $z \in C_1 \cap C_2$, we also have for any $m > 0$ that $e_i = (f_i + mz) - (g_i + mz)$ with $f_i + mz \in C_1$ and $g_i + mz \in C_2$. The convex hull of the vectors $\{f_i + mz\}$ simply translates the convex hull of the original vectors $\{f_i\}$ in $\mathbb{R}^n$ by mz. But this convex hull is a compact set and therefore without loss of generality we may assume that the original vectors f_i are such that 0 does not lie in the convex hull of the vectors $\{f_0, \ldots, f_n\}$.

As in the proof of Theorem 3.1 set $e_0 = -(e_1 + \cdots + e_n)$ and let Σ be the convex hull of $e_0, e_1, \ldots, e_n$, i.e., the n-dimensional unit simplex in $\mathbb{R}^n$. Again write elements $\sigma \in \Sigma$ in a unique way as $\sigma = z_0 e_0 + \cdots + z_n e_n$ with $z_0 + \cdots + z_n = 1$ and let $\vec{z} = (z_0, \ldots, z_n)$ denote this vector of affine coordinates. Since C_1 and C_2 are approximating cones for A_1 and A_2 at 0, given $\epsilon > 0$ there exist vectors $f'_0, \ldots, f'_n$ and $g'_0, \ldots, g'_n$ such that $||f_i - f'_i|| < \frac{\epsilon}{2}$ and $||g_i - g'_i|| < \frac{\epsilon}{2}$ for all $i = 0, \ldots, n$, and for $j = 0, 1$, there exist continuous maps

$$\Psi_j : Q^{n+1}(\delta) \to A_j \cup \{0\}$$

such that

$$\Psi_1(\vec{z}) = z_0 f'_0 + \cdots z_n f'_n + o(||z||)$$

and

$$\Psi_2(\vec{z}) = z_0 g'_0 + \cdots z_n g'_n + o(||z||)$$

as $z \to 0$. By choosing ϵ small enough, we can guarantee that 0 still does not lie in the convex hull of the vectors $\{f'_0, \ldots, f'_n\}$. Let $B^n(\alpha)$ be a ball contained in the unit simplex Σ and for sufficiently small r define a map $\rho_r : \Sigma \to \mathbb{R}^n$ by

$$\rho_r(\sigma) = \frac{1}{r}\Psi_1(r\vec{z}) - \frac{1}{r}\Psi_2(r\vec{z}).$$

Then we have for sufficiently small ϵ and r that

$$\begin{aligned}
||\rho_r(\sigma) - \sigma|| &= ||\sum_{i=0}^{n} z_i(f'_i - g'_i) + \frac{o(r)}{r} - \sum_{i=0}^{n} z_i e_i|| \\
&\leq ||\sum_{i=0}^{n} z_i(f'_i - f_i) - \sum_{i=0}^{n} z_i(g'_i - g_i) + \frac{o(r)}{r}|| \\
&\leq \epsilon + \frac{o(r)}{r} \leq \frac{\alpha}{2}.
\end{aligned}$$

Thus it follows as in the proof of Theorem 3.1 that $\rho_r(\Sigma)$ contains a neighborhood of zero. Especially, $0 \in \rho_r(\Sigma)$ and so there exists a point $z^r = (z_0^r, \ldots, z_n^r) \in \Sigma$ such that $\Psi_1(rz^r) = \Psi_2(rz^r)$. Since 0 does not lie in the convex hull of the vectors $\{f'_0, \ldots, f'_n\}$, $\Psi_1(rz^r) \neq 0$, and thus we have $\Psi_1(rz^r) = \Psi_2(rz^r) \in A_1 \cap A_2$. Hence for every sufficiently small r there exists a nonzero point in $A_1 \cap A_2$. *QED*

We give a simple sufficient condition for convex cones to be strongly transversal which will suffice for the purpose of these notes.

Definition 3.3 (seperation of convex cones) *Let C_1 and C_2 be convex cones in a (finite-dimensional) vector space Z. We say C_1 and C_2 are seperated if there exists a nontrivial linear functional λ such that*

$$\langle \lambda, x\rangle \ \geq 0 \ \textit{ for all } x \in C_1 \quad \textit{and} \quad \langle \lambda, y\rangle \ \leq 0 \ \textit{ for all } y \in C_2$$

Lemma 3.1 *Let C_1 and C_2 be non-empty convex cones which are separated and suppose C_2 is not a linear subspace. Then C_1 and C_2 are strongly transversal.*

Proof: We first show that C_1 and C_2 are transversal. Observe that $C_1 - C_2$ is also a convex cone. Hence, if $C_1 - C_2 \neq \mathbb{R}^n$, then there exists a nontrivial linear functional λ such that $\langle \lambda, z\rangle \geq 0$ for all $z \in C_1 - C_2$. Since cones are invariant under positive scalings this implies

$$\langle \lambda, x\rangle \geq r\langle \lambda, y\rangle \text{ for all } x \in C_1,\ y \in C_2 \text{ and } r > 0.$$

Taking the limit $r \to 0$ we get $\langle \lambda, x\rangle \geq 0$ for all $x \in C_1$. Reversing the roles of C_1 and C_2, it follows that $\langle \lambda, y\rangle \leq 0$ for all $y \in C_2$. Hence C_1 and C_2 are separated. Contradiction.

But then C_1 and C_2 intersect at least along a nontrivial halfline. For, suppose $C_1 \cap C_2 = \{0\}$. Pick any nonzero $z \in C_2$ and write $z = x - y$ with $x \in C_1$ and $y \in C_2$. Thus $z + y = x \in C_1 \cap C_2 = \{0\}$ and so $z = -y$. Hence also $-z$ lies in C_2. But this holds for any $z \in C_2$ and so C_2 is a linear subspace. Contradiction. Thus C_1 and C_2 are strongly transversal. *QED*

3.2 Control Systems and Assumptions. In this section we formalize a definition of control systems as a collection of time-dependent vector fields on a differentiable manifold similar to the one given in Grasse's notes in this volume. Differential geometric concepts provide a powerful and adequate framework for the construction which will be carried out. But in these lecture notes we do not strive for generality. Instead our aim is to present the key ideas and constructions. Therefore we employ a set of simplifying assumptions which lead to what sometimes is referred to as the *smooth Maximum Principle* for time-invariant systems. In these notes, a control system will be a 4-tuple $\Sigma = (M, U, f, \mathcal{U})$ consisting of the following objects:

1. The *state space* M is a connected, finite-dimensional, differentiable manifold at least of class C^2 and we denote the dimension of M by n. Throughout these notes we also use the term *manifold* to imply that M is second-countable and Hausdorff. Under these conditions M is then a metrizable topological space. Also the tangent-bundle and cotangent bundle are denoted by TM and T^*M respectively and they are differentiable manifolds at least of class C^1; T_qM respectively T_q^*M denote the tangent respectively cotangent spaces at a point $q \in M$.

2. The *control set* U is an arbitrary subset of $\mathbb{R}^m$.

3. The *dynamics* or dynamical law assigns to every point $(q, u) \in M \times U$ a tangent vector $f(q, u) \in T_qM$. We assume that this map has the following regularity property: for every coordinate chart $\phi : V \to \mathbb{R}^n$ on M with domain V the coordinate representation f_ϕ of f on $\phi(V)$,

$$\begin{aligned} f_\phi \quad &: \quad \phi(V) \times U \to \mathbb{R}^n, \\ (x, u) \quad &\mapsto \quad f_\phi(x, u) = d\phi(\phi^{-1}(x))f(\phi^{-1}(x), u), \end{aligned}$$

is continuous on $\phi(V) \times U$, continuously differentiable in x for each fixed $u \in U$ and the partial derivatives $\frac{\partial f_\phi}{\partial x}(x,u)$ are continuous on $\phi(V) \times U$.

4. The class $\mathcal{U}$ of *admissible controls* is given by all locally bounded Lebesgue-measurable functions. Thus if η is an admissible control then η takes values in a compact subset of U almost everywhere on a compact interval I.

As in section 2.2 it follows from these conditions that given an admissible control η defined on an interval J for every coordinate chart $\phi : V \to \mathbb{R}^n$ on M the map $F(t,x) = f_\phi(x, \eta(t))$ satisfies the C^1-Caratheodory conditions (c.f. Grasse's notes in this volume or [28, Example 2.10.]). Hence, by patching together coordinate neighborhoods, it follows that the time dependent vector field $(t,q) \mapsto f(q,\eta(t))$ satisfies the C^1-Caratheodory conditions and thus for every initial condition $(t_0, \bar{\xi}_0) \in J \times M$ the initial value problem $\dot{\xi} = f(\xi, \eta(t)), \xi(0) = \bar{\xi}_0$, has a unique absolutely continuous solution $\xi = \xi(t; \bar{\xi}_0)$ which is defined on a maximal open subinterval which contains t_0 and is continuously differentiable with respect to $\bar{\xi}_0$. This solution is called the corresponding *trajectory*. The collection of all *admissible input-trajectory pairs* then consists of all pairs $(\xi(\cdot), \eta(\cdot))$ where $\eta(\cdot) \in \mathcal{U}$ is an admissible control and $\xi(\cdot)$ is the corresponding trajectory determined by the initial condition.

Remark. More generally, one could define the class of admissible controls by requiring that the time-dependent vector field $F(t,x) = f_\phi(x,\eta(t))$ satisfies the C^1-Caratheodory conditions for every coordinate chart $\phi : V \to \mathbb{R}^n$ on M. However, in the course of the construction of approximations to the reachable set additional properties of the class of admissible controls are needed. Specifically, it needs to be required that for all $u_0 \in U$ the class of admissible controls contains the constant controls $\eta(t) \equiv u_0$ and that the class of admissible controls is closed under the following operations: 1. *restrictions* of admissible controls to compact subintervals are again admissible; 2. *concatenations* of two admissible controls are admissible, i.e., if η_1 is an admissible control on $[t_0 - \epsilon, t_0]$ and η_2 is an admissible control on $[t_0, t_0 + \epsilon]$, then defining $\eta_1 * \eta_2$ by η_1 on $[t_0 - \epsilon, t_0]$ and by η_2 on $(t_0, t_0 + \epsilon]$ again is an admissible control; 3. *time-translations* or shifts again give admissible controls (see, for instance, [4]). All these properties hold trivially for the class $\mathcal{U}$ of locally bounded Lebesgue measurable functions with values in U.

A point $\bar{q}$ is said to be reachable from $\bar{\xi}_0$ in time t if there exists an admissible control η defined over $[0,t]$ such that the corresponding trajectory ξ which starts at $\bar{\xi}_0$ at time 0 satisfies $\bar{q} = \xi(t)$. The set of all points which are reachable in time t is denoted by $Reach_{\Sigma,t}(\bar{\xi}_0)$. The reachable set of the system Σ from $\bar{\xi}_0$, $Reach_\Sigma(\bar{\xi}_0)$, consists of all points which are reachable from $\bar{\xi}_0$ in nonnegative times.

Furthermore, to ease the notation from now on we will use the same notation $f(x,u)$ for both the map on the manifold and its coordinate representation f_ϕ and we will not go into the details regarding coordinate charts.

3.3 Needle Variations. Assume as given a fixed reference trajectory $\bar{\xi}(\cdot)$ defined on $[0,T]$ with initial point $\bar{\xi}(0) = \bar{\xi}_0$ and terminal point $\bar{\xi}(T) = \bar{q}$ corresponding to an admissible control $\bar{\eta}(\cdot)$. The goal in this chapter is to construct a (reasonably large) approximating cone for the reachable set $Reach_\Sigma(\bar{\xi}_0)$ from $\bar{\xi}_0$ at $\bar{q}$. This will be achieved by changing the given trajectory infinitesimally as function of some one-dimensional parameter α, i.e., by making variations in the control. Essential for this construction is the possibility to imbed the given trajectory into a smooth family of trajectories. The precise technical conditions are given in the definition of an imbedding below.

Definition 3.4 (imbedding of a trajectory) *An imbedding E of the reference input-trajectory pair $\Gamma = (\bar{\xi}(\cdot), \bar{\eta}(\cdot))$ defined on $[0,T]$ is a family of admissible input-trajectory pairs $\{\Gamma_r^E = (\xi^E(\cdot;r), \eta^E(\cdot;r)) : \eta^E(\cdot;r) \in \mathcal{U}, r \in N^E\}$, which for points r in a neighborhood N^E of $\bar{\xi}_0$ are defined on the interval $[0,T]$, reduce to the reference input-trajectory pair for the reference value $\bar{\xi}_0$, $\Gamma^E_{\bar{\xi}_0} = \Gamma$, and satisfy the following regularity conditions: for each $t \in [0,T]$ the flow $\Phi_t^E(\cdot)$, defined as $\Phi_t^E(r) = \xi^E(t;r)$ is a C^1-diffeomorphism from N^E to some neighborhood N_t^E of $\bar{\xi}(t)$ and the partial derivatives of $\Phi_t^E(\cdot)$ with respect to r are continuous jointly in (t,r).*

Under our technical assumptions it is possible to construct a *canonical imbedding E of* Γ by simply using the flow induced by the reference control $\bar{\eta}$: The time-dependent vector field $(t,x) \mapsto f(x, \bar{\eta}(t))$ on M satisfies the C^1-Caratheodory conditions. It therefore follows using standard arguments about the continuous dependence on initial conditions that there exists a neighborhood N^E of $\bar{\xi}_0$ such that for every $r \in N^E$ there exists a unique trajectory $\xi^E(\cdot;r)$ corresponding to the control $\bar{\eta}$ defined on all of $[0,T]$ which satisfies $\xi^E(0;r) = r$ and that the general solution $\xi^E(t;r)$ is continuous in (t,r), differentiable in r for fixed t with jointly continuous derivative $\frac{\partial}{\partial r}\xi^E(t;r)$ (c.f. Theorem 2.2 and [28, Proposition 2.23.]). Furthermore, the directional derivative in direction of v, $\frac{\partial}{\partial r}\xi^E(t;r)v$, can be calculated as the solution at time t to the *variational equation along* Γ,

$$\dot{\theta}(t) = \frac{\partial f}{\partial x}(\xi^E(t;r), \bar{\eta}(t))\theta(t), \tag{3.2}$$

with initial condition $\theta(0) = v$. Thus

$$E = \{\Gamma_r^E = (\xi^E(\cdot;r), \bar{\eta}(\cdot)), \quad r \in N^E\}$$

constitutes an imbedding of $\Gamma = (\bar{\xi}, \bar{\eta})$. In this construction the reference control has been used, only the initial data were varied. We refer to E as the *canonical imbedding of* Γ.

Although this seems to be a natural way to construct an imbedding, it is not required that the imbedding be given in this form. In fact, the properties of the imbedding required in Definition 3.4 are crucial for the construction to come, not the properties of the reference vector field $(t,x) \mapsto f(x, \bar{\eta}(t))$. (Only the flow matters to move vectors and covectors along the reference trajectory.) It is possible that a vector field may not satisfy the C^1-Caratheodory conditions, but still an imbedding of the desired smoothness properties exists since flows can be nicer than the vector fields which generate them. Generalizations of the construction presented here in this direction are the topic of Sussmann's paper in this volume.

We now proceed to define *needle variations*. This is the fundamental idea for the proof of the Maximum Principle. Needle variations are a generalization of the variations Weierstrass made in the proof of his side condition in the calculus of variations. Below we give a slightly more general definition of what we call a *point variation*. The classical needle variations used by Pontryagin, Boltyansky, Gamkrelidze and Mishchenko in their proof of the Maximum Principle [47] correspond to the examples CUT and PASTE given later. The framework presented here allows to include additional variations which may allow to take advantage of special structures in the reference control. For instance, the Legendre-Clebsch condition, an additional necessary condition if the reference control takes values in the interior of the control set, could also be derived within this framework which directly applies to high-order variations (see also [38]).

Definition 3.5 (point variation) *Given an imbedding E of Γ, a point variation $\mathcal{V}$ along E is a 7-tuple,*

$$\mathcal{V} = \{\bar{t}, \epsilon, t_1(\cdot), t_2(\cdot), \Delta_+(\cdot), \delta(\cdot,\cdot), \mu(\cdot,\cdot)\}$$

where $0 \leq \bar{t} \leq T$, $\epsilon > 0$, $t_1(\cdot), t_2(\cdot)$ and $\Delta_+(\cdot)$ are continuous real-valued functions of a variable α on $[0, \epsilon)$ such that

$$0 \leq t_1(\alpha) \leq t_2(\alpha) \leq T, \qquad \Delta_+(\alpha) \geq 0 \text{ for } 0 \leq \alpha < \epsilon,$$

$$t_1(0) = t_2(0) = \bar{t}, \quad \text{and} \quad \Delta_+(0) = 0;$$

$\delta(\cdot,\cdot)$ and $\mu(\cdot,\cdot)$ are maps defined on

$$\mathcal{D} = \{(\alpha, s) : 0 \leq \alpha < \epsilon, 0 \leq s \leq \Delta_+(\alpha)\}$$

with values in M and U respectively; $\delta(\cdot,\cdot)$ is continuous, satisfies

$$\delta(\alpha, 0) = \bar{\xi}(t_1(\alpha))$$

for $0 \leq \alpha < \epsilon$ and the pair $(\delta(\alpha,\cdot), \mu(\alpha,\cdot))$ is an admissible input-trajectory pair for $0 \leq \alpha < \epsilon$.

For each parameter $\alpha \in [0, \epsilon)$ a modification of the reference input-trajectory pair is defined: $\bar{t}$ denotes a time at which the variation will be "executed", $t_1(\alpha)$ and $t_2(\alpha)$ respectively denote the times at which a change from the reference trajectory is initiated and when a change back to the reference trajectory in form of the imbedding occurs. The amount of time which is inserted is $\Delta_+(\alpha)$; $\Delta_-(\alpha) = t_2(\alpha) - t_1(\alpha)$ is the time which has been taken away from the reference trajectory. Hence $\Delta(\alpha) = \Delta_+(\alpha) - \Delta_-(\alpha)$ gives the total net change of time in the variation. In particular, variations which satisfy $\Delta(\alpha) \equiv 0$ are "equal-time" variations for $Reach_{\Sigma,T}(\bar{\xi}_0)$. For fixed α, the function $\mu(\alpha,\cdot)$ is an admissible control defined on $[0, \epsilon)$ and $\delta(\alpha,\cdot)$ is the corresponding trajectory.

This particular one-parameter family of variations is used to define new input-trajectories as follows: The control is given by

$$\eta_\alpha^{\mathcal{V}}(t) := \begin{cases} \bar{\eta}(t) & \text{if } 0 \leq t \leq t_1(\alpha) \\ \mu(\alpha, t - t_1(\alpha)) & \text{if } t_1(\alpha) \leq t \leq t_1(\alpha) + \Delta_+(\alpha) = t_2(\alpha) + \Delta(\alpha) \\ \eta^E(t - \Delta(\alpha); \pi^{\mathcal{V}}(\alpha)) & \text{if } t_2(\alpha) + \Delta(\alpha) \leq t \leq T + \Delta(\alpha) \end{cases}$$

with corresponding trajectory

$$\xi_\alpha^{\mathcal{V}}(t) := \begin{cases} \bar{\xi}(t) & \text{if } 0 \leq t \leq t_1(\alpha) \\ \delta(\alpha, t - t_1(\alpha)) & \text{if } t_1(\alpha) \leq t \leq t_1(\alpha) + \Delta_+(\alpha) = t_2(\alpha) + \Delta(\alpha) \\ \xi^E(t - \Delta(\alpha); \pi^{\mathcal{V}}(\alpha)) & \text{if } t_2(\alpha) + \Delta(\alpha) \leq t \leq T + \Delta(\alpha) \end{cases}$$

where the parameter $\pi^{\mathcal{V}}(\alpha) \in N^E$ which determines the trajectory of the imbedding to be followed for the last leg is the unique point z which satisfies

$$\xi^E(t_2(\alpha); z) = \delta(\alpha, \Delta_+(\alpha))$$

(see Fig. 3.1).

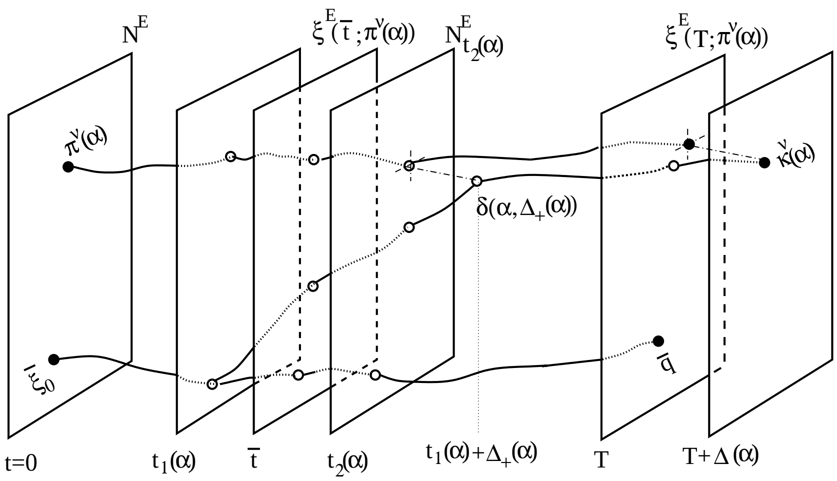

Figure 3.1: Point-variations

Note that $\delta(\alpha, \Delta_+(\alpha))$ is the point to which the variation corresponding to a fixed value α steers the system by the end of the variation. At this point the trajectory switches back to the "reference" trajectory in the sense that from now on it follows the trajectory in the imbedding of Γ which passes through $\delta(\alpha, \Delta_+(\alpha))$ at time $t_2(\alpha)$. Note that we do not just pick up the trajectory of the imbedding which passes through $\delta(\alpha, \Delta_+(\alpha))$ at time $t_1(\alpha) + \Delta_+(\alpha)$, the total time elapsed, but instead the point $\delta(\alpha, \Delta_+(\alpha))$ is taken as initial condition for the trajectory of the imbedding at time $t_2(\alpha)$. This allows to make variations in the terminal time.

Lemma 3.2 *For ϵ sufficiently small, $\pi^{\mathcal{V}}(\alpha)$ exists and is uniquely determined for all α in $[0, \epsilon)$.*

Proof: The set $V = \{(t, r) : 0 < t < T, r \in N^E_t\}$ is the image of $(0, T) \times N^E$ under the C^1-diffeomorphism $(t, r) \mapsto (t, \Phi^E_t(r))$ and thus V is a neighborhood of $\{(t, \bar{\xi}(t)) : 0 < t < T\}$. Since $\lim_{\alpha\to 0} t_2(\alpha) = \bar{t}$ and

$$\lim_{\alpha\to 0} \delta(\alpha, \Delta_+(\alpha)) = \delta(0, 0) = \bar{\xi}(t_1(0)) = \bar{\xi}(\bar{t}),$$

for ϵ sufficiently small, the points $(t_2(\alpha), \delta(\alpha, \Delta_+(\alpha))$ lie in V. Thus $\delta(\alpha, \Delta_+(\alpha)) \in N^E_{t_2(\alpha)}$ and hence this point is of the form

$$\delta(\alpha, \Delta_+(\alpha)) = \Phi^E_{t_2(\alpha)}(r) = \xi^E(t_2(\alpha); r) \tag{3.3}$$

for a unique $r \in N^E$. *QED*

The family of trajectories obtained from the variation $\mathcal{V}$ is then

$$\Gamma^{\mathcal{V}} = \left\{\Gamma^{\mathcal{V}}_\alpha = \left(\xi^{\mathcal{V}}_\alpha(\cdot), \eta^{\mathcal{V}}_\alpha(\cdot)\right), 0 \leq \alpha < \epsilon\right\}$$

and the terminal point

$$\kappa^{\mathcal{V}}(\alpha) = \xi^{\mathcal{V}}_\alpha(T + \Delta(\alpha)) = \xi^E(T; \pi^{\mathcal{V}}(\alpha)), \tag{3.4}$$

represents a new point in the reachable set. Also let

$$\zeta^{\mathcal{V}}(\alpha) = \xi^E(\bar{t}; \pi^{\mathcal{V}}(\alpha)), \tag{3.5}$$

so that $\kappa^{\mathcal{V}}(\alpha)$ is the image of $\zeta^{\mathcal{V}}(\alpha)$ under the flow of the imbedding from time $\bar{t}$ to the terminal time T.

The following two variations were the ones used in the original proof of the Maximum Principle:

Variation $\mathcal{V}_1$ **(CUT)**: Let $\bar{t} \in (0, T]$ and set

$$t_1(\alpha) = \bar{t} - \alpha, \ \ t_2(\alpha) = \bar{t}, \ \text{ and } \ \Delta_+(\alpha) = 0$$

This variation simply removes the part of the reference control on $[\bar{t} - \alpha, \bar{t}]$ (see Fig. 3.2). Hence

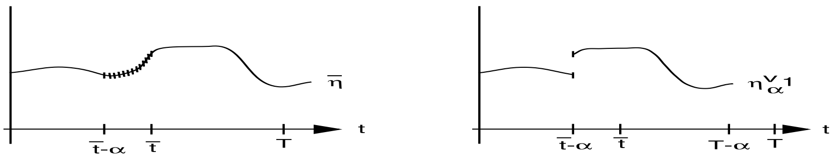

Figure 3.2: Variation CUT

$$\zeta^{\mathcal{V}}(\alpha) = \xi^E(\bar{t}; \pi^{\mathcal{V}}(\alpha)) = \bar{\xi}(\bar{t} - \alpha)$$

and therefore for times $\bar{t}$ where $\bar{\xi}$ is differentiable from the left with derivative $f(\bar{\xi}(\bar{t}), \bar{\eta}(\bar{t}))$, we have that $\zeta^{\mathcal{V}}(\alpha)$ is differentiable at zero with derivative

$$\boxed{\dot{\zeta}^{\mathcal{V}}(0) = -f(\bar{\xi}(\bar{t}), \bar{\eta}(\bar{t})).} \tag{3.6}$$

Since $\bar{\xi}$ is a solution to the ODE $\dot{x} = f(x, \bar{\eta}(t))$ this holds almost everywhere on $[0, T]$.

Variation $\mathcal{V}_2$ **(PASTE)**: Let $\bar{t} \in [0, T]$ and set

$$t_1(\alpha) = t_2(\alpha) = \bar{t}, \ \ \Delta_+(\alpha) = \alpha$$

Such a variation inserts at $\bar{\xi}(\bar{t})$ a trajectory of duration α (see Fig. 3.3). Pick $u_0 \in U$ and let $\delta(\alpha, \cdot)$ be the trajectory corresponding to the constant control $\mu(\alpha, s) \equiv u_0$. Note that for $\alpha_1 \leq \alpha_2$, the trajectory $\delta(\alpha_1, \cdot)$ is just the restriction of the trajectory $\delta(\alpha_2, \cdot)$ to the interval $[0, \alpha_1]$. Therefore the curve

$$\zeta^{\mathcal{V}}(\alpha) = \xi^E(\bar{t}; \pi^{\mathcal{V}}(\alpha)) = \delta(\alpha, \alpha)$$

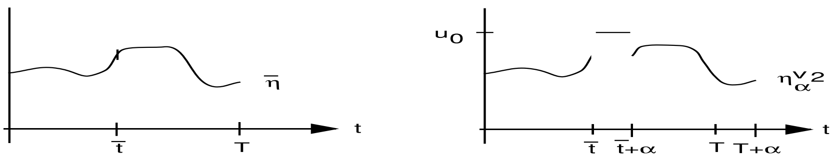

Figure 3.3: Variation PASTE

is differentiable from the right at 0 and

$$\dot{\zeta}^{\mathcal{V}}(\alpha) = f(\zeta^{\mathcal{V}}(\alpha), u_0).$$

In particular, for every $u_0 \in U$ and every $\bar{t} \in [0, T]$

$$\boxed{\dot{\zeta}^{\mathcal{V}}(0) = f(\bar{\xi}(\bar{t}), u_0).} \tag{3.7}$$

The guiding principle behind the construction of variations is to generate as many vectors as possible which are tangent to the reachable set at $\bar{q}$ and can be combined into an approximating cone. Note that the variations CUT and PASTE can *always* be made regardless of the structure of the reference control or trajectory. This is the main difference to the approach of calculus of variations. If the reference control has special structure, then it may be used to make additional variations. Whether this will give additional information of course depends on the structure of the new variations[8].

3.4 Variational Vector and Covector Fields. The curves $\kappa^{\mathcal{V}}$,

$$\begin{aligned} \kappa^{\mathcal{V}} \quad : \quad & [0,\epsilon] \to Reach_{\Sigma}(\bar{\xi}_0) \\ & \alpha \mapsto \kappa^{\mathcal{V}}(\alpha) = \xi^{\mathcal{V}}_{\alpha}(T + \Delta(\alpha)) = \xi^{E}(T; \pi^{\mathcal{V}}(\alpha)), \end{aligned}$$

lie in the reachable set. If the derivative from the right exists and equals a nonzero vector,

$$\left(\frac{d}{d\alpha}_{|\alpha=0}\right) \kappa^{\mathcal{V}}(\alpha) = \dot{\kappa}^{\mathcal{V}}(0) \neq 0,$$

then we call $\dot{\kappa}^{\mathcal{V}}(0)$ a tangent vector generated by the variation $\mathcal{V}$. Note that in general this derivative may fail to exist or be zero. However, for the variations CUT and PASTE considered above it was easy to calculate the derivative $\dot{\zeta}^{\mathcal{V}}(0)$ of the curve $\zeta^{\mathcal{V}}$,

$$\begin{aligned} \zeta^{\mathcal{V}} \quad : \quad & [0,\epsilon] \to M \\ & \alpha \mapsto \zeta^{\mathcal{V}}(\alpha) = \xi^{E}(\bar{t}; \pi^{\mathcal{V}}(\alpha)). \end{aligned}$$

But $\kappa^{\mathcal{V}}(\alpha)$ is the image of $\zeta^{\mathcal{V}}(\alpha)$ under the flow Φ^E of the embedding: if we define

$$\Phi^E_{t,s} : N^E_s \to N^E_t \quad \text{by} \quad \Phi^E_{t,s} = \Phi^E_t \circ (\Phi^E_s)^{-1},$$

[8]For instance, if the reference control takes values in the interior of the control set, then needle variations can be defined which are symmetric around the reference control and in this way the Legendre-Clebsch condition, an independent necessary condition for optimality of so-called singular arcs can be derived [38, 35].

then

$$\kappa^{\mathcal{V}}(\alpha) = \Phi^E_{T,\bar{t}}\left(\zeta^{\mathcal{V}}(\alpha)\right). \tag{3.8}$$

Therefore the tangent vectors are transformed by the corresponding differential[9], i.e.

$$\dot{\kappa}^{\mathcal{V}}(0) = \left(\Phi^E_{T,\bar{t}}\right)_\star \dot{\zeta}^{\mathcal{V}}(0). \tag{3.9}$$

For the canonical imbedding the differential of the flow is calculated as the solution to the variational equation along the reference trajectory: since $\kappa^{\mathcal{V}}(\alpha) = \xi^E(T;\pi^{\mathcal{V}}(\alpha))$ we have

$$\dot{\kappa}^{\mathcal{V}}(0) = \frac{\partial \xi^E}{\partial r}(T;\pi^{\mathcal{V}}(0))\dot{\pi}^{\mathcal{V}}(0) = \frac{\partial \xi^E}{\partial r}(T;\bar{\xi}_0)\dot{\pi}^{\mathcal{V}}(0).$$

Thus $\dot{\kappa}^{\mathcal{V}}(0)$ is the solution at time T to the variational equation (3.2) with initial condition $\dot{\pi}^{\mathcal{V}}(0)$ at time $t=0$. But also

$$\dot{\zeta}^{\mathcal{V}}(0) = \frac{\partial \xi^E}{\partial r}(\bar{t};\pi^{\mathcal{V}}(0))\dot{\pi}^{\mathcal{V}}(0)$$

and thus $\dot{\zeta}^{\mathcal{V}}(0)$ is the value of this solution at time $\bar{t}$. Hence $\dot{\kappa}^{\mathcal{V}}(0)$ can be calculated as the solution θ of the variational equation at time T with initial condition $\dot{\zeta}^{\mathcal{V}}(0)$ at time $\bar{t}$,

$$\dot{\theta}(t) = \frac{\partial f}{\partial x}(\bar{\xi}(t),\bar{\eta}(t))\theta(t), \qquad \theta(\bar{t}) = \dot{\zeta}^{\mathcal{V}}(0). \tag{3.10}$$

Thus for the variation PASTE the vector $\dot{\kappa}^{\mathcal{V}_2}(0)$ is obtained as the value at T of the solution of the variational equation along Γ with initial condition $f(\bar{\xi}(\bar{t}),u_0)$ at time $\bar{t}$. The same holds for variation CUT except that the initial condition is $-f(\bar{\xi}(\bar{t}),\bar{\eta}(\bar{t}))$ and only times $\bar{t}$ where $\bar{\xi}(\cdot)$ is differentiable from the left with derivative $-f(\bar{\xi}(\bar{t}),\bar{\eta}(\bar{t}))$ are permissible.

More generally, tangent vectors are moved along the reference trajectory with variational vector fields:

Definition 3.6 (variational vector field) *Given an admissible input-trajectory pair $\Gamma = (\bar{\xi}(\cdot),\bar{\eta}(\cdot))$ defined on $[0,T]$ and an imbedding E of Γ, a variational vector field along (Γ,E) is a map $\theta(\cdot):[0,T]\to TM$ such that $\theta(t)\in T_{\bar{\xi}(t)}M$ for all t and*

$$\theta(t) = \left(\Phi^E_{t,s}\right)_\star \theta(s) \quad \textit{for all } s,t\in[0,T]. \tag{3.11}$$

Note that given $\bar{t}\in[0,T]$ and $\bar{\theta}\in T_{\bar{\xi}(\bar{t})}M$, there exists a unique variational vector field which satisfies $\theta(\bar{t}) = \bar{\theta}$ given by $\theta(t) = \left(\Phi^E_{t,\bar{t}}\right)_\star \bar{\theta}$.

We shall also need to move covectors along the reference trajectory and this is done using the dual notion of a variational covector field:

[9] If $F: M\to N$ is a map between manifolds, then the differential $F_*: TM\to TN$ is the induced map on the tangent bundle, i.e., if $F(x)=y$ and $v\in T_xM$, then $F_*v\in T_yN$ is the tangent vector who acts on smooth functions ϕ defined near y according to $(F_*v)(\phi) = v(\phi\circ F)$.

Definition 3.7 (variational covector field) *Given an admissible input-trajectory pair $\Gamma = (\bar{\xi}(\cdot), \bar{\eta}(\cdot))$ defined on $[0,T]$ and an imbedding E of Γ, a variational covector field along (Γ, E) is a map $\psi(\cdot) : [0,T] \to T^*M$ such that $\psi(t) \in T^*_{\bar{\xi}(t)}(M)$ for all t and*

$$\psi(t) \left(\Phi^E_{t,s}\right)^\star = \psi(s) \quad \textit{for all } s,t \in [0,T]. \tag{3.12}$$

Here $\left(\Phi^E_{t,s}\right)^\star$ denotes the pullback[10] of the flow, or equivalently, the adjoint map to $\left(\Phi^E_{t,s}\right)_\star$. We prefer to write it so that it acts on the right. The following statement immediately follows from the definitions.

Proposition 3.1 *If θ is a variational vector field along Γ and ψ is a variational covector field along Γ, then the function $h : [0,T] \to \mathbb{R}, t \mapsto \langle \psi(t), \theta(t) \rangle$ is constant.*

Proof: For all $s,t \in [0,T]$ we have

$$\langle \psi(t), \theta(t) \rangle = \langle \psi(s) \left(\Phi^E_{s,t}\right)^\star, \theta(t) \rangle = \langle \psi(s), \left(\Phi^E_{s,t}\right)_\star \theta(t) \rangle = \langle \psi(s), \theta(s) \rangle. \quad QED$$

It can easily be seen from this that the pullback for the canonical imbedding can be calculated by integrating the adjoint equation to the variational equation. For, in coordinates we have for all variational vector fields θ that

$$\begin{aligned} 0 \quad &\equiv \quad \frac{d}{dt}\langle \psi(t), \theta(t) \rangle = \langle \dot{\psi}(t), \theta(t) \rangle + \langle \psi(t), \dot{\theta}(t) \rangle \\ &= \quad \langle \dot{\psi}(t), \theta(t) \rangle + \langle \psi(t), \frac{\partial f}{\partial x}(\bar{\xi}(t), \bar{\eta}(t))\theta(t) \rangle \\ &= \quad \langle \dot{\psi}(t) + \psi(t)\frac{\partial f}{\partial x}(\bar{\xi}(t), \bar{\eta}(t)), \theta(t) \rangle \end{aligned}$$

and thus

$$\dot{\psi}(t) = -\psi(t)\frac{\partial f}{\partial x}(\bar{\xi}(t), \bar{\eta}(t)). \tag{3.13}$$

Note that it is not necessary to take transposes in this equation since we write covectors as row vectors.

3.5 Construction of an Approximating Cone. We now construct an approximating cone for the reachable set from $\bar{\xi}_0$, $Reach_\Sigma(\bar{\xi}_0)$, at $\bar{q}$. So far we only constructed curves in the reachable set and took their tangent vectors. As seen in section 3.1 this by itself is not sufficient to ensure a result like Theorem 3.1. We need to be able to combine finitely many tangent vectors generated by different variations into an approximating map. For this the definition of point variation given in section 3.2 is not strong enough, but we also need to require that these variations or rather the corresponding flow of trajectories extends into a neighborhood of $\bar{\xi}(\bar{t})$ so that it is possible to apply these maps to closeby points. The reason for this will become clear in the proof of the main theorem. Hence we make the following definition:

[10]If $F : M \to N$ is a map between manifolds, then the pullback $F^* : T^*N \to T^*M$ is the dual (or adjoint or transpose) map to the differential $F_* : TM \to TN$. Hence, if $F(x) = y$, $v \in T_xM$, and $\psi \in T^*_yN$, then we have $\langle F^*(\psi), v \rangle = \langle \psi, F_*(v) \rangle$.

Definition 3.8 (C^1-extendable point variation) *We call a point variation $\mathcal{V}$ C^1-extendable if there exist a neighborhood W of $\bar{\xi}(\bar{t})$ and a map $\tilde{\delta}$ defined on the set*

$$\tilde{D} = \{(x, \alpha, s) : x \in W, 0 \le \alpha < \epsilon, 0 \le s \le \Delta_+(\alpha)\}$$

with values in M such that the following conditions are satisfied:

1. *For (x, α) fixed, $\tilde{\delta}(x, \alpha, \cdot)$ is an admissible trajectory of the system Σ which starts at the point x, $\tilde{\delta}(x, \alpha, 0) = x$.*

2. *If the initial point x corresponds to the initial point of the variation $\mathcal{V}$ for α, i.e., if $x = \bar{\xi}(t_1(\alpha))$, then $\tilde{\delta}(x, \alpha, \cdot)$ reduces to the trajectory $\delta(\alpha, \cdot)$ of the variation:*

$$\tilde{\delta}(\bar{\xi}(t_1(\alpha)), \alpha, s) = \delta(\alpha, s).$$

3. *For (α, s) fixed, $\tilde{\delta}(\cdot, \alpha, s)$ is continuously differentiable on W and both $\tilde{\delta}$ and $\frac{\partial \tilde{\delta}}{\partial x}$ are continuous on $\tilde{D}$.*

Both variations CUT and PASTE are C^1-extendable: For CUT this is trivial. Since $\Delta_+(\alpha) \equiv 0$, we can simply take $\tilde{\delta}(x, \alpha, 0) \equiv x$. For PASTE we choose as $\tilde{\delta}$ the obvious map: $\tilde{\delta}(x, \alpha, s)$ is the value at time s of the trajectory corresponding to the constant control u_0 starting at x at time 0. The required smoothness properties of $\tilde{\delta}$ then are immediate consequences of the C^1-dependence of the solution on initial conditions.

We now use the function $\tilde{\delta}$ to extend the map $\bar{\xi}(\bar{t}) \mapsto \zeta^{\mathcal{V}}(\alpha) = \xi^E(\bar{t}; \pi^{\mathcal{V}}(\alpha))$ into the neighborhood W of $\bar{\xi}(\bar{t})$: for $\alpha \in [0, \epsilon)$ define $\Psi_\alpha^{\mathcal{V}} : W \to M$ by

$$\Psi_\alpha^{\mathcal{V}}(x) = \Phi^E_{\bar{t}, t_2(\alpha)} \circ \tilde{\delta}\left(\Phi^E_{t_1(\alpha), \bar{t}}(x), \alpha, \Delta_+(\alpha)\right). \tag{3.14}$$

Geometrically, starting at the point $x \in W$, the flow of the imbedding is used to move x from time $\bar{t}$ to time $t_1(\alpha)$. Then the extension of the variation corresponding to α is run for the full time $\Delta_+(\alpha)$. The resulting point $\tilde{\delta}$ is then moved to time $\bar{t}$ by picking up the trajectory of the imbedding which passes through $\tilde{\delta}$ at time $t_2(\alpha)$ (see Fig. 3.4).

Proposition 3.2 *The function $\Psi_\alpha^{\mathcal{V}}$ is continuous in (x, α) on $W \times [0, \epsilon)$, continuously differentiable in x for fixed α and the partial derivative $\frac{\partial}{\partial x}\Psi_\alpha^{\mathcal{V}}$ is continuous in (x, α) on $W \times [0, \epsilon)$. For $x, y \in W$ and $\alpha \in [0, \epsilon)$ define a function $r = r(x, y, \alpha)$ by*

$$\Psi_\alpha^{\mathcal{V}}(x) - \Psi_\alpha^{\mathcal{V}}(y) = x - y + r(x, y, \alpha). \tag{3.15}$$

Then there exist a convex coordinate neighborhood $W' \subset W$ of $\bar{\xi}(\bar{t})$ with compact closure and a function $C : [0, \epsilon) \to \mathbb{R}_+$ such that for x and y in W'

$$||r(x, y, \alpha)|| \le C(\alpha)||x - y|| \tag{3.16}$$

and

$$\lim_{\alpha \to 0+} C(\alpha) = 0. \tag{3.17}$$

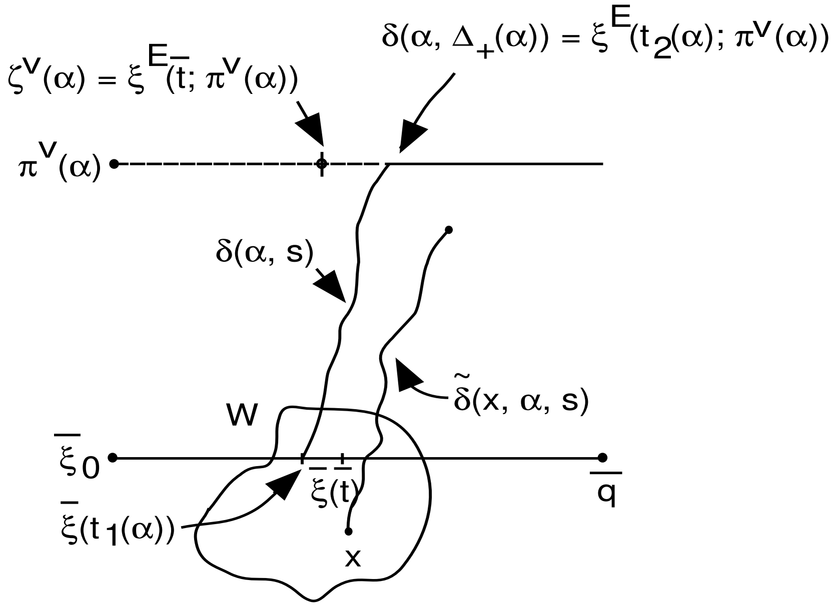

Figure 3.4: Extension of δ

Proof: The smoothness properties of $\Psi^{\mathcal{V}}_\alpha$ are a direct consequence of the definitions. Since the statement is local, without loss of generality we may choose coordinates and take a smaller coordinate neighborhood which is convex and with compact closure. We need to show that the reminder r is of order $o(||x-y||)$ uniformly for x, y. Note that

$$\begin{aligned}
\Psi^{\mathcal{V}}_\alpha(x) - \Psi^{\mathcal{V}}_\alpha(y) &= \int_0^1 \frac{d}{ds}\left(\Psi^{\mathcal{V}}_\alpha(y + s(x-y))\right) ds \\
&= \left(\int_0^1 \frac{\partial \Psi^{\mathcal{V}}_\alpha}{\partial x}(y + s(x-y)) ds\right)(x-y) \\
&= x - y + \left(\int_0^1 \left[\frac{\partial \Psi^{\mathcal{V}}_\alpha}{\partial x}(y + s(x-y)) - 1\right] ds\right)(x-y) \\
&= x - y + \tilde{r}(x, y, \alpha)(x-y)
\end{aligned}$$

where the last line defines $\tilde{r}$. It therefore suffices to show that $||\tilde{r}(x, y, \alpha)|| \le C(\alpha)$ while $C(\alpha)$ satisfies (3.17). By the explicit form given above $\tilde{r}$ is continuous for $x, y \in W$. Thus, by restricting to a compact neighborhood V of $\bar{\xi}(\bar{t})$ in W, for α fixed r satisfies a bound (3.16) where $C(\alpha) = \max_{x,y\in V} \tilde{r}(x, y, \alpha)$. But since $\Psi^{\mathcal{V}}_0(x) = \tilde{\delta}(x, 0, 0) = x$, we also have that $\tilde{r}(x, y, 0) \equiv 0$. Since a continuous function on a compact set is uniformly continuous, it therefore follows that $\lim_{\alpha\to 0+} C(\alpha) = 0$. *QED*

We are now ready to formulate and state the result of this construction. Let

Q$(\bar{t})$ be the set of all vectors $\dot{\zeta}^{\mathcal{V}}(0)$ generated by C^1-extendable point variations at time $\bar{t}$ for which this derivative exists and let

$\mathbf{P}(\bar{t})$ be the set of all vectors $w \in T_{\bar{\xi}(\bar{t})}M$ for which there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n \neq \bar{t}$, converging to $\bar{t}$ and a sequence of vectors $\{w_n\}_{n\in\mathbb{N}}$, $w_n \in Q(t_n)$, which converge to w as $n \to \infty$.

Thus in some sense the vectors in $P(\bar{t})$ correspond to approximating directions which can be generated not only at some specific time $\bar{t}$, but also at closeby points on the reference trajectory (see also [36]). The directions generated by the variations CUT and PASTE have this property.

Lemma 3.3

$$-f(\bar{\xi}(\bar{t})), \bar{\eta}(\bar{t})) \in P(\bar{t}) \qquad \textit{for a.e. } \bar{t} \in [0,T] \tag{3.18}$$

Proof: We know that $-f(\bar{\xi}(\bar{t})), \bar{\eta}(\bar{t})) \in Q(\bar{t})$ holds for times $\bar{t} \in [0,T]$ in a set F of full measure. Given $\varepsilon > 0$, by Lusin's theorem[11] there exists a closed subset F_ε of $[0,T]$ of Lebesgue measure $\mu(F_\varepsilon) > T - \varepsilon$ such that the restriction of the function $\bar{\phi} : [0,T] \to \mathbb{R}^n, t \mapsto f(\bar{\xi}(t), \bar{\eta}(t))$ to F_ε is continuous. Let $\tilde{F}_\varepsilon$ be the set of limit points[12] of $F \cap F_\varepsilon$ in $F \cap F_\varepsilon$. The complement of $\tilde{F}_\varepsilon$ in $F \cap F_\varepsilon$ is a discrete set and hence at most countable. Thus $\tilde{F}_\varepsilon$ has the same Lebesgue measure as $F \cap F_\varepsilon$. For $\bar{t} \in \tilde{F}_\varepsilon$ choose a sequence $\{t_n\}_{n\in\mathbb{N}} \subset F \cap F_\varepsilon$, $t_n \neq \bar{t}$. Since $\bar{\phi}$ is continuous on F_ε, it follows that $f(\bar{\xi}(t_n)), \bar{\eta}(t_n)) \to f(\bar{\xi}(\bar{t})), \bar{\eta}(\bar{t}))$ as $n \to \infty$. Hence $-f(\bar{\xi}(\bar{t})), \bar{\eta}(\bar{t})) \in P(\bar{t})$ for all $\bar{t} \in \tilde{F}_\varepsilon$. Since ε is arbitrary this holds a.e. on $[0,T]$. *QED*

The corresponding Lemma for the variation PASTE is immediate since for any fixed control $u_0 \in U$ we have $\lim_{t_n \to \bar{t}} f(\bar{\xi}(t_n), u_0) = f(\bar{\xi}(\bar{t}), u_0)$ by continuity. Thus:

Lemma 3.4 *For every $u \in U$ and every $\bar{t} \in [0,T]$ we have*

$$f(\bar{\xi}(\bar{t})), u) \in P(\bar{t}) \tag{3.19}$$

Finally, moving the vectors in $P(\bar{t})$ forward to the terminal point using the imbedding we generate tangent directions to the reachable set at $\bar{q}$. Let

$$P^*(\bar{t}) = \left\{ \left(\Phi^E_{T,\bar{t}}\right)_* w : w \in P(\bar{t}) \right\}$$

and let $\mathcal{K}$ be the convex hull generated by all possible vectors in $P^*(\bar{t})$,

$$\mathcal{K} = co\left(\bigcup_{0 \le \bar{t} \le T} P^*(\bar{t}) \right).$$

Then we have

Theorem 3.3 *$\mathcal{K}$ is an approximating cone to $Reach_\Sigma(\bar{\xi}_0)$ at $\bar{q}$.*

[11]Lusin's Theorem: If $f : [0,T] \to \mathbb{R}^n$ is Lebesgue measurable, then for every $\varepsilon > 0$ there exists a measurable subset $F_\varepsilon \subset [0,T]$ of measure greater than $T - \varepsilon$ such that the restriction of f to F_ε is continuous.

[12]A point z is a limit point of a set Z if there exist points $z_n \in Z$, $z_n \neq z$, such that $z_n \to z$.

Proof: Since the convex hull of an approximating cone is again an approximating cone, it suffices to show that $\bigcup_{0\le\bar{t}\le T} P^*(\bar{t})$ is an approximating cone. For this, given any vectors $v_1, \ldots, v_r$, $v_i \in P^*(t_i)$ for some t_i, we need to construct an approximating map.

We first show how this can be done if the vectors v_i are of the form $v_i = \left(\Phi^E_{T,t_i}\right)_* w_i$ with $w_i \in Q(t_i)$ for $t_1 < \ldots < t_r$. Let $\mathcal{V}_i$ be C^1-extendable point variations which generate the w_i. In this case we show that the map

$$\Xi(\alpha_1, \ldots, \alpha_r) = \Phi^E_{T,t_r} \circ \Psi^{\mathcal{V}_r}_{\alpha_r} \circ \Phi^E_{t_r,t_{r-1}} \circ \cdots \circ \Phi^E_{t_2,t_1} \circ \Psi^{\mathcal{V}_1}_{\alpha_1} \circ \Phi^E_{t_1,0}(\bar{\xi}_0) \tag{3.20}$$

is an approximating map as defined in Definition 3.1 for $v_i' = v_i$.

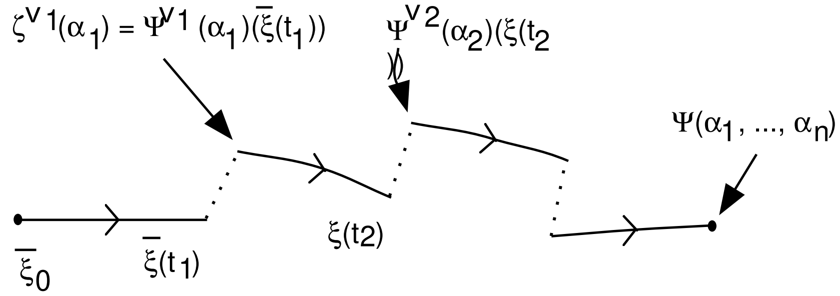

Figure 3.5: Point-variations at different times

The construction of the map Ξ is illustrated in Fig. 3.5. Making a first variation at time t_1 steers the system to the point $\zeta^{\mathcal{V}_1}(\alpha_1) = \Psi^{\mathcal{V}_1}_{\alpha_1}(\bar{\xi}(t_1))$. Then the reference control is being picked up again in form of the trajectory of the imbedding which passes through $\zeta^{\mathcal{V}_1}(\alpha_1)$ at time t_1 and is used to control the system over $[t_1, t_2]$. At that time a second variation is executed. Note, however, that this variation is not executed from the reference trajectory, but rather from this new close-by trajectory of the imbedding. This is the reason for considering only C^1-extendable variations. This step is then repeated finitely many times.

We now verify that Ξ is an approximating map. This is shown by induction on r. For the case $r = 1$ it is simply a consequence of the differentiability of $\zeta^{\mathcal{V}_1}(\alpha_1)$ at $\alpha = 0$ from the right. To simplify the notation, we carry out the inductive step only for $r = 2$. The general argument is the same.

It is assumed that the vectors w_1 and w_2 are generated by C^1-extendable variations $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., in the notation introduced in section 3.3 we have

$$v_i = \dot{\kappa}^{\mathcal{V}_i}(0) = \left(\Phi^E_{T,t_i}\right)_* \dot{\zeta}^{\mathcal{V}_i}(0) = \left(\Phi^E_{T,t_i}\right)_* w_i$$

where $\zeta^{\mathcal{V}_i}(\alpha_i) = \Psi^{\mathcal{V}_i}_{\alpha_i}(\bar{\xi}(t_i))$. By assumption the derivatives $\dot{\zeta}^{\mathcal{V}_i}(0)$ exist. Thus by Taylor's theorem we have

$$\Psi^{\mathcal{V}_1}_{\alpha_1}(\bar{\xi}(t_1)) = \zeta^{\mathcal{V}_1}(\alpha_1) = \bar{\xi}(t_1) + \alpha_1 \dot{\zeta}^{\mathcal{V}_1}(0) + o(\alpha_1). \tag{3.21}$$

Here and in the following we use the standard Landau notation $o(z)$ to denote (changing) functions ρ which have the property that $\lim_{z\to 0} \frac{\rho(z)}{z} = 0$ and $O(z)$ to denote functions ρ which remain bounded after division by $||z||$, i.e., $\limsup_{z\to 0}\frac{\rho(z)}{z}$ is finite. Applying the diffeomorphism $\Phi^E_{t_2,t_1}$ to (3.21) then yields

$$\Phi^E_{t_2,t_1}\left(\Psi^{\mathcal{V}_1}_{\alpha_1}(\bar{\xi}(t_1))\right)=\bar{\xi}(t_2)+\alpha_1\left(\Phi^E_{t_2,t_1}\right)_*\dot{\zeta}^{\mathcal{V}_1}(0)+o(\alpha_1). \tag{3.22}$$

Now the second variation $\mathcal{V}_2$ is executed, but starting from this point, *not* from the point $\bar{\xi}(\bar{t}_2)$ on the reference trajectory. Proposition 3.2 above provides the required control over the error made in this process. For α_1 sufficiently small the point $\bar{\xi}(t_2)+\alpha_1\left(\Phi^E_{t_2,t_1}\right)_*\dot{\zeta}^{\mathcal{V}_1}(0)+o(\alpha_1)$ lies in a prescribed compact neighborhood of $\bar{\xi}(t_2)$ where the proposition will apply. Taking $x=\bar{\xi}(t_2)+\alpha_1\left(\Phi^E_{t_2,t_1}\right)_*\dot{\zeta}^{\mathcal{V}_1}(0)+o(\alpha_1)$ and $y=\bar{\xi}(t_2)$ in this proposition, it follows that

$$\begin{aligned}\Psi^{\mathcal{V}_2}_{\alpha_2}\left(\Phi^E_{t_2,t_1}(\Psi^{\mathcal{V}_1}_{\alpha_1}(\bar{\xi}(t_1)))\right) &= \Psi^{\mathcal{V}_2}_{\alpha_2}\left(\bar{\xi}(t_2)+\alpha_1\left(\Phi^E_{t_2,t_1}\right)_*\dot{\zeta}^{\mathcal{V}_1}(0)+o(\alpha_1)\right)\\ &= \Psi^{\mathcal{V}_2}_{\alpha_2}(\bar{\xi}(t_2))+\alpha_1\left(\Phi^E_{t_2,t_1}\right)_*\dot{\zeta}^{\mathcal{V}_1}(0)\\ &\qquad +o(\alpha_1)+o(1)\cdot O(\alpha_1)\end{aligned} \tag{3.23}$$

In (3.23) the first remainder term $o(\alpha_1)$ is generated by moving the remainder in (3.22) forward along the flow while the second contribution to the remainder is generated by the term

$$C(\alpha_2)||\alpha_1\left(\Phi^E_{t_2,t_1}\right)_*\dot{\zeta}^{\mathcal{V}_1}(0)+o(\alpha_1)||$$

arising in the bound (3.16) in Propositon 3.2. The error made in switching back to the reference trajectory from the new trajectory reached after the first variation is thus of order $O(\alpha_1)$ and it is only propagated with order $o(1)$ as $\alpha_2\to 0$. Since we still have

$$\frac{o(1)\cdot O(\alpha_1)}{\alpha_1+\alpha_2}\to 0,$$

it follows that these remainders together are of order $o(\alpha_1+\alpha_2)$. Furthermore, we also have

$$\Psi^{\mathcal{V}_2}_{\alpha_2}(\bar{\xi}(t_2))=\zeta^{\mathcal{V}_2}(\alpha_2)=\bar{\xi}(t_2)+\alpha_2\dot{\zeta}^{\mathcal{V}_2}(0)+o(\alpha_2). \tag{3.24}$$

Combining these equations yields

$$\Psi^{\mathcal{V}_2}_{\alpha_2}\left(\Phi^E_{t_2,t_1}\left(\Psi^{\mathcal{V}_1}_{\alpha_1}(\bar{\xi}(t_1))\right)\right)=\bar{\xi}(\bar{t}_2)+\alpha_2\dot{\zeta}^{\mathcal{V}_2}(0)+\alpha_1\left(\Phi^E_{t_2,t_1}\right)_*\dot{\zeta}^{\mathcal{V}_1}(0)+o(\alpha_1+\alpha_2). \tag{3.25}$$

Finally, moving all the vectors with the flow Φ^E_{T,t_2} to the terminal point we get

$$\begin{aligned}\Xi(\alpha_1,\alpha_2) &= \bar{q}+\alpha_1\dot{\kappa}^{\mathcal{V}_1}(0)+\alpha_2\dot{\kappa}^{\mathcal{V}_2}(0)+o(\alpha_1+\alpha_2)\\ &= \bar{q}+\alpha_1 v_1+\alpha_2 v_2+o(\alpha_1+\alpha_2).\end{aligned}$$

Thus Ξ is an approximating map for $v_1=\dot{\kappa}^{\mathcal{V}_1}(0)$ and $v_2=\dot{\kappa}^{\mathcal{V}_2}(0)$.

Now consider the general case: suppose $v_1,\ldots,v_r$ are arbitrary vectors from $\bigcup_{0\le\bar{t}\le T}P^*(\bar{t})$. There exist times $t_i\in[0,T]$, not necessarily distinct, and vectors $w_i\in P(t_i)$ such that $v_i=\left(\Phi^E_{T,t_i}\right)_* w_i$. By definition of $P(t_i)$ there exist sequences $\{t_{i,n}\}_{n\in\mathbb{N}}$, $t_{i,n}\neq t_i$, $t_{i,n}\to t_i$, and vectors $w_{i,n}\in Q(t_{i,n})$ converging to w_i. Given $\epsilon>0$ choose integers $n_1,\ldots,n_r$ such that the times t_{i,n_i} are all distinct and such that $||w_i - w_{i,n_i}|| < \epsilon$. By the first step of the proof there exists an approximating map Ξ for the vectors v_i using $v_i' := \left(\Phi^E_{T,t_i}\right)_* w_{i,n_i}$. Hence $\bigcup_{0\le\bar{t}\le T} P^*(\bar{t})$ and thus also $\mathcal{K}$ is an approximating cone. *QED*

The last step in the proof in conjunction with the definition of the set $P(\bar{t})$ is the reason for the somewhat odd definition of approximating cone allowing to approximate nearby vectors. These definitions have the advantage that we do not need to address the highly nontrivial issue of whether and when point-variations made at the same time $\bar{t}$ can be combined. This is not true in general (see, for instance [36, 4]).

3.6. Necessary Conditions for Boundary Trajectories. Theorem 3.3 immediately implies necessary conditions for a point to lie in the boundary of the reachable set. For, if $\bar{q} \in \partial Reach_\Sigma(\bar{\xi}_0)$, then by Proposition 3.1 $\mathcal{K}$ cannot be the full tangent space $T_{\bar{q}}M$. Hence there exists a non-zero covector $\bar{\psi} \in T^*_{\bar{q}}M$ such that

$$\langle \bar{\psi}, v\rangle \le 0 \text{ for all } v \in \mathcal{K}.$$

In the construction of $\mathcal{K}$ the tangent vectors $w \in P(\bar{t})$ are moved forward along the reference trajectory to the endpoint $\bar{q}$. Rather than moving all these tangent vectors, we can instead move the covector $\bar{\psi}$ backward along the trajectory to $\bar{\xi}(\bar{t})$. This is simply done by the variational covector field with terminal condition $\psi(T) = \bar{\psi}$. Then we get:

Corollary 3.1 *If $\bar{q} \in \partial \mathrm{Reach}_\Sigma(\bar{\xi}_0)$, then there exists a nontrivial variational covector field $\psi : [0,T] \to T^*M$ such that*

$$\langle \psi(t), w\rangle \le 0 \textit{ for all } w \in P(t). \tag{3.26}$$

By considering the variations CUT and PASTE along with the canonical imbedding this abstract formulation can be made more specific. Recall from section 3.3 that, in local coordinates, this variational covector field is the solution to the adjoint equation,

$$\dot{\psi}(t) = -\psi(t)\frac{\partial f}{\partial x}(\bar{\xi}(t), \bar{\eta}(t)), \qquad \psi(T) \ne 0.$$

It follows from Lemmas 3.3 and 3.4 that we have

$$\langle \psi(t), f(\bar{\xi}(t), u)\rangle \le 0 \qquad \text{for all } t \in [0,T] \text{ and } u \in U$$

and

$$\langle \psi(t), f(\bar{\xi}(t), \bar{\eta}(t))\rangle \ge 0 \qquad \text{a.e. on } [0,T].$$

If we introduce the Hamiltonian as

$$H(p,x,u) = \langle p, f(x,u)\rangle, \tag{3.27}$$

then the following result follows:

Theorem 3.4 [47] (Maximum Principle for boundary trajectories) *Let $\Gamma = (\bar{\xi}(\cdot), \bar{\eta}(\cdot))$ be an admissible trajectory defined on $[0,T]$ with initial point $\bar{\xi}(0) = \bar{\xi}_0$ and terminal point $\bar{\xi}(T) = \bar{q}$. If $\bar{q} \in \partial \mathrm{Reach}_\Sigma(\bar{\xi}_0)$, then there exists a nontrivial variational covector field along Γ (or adjoint vector) $\psi : [0,T] \to T^*M$,*

$$\dot{\psi}(t) = -\psi(t)\frac{\partial f}{\partial x}(\bar{\xi}(t), \bar{\eta}(t)), \qquad \psi(T) \neq 0 \tag{3.28}$$

such that almost everywhere on $[0,T]$

$$0 = H(\psi(t), \bar{\xi}(t), \bar{\eta}(t)) = \max_{u \in U} H(\psi(t), \bar{\xi}(t), u). \tag{3.29}$$

3.7 Necessary Conditions for Optimality. Also necessary conditions for optimality for an optimal control problem easily follow from Theorem 3.3. We consider the problem **(OC)** to minimize the functional

$$\mathcal{J}(\eta) = \int_0^T L(\xi(t), \eta(t))dt + \varphi(\xi(T)) \tag{3.30}$$

over all admissible controls η subject to the dynamics $\dot{x} = f(x, \eta(t))$ with fixed initial condition $\xi(0) = \xi_0$, terminal condition $\xi(T) \in N$, and control constraint $U \subset \mathbb{R}^m$. The terminal time T is free.

We make the following regularity **assumptions** on the data:

- The functions $f : M \times U \to TM$, $(q,u) \mapsto f(q,u) \in T_qM$ and $L : M \times U \to \mathbb{R}$, $(q,u) \mapsto L(q,u)$ have the following regularity property: for every coordinate chart $\phi : V \to \mathbb{R}^n$ with domain V the coordinate representations f_ϕ of f on $\phi(V)$,

$$\begin{aligned} f_\phi &: \phi(V) \times U \to \mathbb{R}^n, \\ (x,u) &\mapsto f_\phi(x,u) = d\phi(\phi^{-1}(x)) f(\phi^{-1}(x), u), \end{aligned}$$

  and L_ϕ of L on $\phi(V)$,

$$\begin{aligned} L_\phi &: \phi(V) \times U \to \mathbb{R}, \\ (x,u) &\mapsto L_\phi(x,u) = d\phi(\phi^{-1}(x)) L(\phi^{-1}(x), u), \end{aligned}$$

  are continuous on $\phi(V) \times U$, continuously differentiable in x for each fixed $u \in U$ and the partial derivatives with respect to x are continuous on $\phi(V) \times U$. As before, to ease the notation we will use the same notation $f(x,u)$ and $L(x,u)$ for both the maps on the manifold and their coordinate representations f_ϕ respectively L_ϕ.

- N is a k-dimensional imbedded submanifold of the state space M and $\varphi : N \to \mathbb{R}$ is continuously differentiable.

As before, admissible controls are Lebesgue-measurable functions defined on a compact interval and with values in a compact subset of U.

It is possible to phrase optimality of a trajectory in terms of the reachable set for a system with augmented state space where the cost has been added as extra variable. Let $\widetilde{\Sigma}$ be the system with state space $\widetilde{M} = M \times \mathbb{R}$, the same control set U, and dynamical law $\widetilde{f} : \widetilde{M} \times U \to \mathbb{R}^{n+1}$, $(x, y, u) \mapsto \widetilde{f}(x, y, u) = (f(x, u), L(x, u))$ with initial condition $\widetilde{\xi}_0 = (\xi_0, 0)$. Thus the dynamical equations are given by

$$\begin{aligned} \dot{x} &= f(x, \eta(t)), \qquad x(0) = \xi_0 \\ \dot{y} &= L(x, \eta(t)), \qquad y(0) = 0. \end{aligned}$$

The terminal manifold remains the same, but is now considered an imbedded submanifold of $\widetilde{M}$. Thus to each admissible input-trajectory pair (ξ, η) of Σ defined over the interval $[0, T]$, there corresponds a unique trajectory (ξ, ρ) of $\widetilde{\Sigma}$ with ρ given as

$$\rho(t) = \int_0^t L(\xi(s), \eta(s)) ds.$$

The corresponding value of the objective is therefore $\rho(T) + \varphi(\xi(T))$. On $N \times \mathbb{R}$ let the function $\omega : (x, y) \mapsto \omega(x, y) = \varphi(x) + y$ denote this expression. Given a reference trajectory $\bar{\xi}$ defined over the interval $[0, T]$ which steers $\bar{\xi}_0$ into $\bar{q}$, let $\bar{\omega} = \omega(\bar{\xi}(T), \bar{\rho}(T))$ be the value of the objective. Define

$$\widetilde{A} = \{(x, y) \in N \times \mathbb{R} : \omega(x, y) = y + \varphi(x) < \bar{\omega}\}, \tag{3.31}$$

as the set of all possible terminal points which have a better value for the objective. An approximating cone to this set is easily constructed:

Lemma 3.5

$$\widetilde{\mathcal{J}} = \{(v, \alpha) \in T_{\bar{q}} N \times \mathbb{R} : \langle d\varphi(\bar{q}), v \rangle + \alpha < 0\} \tag{3.32}$$

is a nonempty approximating cone for $\widetilde{A}$ at $\widetilde{q}$ which is not a linear subspace.

Proof: It is clear that $\widetilde{\mathcal{J}}$ is a nonempty convex cone which is not a linear subspace (e.g. $(0, -1) \in \widetilde{J}$, but $(0, 1) \notin \widetilde{J}$). We will show that it is an approximating cone. Let $(v_1, \alpha_1), \ldots, (v_n, \alpha_n) \in \widetilde{\mathcal{J}}$. Since the vectors v_i are tangent vectors to N at $\bar{q}$, there exists an approximating map $\Psi : Q^n(\delta) \to N$,

$$\Psi(z_1, \ldots, z_n) = \bar{q} + \sum_{i=1}^{n} z_i v_i + o(\|z\|).$$

for these vectors. Define a map $\widetilde{\Psi}$ by

$$\widetilde{\Psi}(z_1, \ldots, z_n) = (\Psi(z_1, \ldots, z_n), \bar{\omega} + \sum_{i=1}^{n} \alpha_i z_i).$$

It is clear that $\widetilde{\Psi}$ satisfies (3.1) with $(v_i', \alpha_i') = (v_i, \alpha_i)$. By definition of $\widetilde{\mathcal{J}}$, $\widetilde{\Psi}$ also takes values in $\widetilde{A}$ for small $\|z\|$:

$$\begin{aligned}
\omega(\widetilde{\Psi}(z_1,\ldots,z_n)) &= \varphi\left(\Psi(z_1,\ldots,z_n)\right)+\bar{\omega}+\sum_{i=1}^{n}\alpha_i z_i \\
&= \varphi(\bar{q})+\sum_{i=1}^{n}\langle d\varphi(\bar{q}),v_i\rangle z_i+o(\|z\|)+\bar{\omega}+\sum_{i=1}^{n}\alpha_i z_i \\
&= \varphi(\bar{q})+\bar{\omega}+\sum_{i=1}^{n}\left(\langle d\varphi(\bar{q}),v_i\rangle+\alpha_i\right)z_i+o(\|z\|) \\
&< \varphi(\bar{q})+\bar{\omega}.
\end{aligned}$$

The last inequality follows since $\langle d\varphi(\bar{q}),v_i\rangle+\alpha_i$ is negative by definition of $\widetilde{\mathcal{J}}$ and thus dominates the remainder $o(\|z\|)$ for $\|z\|$ sufficiently small. Hence $\widetilde{\Psi}$ takes values in $\widetilde{A}$ and thus is an approximating map. *QED*

Now suppose Γ is optimal. Then no point in $\widetilde{A}$ can be reachable and thus the reachable set of $\widetilde{\Sigma}$ does not intersect $\widetilde{A}$,

$$Reach_{\widetilde{\Sigma}}(\widetilde{\xi_0})\cap\widetilde{A}=\emptyset. \tag{3.33}$$

In other words, $\widetilde{q}$ is an isolated point of $Reach_{\widetilde{\Sigma}}(\widetilde{\xi_0})\cap(\widetilde{A}\cup\{\widetilde{q}\})$. Since $\widetilde{\mathcal{K}}$ and $\widetilde{J}$ are strongly transversal by Lemma 3.1, it follows from Theorem 3.2 that the convex approximating cone $\widetilde{\mathcal{K}}$ to the reachable set of $\widetilde{\Sigma}$ at $(\bar{\xi}(T),\bar{\rho}(T))$ can be seperated from $\widetilde{J}$. Thus there exists a nontrivial covector $\widetilde{\widetilde{\psi}}\in T^*_{\widetilde{q}}\widetilde{M}$ such that

$$\langle\widetilde{\widetilde{\psi}},\widetilde{v}\rangle\leq 0 \text{ for all } \widetilde{v}\in\widetilde{\mathcal{K}} \tag{3.34}$$

and

$$\langle\widetilde{\widetilde{\psi}},\widetilde{j}\rangle\geq 0 \text{ for all } \widetilde{j}\in\widetilde{\mathcal{J}} \tag{3.35}$$

Again, by moving the covector back along the reference trajectory with the flow, it follows that there exists a nontrivial variational covector field $\widetilde{\psi}:[0,T]\to T^*(M\times\mathbb{R})=T^*M\times\mathbb{R}$, $\widetilde{\psi}(T)=\widetilde{\widetilde{\psi}}$, such that

$$\langle\widetilde{\psi}(t),w\rangle\leq 0 \text{ for all } w\in P(t). \tag{3.36}$$

Using the canonical imbedding and the variations CUT and PASTE as in section 3.5 this implies that there exists a nontrivial solution $\widetilde{\psi}=(\psi,-\psi_0)$ to the adjoint equation such that we have almost everywhere on $[0,T]$

$$0\equiv\langle\widetilde{\psi}(t),\widetilde{f}(\bar{\xi}(t),\bar{\rho}(t),\bar{\eta}(t))\rangle=\max_{u\in U}\langle\widetilde{\psi}(t),\widetilde{f}(\bar{\xi}(t),\bar{\rho}(t),u)\rangle.$$

In components the adjoint equation reads

$$\begin{aligned}
\dot{\psi}(t) &= -\psi(t)\frac{\partial f}{\partial x}(\bar{\xi}(t),\bar{\eta}(t))+\psi_0(t)\frac{\partial L}{\partial x}(\bar{\xi}(t),\bar{\eta}(t)) \\
\dot{\psi}_0(t) &\equiv 0,
\end{aligned}$$

i.e., ψ_0 is constant. If we define the Hamiltonian H,

$$H(p_0, p, x, u) = -p_0 L(x, u) + p f(x, u),$$

then the adjoint equation for ψ can be restated as

$$\dot{\psi}(t) = -\frac{\partial H}{\partial x}(\psi_0, \psi(t), \bar{\xi}(t), \bar{\eta}(t))$$

and the maximum condition becomes

$$0 = H(\psi_0, \psi(t), \bar{\xi}(t), \bar{\eta}(t)) = \max_{u \in U} H(\psi_0, \psi(t), \bar{\xi}(t), u).$$

We also have the nontrivality condition $(\psi_0, \psi(t)) \neq 0$ for $t \in [0, T]$ by the nontriviality of $\widetilde{\widetilde{\psi}}$. Condition (3.36) furthermore gives that

$$\langle \psi(T), v \rangle - \psi_0 \alpha \geq 0 \quad \text{for all } (v, \alpha) \in \widetilde{J}.$$

Taking $(0, -1) \in \widetilde{J}$ it follows that $\psi_0 \geq 0$. Furthermore, in $\widetilde{\mathcal{J}}$ we may let $\alpha \to -\langle d\varphi(\bar{q}), v \rangle$ and this implies

$$\langle \psi(T), v \rangle \geq -\psi_0 \langle d\varphi(\bar{q}), v \rangle \text{ for all } v \in T_{\bar{q}} N.$$

But $T_{\bar{q}} N$ is a linear subspace and thus also $-v$ lies in $T_{\bar{q}} N$ if v does. Therefore this inequality is indeed an equality, i.e., the restriction of $\psi(T)$ to the tangent space of N at $\bar{q}$ is given by $-\psi_0 d\varphi(\bar{q})$. This is the *transversality condition* of the Maximum Principle. Summarizing, we have shown the following theorem:

Theorem 3.5 [47] (Maximum Principle for optimal control problems) *Suppose the admissible input-trajectory pair $\Gamma = (\bar{\xi}(\cdot), \bar{\eta}(\cdot))$ defined on $[0, T]$ with initial point $\xi(0) = \bar{\xi}_0$ and terminal point $\bar{\xi}(T) = \bar{q}$ is optimal for the optimal contol problem (OC). Then there exist a constant $\psi_0 \geq 0$ and a solution $\psi : [0, T] \to T^* M$ of the adjoint equation along Γ,*

$$\dot{\psi}(t) = \psi_0 \frac{\partial L}{\partial x}(\bar{\xi}(t), \bar{\eta}(t)) - \psi(t) \frac{\partial f}{\partial x}(\bar{\xi}(t), \bar{\eta}(t)), \tag{3.37}$$

which satisfies the terminal condition

$$\langle \psi(T), v \rangle = -\psi_0 \langle d\varphi(\bar{q}), v \rangle \quad \textit{for all} \quad v \in T_{\bar{q}} N, \tag{3.38}$$

such that

$$(\psi_0, \psi(t)) \neq 0 \textit{ for all } t \in [0, T], \tag{3.39}$$

and almost everywhere on $[0, T]$ we have

$$0 = H(\psi_0, \psi(t), \bar{\xi}(t), \bar{\eta}(t)) = \max_{u \in U} H(\psi_0, \psi(t), \bar{\xi}(t), u). \tag{3.40}$$

4 Small-Time Reachable Sets in Low Dimension

In this chapter we illustrate how the Maximum Principle can be used to obtain information about the structure of reachable sets in small time. Recall that given an admissible control and an initial condition ξ_0 there exists a unique corresponding trajectory defined on some maximal interval. A point q is reachable from ξ_0 in time t if there exists an admissible control η such that the corresponding trajectory ξ with initial condition $\xi(0) = \xi_0$ satisfies $q = \xi(t)$. If q is not reachable in smaller time, then the trajectory is called *time-optimal*. Clearly there are close connections between boundary trajectories and time-optimal trajectories. Let $Reach_{\Sigma,t}(\xi_0)$ denote the set of all points which are reachable in time t. If ξ is a time-optimal trajectory defined over $[0,T]$, then $\xi(t) \in \partial Reach_{\Sigma,t}(\xi_0)$ for all times $t \in [0,T)$. (For, if $\xi(\tau) \in int Reach_{\Sigma,\tau}(\xi_0)$, then there exists an $\epsilon > 0$ such that $\xi(\tau+\epsilon)$ is also reachable in time τ. Concatenating the corresponding control with the original trajectory when $\xi(\tau+\epsilon)$ is reached steers ξ_0 into $\xi(T)$ in time $T-\epsilon$.) Thus an analysis of the boundary of the time-t-reachable sets gives necessary conditions for time-optimal trajectories. At the expense of adding one extra dimension to the state variable we can always achieve that boundary trajectories also lie in the boundary of the time-t-reachable sets by adjoining the trivial dynamics $\dot{t} \equiv 1$. Hence an analysis of boundary trajectories has direct implications on the structure of time-optimal controls [35]. We will only comment on these relations briefly in section 4.5, but keep our attention focussed on the structure of reachable sets of the system Σ from ξ_0, $Reach_\Sigma(\xi_0)$, and $Reach_{\Sigma,\leq T}(\xi_0)$, those points which are reachable in times t, $0 \leq t \leq T$. If T is "small", we call this set the *small-time reachable set*.

In this section we exclusively consider systems of the form Σ

$$\dot{x} = f(x) + ug(x), \quad |u| \leq 1, \quad x \in M \subseteq \mathbb{R}^n, \tag{4.1}$$

where M is a sufficiently small open neighborhood of some reference point which henceforth we prefer to call q_0, f and g are smooth (sufficiently often continuously differentiable) vector fields on M and admissible controls are Lebesgue measurable functions η with values in the closed interval $[-1,1]$ almost everywhere. In general, even for systems of the form (4.1), the structure of the small-time reachable set can be arbitrarily complicated making a general analysis prohibitively difficult. But it is reasonable, and this is our guiding principle, to proceed from the "general" to the "special". Like the local behavior of an analytic function near a point is determined by its Taylor coefficients, the local behavior of an analytic system is determined by the values of the drift vector field f, the control vector field g and their Lie-brackets at q_0 [64]. The Lie bracket of two vector fields X and Y on M is another vector field which in local coordinates can be defined as

$$[X,Y](z) = DY(z)X(z) - DX(z)Y(z) \tag{4.2}$$

where D denotes the Jacobian matrix[13]. Important properties of the Lie-bracket

[13] On a manifold M vector fields X can also be defined by their action of differentiable functions: If $\mathcal{F}(M)$ denote the class of real-valued C^∞-functions on M, then the vector field X acts on $\phi \in \mathcal{F}(M)$ by taking the directional derivative of ϕ at q in direction of the vector X_q, $(X\phi)(q_0) = X_q\phi$. The Lie bracket of X and another vector field Y is then defined coordinate free as the commutator $[X,Y]\phi = X(Y\phi) - Y(X\phi)$. An easy calculation shows that this indeed defines a first-order differential operator, i.e., a vector field, and that its coordinate representation is given by (4.2).

are anti-symmetry, $[X, Y] = -[Y, X]$, and the Jacobi identity: for vector fields X, Y and Z we have $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] \equiv 0$. In this sense, the most general situation arises then if all the vector fields f, g and their relevant (typically low-order) Lie brackets are in general position, that is, satisfy no linear dependencies beyond those required by anti-symmetry and the Jacobi identity. We refer to this as *codimension 0* conditions and depending on how many nontrivial independent equality relations exist, we distinguish cases of positive codimension. The concept of *codimension* is simply used to organize the Lie bracket conditions into groups of increasing degrees of degeneracy. We loosely call the dependencies of the vector fields and their Lie brackets at a point q_0 the *Lie bracket configuration* of the system at q_0. (For precise definitions we refer the reader to [64]). In these notes we only consider codimension 0 cases for which we analyze the geometric structure of the small-time reachable set in increasing low dimensions.

The necessary conditions of the Pontryagin Maximum Principle were derived using convex approximations to the reachable set. It is intuitively clear that although such an approximation may be sufficient to obtain complete results for some systems, in general it will be inconclusive. We shall see that under general position type assumptions the structure of the small-time reachable set is trivial in $\mathbb{R}^2$, that it can be determined completely from the conditions of the Maximum Principle in $\mathbb{R}^3$, but that these conditions alone are inconclusive in $\mathbb{R}^4$ and need to be supplemented by further considerations. In these notes we use geometric constructions to determine the small-time reachable set exactly. Although this appears a difficult task in general, it is feasible in low dimensions. Our procedure is to first analyze extremal trajectories corresponding to simple structures suggested by the necessary conditions of the Maximum Principle, and then to choose increasingly more complex structures while analysing the geometry of the resulting surfaces. Obviously the heuristics behind this approach is that we expect simple structures rather than complicated ones. The structure should be as simple as possible, but not more so.

We shall see that the choice of a "good" set of coordinates is extremly important in carrying out these constructions. Here, even for systems considered locally in $\mathbb{R}^n$, differential geometric concepts give the answers and quite frankly, as we shall see, dictate the choice of coordinates. In recent years a computational framework based on a Lie-algebraic formalism has been developed which allows to perform the necessary calculations efficiently [65, 70]. We describe this formalism to select these so-called canonical coordinates (of the second kind) in section 4.2, but without going into the algebraic details. In section 4.1 we first analyze what the conditions of the Maximum Principle imply about the structure of boundary trajectories for systems of the form (4.1). After some algebraic tools and the framework for the geometric constructions have been established in section 4.2, in section 4.3 we analyze the 2- and 3-dimensional cases. Section 4.4 contains a derivation of the structure of the small-time reachable set in dimension 4 and its implication on time-optimal control in dimension three. Our presentation here follows [56], but is more tutorial.

4.1 A Preliminary Analysis of Boundary Trajectories. Let $\Gamma = (\xi(\cdot), \eta(\cdot))$ be an admissible input-trajectory pair defined over the interval $[0, T]$ and suppose $\xi(T) \in \partial Reach_{\Sigma}(\xi_0)$. Then by Theorem 3.4 there exists an absolutely continuous curve $\psi(\cdot) : [0, T] \to (\mathbb{R}^n)^*$, called the adjoint variable, which is not identically zero, such that in local coordinates we have almost everywhere on $[0, T]$

$$\dot{\psi}(t) = -\psi(t)\left(Df(\xi(t)) + Dg(\xi(t))\eta(t)\right) \tag{4.3}$$

$$\langle \psi(t), g(\xi(t))\rangle \eta(t) \equiv \max_{|u|\leq 1} \langle \psi(t), g(\xi(t))\rangle u \tag{4.4}$$

$$H = \langle \psi(t), f(\xi(t)) + \eta(t) g(\xi(t))\rangle \equiv 0. \tag{4.5}$$

Note that the absolutely continuous function

$$\Phi_\Gamma(t) = \langle \psi(t), g(\xi(t))\rangle \tag{4.6}$$

uniquely determines the control as

$$\eta(t) = -sgn\ \Phi_\Gamma(t) \tag{4.7}$$

away from the set $N(\Gamma) = \{t \in [0, T] : \Phi_\Gamma(t) = 0\}$. Therefore Φ_Γ is called the *switching function*. But all that can be said about $N(\Gamma)$ is that it is a closed subset of $[0, T]$. In principle it can be *any* closed subset[14]. This leaves questions about regularity properties of η wide open. For example, the Fuller phenomenon shows that even for an analytic system optimal controls need not be piecewise continuous [24, 41]. In this example optimal controls have infinitely many switchings which accumulate at the terminal time generating so-called *chattering controls*. In order to investigate the structure of the set $N(\Gamma)$ further, it is a natural next step to consider the derivatives of the switching function. For instance, if $\Phi_\Gamma(\bar{t}) = 0$, but $\dot{\Phi}_\Gamma(\bar{t}) \neq 0$, then $\bar{t}$ is an isolated point of $N(\Gamma)$ and the control has a bang-bang switch at time $\bar{t}$.

Lemma 4.1 *Let $\Gamma = (\xi(\cdot), \eta(\cdot)) : [0, T] \to M \times U$ be an extremal input-trajectory pair with adjoint variable ψ. Let Z be a smooth vector field and define $\Psi_\Gamma(t) = \langle \psi(t), Z(\xi(t))\rangle$. Then*

$$\dot{\Psi}_\Gamma(t) = \langle \psi(t), [f, Z](\xi(t)) + \eta(t)[g, Z](\xi(t))\rangle. \tag{4.8}$$

Proof: The proof is by a direct computation. Omitting the variable t and writing the inner product as dot product, we have

$$\begin{aligned}
\dot{\Psi}_\Gamma &= \dot{\psi} Z(\xi) + \psi DZ(\xi)\dot{\xi} \\
&= -\psi(Df(\xi) + \eta Dg(\xi))Z(\xi) + \psi DZ(\xi)(f(\xi) + \eta g(\xi)) \\
&= \psi(DZ(\xi)f(\xi) - Df(\xi)Z(\xi)) + \eta\psi(DZ(\xi)g(\xi) - Dg(\xi)Z(\xi)) \\
&= \psi[f, Z](\xi) + \eta\psi[g, Z](\xi) \\
&= \langle \psi(t), [f + \eta(t) g, Z](\xi(t))\rangle. \qquad\qquad QED
\end{aligned}$$

Thus we get for the switching function Φ_Γ

$$\dot{\Phi}_\Gamma(t) = \langle \psi(t), [f, g](\xi(t)) + \eta(t)[g, g](\xi(t))\rangle = \langle \psi(t), [f, g](\xi(t))\rangle \tag{4.9}$$

$$\ddot{\Phi}_\Gamma(t) = \langle \psi(t), [f, [f, g]](\xi(t)) + \eta(t)[g, [f, g]](\xi(t))\rangle. \tag{4.10}$$

[14] Given any closed subset F of $\mathbb{R}^n$ there exists a C^∞-function $f : \mathbb{R}^n \to \mathbb{R}$ such that $F = \{x \in \mathbb{R}^n : f(x) = 0\}$ (see, for instance, [27]).

Even though the control η is in principle undetermined on $N(\Gamma)$, if $N(\Gamma)$ contains an open interval I, then also all the derivatives of the switching function vanish on I and this may determine the control. For instance, if in equation (4.10) the quantity $\langle\psi(t), [g, [f, g]](\xi(t))\rangle$ does not vanish on I, then this determines the control as

$$\eta(t) = -\frac{\langle\psi(t), [f, [f, g]](\xi(t))\rangle}{\langle\psi(t), [g, [f, g]](\xi(t))\rangle}. \tag{4.11}$$

Controls of this type, which are calculated by equating derivatives of the switching function with zero, are called *singular*. However, this expression only defines an admissible control if $\eta(t)$ takes values in the control set. Once a singular control violates the control constraint, it is no longer admissible and must be terminated. This phenomenon is called *saturation*.

Thus the Maximum Principle indicates as most likely candidates for optimal controls the constant controls $\eta \equiv 1$ and $\eta \equiv -1$ which we call *bang* controls and singular controls. We denote the corresponding vector fields by $X = f - g$, $Y = f + g$ and by S. Note that S generally is time-dependent. However, the precise structure of optimal controls, like a specific concatenation structure of the controls still needs to be established. Here a closer look at the necessary conditions of the Maximum Principle as well as the consideration of possible high-order conditions for optimality become necessary. One of these is the so-called *Legendre-Clebsch condition* for the optimality of singular arcs which we only quote here[15].

Proposition 4.1 [38, 35] *Let $\xi(T) \in \partial Reach_\Sigma(\xi_0)$ and suppose the corresponding admissible control η takes values in the interior of the control set over an open interval $I \subset [0, T]$. Then there exist an adjoint multiplier ψ such that the conditions of the Maximum Principle are satisfied and in addition*

$$\langle\psi(t), [g, [f, g]](\xi(t))\rangle \geq 0 \qquad \text{for all } t \in I. \tag{4.12}$$

Many more necessary conditions for optimality exist such as the *Goh-condition* for multi-input control systems [26], a generalization of the *theory of envelopes* from the calculus of variations to optimal control by Sussmann [69, 70] or a general *index theory* by Agrachev and Gamkrelidze [1, 2] based on methods of symplectic geometry. In these notes, however, we will not rely on these results. Instead we take the position that a direct geometric construction of the small-time reachable set is a natural way to complement variational arguments in the solution to time-optimal control problems and that it can actually make some of the more technical and difficult variational arguments unnecessary.

4.2 Algebraic and Geometric Preliminaries. We first develop an adequate algebraic framework for choosing a "good" set of coordinates and the proper way to do calculations in these so-called *canonical coordinates*. We want to stress the importance of this choice. Depending on the choice of coordinates the problem may become simple or extremely complicated. It is here that the Lie bracket configuration for the system at q_0 matters. In nondegenerate situations the choice of canonical coordinates is plainly dictated by which vector fields are linearly independent and the flows along these basis vector fields define the coordinates.

[15]This necessary condition for optimality can be derived within the framework presented in section 3 by making suitable point variations which are symmetric around the reference control (see, for instance [38, 35]).

We use exponential notation for the flow of a vector field, i.e., we write $q_t = q_0 e^{tZ}$ for the point at time t on the integral curve of the vector field Z which starts at q_0 at time 0. Note that we now let the diffeomorphisms act on the right. This agrees with standard Lie-algebraic conventions [32] and simplifies formal calculations. Accordingly, we use the same notation for the flow of Z, that is $q_0 X e^{tZ}$ denotes the vector X transported from q_0 to q_t along the flow of Z. In the earlier terminology $q_0 X e^{tZ}$ is the solution at time t of the variational equation along the integral curve $s \mapsto q_0 e^{sZ}$ with initial condition $X(q_0)$. We freely use $X(q_0)$ for $q_0 X$ and we switch between $q_0 e^{tZ} X e^{-tZ}$ and the more familiar notation $(e^{t\mathrm{ad}Z} X)(q_0)$. It is well-known that the operator $e^{t\mathrm{ad}Z} X$ has the asymptotic Taylor-series representation [11]

$$e^{t\mathrm{ad}Z} X \asymp \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{ad}^n Z(X). \tag{4.13}$$

where $adZ(X) = [Z, X]$. Thus for small t the vector $(e^{t\mathrm{ad}Z} X)(q_0)$ can be approximated by the terms in this series when all the vector fields are evaluated at q_0. Equation (4.13) is a direct corollary of the following fundamental relation:

Lemma 4.2

$$\frac{d}{dt}\left(q e^{tZ} X e^{-tZ}\right) = q e^{tZ} [Z, X] e^{-tZ} \tag{4.14}$$

Hence it follows inductively that for all $n \in \mathbb{N}$ we have

$$\frac{d^n}{dt^n}_{|t=0}\left(q e^{tZ} X e^{-tZ}\right) = ad^n Z(X)(q)$$

which implies the Taylor representation (4.13). A derivation of (4.14) can easily be given by differentiating the left-hand side formally using the product-rule:

$$\begin{aligned} \frac{d}{dt}\left(q e^{tZ} X e^{-tZ}\right) &= q e^{tZ}(ZX) e^{-tZ} - q e^{tZ}(XZ) e^{-tZ} \\ &= q e^{tZ}[Z, X] e^{-tZ}. \end{aligned} \tag{4.15}$$

The proof gives a rigorous justification for these calculations: To distinguish it from the formal derivation, for the moment we use the standard notation and denote the flow of the vector field Z by $\Phi_t^Z(\cdot)$. Thus

$$q e^{tZ} X e^{-tZ} = \left(\Phi_{-t}^Z\right)_* X\left(\Phi_t^Z(q)\right) = V(t) \tag{4.16}$$

where $(\Phi_t^Z)_*$ denotes the differential of the flow. Thus $V(t)$ is the solution to the variational equation along Z at time $s = 0$ with initial condition $X(\Phi_t^Z(q))$ at time $s = t$. If $\Psi(s, t)$ denotes the fundamental matrix of the variational equation along Z, then we have

$$V(t) = \Psi(0, t) X(\Phi_t^Z(q)) \tag{4.17}$$

and thus

$$\dot{V}(t) = \frac{d}{dt}\left(\Psi(0, t)\right) \cdot X(\Phi_t^Z(q)) + \Psi(0, t) \cdot \frac{d}{dt}\left(X(\Phi_t^Z(q))\right). \tag{4.18}$$

Differentiating $\Psi(t,0)\Psi(0,t) \equiv Id$ gives

$$DZ(\Phi_t^Z(q))\Psi(t,0)\Psi(0,t) + \Psi(t,0)\frac{d}{dt}\Psi(0,t) \equiv 0$$

and therefore

$$\frac{d}{dt}\Psi(0,t) = -\Psi(t,0)^{-1}DZ(\Phi_t^Z(q)) = -\Psi(0,t)DZ(\Phi_t^Z(q)). \tag{4.19}$$

Hence

$$\begin{aligned}\dot{V}(t) &= \Psi(0,t)\left\{-(DZ\cdot X)(\Phi_t^Z(q)) + (DX\cdot Z)(\Phi_t^Z(q))\right\}\\ &= \Psi(0,t)[Z,X](\Phi_t^Z(q)) = \left(\Phi_{-t}^Z\right)_*[Z,X](\Phi_t^Z(q))\end{aligned} \tag{4.20}$$

verifying (4.14). Comparing this calculation with (4.15) should convince the reader of the efficiency of the exponential notation. Henceforth we freely use the product rule to differentiate exponentials.

We also need an asymptotic product expansion for exponential Lie-series due to Sussmann [65][16]: Let $\mathcal{X} = \{X_1,\ldots,X_r\}$ be a family of noncommutative indeterminates and let $L(\mathcal{X})$ denote the Lie-subalgebra of the free associative algebra generated by $X_1,\ldots,X_r$. Let v_i be integrable functions of a scalar variable t defined over $[0,T]$. It is shown in [65] that the solution $S(t)$ to the initial value problem

$$\dot{S}(t) = S(t)\left(\sum_{i=1}^r v_i(t)X_i\right), \quad S(0) = Id \tag{4.21}$$

has a welldefined infinite product expansion in terms of exponentials of elements in a Hall basis [11] of $L(\mathcal{X})$ with coefficients which can be computed by successive integrations and algebraic manipulations. We refer the reader to [65] for a proof of these algebraic constructions or a definition of a Hall basis. An explicit form of this expansion in terms of iterated Lie-brackets can easily be calculated using (4.13) as is shown below in the proof of Lemma 4.3 which derives this expansion for the case $r=2$ taking into account all brackets of order ≤ 3. This formula will be sufficient for our purpose.

Lemma 4.3 *Let A and B be noncommutative indeterminates. Let v and w be integrable functions of t on $[0,T]$ and set*

$$V(t) = \int_0^t v(s)ds \quad and \quad W(t) = \int_0^t w(s)ds.$$

Let $S(t)$ be the solution of the initial value problem

$$\dot{S} = S(vA + wB), \quad S(0) = Id. \tag{4.22}$$

Then S has the asymptotic product expansion

$$\begin{aligned}S(t) &= \cdots e^{(\int_0^t wVW ds)[B,[A,B]]}\, e^{(\frac{1}{2}\int_0^t wV^2 ds)[A,[A,B]]}\\ &\quad e^{(\int_0^t wV ds)[A,B]}\, e^{WB} e^{VA}\end{aligned} \tag{4.23}$$

where the remainder is an infinite product of exponentials of Lie-brackets of A and B of orders ≥ 4.

[16]We ask the reader to consult this paper or any book on Lie-algebra such as [32] for the algebraic definitions. They are not essential for the further development of these lecture notes.

Proof: By writing S in the form $S = S_1 e^{VA}$ and differentiating in t we get that

$$\dot{S} = \dot{S}_1 e^{VA} + S_1 e^{VA}(vA) = \dot{S}_1 e^{VA} + SvA.$$

Hence

$$\begin{aligned}\dot{S}_1 &= (\dot{S} - SvA)e^{-VA} = SwBe^{-VA} = S_1 e^{VadA}(wB) = \\ &= S_1 \left(wB + wV[A,B] + \frac{1}{2}wV^2[A,[A,B]] + \cdots \right)\end{aligned}$$

where we use the asymptotic series (4.13). Now write $S_1 = S_2 e^{WB}$ and repeat these steps to get

$$\begin{aligned}\dot{S}_2 &= (\dot{S}_1 - S_1 wB)e^{-WB} \\ &= S_2 e^{WadB} \left(wV[A,B] + \frac{1}{2}wV^2[A,[A,B]] + \cdots \right) \\ &= S_2 \left(wV[A,B] + \frac{1}{2}wV^2[A,[A,B]] + wVW[B,[A,B]] + \cdots \right).\end{aligned}$$

This equation can formally be solved simply by integrating and we obtain

$$\begin{aligned}S_2(t) &= exp\left((\int_0^t wVds)\ [A,B] + (\int_0^t \frac{1}{2}wV^2 ds)\ [A,[A,B]] + \right. \\ &\qquad \left. (\int_0^t wVWds)\ [B,[A,B]] + \cdots \right)\end{aligned}$$

We now write the exponential of the sum as a product of exponentials. Since $S(t) = S_2(t)e^{WB}e^{VA}$ It already follows from the calculation done so far that for any vector fields X and Y we have that $e^{X+Y} = e^{r(X,Y)}e^Y e^X$ where the remainder $r(X,Y)$ consists of commutator brackets at least of order 2. Thus when we write $S_2(t)$ as a product of exponentials all commutator terms here are brackets of order 5 at least and thus do not contribute to the terms we want to calculate. Hence modulo terms of higher order we can write $S_2(t)$ as the product of the exponentials of the terms in the sum,

$$\begin{aligned}S_2(t) &= \ldots exp\left((\int_0^t wVWds)\ [B,[A,B]] \right) \\ &\qquad exp\left((\int_0^t \frac{1}{2}wV^2 ds)\ [A,[A,B]] \right) exp\left((\int_0^t wVds)\ [A,B] \right)\end{aligned}$$

Equation (4.23) then follows from $S(t) = S_2(t)e^{W(t)B}e^{V(t)A}$. *QED*

Corollary 4.1 *(Commutator formula)*

$$e^{rA}e^{sB} = \cdots e^{(\frac{1}{2}rs^2[B,[A,B]])}e^{(\frac{1}{2}r^2s[A,[A,B]])}e^{rs[A,B]}e^{sB}e^{rA} \tag{4.24}$$

Proof: In Lemma (4.3) take

$$v(t) = \begin{cases} 1 & \text{if } 0 \le t < r \\ 0 & \text{if } r \le t \le r+s \end{cases} \qquad \text{and} \qquad w(t) = \begin{cases} 0 & \text{if } 0 \le t < r \\ 1 & \text{if } r \le t \le r+s \end{cases}$$

QED

Corollary 4.2 *(expansion of the sum)*

$$e^{r(A+B)} = \cdots e^{(\frac{1}{3}r^3[B,[A,B]])} e^{(\frac{1}{6}r^3[A,[A,B]])} e^{(\frac{1}{2}r^2[A,B])} e^{rB} e^{rA} \tag{4.25}$$

Proof: In Lemma (4.3) take $v(t) \equiv w(t) \equiv 1$ for $0 \leq t \leq r$. *QED*

We use this formalism to set up so-called canonical coordinates of the second kind. Since admissible controls are convex combinations of $u = -1$ and $u = +1$, there exists a time $T > 0$ with the property that every trajectory corresponding to an admissible control defined on $[0, T]$ exists on $[0, T]$ and lies in M. Henceforth we will assume that the time along our trajectories is no longer. We construct the small-time reachable set for Σ in increasing dimensions under codimension 0 assumptions, i.e., we assume that the vector fields f, g and relevant low order Lie brackets of f and g are in general position. If $X, Y, Z_2, \ldots, Z_n$ are vector fields whose values at q_0 are linearly independent, then, by shrinking M, if necessary, we can define a coordinate system on M centered at q_0 by

$$\Phi : B^{n+1}(\varepsilon) \to M, \quad (\alpha_0, \ldots, \alpha_n) \mapsto q_0 e^{\alpha_n Z_n} \cdots e^{\alpha_2 Z_2} e^{\alpha_1 Y} e^{\alpha_0 X}. \tag{4.26}$$

Note that the point q_0 is mapped into the origin and that the columns of the Jacobian matrix at 0 are given by the partial derivatives with respect to α_i, $i = 0, 1, \ldots, n$, at the origin. But

$$\frac{\partial \Phi}{\partial \alpha_i}(0, \ldots, 0) = \frac{\partial}{\partial \alpha_i} q_0 e^{\alpha_i Z_i} = Z_i(q_0)$$

and since these vectors are linearly independent it follows that Φ is a well-defined change of coordinates near q_0. In Lie theory these coordinates are called *canonical coordinates of the second kind*. We shall always employ a coordinate system of this type and Proposition 4.3 and its corollaries provide us with the required calculus. We take M as a sufficiently small cube

$$M = M(\rho) = \{(\alpha_0, \ldots, \alpha_n) : |\alpha_i| \leq \rho, \quad i = 0, \ldots, n\},$$

which we will shrink whenever necessary for the argument. Without loss of generality we may therefore assume that any linear independence relations which hold at q_0 are valid on all of M.

Finally, we once and for all fix our notation for concatenations of trajectories. For concatenations of bang and singular arcs we simply write the corresponding sequence of letters, i.e., $XYSX$ denotes an arc which starts with control $\eta \equiv -1$, then switches to $\eta \equiv 1$ followed by a singular arc and ends again with control $\eta \equiv -1$. Furthermore, let

$$\mathcal{C}_0 = \mathcal{S}_0 = \{q_0\},$$

$$\mathcal{C}_X = \{q_0 e^{sX} : s \geq 0\}, \ \ \mathcal{S}_X = \{q_0 e^{sX} : s > 0\},$$

$$\mathcal{C}_Y = \{q_0 e^{tY} : t \geq 0\}, \ \ \mathcal{S}_Y = \{q_0 e^{tY} : t > 0\},$$

$$\mathcal{C}_{XY} = \{q_0 e^{sX} e^{tY} : s, t \geq 0\}, \ \ \mathcal{S}_{XY} = \{q_0 e^{sX} e^{tY} : s, t > 0\},$$

$$\mathcal{C}_{YX} = \{q_0 e^{tY} e^{sX} : s, t \geq 0\}, \;\; \mathcal{S}_{YX} = \{q_0 e^{tY} e^{sX} : s, t > 0\}$$

and analogously for concatenations of more pieces or with singular arcs S. Also we denote intersections of sets with the halfspaces $\{\alpha_0 \leq \epsilon\}$ and $\{\alpha_1 \leq \epsilon\}$ by a superscript ϵ, e.g. $\mathcal{S}_X^\epsilon = \mathcal{S}_X \cap \{\alpha_0 \leq \epsilon\} \cap \{\alpha_1 \leq \epsilon\}$.

4.3 Lobry's Example. We motivate our constructions in higher dimensions with a simple rederivation of a classical result of Lobry [43] about the structure of the small-time reachable set in dimension 3 under codimension 0 conditions. We start by describing the codimension 0 case in dimension 2. Assuming that X and Y are linearly independent, the small-time reachable set from q_0 is trivial. It has in its boundary the X and Y trajectories emanating from q_0 and its interior is the region in between (see Fig. 4.1). This is obvious with a good set of coordinates, namely

$$(\alpha_0, \alpha_1) \mapsto q_0 e^{\alpha_1 Y} e^{\alpha_0 X}. \tag{4.27}$$

Clearly, $\mathcal{C}_X^\epsilon = \{(\alpha_0, 0) : 0 \leq \alpha_0 \leq \epsilon\}$, $\mathcal{C}_Y^\epsilon = \{(0, \alpha_1) : 0 \leq \alpha_1 \leq \epsilon\}$, $\mathcal{C}_{YX}^\epsilon = \{(\alpha_0, \alpha_1) : 0 \leq \alpha_i \leq \epsilon, i = 1, 2\}$ and no points outside of the first quadrant are reachable in small time within M^ϵ.

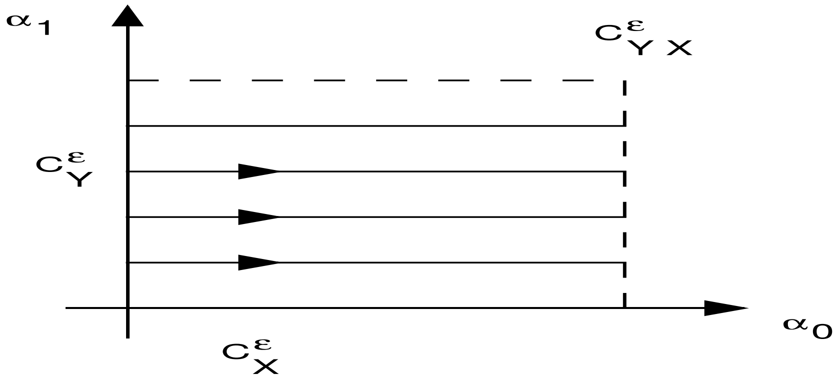

Figure 4.1: $X \wedge Y \neq 0$ in dimension 2

Lobry's example gives the small-time reachable set in $\mathbb{R}^3$ when

(Lobry) the vector fields X, Y and their Lie bracket $[X, Y]$ are linearly independent on a neighborhood M of a reference point $q_0 \in \mathbb{R}^3$.

Again the geometry becomes readily evident if the correct set of coordinates is chosen, namely

$$(\alpha_0, \alpha_1, \alpha_2) \mapsto q_0 e^{\alpha_2 [X,Y]} e^{\alpha_1 Y} e^{\alpha_0 X}. \tag{4.28}$$

In these coordinates the cells $\mathcal{C}_X^\epsilon$, $\mathcal{C}_Y^\epsilon$ and $\mathcal{C}_{YX}^\epsilon$ have the same coordinate descriptions as in the two-dimensional problem and the coordinate expression for $\mathcal{C}_{XY}^\epsilon$ follows from the commutator formula in Corollary 4.1:

$$q_0 e^{sX} e^{tY} = q_0 e^{st(1+O(T))[X,Y]} e^{t(1+O(T^2))Y} e^{s(1+O(T^2))X} \tag{4.29}$$

where $O(T^k)$ denotes terms which can be bounded by CT^k for some constant $C < \infty$. Thus

$$\alpha_0 = s(1 + O(T^2)), \quad \alpha_1 = t(1 + O(T^2)), \quad \alpha_2 = st(1 + O(T)). \tag{4.30}$$

The equations for α_0 and α_1 can be solved for s and t near $(0,0)$ and hence for ϵ sufficiently small we can write α_2 as a function of α_0 and α_1 on $D_2 = \{(\alpha_0, \alpha_1) : 0 \leq \alpha_i \leq \epsilon, i = 1, 2\}$. In fact,

$$\alpha_2 = \alpha_0 \alpha_1 (1 + O(\alpha_0, \alpha_1)) \tag{4.31}$$

and this expression is positive for $0 < \alpha_i \leq \epsilon$, i=1,2. Hence the surface $\mathcal{S}^{\epsilon}_{XY}$ can be described as the graph of a smooth function $\alpha_2^{XY}(\alpha_0, \alpha_1)$, $(\alpha_0, \alpha_1) \in D_2$, which entirely lies above $\mathcal{S}^{\epsilon}_{YX}$. Note that the latter is given as the graph over D_2 of the trivial function $\alpha_2^{YX}(\alpha_0, \alpha_1) \equiv 0$. We claim that for sufficiently small ϵ the restriction of the small-time reachable set $Reach_{\Sigma, \leq T}(q_0)$ to $\{\alpha_0 \leq \epsilon\} \cap \{\alpha_1 \leq \epsilon\}$ is given by

$$\boxed{Reach^{\epsilon}_{\Sigma, \leq T}(\alpha_0) = \{(\alpha_0, \alpha_1, \alpha_2) : (\alpha_0, \alpha_1) \in D_2,\ 0 \leq \alpha_2 \leq \alpha_2^{XY}(\alpha_0, \alpha_1)\}.} \tag{4.32}$$

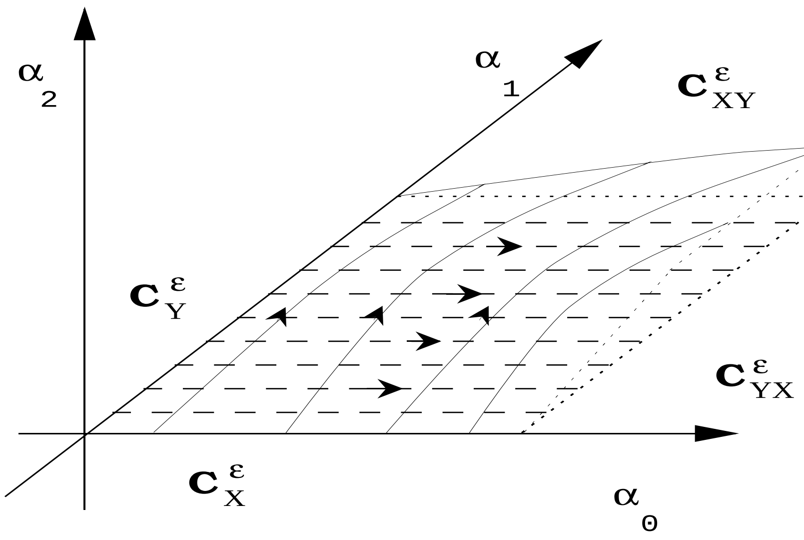

Figure 4.2: $X \wedge Y \wedge [X, Y] \neq 0$ in dimension 3

One way to see this is geometrically. Temporarily call this set D_3 (see Fig. 4.2). It is easy to verify (using the explicit formulas) that X points inside D_3 at points in $\mathcal{S}^{\epsilon}_{XY} \setminus \{\alpha_0 = \epsilon\} \cup \{\alpha_1 = \epsilon\}$ and that Y points inside D_3 at points in $\mathcal{S}^{\epsilon}_{YX} \setminus \{\alpha_0 = \epsilon\} \cup \{\alpha_1 = \epsilon\}$. Since admissible controls are convex combinations of ± 1, this implies that any admissible trajectory which starts at a boundary point of $D_3 \setminus \{\alpha_0 = \epsilon\} \cup \{\alpha_1 = \epsilon\}$ either is tangent to or enters into the interior of D_3. Conversely, given any point $(\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2) \in D_3$, integrate the vector field $X = (1, 0, 0)^T$ backward in time from $T = 0$. This trajectory $\xi(t) = (\alpha_0(t), \alpha_1(t), \alpha_2(t))$, $t \leq 0$, is given by $\alpha_0(t) = \bar{\alpha}_0 + t$, $\alpha_1(t) = \bar{\alpha}_1$ and $\alpha_2 = \bar{\alpha}_2$ for $t \leq 0$. Hence there exists a time $\tau < 0$ so that $\xi(\tau) \in \mathcal{S}^{\epsilon}_{XY}$ and $\xi(t) \in D_3$ for $\tau < t \leq 0$. Concatenating the XY-control which steers q_0 into $\xi(\tau)$ with this X-trajectory shows that $(\bar{\alpha}_0, \bar{\alpha}_1, \bar{\alpha}_2) \in \mathcal{S}^{\epsilon}_{XYX}$ and therefore actually $D_3 = \mathcal{C}^{\epsilon}_{XYX}$. Similarly it can also be shown that $D_3 = \mathcal{C}^{\epsilon}_{YXY}$.

This geometric argument can be complemented by an alternative and more analytical approach which proves that the construction exhausts all boundary trajectories. This will be useful in higher dimensions. Suppose $\Gamma = (\xi(\cdot), \eta(\cdot))$ is a boundary input-trajectory pair and let $\Phi_\Gamma(t_1) = 0$. It then follows from equation (4.5) that also $\langle \psi(t_1), f(\xi(t_1)) \rangle = 0$ and thus

$$\langle \psi(t_1), X(\xi(t_1)) \rangle = 0 \quad \text{and} \quad \langle \psi(t_1), Y(\xi(t_1)) \rangle = 0. \tag{4.33}$$

Hence $\psi(t_1)$ also vanishes against $f(\xi(t_1))$ and $g(\xi(t_1))$. Since f, g, and $[f,g]$ are linearly independent on M and $\psi(t_1)$ is nonzero, it cannot vanish against $[f,g]$ at $\xi(t_1)$. Hence $\dot{\Phi}_\Gamma(t_1) \neq 0$ and t_1 is a bang-bang switch. Thus boundary trajectories are bang-bang with isolated switchings. Suppose there is a second switching at time $t_2 > t_1$ and without loss of generality assume that $\eta(t) \equiv -1$ on (t_1, t_2). Then, as in equation (4.33), $\psi(t_2)$ vanishes against both X and Y at $\xi(t_2)$. Let $q_1 = \xi(t_1)$, $q_2 = \xi(t_2)$ and move the vector $Y(q_2)$ back to q_1 along the flow of X. In exponential notation this simply reads

$$\begin{aligned} q_2 Y e^{(t_1-t_2)X} &= q_1 e^{(t_2-t_1)X} Y e^{(t_1-t_2)Y} = (e^{(t_2-t_1)adX} Y)(q_1) \\ &= Y(q_1) + (t_2 - t_1)[X,Y](q_1) + O((t_1-t_2)^2) \end{aligned} \tag{4.34}$$

where we made use of the asymptotic expansion (4.13). The covector ψ is moved backward along the flow of X by integrating the covariational equation. But by (3.13) this is the adjoint equation and so we simply get $\psi(t_1)$. Furthermore (see Proposition 3.1 or simply use the definition of the adjoint operator),

$$< \psi(t_1), q_2 Y e^{(t_2-t_1)X} >=< \psi(t_1)(e^{(t_2-t_1)X})^*, q_2 Y >=< \psi(t_2), Y(q_2) >= 0. \tag{4.35}$$

Hence $\psi(t_1)$ vanishes against X, Y and $e^{(t_2-t_1)adX}Y$, all evaluated at q_1. By the nontriviality of ψ these vectors must therefore be linearly dependent. Hence[17]

$$\begin{aligned} 0 &= X \wedge Y \wedge e^{(t_2-t_1)adX} Y \\ &= X \wedge Y \wedge Y + (t_2-t_1)[X,Y] + O((t_2-t_1)^2) \\ &= (t_2-t_1)(1 + O(t_2-t_1))\,(X \wedge Y \wedge [X,Y])\,. \end{aligned} \tag{4.36}$$

But this contradicts the linear independence of X, Y and $[X,Y]$ or, equivalently, of f, g and $[f,g]$. Hence boundary trajectories are bang-bang with at most one switching. Thus D_3 exhausts all possible boundary trajectories and no points other than those in D_3 are reachable within M^ϵ.

This 3-dimensional example was first analyzed by Lobry in a landmark paper on nonlinear control [43]. The proof given here shows the power of using a good set

[17] We use a wedge product to denote the determinant of the matrix whose columns are given by the corresponding vectors. In particular, the wedge product is a multi-linear form which is zero if and only if the vectors are linearly dependent.

of coordinates. This is a pervasive feature in all constructions. Another one is the inductive nature of the construction. Lobry's example contains the structure of the trivial 2-dimensional case in the form of $\mathcal{C}_{XY}$ and the 3-dimensional reachable set is constructed on top of this structure. Similar reductions occur in every step to a higher dimension [39, 55]. This is particularly useful since it allows to utilise earlier results as the dimension is increased. This feature is also the basis for solving the problem under codimension 0 conditions in dimension 4.

4.4 The Four-Dimensional Case. The second argument presented above illustrates a general line of reasoning which becomes important in higher dimensions as a means to use the conditions of the Maximum Principle to restrict possible candidates for extremals. In general, if a boundary trajectory is bang-bang and has n switchings at times $t_1 < t_2 < \ldots < t_n$, then $\psi(t_i)$ vanishes against both X and Y evaluated at $q_i = \xi(t_i)$ for $i = 1, \ldots, n$. Moving these vectors to one point, say q_1, gives $(n+1)$ conditions of the type

$$\langle \psi(t_1), Z_i \rangle = 0 \qquad \text{for } i = 0, 1, \ldots, n \tag{4.37}$$

with $Z_0 = X(q_1)$ and $Z_1 = Y(q_1)$. For instance, if $\eta(t) \equiv -1$ on (t_1, t_2), then $Z_2 = (e^{(t_2-t_1)adX}Y)(q_1)$ as calculated above. Note that when the vector X is transported backward from q_2 to q_1 along the X-trajectory, this gives no additional information since $e^{(t_2-t_1)adX}X = X$. Hence every extra junction adds one more condition. Again, by the nontriviality of $\psi(t_1)$ the vectors $X, Y, Z_2, \ldots, Z_n$ must be linearly dependent:

$$X \wedge Y \wedge Z_2 \wedge \ldots \wedge Z_n = 0. \tag{4.38}$$

This puts restrictions on the times $t_1, \ldots, t_n$. Following [69] we call the points $(q_1, \ldots, q_n)$ a *conjugate n-tuple* and equation (4.38) a *conjugate relation.* In this terminology, our calculation above shows that locally there exist no conjugate 3-tuples in dimension 3 if f, g and $[f, g]$ are linearly independent.

The 2- and 3-dimensional examples are easy to analyse because the strata $\mathcal{S}_X$, $\mathcal{S}_Y$, and $\mathcal{S}_{YX}$ become trivial with the choice of coordinates. In higher dimensions longer concatenations must be considered and it is here where the Lie-algebraic framework shows its full power. We illustrate this for the four-dimensional case rederiving some of the results presented first in [39]. We **assume**

(A) the vector fields X, Y, $[X, Y]$ and $[Y, [X, Y]]$ are linearly independent on M

and choose as coordinates

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \mapsto q_0 e^{\alpha_3[Y,[X,Y]]} e^{\alpha_2[X,Y]} e^{\alpha_1 Y} e^{\alpha_0 X}. \tag{4.39}$$

The projection onto $\{\alpha_3 = 0\}$ reduces to Lobry's example analyzed above. It therefore follows that in these coordinates

$$E = \mathcal{C}_{XY} \cup \mathcal{C}_{YX} = \mathcal{S}_0 \cup \mathcal{S}_X \cup \mathcal{S}_Y \cup \mathcal{S}_{XY} \cup \mathcal{S}_{YX} \tag{4.40}$$

has the same geometric structure as the small-time reachable set for Lobry's example. Let

$$D_3 = \{(\alpha_0, \alpha_1, \alpha_2) : (\alpha_0, \alpha_1) \in D_2, 0 \le \alpha_2 \le \alpha_2^{XY}(\alpha_0, \alpha_1)\}. \tag{4.41}$$

We have seen above that the small-time reachable set for Lobry's example can be described as either $\mathcal{C}_{XYX}$ or as $\mathcal{C}_{YXY}$. For the 4-dimensional problem these surfaces will now be seperated in the direction of α_3. We continue our inductive construction starting with E and we show first that both $\mathcal{S}_{XYX}$ and $\mathcal{S}_{YXY}$ can be parametrized as graphs over D_3.

Lemma 4.4 *The cells $\mathcal{C}^{\epsilon}_{XYX}$ and $\mathcal{C}^{\epsilon}_{YXY}$ are graphs of continuous functions $\phi_3^{\mp} : D_3 \to M$. These functions are smooth on the relative interior of D_3, have smooth extensions to the boundary strata $\mathcal{S}_{XY}$ and $\mathcal{S}_{YX}$, and also to $\mathcal{S}_Y$ for $\mathcal{S}_{XYX}$, respectively to $\mathcal{S}_X$ for $\mathcal{S}_{YXY}$.*

Proof: Without loss of generality we consider the hypersurface $\mathcal{S}_{XYX}$. Let $q = q_0 e^{s_1 X} e^{s_2 Y} e^{s_3 X}$ be a point on $\mathcal{S}_{XYX}$ and set $q_1 = q_0 e^{s_1 X}$ and $q_2 = q_1 e^{s_2 Y}$. We show simultaneously that $\mathcal{S}_{XYX}$ is a 3-dimensional manifold and that the coordinate vector field $\frac{\partial}{\partial \alpha_3}$ always points to one side of $\mathcal{S}_{XYX}$. Hence $\mathcal{S}_{XYX}$ passes the vertical line test in canonical coordinates and thus can be represented as the graph of some function ϕ_3^-. The partial derivatives of q with respect to s_1, s_2, and s_3, are given by

$$\begin{aligned} \frac{\partial q}{\partial s_1} &= q_0 e^{s_1 X} X e^{s_2 Y} e^{s_3 X} = (e^{-s_3 adX} e^{-s_2 adY} X)(q) && (4.42) \\ \frac{\partial q}{\partial s_2} &= q_0 e^{s_1 X} e^{s_2 Y} Y e^{s_3 X} = (e^{-s_3 adX} Y)(q) && (4.43) \\ \frac{\partial q}{\partial s_3} &= q_0 e^{s_1 X} e^{s_2 Y} e^{s_3 X} X = X(q) && (4.44) \end{aligned}$$

and are tangent vectors to $\mathcal{S}_{XYX}$ at q. To prove that $\frac{\partial}{\partial \alpha_3}$ points to one side of $\mathcal{S}_{XYX}$, it suffices to verify that the wedge product of these vectors with $\frac{\partial}{\partial \alpha_3}$ has constant nonzero sign. In particular, this also proves the linear independence of these vectors and hence they span the tangent space to $\mathcal{S}_{XYX}$ at q. Now, rather than calculating the wedge product at q, we move all vectors backward along the trajectory to the point q_1. This generates the vectors

$$\begin{aligned} \frac{\partial q}{\partial s_1} e^{-s_3 X} e^{-s_2 Y} &= q_1 X && (4.45) \\ \frac{\partial q}{\partial s_2} e^{-s_3 X} e^{-s_2 Y} &= q_1 Y && (4.46) \\ \frac{\partial q}{\partial s_3} e^{-s_3 X} e^{-s_2 Y} &= q_1 e^{s_2 Y} X e^{-s_2 Y} = (e^{s_2 adY} X)(q_1) && (4.47) \end{aligned}$$

In the wedge product of these vectors, the constant term X in $e^{s_2 adY} X = X + s_2 [Y, X] + O(s_2^2)$ is cancelled by the vector $X(q_1)$. Hence it is actually possible to factor s_2 and the resulting vector field is given by $(\frac{e^{s_2 adY}-1}{s_2})X$. Thus we get

$$\begin{aligned}
X \quad \wedge \quad & Y \wedge e^{s_2 adY} X \wedge e^{s_2 adY} e^{s_3 adX} \frac{\partial}{\partial \alpha_3} = \\
= \quad & s_2 \left(X \wedge Y \wedge \frac{e^{s_2 adY}-1}{s_2} X \wedge e^{s_2 adY} e^{s_3 adX} \frac{\partial}{\partial \alpha_3} \right) \\
= \quad & s_2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & -1 & O(T) \\ * & * & O(T) & 1+O(T) \end{vmatrix} (X \wedge Y \wedge [X,Y] \wedge [Y,[X,Y]]) \\
= \quad & s_2 \left(-1 + O(T)\right) \left(X \wedge Y \wedge [X,Y] \wedge [Y,[X,Y]]\right). \qquad (4.48)
\end{aligned}$$

Since moving the vectors along the flow is a diffeomorphism, the determinant stays negative if we move the vectors forward from q_1 to q. Therefore the tangent vectors

$$q_1 X e^{s_2 Y} e^{s_3 X} = \frac{\partial}{\partial \alpha_0}\frac{\partial \alpha_0}{\partial s_1} + \frac{\partial}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial s_1} + \frac{\partial}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial s_1} + \frac{\partial}{\partial \alpha_3}\frac{\partial \alpha_3}{\partial s_1} \qquad (4.49)$$

$$q_2 Y e^{s_3 X} = \frac{\partial}{\partial \alpha_0}\frac{\partial \alpha_0}{\partial s_2} + \frac{\partial}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial s_2} + \frac{\partial}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial s_2} + \frac{\partial}{\partial \alpha_3}\frac{\partial \alpha_3}{\partial s_2} \qquad (4.50)$$

$$qX = \frac{\partial}{\partial \alpha_0}\frac{\partial \alpha_0}{\partial s_3} + \frac{\partial}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial s_3} + \frac{\partial}{\partial \alpha_2}\frac{\partial \alpha_2}{\partial s_3} + \frac{\partial}{\partial \alpha_3}\frac{\partial \alpha_3}{\partial s_3} \qquad (4.51)$$

and $\frac{\partial}{\partial \alpha_3}$ are also linearly independent. Hence the matrix

$$\begin{pmatrix} \frac{\partial \alpha_0}{\partial s_1} & \frac{\partial \alpha_1}{\partial s_1} & \frac{\partial \alpha_2}{\partial s_1} & \frac{\partial \alpha_3}{\partial s_1} \\ \frac{\partial \alpha_0}{\partial s_2} & \frac{\partial \alpha_1}{\partial s_2} & \frac{\partial \alpha_2}{\partial s_2} & \frac{\partial \alpha_3}{\partial s_2} \\ \frac{\partial \alpha_0}{\partial s_3} & \frac{\partial \alpha_1}{\partial s_3} & \frac{\partial \alpha_2}{\partial s_3} & \frac{\partial \alpha_3}{\partial s_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is nonsingular and so is then the principal 3×3 minor. By the implicit function theorem it is therefore possible to solve for the times (s_1, s_2, s_3) as smooth functions of $(\alpha_0, \alpha_1, \alpha_2)$ near q provided $s_2 \neq 0$. Since $\frac{\partial}{\partial \alpha_3}$ is transversal to $\mathcal{S}_{XYX}$ everywhere, the entire 3-manifold is the graph of a smooth function ϕ_3^-. Furthermore, these calculations show that ϕ_3^- has a smooth extension to a neighborhood of $s_1 = 0$ and $s_3 = 0$ provided $s_2 > 0$. It is clear that these extensions must agree with $\mathcal{S}_Y$, $\mathcal{S}_{XY}$ and $\mathcal{S}_{YX}$. The vertical line test breaks down only along the X-trajectory. But for $\{s_2 = 0\}$ the hypersurface $\mathcal{S}_{XYX}$ simply reduces to $\mathcal{S}_X$ and therefore ϕ_3^- extends continuously to $\mathcal{S}_X = \{(\alpha_0, 0, 0, 0) : \alpha_0 > 0\}$. This proves the Lemma for $\mathcal{S}_{XYX}$. The argument runs completely parallel for the case of $\mathcal{S}_{YXY}$. (It is clear from the construction that the domain of the maps $\phi_3^\pm$ is given by D_3.) *QED*

So far we have considered all bang-bang trajectories with at most two switchings regardless of whether they are extremals or not. Also, at this point in the construction, there is no reason to believe that bang-bang trajectories with more switchings are not extremal. Investigations into whether or not bang-bang trajectories with two switchings are extremal lead us back to the concept of conjugate relations.

Consider an arbitrary XY-trajectory of the form $q_2 = q_0 e^{sX} e^{tY}$ which is extremal and has junctions at q_0, $q_1 = q_0 e^{sX}$ and at q_2. Then there exists an adjoint vector ψ which vanishes against X and Y at every junction. Analogous to the calculations in section 4.4 transporting all vectors to q_0, it follows that ψ also vanishes against the vectors $q_1 Y e^{-sX} = (e^{sadX} Y)(q_0)$ and $q_2 X e^{-tY} e^{-sX} = (e^{sadX} e^{tadY} X)(q_0)$ at q_0. Again, by the nontriviality of ψ, these four vectors are linearly dependent and thus

$$\begin{aligned} 0 &= X \wedge Y \wedge e^{sadX} Y \wedge e^{sadX} e^{tadY} X \\ &= st \left(X \wedge Y \wedge \frac{e^{sadX} - 1}{s} Y \wedge e^{sadX} \left(\frac{e^{tadY} - 1}{t} \right) X \right) \\ &= st \Psi_{XY}(q_0; s, t)(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]]) \end{aligned} \tag{4.52}$$

where the last equation defines $\Psi_{XY}(\cdot; s, t)$ as the quotient of two 4-vectors. The conjugate relation thus becomes

$$\boxed{\Psi_{XY}(q_0; s, t) = 0} \tag{4.53}$$

where q_0 denotes the first junction. By using equation (4.13), an expansion for Ψ_{XY} in terms of s and t can readily be calculated. Write

$$[X, [X, Y]] = \alpha X + \beta Y + \gamma [X, Y] + \delta [Y, [X, Y]] \tag{4.54}$$

where α, β, γ and δ are continuous functions on M. Then

$$\begin{aligned} X \quad &\wedge \quad Y \wedge \frac{e^{sadX} - 1}{s} Y \wedge e^{sadX} \left(\frac{e^{tadY} - 1}{t} \right) X = \\ &= \quad X \wedge Y \wedge [X, Y] + \frac{1}{2} s [X, [X, Y]] + \cdots \\ &\qquad \wedge - [X, Y] - \frac{1}{2} t [Y, [X, Y]] + \cdots - s [X, [X, Y]] + \cdots \\ &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 + O(T) & \frac{1}{2} s\delta + O(T^2) \\ * & * & -1 + O(T) & -\frac{1}{2} t - s\delta + O(T^2) \end{vmatrix} \\ &\qquad \times (X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]]) \end{aligned}$$

and therefore

$$\Psi_{XY}(q_0; s, t) = -\frac{1}{2} t - \frac{1}{2} s\delta + O(T^2) \tag{4.55}$$

Let us now also **assume** that

(B) δ does not vanish on M.

Then, by shrinking M if necessary, we may assume that $|\delta(q)|$ is bounded away from zero and that it dominates the remainder terms which are of order $O(T)$. Then the linear terms in this expansion dominate the remainders and therefore we obtain

- if $\delta > 0$, then for all $q_0 \in M$ and all $s, t > 0$ we have $\Psi_{XY}(q_0; s, t) < 0$ contradicting equation (4.53). Hence $\cdot XY \cdot$ trajectories cannot be extremal. (The dot denotes a junction at the beginning and end of the trajectory).

- if $\delta < 0$, then the equation $\Psi_{XY}(q_0; s, t) = 0$ uniquely determines the time t along an $\cdot XY\cdot$ extremal as a function of q_0 and s, namely $t = -\delta(q_0)s + O(s^2)$.

Repeating the calculations for a $\cdot YX\cdot$ trajectory, the analogous result follows. In particular, we have:

Lemma 4.5 *If δ is positive on M, then bang-bang trajectories with more than two switchings are not boundary trajectories. All bang-bang trajectories with at most two switchings are extremals.*

If $\delta > 0$, then it can indeed be shown that if $q \in \mathcal{S}_{XYX}$ and $r \in \mathcal{S}_{YXY}$ have the same $(\alpha_0, \alpha_1, \alpha_2)$ coordinates, then the α_3 coordinate of r is always smaller than the α_3 coordinate of q. (This can also be seen from the cut-locus calculation given below.) Therefore $\mathcal{S}_{YXY}$ entirely lies below $\mathcal{S}_{XYX}$. It is not difficult to verify that in this case the structure of the vvsmall-time reachable set is given by a direct extension of Lobry's 3-dimensional example to dimension 4 [39].

¿From now on we assume $\delta < 0$. Then the conjugate relations determine the lengths of successive arcs. Even in this case it follows from results of Agrachev and Gamkrelidze [1, 2] or of Sussmann [70] that bang-bang trajectories with more than two switchings are not locally optimal and so $\mathcal{C}_{XYX}$ and $\mathcal{C}_{YXY}$ exhaust all possible bang-bang boundary trajectories. For the 3-dimensional time-optimal control problem to stabilize an equilibrium this was also shown earlier by Bressan in [12]. In our construction, however, we need not utilize these results. We now investigate the geometric structure of these surfaces further, in particular, whether there is a nontrivial intersection of these surfaces. We will see that this is indeed the case if $\delta < 0$ and we call this set the *cut-locus*. It is determined by the (nontrivial) nonnegative solutions to the equation

$$q_0 e^{s_1 X} e^{s_2 Y} e^{s_3 X} = q_0 e^{t_1 Y} e^{t_2 X} e^{t_3 Y}. \tag{4.56}$$

(Trivial solutions arise if some of the times are zero and they correspond to trajectories in $E = \mathcal{C}_{XY} \cup \mathcal{C}_{YX}$).

We first indicate a geometric reasoning to show that the presence of conjugate triples implies the existence of a nontrivial cut-locus [51]. By the implicit function theorem, equation (4.56) can be solved uniquely in terms of (s_1, s_2) near a point q on the cut-locus if the Jacobian with respect to $(s_3; t_1, t_2, t_3)$ is nonsingular. Let $q_1 = q_0 e^{t_1 Y}$, $q_2 = q_1 e^{t_2 X}$ and $q = q_3 = q_2 e^{t_3 Y}$. Then these partial derivatives are given by

$$\frac{\partial q}{\partial t_1} = q_1 Y e^{t_2 X} e^{t_3 Y}, \; \frac{\partial q}{\partial t_2} = q_2 X e^{t_3 Y}, \; \frac{\partial q}{\partial t_3} = q_3 Y, \text{ and } \frac{\partial q}{\partial s_3} = q_3 X. \tag{4.57}$$

If we move all vectors back to q_1 along the flow of the YXY-trajectory, their wedge-product becomes

$$\begin{aligned} &X \wedge Y \wedge e^{t_2 adX} Y \wedge e^{t_2 adX} e^{t_3 adY} X \\ &\quad = t_2 t_3 \Psi_{XY}(q_1; t_2, t_3) \left(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]]\right)(q_1). \end{aligned} \tag{4.58}$$

This expression is zero if and only if the points (q_1, q_2, q_3) are a conjugate triple. Hence, by the implicit function theorem, the cut-locus equation (4.56) has a unique (and hence only the trivial) solution everywhere on $\mathcal{S}_{XY}$ except possibly along a curve Γ_{XY} defined as the solution to $\Psi_{XY}(q_1; t_2, t_3) = 0$. And along this curve indeed a nontrivial solution to equation (4.56) bifurcates. This can be seen geometrically by considering the tangent space to $\mathcal{S}_{YXY}$ at q. If we transport the tangent space back to q_1, then it is spanned by the vectors X, Y and $e^{t_2 adX}Y$. But

$$X \wedge Y \wedge e^{t_2 adX} Y \wedge [Y, [X, Y]] \equiv t_2(1 + O(t_2)) \left(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]] \right) \tag{4.59}$$

and together with equation (4.58) this implies that $[Y, [X, Y]]$ and X point to the same side of $T_q(\mathcal{S}_{YXY})$ if $\Psi_{XY} > 0$ and to opposite sides if $\Psi_{XY} < 0$. This implies that the surfaces $\mathcal{S}_{YXY}$ and $\mathcal{S}_{XYX}$ cross each other near $\Gamma_{XY} \subset \mathcal{S}_{XY}$ [51]. Elementary geometric considerations show that trajectories which maximize the coordinate α_3 near $\mathcal{S}_{XY}$ are of the form YXY if $\Psi_{XY} > 0$ and of the form XYX if $\Psi_{XY} < 0$. The fact that the cut-locus bifurcates from Γ_{XY} can also be seen from a recent result of Shin [58].

This cut-locus can be calculated explicitely using Corollary 4.1. We have

$$\begin{aligned} q_0 e^{s_1 X} e^{s_2 Y} e^{s_3 X} &= q_0 \cdots e^{\frac{1}{2} s_1 s_2^2 [Y,[X,Y]]} e^{\frac{1}{2} s_1^2 s_2 [X,[X,Y]]} e^{\frac{1}{2} s_1 s_2 [X,Y]]} e^{s_2 Y} e^{(s_1+s_3)X} \\ &= q_0 e^{\frac{1}{2} s_1 s_2 (s_1 \delta + s_2 + \cdots)[Y,[X,Y]]} \\ &\qquad e^{\frac{1}{2} s_1 s_2 (1+\cdots)[X,Y]]} e^{(s_2+\cdots)Y} e^{(s_1+s_3+\cdots)X} \end{aligned}$$

and

$$\begin{aligned} q_0 e^{t_1 Y} e^{t_2 X} e^{t_3 Y} &= q_0 e^{t_1 Y} \cdots e^{\frac{1}{2} t_2 t_3^2 [Y,[X,Y]]} e^{\frac{1}{2} t_2^2 t_3 [X,[X,Y]]} e^{t_2 t_3 [X,Y]} e^{t_3 Y} e^{t_2 X} \\ &= q_0 \cdots e^{\frac{1}{2} t_2 t_3^2 [Y,[X,Y]]} e^{\frac{1}{2} t_2^2 t_3 [X,[X,Y]]} \\ &\qquad \cdots e^{t_1 t_2 t_3 [Y,[X,Y]]} e^{t_2 t_3 [X,Y]} e^{(t_1+t_3)Y} e^{t_2 X} \\ &= q_0 e^{\frac{1}{2} t_2 t_3 (2t_1 + t_2 \delta + t_3 + \cdots)[Y,[X,Y]]} \\ &\qquad e^{t_2 t_3 (1+\cdots)[X,Y]} e^{(t_1+t_3+\cdots)Y} e^{(t_2+\cdots)X}. \end{aligned}$$

Equating coordinates gives the following equations:

$$(\alpha_0) \quad s_1 + s_3 + O(S^3) = t_2 + O(T^3) \tag{4.60}$$
$$(\alpha_1) \quad s_2 + O(S^3) = t_1 + t_3 + O(T^3) \tag{4.61}$$
$$(\alpha_2) \quad s_1 s_2 (1 + O(S))) = t_2 t_3 (1 + O(T)) \tag{4.62}$$
$$(\alpha_3) \quad \frac{1}{2} s_1 s_2 (s_1 \delta + s_2 + O(S)) = \frac{1}{2} t_2 t_3 (2t_1 + t_2 \delta + t_3 + O(T)) \tag{4.63}$$

where $S = s_1 + s_2 + s_3$ and $T = t_1 + t_2 + t_3$. (It can be seen from the proof of Lemma 4.3 that $s_1 s_2$ actually divides the left-hand side of the equation for α_3 and that $t_2 t_3$ divides the right-hand side.) Dividing α_3 by α_2 we therefore get

$$s_1 \delta + s_2 + O(S) = 2t_1 + t_2 \delta + t_3 + O(T) \tag{4.64}$$

Equations (4.60), (4.61) and (4.64) have dominant linear terms and can be solved uniquely in terms of s or t. We have, for instance,

$$s_1 = \frac{1}{\delta} t_1 + t_2 + O(T^2) \tag{4.65}$$
$$s_2 = t_1 + t_3 + O(T^3) \tag{4.66}$$
$$s_3 = -\frac{1}{\delta} t_1 + O(T^2) \tag{4.67}$$

The conjugate point relations imply that all these times (and also t_3 calculated below) are nonnegative for extremals. Now substitute these functions for s into equation (4.62) to obtain

$$t_1 \; (\frac{1}{\delta}t_1 + t_2 + \frac{1}{\delta}t_3) \; + \; O(T^3) \; = 0 \tag{4.68}$$

In general, the quadratic terms need not dominate the cubic remainders. However, if the times t_i satisfy a relation of the type

$$t_1 \geq \varepsilon T = \varepsilon(t_1 + t_2 + t_3) \quad \Leftrightarrow \quad t_1 \geq \frac{\varepsilon}{1-\varepsilon} \; (t_2 + t_3) \; , \tag{4.69}$$

then this equation can be solved for t_3 as

$$t_3 = -t_1 - \delta t_2 + O(T^2). \tag{4.70}$$

Note that this solution is well-defined near $\{t_3 = 0\}$ (i.e., Γ_{YX}) and thus a nontrivial cut-locus Γ,

$$\Gamma = \mathcal{S}_{YXY} \cap \mathcal{S}_{XYX}, \tag{4.71}$$

extends beyond $\mathcal{S}_{YX}$ at Γ_{YX}. Similarly, by solving equations (4.60), (4.61) and (4.64) for t as a function of s, we can show that Γ also extends beyond $\mathcal{S}_{XY}$ at Γ_{XY}. The curves Γ_{XY} and Γ_{YX} of intersection of Γ with $\mathcal{S}_{YX}$ and $\mathcal{S}_{XY}$ are precisely the curves of conjugate triples. Except for the stated transversality, which is easily verifiable, we therefore have shown the following result:

Proposition 4.2 *If $\delta < 0$, then $\mathcal{S}_{YXY}$ and $\mathcal{S}_{XYX}$ intersect transversally along a 2-dimensional surface Γ. This surface extends smoothly across $\mathcal{S}_{YX}$ and $\mathcal{S}_{XY}$ and the intersections with these surfaces are the curves Γ_{YX} and Γ_{XY} of conjugate triples.*

The cut-locus Γ is the decisive structure for local optimality of bang-bang trajectories. Note that, as subset of $\mathcal{S}_{XYX}$ (or $\mathcal{S}_{YXY}$), Γ can be described as the graph of the function $\alpha_3 = \alpha_3^{XYX}(\alpha_0, \alpha_1, \alpha_2)$ which describes $\mathcal{S}_{XYX}$ over a 2-dimensional submanifold D_Γ of D_3. D_Γ divides D_3 into two connected components $D_{3,+}$ and $D_{3,-}$ which have the property that $\mathcal{S}_{XYX}$ lies above $\mathcal{S}_{YXY}$ in direction of α_3 over $D_{3,-}$ and below $\mathcal{S}_{YXY}$ over $D_{3,+}$. If we denote the corresponding substrata by a superscript $\pm$, then only the trajectories in

$$\mathcal{N} = \mathcal{S}^-_{XYX} \cap \Gamma \cap \mathcal{S}^+_{YXY} \tag{4.72}$$

maximize the α_3-coordinate over D_3 (see Fig. 4.3).

The surfaces $\mathcal{S}^+_{XYX}$ and $\mathcal{S}^-_{YXY}$ in fact lie in the interior of the small-time reachable set and therefore cannot be time-optimal. Within our construction the natural way to see this is to complement the construction with a stratified hypersurface $\mathcal{S}$ which closes the small-time reachable set with upper hemisphere $\mathcal{N}$ and lower hemisphere $\mathcal{S}$ in the coordinates $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. This is indeed possible, but unlike the case $\delta > 0$, now bang-bang trajectories no longer suffice, but $\mathcal{S}$ consists of concatenations with singular arcs.

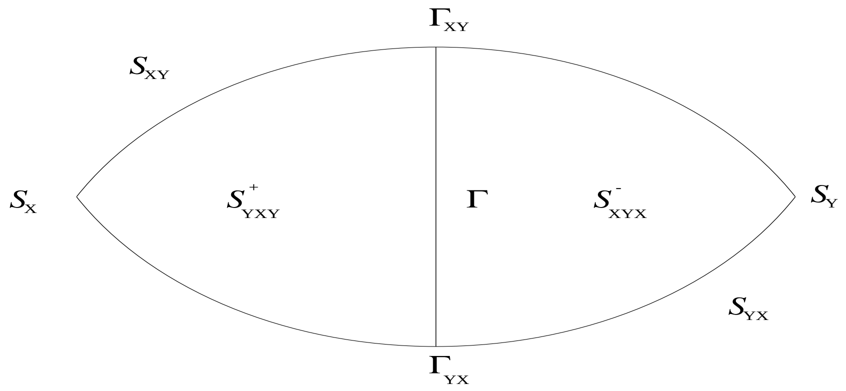

Figure 4.3: Northern hemisphere for $\delta < 0$

Lemma 4.6 *If* $\Gamma = (\xi(\cdot), \eta(\cdot))$ *is an extremal input-trajectory pair defined over the interval* $[0, T]$ *with the property that the control* η *is singular on an open subinterval* I*, then necessarily*

$$\eta(t) = -\frac{\delta(\xi(t)) + 1}{\delta(\xi(t)) - 1} \quad \textit{for} \ \ t \in I. \tag{4.73}$$

Proof: Since η is singular on I, the switching function $\Phi_\Gamma(t) = \langle \psi(t), g(\xi(t)) \rangle$ and all its derivatives vanish on I. Hence for $t \in I$ ψ vanishes against X, Y and $[X, Y]$ along $\xi(t)$. Since X, Y, $[X, Y]$ and $[Y, [X, Y]]$ are linearly independent, and since ψ is nontrivial, ψ cannot vanish against $[Y, [X, Y]]$ and thus

$$\begin{aligned} 2\langle \psi(t), [g, [f, g]](\xi(t)) \rangle &= \langle \psi(t), [Y - X, [X, Y]](\xi(t)) \rangle \\ &= (1 - \delta(\xi(t))) \langle \psi(t), [Y, [X, Y]](\xi(t)) \rangle \neq 0. \end{aligned}$$

Hence equation (4.11) gives

$$\eta(t) = -\frac{\langle \psi(t), [Y + X, [X, Y]](\xi(t)) \rangle}{\langle \psi(t), [Y - X, [X, Y]](\xi(t)) \rangle} = -\frac{\delta(\xi(t)) + 1}{\delta(\xi(t)) - 1}$$

as desired. *QED*

The singular control (4.73) gives rise to a smooth feedback control

$$u_{sin}(x) = \frac{1 + \delta(x)}{1 - \delta(x)} \tag{4.74}$$

defined on M. Note that this defines an admissible control for $\delta < 0$. Also, in a sufficiently small neighborhood of q_0, the quotient $(1 + \delta)/(1 - \delta)$ is bounded away from ± 1 and so no saturation is possible in M.

The concept of conjugate relations can be invoked to show that singular controls cannot be concatenated with bang-bang trajecories which have additional switchings. Consider an arbitrary SX-trajectory of the form $q_2 = q_0 e^{rS} e^{sX}$ and let $q_1 = q_0 e^{rS}$ denote the junction. If the trajectory is extremal, then there exists an adjoint multiplier which vanishes against X and Y, but also against $[X, Y]$ at q_1. The latter equation follows since $\dot{\Phi}_\Gamma(r) = 0$. We call any point q_1 where this holds a *singular junction* regardless of whether a singular arc is actually present or not. If q_2 is another switching, then ψ vanishes also against X and Y at q_2. By moving Y backward along the trajectory to q_1, again four conditions are imposed on ψ. By the nontriviality of the multiplier these four vectors must be linearly dependent. Thus we get

$$\begin{aligned} 0 &= X \wedge Y \wedge [X, Y] \wedge e^{sadX} Y \\ &= \frac{1}{2} s^2 (\delta + O(s)) \left(X \wedge Y \wedge [X, Y] \wedge [Y, [X, Y]] \right). \end{aligned}$$

But this contradicts the linear independence of X, Y, $[X, Y]$ and $[Y, [X, Y]]$. Similar calculations verify that extremal bang-bang trajectories which contain a singular subarc can at most have the structure BSB where B stands for either X or Y.

Having established this structure, the remaining step is to show that the surfaces $\mathcal{S}_{XSX}$, $\mathcal{S}_{XSY}$, $\mathcal{S}_{YSX}$ and $\mathcal{S}_{YSY}$ form a stratified hypersurface which can be described as the graph of a piecewise smooth function $\phi_3 = \phi_3(\alpha_0, \alpha_1, \alpha_2)$ with domain D_3. In principle this verification is exactly the same as for bang-bang trajectories, but the fact that the singular control is a smooth feedback must be taken into account in the calculations. We leave it to the attentive reader to supply the details or to consult [53] where a similar calculation is carried out. It then follows easily that

$$\mathcal{S} = \mathcal{C}_{XSX} \cup \mathcal{C}_{XSY} \cup \mathcal{C}_{YSX} \cup \mathcal{C}_{YSY} \tag{4.75}$$

provides the lower hemisphere to the reachable set (see Fig. 4.4). This concludes the construction of the small-time reachable set for the nondegenerate 4-dimensional case.

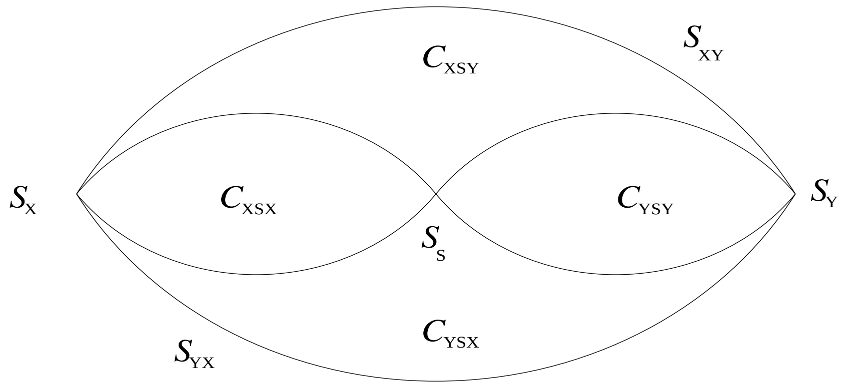

Figure 4.4: Southern hemisphere for $\delta < 0$

4.5 Concluding Remarks: Connection with Time-Optimal Control. We illustrated how the conditions of the Maximum Principle can be used in low dimensions to construct the small-time reachable set for a system Σ of the form (4.1) under the generic assumption that the vector fields f, g and some relevant low-order Lie brackets are in general position. An important feature of these constructions which simplifies the analysis is that the small-time reachable set for the lower-dimensional systems shows up as a segment in the higher-dimensional reachable set. As seen, these constructions become increasingly more complex and eventually require to analyse features for which the Maximum Principle alone remains inconclusive. This shows up for the first time in dimension four with the cut-locus of bang-bang extremals. The construction in $\mathbb{R}^4$ was carried out for the first time in joint work with A. Krener in [39]. While a cut-locus only exists for the four-dimensional system when $\delta < 0$, in the five-dimensional system a cut-locus always limits the optimality of bang-bang trajectories with three switchings near q_0. In addition, saturation phenomena on the singular control need to be analyzed. These 5-dimensional results were obtained by M. Jankovic in her thesis [33]. The same phenomenon can be observed if rather than increasing the dimension, situations of higher codimension are analyzed. In this sense, the results in [39] give the precise structure of the small-time reachable set from a reference point q_0 in dimension 4 under codimension 0 conditions and in [55] the codimension 1 case is analyzed giving the structure of the small-time reachable set from a point q_0 where $\delta(q_0) = 0$ under additional linear independence assumptions on higher order Lie-brackets. In this scenario saturation of singular arcs occurs and this makes the calculations more complicated since higher order terms in the expansions need to be tracked carefully near saturation points. Still the precise structure of the small-time reachable set and the corresponding time-optimal synthesis near q_0 can be established. We refer the reader to [55] for these results. This saturation phenomenon for singular arcs is also analyzed in papers by B. Bonnard, J.P. Gauthier and J. de Morant [8, 9] for the control of a chemical batch reactor and a complete solution is given there as well.

Our results on the structure of small-time reachable sets have immediate implications on the local structure of time-optimal controls. For, given a system Σ of the form (4.1) in $\mathbb{R}^n$, construct an extended system Σ^e in $\mathbb{R}^{n+1}$ by adding time as a state-variable x_{n+1} with the trivial dynamics $\dot{x}_{n+1} \equiv 1$ and initial condition $x_{n+1}(0) = 0$. Then time-optimal trajectories ξ for Σ have extended trajectories $\xi^e = (\xi, \xi_{n+1})$ which lie in the boundary of the reachable set $Reach_{\Sigma^e}$. In particular, none of the original trajectories can be time-optimal once its extended trajectory enters the interior of $Reach_{\Sigma^e}$ after the cut-locus. Indeed, for each of the examples considered here, a time-optimal synthesis can be constructed by projecting the extended trajectories back into the original state-space. For Lobry's example this gives the textbook example of time-optimal control for the double integrator. The three-dimensional time-optimal synthesis which corresponds to the four-dimensional codimension 0 small-time reachable set is constructed in [53] and for the codimension 1 case the result is stated in [55]. The synthesis for the codimension 0 case in $\mathbb{R}^4$ implies the following result:

Proposition 4.3 [54] *Suppose $\delta < 0$ and let M be a sufficiently small neighborhood of q_0. Then all extremal bang-bang trajectories which lie in M are strong relative extrema in the sense of the calculus of variations upto the third switching time. However, in M they are only time-optimal until the time determined by the cut-locus is reached. This occurs after the second, but prior to the third switching. After this time the trajectory lies in the interior of the small-time reachable set.*

In particular our results apply to the problem of stabilizing an equilibrium point q_0 since the trivial dynamics $\dot{x}_0 \equiv 1$ generates the required independent direction in f. Therefore our results include the results of Bressan [12] in dimension three or the results of Shin [57] in dimension four. Indeed they improve on these results in the sense that they not only limit the structure of extremal bang-bang trajectories to have at most two respectively three switchings, but exclude the portions on trajectories after the cut-locus from optimality near q_0. The construction of the small-time reachable set allows to distinguish global from local optimality on the level of the trajectories since these explicit calculations give the exact structure of time-optimal trajectories.

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