A Tutorial on the Positive Realization Problem

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***Abstract*— This paper is a tutorial on the positive realization problem, that is the problem of finding a positive state–space representation of a given transfer function and characterizing existence and minimality of such representation. This problem goes back to the 50's and was first related to the identifiability problem for Hidden Markov Models, then to the determination of internal structures for compartmental systems and later embedded in the more general framework of positive systems theory. Within this framework, developing some ideas sprang in the 60's, during the 80's, the positive realization problem was reformulated in terms of a geometric condition which was recently exploited as a tool for finding the solution to the existence problem and providing partial answers to the minimality problem.**

In this paper the reader is carried through the key ideas which have proved to be useful in order to tackle this problem. In order to illustrate main results, contributions and open problems, several motivating examples and a comprehensive bibliography on positive systems organized by topics are provided.

***Index Terms*— positive systems, realizations, nonnegative matrices, polyhedral cones.**

I. INTRODUCTION

MATHEMATICAL modelling is concerned with choosing the relevant variables of the phenomenon at hand and revealing the relationships among those. *Positivity* of the variables often emerges as the immediate consequence of the nature of the phenomenon itself. A huge number of evidences are just before our eyes: any variable representing any possible type of resource measured by a quantity such as time [14], [35], money and goods [20], [21], buffer size and queues [31], data packets flowing in a network [24], human, animal and plant populations [22], [25], concentration of any conceivable substance you may think of [18], [19], [32] and also – if you haven't conceived this - mRNAs, proteins and molecules [13], electric charge [7], [9], [15] and light intensity levels [10], [23]. Moreover, also probabilities are positive quantities, so that one has to mention in this list also hidden Markov models [6], [30] and phase–type distributions models [28], [29].

In this paper we focus on systems whose state variables are positive (or at least nonnegative) in value for all times and consider the class of models in which the relationships among variables are described by difference equations. Such systems, known as *positive systems* (see for general references [1]–[5]), have the peculiar property that any nonnegative input and nonnegative initial state generates a nonnegative state trajectory and output for all times.

For linear systems, positivity results in a specific sign pattern on the entries of the system's matrices. In particular, a discrete–time system is described by nonnegative matrices which have been the subject of study in the mathematical sciences from long ago (see for general references [82]–[91]). The main results regard the characterization of the eigenvalue location and are due to Perron [90], Frobenius [85] and Karpelevich [84], [88].

The system theoretic approach to positive linear systems has been initiated by Luenberger in his seminal work [5] during the 80's. From that time on, an impressive number of theoretical contributions to this field has appeared. Some of the topics are: reachability and controllability [54], [56], [58], [59], [66], [67], [71], [72], observability [41], realizability [92]–[127], stability [38]–[40], [44], [50], [51] and stabilization [64], [65], pole/zero pattern and pole assignment [45], [48], [49], [63], identification [96], [97], 2D systems and behavioral approach [73]–[80]. Moreover, there is an extensive literature on nonlinear positive systems, also known as monotone cooperative systems [2],[33].

In this paper we focus on the realizability problem for discrete–time SISO positive linear systems, that is, as described in Section II, the problem of finding a triple $\{A_+, b_+, c_+\}$ with nonnegative entries (called *positive realization*) realizing a given transfer function. This problem goes back to the 50's and was first related to the identifiability problem for hidden Markov models (known also as functions of finite Markov chains) [12]. Moreover, during the 60's, several publications appeared that provide a necessary and sufficient condition for the existence of a finite stochastic realization of a given distribution (see the references given in [114]). In the 70's such problem arose in the context of the determination of internal structure for compartmental systems [108] and later was embedded in the more general framework of positive systems theory [5]. More precisely [1], when a transfer function is given, the following issues appear to be fundamental: find conditions on the transfer function for the existence of a positive realization and provide an algorithm to construct such a realization (*existence problem*), determine its minimal allowed order (*minimality problem*) and find how all minimal positive realizations are related to each other (*generation problem*).

An important geometric interpretation of the positive realization problem is due to Furstenberg [104] and Picci [114] and was recently exploited as a tool for finding a complete solution to the existence problem [92], [101], [107] and for providing partial answers to the minimality problem [119]–[127]. Another geometrical interpretation is given in [16], [115], [124], [125].

Moreover, in [114] a characterization of the positive realization problem is given in terms of positive factorizability of the Hankel matrix. Finally, an extension of the results given in [92], [101], [107] to the case of a positive realization having the matrix A_+ irreducible, strictly positive or primitive, is given in [103].

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In this tutorial paper we review, by means of several examples, the development of the key ideas leading to the complete solution of the existence problem and giving new insight into the minimality and generation problem. The main difference of this work w.r.t. the tutorial references [1], [3] is that only some sufficient conditions for the existence problem are there considered; moreover, only few proofs are there sketched as well examples. Finally, note that, all results herein presented have a corresponding continuous–time formulation which can be easily derived [92], [95], [101], [113], [124]. In this paper we will draw particularly from references [60], [92], [101], [109], [110], [119] and [120].

The paper is organized as follows. In Section II the positive realization problem is formally stated together with some basic definitions and notations. Section III contains well known basic results on the spectrum of nonnegative matrices useful in the subsequent sections. Sections IV and VI are collections of examples devoted to introduce and illustrate the key issues related to the existence and minimality problem, respectively. The complete solution of the existence problem is given in Section V and some preliminary results on minimality are illustrated in Section VII. The problem of finding how different minimal positive realization are related each other is briefly discussed in Section VIII and open problems and directions are addressed in Section IX.

The bibliographic section does not contain only references strictly related to the positive realization problem but to the more general area of positive systems theory and applications. In particular, the references have been grouped by topics as follows: basics of positive systems theory (the absolute beginner may start with reference [5] and [3]), applications in different fields (economy, biology, electronics, ...), state–space issues (stability, invariance, ...), control of positive systems and that of general systems with a positive control (reachability, controllability, ...), multidimensional positive systems and the behavioral approach (asymptotic behavior, realization, ...), basics of nonnegative matrix theory (the interested reader may begin with [89]), positive realization problem (existence and minimality).

II. Problem formulation

In this section we provide the formulation of the positive realization problem together with some basic definitions. Moreover, the geometric reformulation in terms of invariant cones, mentioned in the Introduction, is given at the end of this section.

We begin with some definitions: $\mathbb{R}_+$ denotes the set of nonnegative real numbers; given a matrix A, $\sigma(A)$ denotes its spectrum and $\deg\lambda$, with $\lambda \in \sigma(A)$, is the size of the largest block containing λ in the Jordan canonical form of A. A matrix (or a vector) A is said to be *nonnegative* if all its entries are nonnegative and at least one is positive (so to avoid the trivial case of an all–zero matrix). Any eigenvalue λ of a nonnegative matrix A such that $|\lambda| = \rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ will be called a *dominant eigenvalue* of A and $\rho(A)$ will be called the *spectral radius* of A. Given a transfer function $H(z)$[1], any pole of maximal modulus will be called a *dominant pole* of $H(z)$, and its modulus will be denoted by $\rho(H(z))$. Moreover $\rho_-(H(z))$ denotes the maximal modulus of the poles apart from the dominant ones, if any. Any pole of modulus $\rho_-(H(z))$ will be called a *subdominant pole* of $H(z)$.

Definition 1: A discrete–time SISO linear system of the form

$$\begin{aligned} x_{k+1} &= Ax_k + bu_k \\ y_k &= cx_k \qquad\qquad k = 0, 1, \dots \end{aligned} \tag{1}$$

is said to be a *positive system* provided that for any nonnegative input sequence u_k and nonnegative initial state x_0, the state trajectory x_k and the output y_k are always nonnegative.

The following theorem characterizes positive systems in terms of the sign pattern of the system's matrices:

Theorem 1: [3] A system of the form (1) is a positive system if and only if A, b and c are nonnegative. ■

From Definition 1 immediately follows that the impulse response h_k ($h_0 = 0$) of a positive system is nonnegative for all times $k = 1, 2, \dots$. Hence, in the sequel we will consider only systems with a nonnegative impulse response. We are now ready to give a formal statement of the *positive realization problem*, that is the realization problem for positive systems [1], [93], [110]: given a strictly proper rational transfer function $H(z)$ with nonnegative impulse response h_k, the triple $\{A_+, b_+, c_+\}$ is said to be a *positive realization* if $H(z) = \sum_{k\geq 1} h_k z^{-k} = c_+(zI - A_+)^{-1}b_+$ with A_+, b_+, c_+ nonnegative. The positive realization problem consists of providing answers to the questions:

- *The existence problem*: Is there a positive realization $\{A_+, b_+, c_+\}$ of some finite dimension N and how may it be found?
- *The minimality problem*: What is the minimal value for N?
- *The generation problem*: How can we generate all possible positive minimal realizations?

It is worth noting that when no specific sign pattern is required for the system's matrices, the above problems have a well known solution: existence is always guaranteed, the minimal order of a realization equals the order of the transfer function and all minimal realizations can be generated by using any invertible change of coordinates. It will be clear in the sequel how positivity dramatically changes this situation leading to an intriguing scenario where the solutions are far from trivial. It is plain that, as stated in reference [94], we consider the positive realization problem *solved* only when an algorithm for checking the existence of solutions or allowing their construction is given.

The geometrical interpretation of the positive realization problem provided in the sequel, requires some basic definitions from cone theory. A set $\mathcal{K} \subset \mathbb{R}^n$ is said to be a *cone* provided that $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$; if $\mathcal{K}$ contains an open ball of $\mathbb{R}^n$ then $\mathcal{K}$ is said to be *solid*; if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$ then $\mathcal{K}$ is said to be *pointed*. A cone which is closed, convex, solid

[1]Without loss of generality, we consider only strictly proper transfer functions of finite order in which the numerator and denominator are coprime polynomials.

and pointed will be called a *proper cone*. A cone $\mathcal{K}$ is said to be *polyhedral* if it is expressible as the intersection of a finite family of closed half-spaces. The notation $\mathrm{cone}(v_1,\ldots,v_M)$ indicates the polyhedral closed convex cone consisting of all nonnegative linear combinations of vectors $v_1,\ldots,v_M$, with M possibly infinite.

The following is a reformulation presented in [109] of a theorem due to Furstenberg [104] and Picci [114] which gives a geometrical interpretation of the positive realization problem.

Theorem 2: Let $H(z)$ be a strictly proper rational transfer function of order n and let $\{A,b,c\}$, with $A\in\mathbb{R}^{n\times n}$ and $b,c^T\in\mathbb{R}^n$ be a minimal (i.e. jointly reachable and observable) realization of $H(z)$. Then, $H(z)$ has a positive realization if and only if there exists a polyhedral proper cone $\mathcal{K}$ such that

1) $A\mathcal{K}\subset\mathcal{K}$, i.e. $\mathcal{K}$ is A-invariant;
2) $\mathcal{K}\subset\mathcal{O}$;
3) $b\in\mathcal{K}$

where $\mathcal{O}=\{x\in\mathbb{R}^n|cA^{k-1}x\geq 0, k=1,2,\ldots\}$ is called the observability cone. ■

It is worth noting that, once $\mathcal{K}=\mathrm{cone}\,(K)$ has been found[2], where the columns of K are the extremal vectors (see [82]) of $\mathcal{K}$, a positive realization $\{A_+,b_+,c_+\}$ can be obtained by solving

$$AK=KA_+ \qquad b=Kb_+ \qquad c_+=cK \tag{2}$$

Hence, the number of extremal vectors of the cone $\mathcal{K}$ equals the dimension of the positive realization. This fact amounts to saying that polyhedrality of $\mathcal{K}$ corresponds to a finite dimension of the positive realization. Moreover, the MIMO case with m inputs and p outputs can be easily handled by replacing conditions 2 and 3 of the above theorem by the following:

2) $\mathcal{K}\subset\mathcal{O}_j$, $\forall j=1,\ldots,p$;
3) $b_i\in\mathcal{K}$, $\forall i=1,\ldots,m$

where b_i $[c_i]$ is the i–th column [row] of B $[C]$ and $\mathcal{O}_j=\{x\in\mathbb{R}^n|c_jA^{k-1}x\geq 0,k=1,2,\ldots\}$.

Another interesting geometric interpretation of the positive realization problem can be found in [16], [115], [124], [125]. This interpretation refers to the impulse response h_k instead of the transfer function $H(z)$ and relies on the concept of *shift invariance* of a cone:

Theorem 3: [16] Consider a nonnegative impulse response $H_k:\mathbb{N}\to\mathbb{R}_+^{p\times m}$. Then, H_k has a positive realization[3] if and only if there exists a polyhedral cone $\mathcal{C}\subseteq\mathbb{R}_+^\infty$ such that

1) $\mathrm{cone}\left((H_1^T\,H_2^T\,H_3^T\ldots)^T\right)\subseteq\mathcal{C}$;
2) $\mathcal{C}$ is p–shift invariant;

where a cone $\mathcal{C}$ is said to be p–shift invariant if $z=(z_1\ \ldots z_p\,z_{p+1}\ldots)^T\in\mathcal{C}$ implies $(z_{p+1}\,z_{p+2}\ldots)^T\in\mathcal{C}$. ■
Note that, the conditions of Theorem 2 are given in terms of A–invariance of a cone $\mathcal{K}\subseteq\mathbb{R}^n$, where A is such that $\{A,b,c\}$ is a minimal realization of $H(z)$ while those of Theorem 3 are given only in terms of the impulse response H_k and of shift invariance of a cone $\mathcal{C}\subseteq\mathbb{R}_+^\infty$. The main advantage of this last approach is that it provides a theoretical framework for the positive realization problem which may give new insights and new directions for the solution of those problems which are currently unsolved, such as minimality and generation. In fact, this approach leads naturally to the concept of *positive system rank* [16] which allows the reformulation of the minimality problem in terms of positive factorization (see [115]). However, the evaluation of the positive system rank involves an infinite test so that, at the moment, such result is mainly theoretical.

In this paper, we will refer to the approach considered in Theorem 2.

III. Eigenvalue locations for positive systems

In this section we present those results related to nonnegative matrices which can be used, in view of Theorem 1, to characterize the eigenvalues location for positive systems.

We first state the celebrated Perron–Frobenius Theorem [85], [86], [87], [89], [90] in a suitable reformulation:

Theorem 4: [Perron–Frobenius] The dominant eigenvalues of a nonnegative matrix A of dimension n are all the roots of $\lambda^k-\rho(A)^k=0$ for some (possibly more than one) values of $k=1,...,n$. In particular, one of the dominant eigenvalues is positive real, i.e. $\rho(A)\in\sigma(A)$. Moreover, $\deg\rho(A)\geq\deg\lambda$ for any dominant eigenvalue λ. ■

The following theorem (whose statement presented below is due to Ito [84] and in view of Theorem 1.2 at page 168 in reference [89]) is the celebrated Karpelevich theorem [88], [89] that completely characterizes the regions Θ_n^ρ of points in the complex plane that are eigenvalues of nonnegative $n\times n$ matrices with spectral radius ρ. In fact, $\Theta_n^\rho=\rho\Theta_n^1$ and the following holds.

Theorem 5: [Karpelevich] The region Θ_n^1 is symmetric relative to the real axis, is included in the disc $|z|\leq 1$, and intersects the circle $|z|=1$ at points $e^{2\pi ia/b}$, where a and b run over the relatively prime integers satisfying $0\leq a\leq b\leq n$. The boundary of Θ_n^1 consists of these points and of curvilinear arcs connecting them in circular order. Let the endpoints of an arc be $e^{2\pi ia_1/b_1}$ and $e^{2\pi ia_2/b_2}$ $(b_1\leq b_2)$. Each of these arcs is given by the following parametric equation:

$$\lambda^{b_2}(\lambda^{b_1}-s)^{[n/b_1]}=(1-s)^{[n/b_1]}\lambda^{b_1\,[n/b_1]}$$

where the real parameter s runs over the interval $0\leq s\leq 1$ and $[x]$ denotes the nearest integer to x. ■

For the sake of illustration, the regions Θ_3^ρ and Θ_4^ρ are depicted in Figure 1.

We end this section with two technical lemmas which will be used in the sequel.

Lemma 6: [92] Let $H(z)$ be a strictly proper rational transfer function and let $\{A_+,b_+,c_+\}$ be a positive realization of $H(z)$. If $\rho(H(z))<\rho(A_+)$, then there exist another positive realization of lesser dimension $\{\bar{A}_+,\bar{b}_+,\bar{c}_+\}$ of $H(z)$ such that $\rho(H(z))=\rho(\bar{A}_+)$. ■

Lemma 7: Let $H(z)$ be a strictly proper rational transfer function with nonnegative impulse response $(h_k\geq 0, k=0,1,\ldots)$. Then $\rho(H(z))$ is a pole of $H(z)$ and has maximal multiplicity among all the dominant poles. ■

[2]Obviously, the notation $\mathrm{cone}\,(K)$ indicates the cone generated by the columns of the matrix K.

[3]A nonnegative impulse response H_k of a MIMO system is said to have a *positive realization* if there exists a nonnegative triple $\{A_+,B_+,C_+\}$ such that $H_k=C_+A_+^{k-1}B_+$.

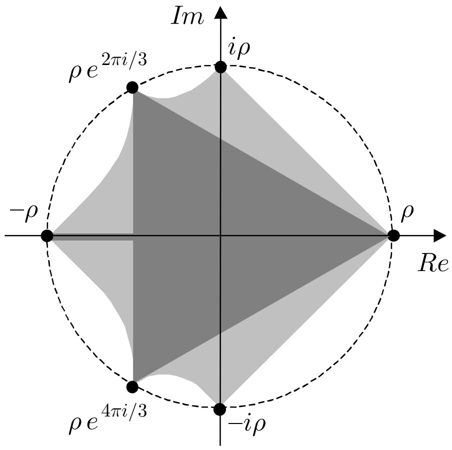

Fig. 1. The Karpelevich regions Θ_3^ρ (dark gray) and Θ_4^ρ (dark and light gray)

IV. EXISTENCE: A PROLOGUE VIA EXAMPLES

This section is devoted to introducing the key ideas and pitfalls regarding the existence problem for positive systems by means of several examples. First note that since $\mathcal{O} = \{x \in \mathbb{R}^n | cA^{k-1}x \geq 0, k = 1, 2, \ldots\}$, then it is immediate to see by direct substitution that nonnegativity of the impulse response $h_k = cA^{k-1}b$ is equivalent to the following cone condition

$$\mathcal{R} := \text{cone}\left(b, Ab, A^2b, \ldots\right) \subset \mathcal{O}$$

which defines the *reachability cone* $\mathcal{R}$. In view of the geometrical interpretation of the positive realization problem, *i.e.* Theorem 2, one needs to find a polyhedral proper cone satisfying conditions $1-3$. By construction and from Lemma 7 (see [113]), the reachability cone $\mathcal{R}$ fullfils these conditions apart from polyhedrality, so that it is worth investigating conditions for polyhedrality of $\mathcal{R}$. Let's explore this possibility by considering the following example.

Example 1: Consider the system with transfer function

$$H(z) = \frac{2}{z-1} + \frac{1}{z+0.4} + \frac{1}{z+0.8}$$

Its impulse response

$$h_k = 2 + (-0.4)^{k-1} + (-0.8)^{k-1} \qquad k \geq 1$$

is clearly nonnegative for all k. Consider then the minimal realization in Jordan canonical form

$$A = \text{diag}\,(1, -0.4, -0.8),\ b^T = (1\ 1\ 1),\ c = (2\ 1\ 1)$$

The reachability cone $\mathcal{R}$, as shown on the left hand side of Figure 2, is polyhedral with 5 extremal vectors and, in fact,

$$\mathcal{R} := \text{cone}\left(b, Ab, A^2b, \ldots\right) = \text{cone}\left(b, Ab, A^2b, A^3b, A^4b\right)$$

since A^5b can be expressed as a nonnegative linear combination of vectors b, Ab, A^2b, A^3b and A^4b. For example,

$$A^5b = 0.2304 \cdot b + 0.5696 \cdot Ab + 0.2 \cdot A^3b$$

A positive realization of order 5 can then be found by solving equations (2) with $K = (b, Ab,\ A^2b, A^3b, A^4b)$ thus obtaining:

$$A_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.2304 \\ 1 & 0 & 0 & 0 & 0.5696 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},\ b_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_+ = (4\ \ 0.8\ \ 2.8\ \ 1.424\ \ 2.4352)$$

■

The previous example shows that the reachability cone can play the role of the cone $\mathcal{K}$ in Theorem 2 being polyhedral. Is it always the case? That is, can one always use the reachability cone in order to find a positive realization? The next two examples illustrate this point.

Example 2: Consider the system with transfer function

$$H(z) = \frac{1}{z-1} + \frac{1}{z-0.8} + \frac{1}{z+0.8}$$

Its impulse response

$$h_k = 1 + (0.8)^{k-1} + (-0.8)^{k-1} \qquad k \geq 1$$

is clearly nonnegative for all k. Consider then the minimal realization in Jordan canonical form

$$A = \text{diag}\,(1, 0.8, -0.8),\ b^T = (1\ 1\ 1),\ c = (1\ 1\ 1)$$

The reachability cone $\mathcal{R}$, as the picture in the middle of Fig.2 shows, is not polyhedral. In fact, for every $k > 0$ one has

$$A^kb \notin \text{cone}\left(b, Ab, A^2b, \ldots, A^{k-1}b\right)$$

This implies that the vector

$$v_\infty := \lim_{k\to\infty} \frac{A^kb}{||A^kb||} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$$

belongs to the boundary of $\mathcal{R}$ but not to $\mathcal{R}$. Accordingly, the closure $\overline{\mathcal{R}}$ of $\mathcal{R}$ is given by

$$\overline{\mathcal{R}} = \text{cone}\left(v_\infty, b, Ab, A^2b, \ldots\right)$$

In particular, in this example,

$$\overline{\mathcal{R}} = \text{cone}\,(v_\infty, b, Ab)$$

is A–invariant and is polyhedral with 3 extremal vectors. A positive realization of order 3 can then be found by solving equations (2) with $K = (v_\infty, b, Ab)$ thus obtaining:

$$A_+ = \begin{pmatrix} 1 & 0 & 0.36 \\ 0 & 0 & 0.64 \\ 0 & 1 & 0 \end{pmatrix}, b_+ = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, c_+^T = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

■

The previous example makes clear that the reachability cone $\mathcal{R}$ can be nonpolyhedral but its closure $\overline{\mathcal{R}}$ may be such. Hence, a positive realization can be found by using the cone $\overline{\mathcal{R}}$ instead of $\mathcal{R}$. Unfortunately, this is not always the case as shown in the next example.

Example 3: Consider the system with transfer function

$$H(z) = \frac{1}{z-1} + \frac{1}{z-0.9} + \frac{1}{z-0.8}$$

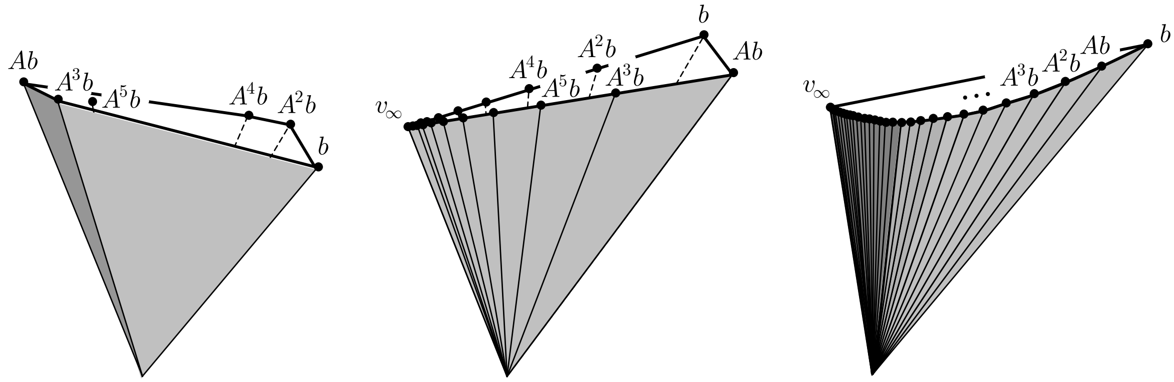

Fig. 2. The reachability cone $\mathcal{R}$ considered in example 1 (left), in example 2 (center), and in examples 3 and 5 (right).

Its impulse response

$$h_k = 1 + (0.9)^{k-1} + (0.8)^{k-1} \qquad k \geq 1$$

is clearly nonnegative for all k*. Consider then the minimal realization in Jordan canonical form*

$$A = \mathrm{diag}(1, 0.9, 0.8),\ b^T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},\ c = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

which, in this case, is a positive realization. Nevertheless, the reachability cone and its closure as well, depicted at the right hand side of Fig.2, are nonpolyhedral, i.e. they have an infinite number of extremal vectors. Hence, neither the reachability cone $\mathcal{R}$ *nor its closure can function as the cone* $\mathcal{K}$ *in Theorem 2, while, in this case, the positive orthant can.* ■

The examples considered so far show that one cannot consider only the reachability cone (or its closure) when searching for the cone $\mathcal{K}$ of Theorem 2. Consequently, on one hand, one has to find conditions assuring polyhedrality of the reachability cone (or of its closure), while on the other hand it is necessary to develop more general methods in order to find such a cone $\mathcal{K}$. However, it is worth noting that even if the impulse response is nonnegative, a cone $\mathcal{K}$ satisfying the conditions of Theorem 2 may not exist at all, that is nonnegativity of the impulse response alone is not a sufficient condition for a transfer function to have a positive realization. The next example illustrates this situation.

Example 4: *Consider the system with transfer function*

$$H(z) = \frac{1}{z-1} + \frac{z - \cos\varphi}{z^2 - (2\cos\varphi)\, z + 1}$$

whose poles are 1 *and* $e^{\pm i\varphi}$*. Its impulse response*

$$h_k = 1 + \cos[(k-1)\,\varphi] \qquad k \geq 1$$

is clearly nonnegative for all k*. Consider then the minimal realization in Jordan canonical form*

$$A = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ c^T = \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix}$$

If φ/π *is an irrational number, then neither the reachability cone* $\mathcal{R}$ *nor its closure and, indeed, any other* A*–invariant proper cone, can be polyhedral. Hence, a positive realization of finite order does not exist. To show this, suppose there exists an invariant polyhedral proper cone and consider any of its extremal vectors* v*. Since the third component of* v *remains unchanged under* A *and the first two components are rotated by an angle* φ *in the* x_1*–*x_2 *plane, then it is easily seen that, as* k *goes to infinity, the cone*

$$\mathrm{cone}\left(v, Av, A^2v, \ldots, A^kv\right)$$

is an ice–cream cone (see [82] p. 2), thus contradicting the polyhedrality hypothesis. This conclusion can also be derived by using the Perron–Frobenius theorem and Lemma 6. In fact, from Lemma 6 it follows that, without loss of generality, one can consider a positive realization $\{A_+, b_+, c_+\}$ *of* $H(z)$ *with spectral radius* $\rho(A_+) = \rho(H(z))$ *which, in this case, equals* 1*. Hence, from the Perron–Frobenius theorem, the dominant eigenvalues of* A_+ *would be roots of* $\lambda^k - 1 = 0$ *for some* $k > 0$*. This is obviously not the case since, when* φ/π *is an irrational number, then there is no integer* k *such that* $e^{ik\varphi} - 1 = 0$ *holds. Finally, observe that the above conclusion does hold independently of the specific value for the input and output vectors* b *and* c*, since it relies only on the Perron–Frobenius theorem.* ■

This last example reveals that the location of the dominant poles plays an important role in the positive realization problem. In fact, in view of Lemma 6, the Perron–Frobenius theorem imposes a specific poles pattern on the dominant poles of $H(z)$ and defines a new necessary condition together with nonnegativity of the impulse response. Are those conditions sufficient? The next section provides a negative answer to this intriguing question thus revealing that the situation is far more complicated than one may expect.

V. Existence: the epilogue via theorems

In this section we will provide necessary and sufficient conditions for the existence of a positive realization of a given transfer function with nonnegative impulse response. In particular, we begin the section by giving necessary and sufficient conditions for the reachability cone $\mathcal{R}$ (or its closure) to be polyhedral (Theorems 8, 9 and 10). As mentioned above, polyhedrality of $\mathcal{R}$, or of its closure, together with nonnegativity of the impulse response allow to construct a positive realization by using $\mathcal{K} = \mathcal{R}$ (or $\overline{\mathcal{R}}$) in Theorem 2. These results provide a complete answer to the questions raised by Examples 1, 2 and 3. Detailed proofs of Theorems 9 and 10 are not given here since they are straightforward consequences of Theorems 1 and 2 in reference [60][4]. In fact, roughly speaking, these last theorems provide conditions for polyhedrality in terms of the spectrum of a minimal realization. Hence, in view of Lemma 7, Theorems 1 and 2 in references [60] can be reformulated in terms of the poles of $H(z)$ and this is done in Theorems 9 and 10.

The remaining part of the section deals with necessary and sufficient conditions for a transfer function to have a positive realization when polyhedrality of $\mathcal{R}$, or of its closure, may not be present. Moreover, the constructive procedure to obtain the cone $\mathcal{K}$ is also sketched.

The following theorem considers the simple case of $H(z)$ with all–zero poles.

Theorem 8: Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0,\ k = 0, 1, ...$) and $\rho(H(z)) = 0$. Then the reachability cone of any minimal realization $\{A, b, c\}$ of $H(z)$ is $\mathcal{R} = \mathrm{cone}(b, Ab, ..., A^{n-1}b)$ and is polyhedral with n extremal vectors. Moreover $H(z)$ has a positive realization of order n. ■

Sketch of proof. Condition $\rho(H(z)) = 0$ implies that A is a nilpotent matrix, so that $A^k b = 0 \in \mathcal{R}$ for every $k \geq n$. Moreover, it is immediate to see that the Markov canonical realization (see [3] p. 83) is positive. △

In view of the above theorem, we will consider in the sequel $\rho(H(z)) > 0$ without loss of generality. This case is considered in what follows where $\mathrm{lcm}\{r_1, r_2\}$ denotes the least common multiple between the numbers r_1 and r_2.

Theorem 9: Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0,\ k = 0, 1, ...$) and $\rho(H(z)) > 0$. Then the reachability cone $\mathcal{R}$ of any minimal realization of $H(z)$ is polyhedral if and only if the poles of $H(z)$ satisfy the following conditions:

1) the dominant poles are simple;
2) the dominant poles are among the r–th roots of $\rho(H(z))^r$ for some positive integer r. Moreover, taking the minimal value r_{min} of r, no nonzero non dominant pole can have an argument which is an integer multiple of $2\pi/r_{min}$.

Furthermore, $H(z)$ has a positive realization of some finite order $N \geq n$ where N is the number of extremal vectors of $\mathcal{R}$. ■

Applying the above theorem to Examples 1, 2 and 3 one obtains $r_{min} = 1$, so that polyhedrality is assured provided that there are no positive real poles apart from the dominant one. Hence, the reachability cone is polyhedral only in Example 1. To deal with the situation illustrated by examples 2 and 3, one can resort to the following theorem.

Theorem 10: Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0,\ k = 0, 1, ...$) and $\rho(H(z)) > 0$. If the reachability cone $\mathcal{R}$ of any minimal realization of $H(z)$ is nonpolyhedral, then its closure $\overline{\mathcal{R}}$ is polyhedral if and only if the poles of $H(z)$ satisfy either the following:

1a) the dominant and subdominant poles are simple;
2a) the dominant poles are among the r–th roots of $\rho(H(z))^r$ for some positive integer r and the subdominant poles are among the r^-–th roots of $\rho_-(H(z))^{r^-}$ for some positive integer r^-;
3a) taking the minimal values r_{min}, r^-_{min} of r and r^- respectively, then no nonzero non subdominant pole can have an argument which is an integer multiple of $2\pi/\widetilde{r}$, with $\widetilde{r} := \mathrm{lcm}\{r_{min}, r^-_{min}\}$;

or

1b) $\rho(H(z))$ has multiplicity 2;
2b) the dominant poles are among the r–th roots of $\rho(H(z))^r$ for some positive integer r. Moreover, taking the minimal value r_{min} of r, no nonzero non dominant pole can have an argument which is an integer multiple of $2\pi/r_{min}$.

Furthermore, $H(z)$ has a positive realization of some finite order $N \geq n$ where N is the number of extremal vectors of $\overline{\mathcal{R}}$. ■

As an example of application of the above theorem, consider Examples 2 and 3 where case (a) applies with $r_{min} = r^-_{min} = \tilde{r} = 1$. Then polyhedrality of the closure of $\mathcal{R}$ is assured provided that there are no positive real eigenvalues apart from the dominant and the subdominant ones. Therefore, the closure of the reachability cone is polyhedral only in Example 2.

As a final comment to the previous theorems of this section, we note that the observability cone $\mathcal{O}$ shares the same polyhedrality properties of the reachability cone $\mathcal{R}$ since a duality property holds, as shown in [109]. Moreover, polyhedrality of $\mathcal{R}$ can be interpreted as the property of finite–time reachability. Finally, when using $\mathcal{R}$ for finding a positive realization, the resulting realization has the positive orthant as the reachability cone [121]. This property of positive systems is called *positive reachability* and has been studied by several authors (see [55], [56], [58], [59], [66], [67], [71]).

When the easily testable conditions of the previous theorems do not hold, then one cannot use the reachability cone in order to find a positive realization, so that this scenario calls for a more general methodology for finding the cone $\mathcal{K}$ satisfying the conditions of Theorem 2. As discussed at the end of the previous section, besides nonnegativity of the impulse response, the dominant pole pattern must be consistent with the Perron–Frobenius theorem for a given transfer function $H(z)$ to have a positive realization, that is the dominant poles must have an argument ϕ such that ϕ/π is a rational number.

We begin our discussion by considering the case in which the dominant pole is unique in order to check whether nonnegativity of the impulse response is a sufficient condition when the dominant pole pattern doesn't play any role. The answer is affirmative and is provided by the following result. Moreover, the proof is constructive, so that a methodology for finding a positive realization is also briefly illustrated. Finally, the following result is the "building block" for the general case, as it will clearly appear in the sequel.

Theorem 11: [92] Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0,\ k = 0, 1, ...$) and $\rho(H(z)) > 0$. If $H(z)$ has a unique

[4]Detailed proofs of these theorems can be found in reference [61].

(possibly multiple) dominant pole, then $H(z)$ has a positive realization of some finite order $N \geq n$. ■

The proof of the above theorem relies on the property that whenever nonpolyhedrality of the reachability cone $\mathcal{R}$ occurs, then its extremal vectors accumulate on the vector

$$v_\infty := \lim_{k\to\infty} \frac{A^k b}{||A^k b||}$$

Roughly speaking, nonpolyhedrality shows up in the fact that cone $\mathcal{R}$ has infinitely many extremal vectors approaching v_∞. Then, the key idea of the constructive proof of the previous theorem is:

1) surround v_∞ with an appropriate cone
2) add the vector b to such a cone
3) add appropriate vectors so that the overall cone is invariant under A.

To be more precise, consider any polyhedral proper cone $\mathcal{Q} = \text{cone}(v_1, \ldots, v_n)$ strictly containing v_∞ and contained in $\mathcal{O}$. Such cone surely exists because v_∞ is strictly contained in the observability cone $\mathcal{O}$. In fact, since the realization is observable, then $cA^{k-1}v_\infty > 0$ for any $k \geq 1$. Consider then the cone

$$\widetilde{\mathcal{K}} = \text{cone}(b, v_1, \ldots, v_n) =: \text{cone}(\widetilde{K})$$

which is proper, polyhedral, contained in $\mathcal{O}$ and containing b by construction. Hence, only invariance is required in order to fulfill all the conditions of Theorem 2. To obtain invariance, one can take the cone

$$\mathcal{K} = \text{cone}(\widetilde{K}, A\widetilde{K}, A^2\widetilde{K}, \ldots) \tag{3}$$

which is by construction invariant, is contained[5] in $\mathcal{O}$ and maintains polyhedrality since all the extremal vectors accumulate on v_∞ that is *strictly* contained in the interior of $\widetilde{\mathcal{K}} \subset \mathcal{K}$.

For the sake of illustration, such methodology is applied to the system considered in Example 3 even if, in this case, the Jordan canonical form is already a positive realization:

Example 5: Consider the system as in Example 3 for which the reachability cone and its closure are not polyhedral cones, as shown in Figure 2 on the right. Then, choose $\mathcal{Q} = \text{cone}(v_1, v_2, v_3)$ *as follows:*

$$\mathcal{Q} := \text{cone}\left(\begin{pmatrix} 1 \\ 0.48 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0.1 \end{pmatrix}, \begin{pmatrix} 1 \\ -0.1 \\ -0.1 \end{pmatrix}\right)$$

It easy to verify that with this choice $v_\infty \in \text{int}(\mathcal{Q})$ *and* $\mathcal{Q} \subset \mathcal{O}$. *The cone* $\mathcal{Q}$ *is depicted in white on the left hand side of Figure 3 together with the cone* $\mathcal{R}$. *One has*

$$\widetilde{\mathcal{K}} = \text{cone}(b, v_1, v_2, v_3)$$

$$\mathcal{K} = \text{cone}(\widetilde{K}, A\widetilde{K}, A^2\widetilde{K}, \ldots) = \text{cone}(b, v_1, v_2, v_3, Ab)$$

since, in this case, the vectors Av_1, Av_2, Av_3 *and* A^2b *are contained in* $\mathcal{K}$. *The cone* $\mathcal{K}$ *is depicted on the right hand side of Figure 3. It is worth noting that, in this case, the cone* $\mathcal{Q}$ *is invariant but in general this is not the case; nevertheless this methodology always produces a cone* $\mathcal{K}$ *that is invariant. A positive realization can then be found using equations (2)*

[5]Note that the observability cone $\mathcal{O}$ is A–invariant by definition.

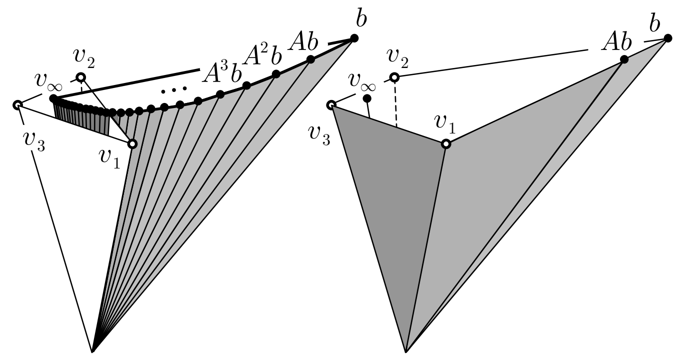

Fig. 3. The reachability cone $\mathcal{R}$ and the cone $\mathcal{Q}$ depicted in white (left) and the final cone $\mathcal{K}$ (right)

thus obtaining:

$$A_+ = \begin{pmatrix} 0 & 0.0091 & 0.0182 & 0 & 0.6405 \\ 0 & 0.9 & 0 & 0 & 0.3542 \\ 0 & 0 & 0.8 & 0.1 & 0 \\ 0 & 0.0909 & 0.1818 & 0.9 & 0.0053 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, b_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_+ = (3 \;\; 1.48 \;\; 1.1 \;\; 0.8 \;\; 2.7)$$

■

When the dominant pole is not unique but the dominant pole pattern is consistent with the Perron–Frobenius theorem, then one can try to extend the previous result. First of all consider the case of simple dominant poles and note that there is a finite set of vectors $v_\infty^1, \ldots, v_\infty^r$ such that in the limit as $k \to \infty$, $A^k b/||A^k b||$ cycles through them. Moreover, r is equal to r_{min} as defined in condition 2 of Theorem 9 and

$$v_\infty^i := \lim_{k\to\infty} \frac{A^{r(k-1)+i}b}{||A^{r(k-1)+i}b||} \qquad i = 1, \ldots, r \tag{4}$$

In order to extend Theorem 11, one has to consider that in this case, even if the realization $\{A, b, c\}$ is minimal and consequently observable, one may have $cv_\infty^i = 0$ for some but not all i's. Hence in order to extend in a straightforward manner the proof of Theorem 11 to the case of nonuniqueness of the dominant pole, one needs the additional assumption that

$$cv_\infty^i > 0 \qquad i = 1, \ldots, r \tag{5}$$

which is equivalent to condition 2 of the following theorem as it will be clear in the sequel:

Theorem 12: [92] Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0$, $k = 0, 1, \ldots$) and $\rho(H(z)) > 0$. If the poles of $H(z)$ satisfy the following conditions:

1) the dominant poles are simple and are among the r–roots of $\rho(H(z))^r$ for some positive integer r;
2) $\lim_{k\to\infty} \inf \rho(H(z))^{-k} h_k > 0$

then $H(z)$ has a positive realization of some finite order $N \geq n$. ■

It is plain that the proof of the previous theorem goes in the same way of that of Theorem 11 by choosing an appropriate polyhedral cone $\mathcal{Q}$ contained in $\mathcal{O}$ and strictly containing all the vectors v_∞^i. The case $cv_\infty^i = 0$, *i.e.* when condition 2 of the above theorem does not hold, implies that the corresponding v_∞^i belongs to the boundary of the observability cone $\mathcal{O}$. Hence, it is not possible anymore to find a polyhedral cone $\mathcal{Q}$ contained in $\mathcal{O}$ and *strictly* containing all the vectors v_∞^i. Because conditions $v_\infty^i \in \mathrm{int}(\mathcal{Q})$ ensure that the associated cone $\mathcal{K}$, defined by (3), is polyhedral and condition $\mathcal{Q} \subset \mathcal{O}$ ensures $\mathcal{K} \subset \mathcal{O}$, then the reasoning used to prove Theorems 11 and 12 cannot be "relaxed" to consider the case $cv_\infty^i = 0$.

In order to gain insight into the case $cv_\infty^i = 0$, note that if $cv_\infty^i = 0$ for some i, then using (4), one has that there exists a value i such that

$$cv_\infty^i = \lim_{k\to\infty} \frac{cA^{r(k-1)+i}b}{||A^{r(k-1)+i}b||} = \lim_{k\to\infty} \frac{h_{r(k-1)+i}}{||A^{r(k-1)+i}b||} = \lim_{k\to\infty} \frac{h_{r(k-1)+i}}{\rho((H(z))^{r(k-1)+i}} = 0 \tag{6}$$

In other words, there exists a subsequence $h_{r(k-1)+i}$ of the impulse response h_k for which equation (6) holds so that condition 2 of Theorem 12 rules out this case. This lead us to try to tackle this situation by changing the focus on the properties of subsequences. Therefore, consider the following procedure $\mathcal{P}$ which can be used in order to decompose the impulse response h_k of a given transfer function $H(z)$ in appropriate subsequences:

Procedure $\mathcal{P}(H^{0,i_1,i_2,\dots,i_j}(z))$

begin

if *$H^{0,i_1,i_2,\dots,i_j}(z)$ has a unique dominant pole or has at least one of the dominant poles p_i with an argument ϕ such that ϕ/π is not a rational number*

then *let $H_m(z) := H^{0,i_1,i_2,\dots,i_j}(z)$ and $m = m + 1$.*

otherwise *let r the minimal positive integer for which the dominant poles of $H^{0,i_1,i_2,\dots,i_j}(z)$ are among the r–th roots of $\rho(H^{0,i_1,i_2,\dots,i_j}(z))^r$. Hence, decompose the impulse response $h_k^{0,i_1,i_2,\dots,i_j}$ in the $r > 1$ subsequences*

$$h_k^{0,i_1,i_2,\dots,i_j,i_{j+1}} = h_{r(k-1)+i_{j+1}}^{0,i_1,i_2,\dots,i_j}$$

with corresponding transfer functions

$$H^{0,i_1,i_2,\dots,i_j,i_{j+1}}(z) = \mathcal{Z}[h_{r(k-1)+i_{j+1}}^{0,i_1,i_2,\dots,i_j}]$$

where $i_{j+1} = 1, \dots, r$.

end

To decompose h_k, let $H^0(z) = H(z)$, $h_k^0 = h_k$ and perform the procedure $\mathcal{P}(H^0(z))$, starting with $m = 1$. Then repeat it iteratively for each new transfer function produced by the procedure itself until only $H_m(z)$'s are produced.

The transfer functions $H_m(z)$ with $m = 1, \dots, n_m$ obtained at the end of this decomposition procedure will have, by construction, either a unique dominant pole or at least a dominant pole having an argument ϕ such that ϕ/π is not a rational number. Moreover, as shown in [101], n_m is finite.

The following example illustrates such decomposition procedure:

Example 6: *Consider the system with transfer function*

$$H(z) = \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z-0.9} + \frac{z - 0.9\cos\varphi}{z^2 - (2\cdot 0.9\cos\varphi)z + (0.9)^2} + \frac{1}{z-0.4}$$

whose impulse response

$$h_k = 1 + (-1)^{k-1} + (0.9)^{k-1}(1 + \cos[(k-1)\varphi]) + (0.4)^{k-1}$$

is nonnegative for every $k \geq 1$. Note moreover that,

$$\lim_{k\to\infty} \inf \rho(H(z))^{-k} h_k = \lim_{k\to\infty} \inf h_k = 0$$

so that Theorem 12 does not apply.

The dominant poles of $H^0(z) = H(z)$ are $p_1 = 1$ and $p_2 = -1$. Since $r = 2$, then the procedure $\mathcal{P}(H^0(z))$ produces the two subsequences $h_k^{0,1} = h_{2k-1}$ and $h_k^{0,2} = h_{2k}$ whose $\mathcal{Z}$–transforms are

$$H^{0,1}(z) := \mathcal{Z}[h_{2k-1}] = \frac{2}{z-1} + \frac{1}{z-(0.9)^2} + \frac{z-(0.9)^2\cos 2\varphi}{z^2 - (2\cdot(0.9)^2\cos 2\varphi)z + (0.9)^4} + \frac{1}{z-(0.4)^2}$$

$$H^{0,2}(z) := \mathcal{Z}[h_{2k}] = \frac{0.9}{z-(0.9)^2} + \frac{0.9\cos\varphi\,(z-(0.9)^2)}{z^2 - (2\cdot(0.9)^2\cos 2\varphi)z + (0.9)^4} + \frac{0.4}{z-(0.4)^2}$$

First note that $H^{0,1}(z)$ has one unique dominant pole. If φ/π is not a rational number, say $\varphi = 1$, then $H^{0,2}(z)$ has dominant poles (which are not dominant poles of $H(z)$!) with argument 2φ such that $2\varphi/\pi$ is not a rational number. In this case, by performing the decomposition procedures $\mathcal{P}(H^{0,1}(z))$ and then $\mathcal{P}(H^{0,2}(z))$, the decomposition ends with $n_m = 2$ and $H_1(z) = H^{0,1}(z)$, $H_2(z) = H^{0,2}(z)$.

Otherwise, that is if φ/π is a rational number, say $\varphi = \pi/3$, then $H^{0,2}(z)$ has three dominant poles, namely $p_1 = 0.81$, $p_2 = 0.81e^{2\pi i/3}$ and $p_3 = 0.81e^{-2\pi i/3}$. Since $r = 3$, then the procedure $\mathcal{P}(H^{0,2}(z))$ decomposes the subsequence $h_k^{0,2} = h_{2k}$ in 3 more subsequences

$$h_k^{0,2,i_2} = h_{3(k-1)+i_2}^{0,2} = h_{2(3(k-1)+i_2)}$$

with $i_2 = 1, 2, 3$, i.e.

$$h_k^{0,2,1} = h_{6k-4}, \quad h_k^{0,2,2} = h_{6k-2}, \quad h_k^{0,2,3} = h_{6k},$$

whose $\mathcal{Z}$–transforms are

$$H^{0,2,1}(z) := \mathcal{Z}[h_{6k-4}] = \frac{3}{2}\,\frac{0.9}{z-(0.9)^6} + \frac{0.4}{z-(0.4)^6}$$

$$H^{0,2,2}(z) := \mathcal{Z}[h_{6k-2}] = \frac{(0.4)^3}{z-(0.4)^6}$$

$$H^{0,2,3}(z) := \mathcal{Z}[h_{6k}] = \frac{3}{2}\,\frac{(0.9)^5}{z-(0.9)^6} + \frac{(0.4)^5}{z-(0.4)^6}$$

The dominant poles of the above transfer functions are unique. Hence, the decomposition procedure stops with $n_m = 4$ and

$$H_1(z) = H^{0,1}(z), \qquad H_2(z) = H^{0,2,1}(z)$$

$$H_3(z) = H^{0,2,2}(z), \quad H_4(z) = H^{0,2,3}(z)$$

■

We are now ready to state the necessary and sufficient conditions for a transfer function $H(z)$ to have a positive realization. Note that, as shown in [101], also the case of multiple poles is encompassed.

Theorem 13: [101] Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0,\ k = 0, 1, ...$) and $\rho(H(z)) > 0$. Then $H(z)$ has a positive realization of some finite order $N \geq n$ if and only if all the $H_m(z)$'s, $m = 1, \ldots, n_m$, obtained by iteratively applying the decomposition procedure $\mathcal{P}$, have a unique (possibly multiple) dominant pole. ■

A simple procedure to obtain a positive realization from the positive realizations of the $H_m(z)$'s is not reported here for the sake of brevity and can be found in reference [101] and [107]. The previous theorem states that, in general, nonnegativity of the impulse response together with a specific pole pattern for the dominant poles, as required by the Perron–Frobenius theorem, are necessary but not sufficient conditions for a transfer function $H(z)$ to have a positive realization. For example, the transfer function considered in Example 6 has dominant poles consistent with the Perron–Frobenius theorem but, from Theorem 13, has not a positive realization in the case φ/π is not a rational number.

An example considering a transfer function for which the conditions of Theorems 8, 9, 10, 11 and 12 are not fulfilled but having a positive realization, that is conditions of Theorem 13 are satisfied, is Example 6 in the case $\varphi = \pi/3$.

In order to check positive realizability one can also resort to the following corollary:

Corollary 14: [101] Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0,\ k = 0, 1, ...$) and $\rho(H(z)) > 0$. Then $H(z)$ has a positive realization of some finite order $N \geq n$ if every pole p_i has the property that $p_i/|p_i|$ is a root of unity. ■

VI. MINIMALITY: A PROLOGUE VIA EXAMPLES

This section deals with the minimality problem for positive systems as defined in Section II. Hereafter it will be shown that the minimal dimension of a positive realization of a given transfer function may be much "larger" than its order and how this can be related to different mechanisms. First note that the poles of a given transfer function are a subset of the eigenvalues of any of its positive realizations. Then, the poles must be a subset of eigenvalues consistent with a nonnegative matrix. Basing on this consideration, the next example considers the mechanism related to the rotational symmetry of the dominant eigenvalues of a nonnegative matrix required by the Perron–Frobenius theorem.

Example 7: Consider the system with transfer function

$$H(z) = \frac{1}{(z-1)(z^2+1)}$$

Its impulse response $h_k = \{0, 0, 1, 1, 0, 0, 1, 1, \ldots\}$ *is clearly nonnegative for all* k. *Note that since* i, $-i$ *and* 1 *are the dominant poles of* $H(z)$, *then from the Perron–Frobenius theorem, all the* 4*–th roots of unity must belong to the spectrum of the matrix* A_+ *of any positive realization. Consequently, the minimal dimension for a positive realization of* $H(z)$ *is not smaller than* 4 *so that the following realization*

$$A_+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, b_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, c_+^T = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

is minimal as a positive linear system. A more general case is the following one parameter family of transfer functions of order $n(m) = 2^m + 1$

$$H(z, m) = \frac{1}{(z-1)(z^{2^m}+1)}$$

It has been shown (see [119]) that for any m, *the minimal positive realization of* $H(z, m)$ *is of dimension* $N(m) = 2^{m+1}$. *Hence, the difference* $N(m) - n(m) = 2^m - 1$ *between the dimension of any minimal positive realization and the order of the transfer function goes to infinity as* m *increases.* ■

The rotational symmetry of the dominant spectrum of a nonnegative matrix, due to the specific dominant poles pattern, is not the only reason for the dimension of any minimal positive realization to be greater than the order of the transfer function. This is illustrated by the following example in which the dominant eigenvalue is unique so that no symmetry of the dominant spectrum is required by the Perron–Frobenius theorem:

Example 8: Consider the system with transfer function

$$H(z) = \frac{1.9z - 0.09}{(z-1)(z^2+0.81)}$$

Its impulse response

$$h_k = 1 - 0.9^{k-1}\left[\cos\frac{k\pi}{2} + \sin\frac{k\pi}{2}\right] \qquad k \geq 1$$

is clearly nonnegative for all k. *Since the poles of* $H(z)$ *are* 1, $\pm 0.9\,i$ *and, as one can easily check from Figure 1, lie outside* Θ_3^1, *then the matrix* A_+ *of any minimal positive realization must be of dimension greater than* 3. *Therefore the following fourth order positive realization*

$$A_+ = \begin{pmatrix} 0 & 0.95 & 0 & 0.05 \\ 0.05 & 0 & 0.95 & 0 \\ 0 & 0.05 & 0 & 0.95 \\ 0.95 & 0 & 0.05 & 0 \end{pmatrix}, b_+ = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c_+ = (0 \quad 0 \quad 1 \quad 1)$$

is minimal as a positive system. ■

This last mechanism, related to a specific poles pattern, is – again – not the only reason for the dimension of any minimal positive realization to be greater than the order of the transfer function even when the dominant eigenvalue is unique. This should be not surprising since positivity of the system implies restrictions not only on the dynamic matrix but on the input and output vectors also. The next example is based on the geometrical point of view given by Theorem 2.

Example 9: Consider the system with transfer function

$$H(z) = \frac{1}{z-1} - \frac{25}{z-0.4} + \frac{75}{z-0.2}$$

Its impulse response

$$h_k = 1 - 25\,(0.4)^{k-1} + 75\,(0.2)^{k-1} \qquad k \geq 1$$

is such that $h_1 = 51$, $h_2 = 6$, $h_3 = h_4 = 0$, *and for* $k > 4$

$$h_k = \frac{5^{k-3} - 2^{k-1} + 3}{5^{k-3}} > \frac{2^{k-2}\left(2^{k-4} - 2\right) + 3}{5^{k-3}} > 0$$

Then it is nonnegative for all k. *Since the impulse response of the system is such that* $h_3 = h_4 = 0$, *then for any minimal realization* $\{A, b, c\}$ *we have*

$$cA^2b = cA^3b = 0 \tag{7}$$

Suppose then that there exists a third order positive realization $\{A_+, b_+, c_+\}$ *of* $H(z)$. *Hence, the cone* $\mathcal{K} = \mathrm{cone}(T)$, *where* T *is the similarity transformation between* $\{A_+, b_+, c_+\}$ *and* $\{A, b, c\}$, *has three extremal vectors and satisfies the conditions of Theorem 2. From conditions 1 and 3, it follows that* $A^k b \in \mathcal{K}$ *for* $k = 0, 1, \ldots$. *Moreover, in view of (7) the following hold:*

$$\begin{array}{rrr} c(b) > 0 & c(Ab) > 0 & c(A^2b) = 0 \\ cA(b) > 0 & cA(Ab) = 0 & cA(A^2b) = 0 \\ cA^2(b) = 0 & cA^2(Ab) = 0 & cA^2(A^2b) > 0 \\ cA^3(b) = 0 & cA^3(Ab) > 0 & cA^3(A^2b) > 0 \\ cA^k(b) > 0 & cA^k(Ab) > 0 & cA^k(A^2b) > 0 \end{array}$$

for $k = 4, 5, \ldots$. *Consequently, the three vectors* b, Ab *and* A^2b *lie on different extremal vectors of the observability cone* $\mathcal{O}$. *Then, from condition 2 of Theorem 2, i.e.* $\mathcal{K} \subset \mathcal{O}$, *the vectors* b, Ab *and* A^2b *are necessarily extremal vectors of* $\mathcal{K}$ *so that* $\mathcal{K}$ *is the polyhedral closed convex cone consisting of all finite nonnegative linear combinations of vectors* b, Ab *and* A^2b, *i.e.*

$$\mathcal{K} = \mathit{cone}(b, Ab, A^2b)$$

Since $A^3b \notin \mathcal{K}$, *then* $\mathcal{K}$ *is not* A*-invariant, thus arriving at a contradiction. Therefore the following fourth order positive realization*

$$A_+ = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & \frac{63+4\sqrt{26}}{85} & 0 & 0 \\ 0 & \frac{22-4\sqrt{26}}{85} & \frac{63-4\sqrt{26}}{85} & 0 \\ 0 & 0 & \frac{22+4\sqrt{26}}{85} & 0 \end{pmatrix}, b_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{8}$$

$$c_+ = (6 \;\; 0 \;\; 0 \;\; 51)$$

is minimal as a positive system. This example has been generalized in reference [126] using a digraph approach. In particular, a family of transfer functions $H(z, m)$ *with four fixed real simple poles is there considered and it is shown that for any* $m \geq 4$, *the order of any minimal positive realization of the fourth order transfer function* $H(z, m)$ *is not less than* $m+1$. *This is quite surprising since the value* m *can be chosen arbitrarily large.* ■

The examples considered in this section reveal that the location of the poles plays an important role in the minimal dimension of a positive realization. In fact, the Perron–Frobenius theorem imposes a specific pattern on the dominant poles of $H(z)$ and the Karpelevich regions on the remaining poles. Moreover the geometric characterization of the realization problem given by Theorem 2 and the digraph approach lead to a different mechanism enforcing the order of a positive realization to be greater than the order of the transfer function $H(z)$. The next section provides partial answers to the minimality problem that, differently from the realization problem, is still an essentially open problem.

VII. Minimality: a partial epilogue via theorems

In this section some results on the minimality problem are presented. First, a lower bound to the minimal order of a positive realization is given basing on the Karpelevich theorem. Then, necessary and sufficient conditions are given for a third order transfer function with distinct real positive poles and nonnegative impulse response to have a (minimal) positive realization of order three.

As examples 7 and 8 make clear, since the set of the eigenvalues of any positive realization of a given transfer function contains the poles of the transfer function itself, then the poles pattern must be consistent with the Karpelevich and Perron–Frobenius theorems. This is formally stated in the following theorem:

Theorem 15: Let $H(z)$ be a strictly proper rational transfer function of order n with nonnegative impulse response ($h_k \geq 0,\ k = 0, 1, \ldots$) and $\rho(H(z)) > 0$. Let p_i, with $i = 1, \ldots, n$, be the poles of $H(z)$, then the minimal order of a positive realization of $H(z)$ is not less than $\max\{n, N\}$ where N is the minimal value such that

$$p_i \in \Theta_N^{\rho(H(z))} \qquad \text{for any } i = 1, \ldots, n$$

■

Other interesting lower and upper bounds for the order of a minimal positive realization can be found in [123] and [126].

On the other hand, as Example 9 shows, the minimal order of a positive realization may be much larger than the lower bound given by Theorem 15. This may happen, as in Example 9, when all the poles of $H(z)$ are real and the Karpelevich theorem does not play any role in the determination of the minimal order of a positive realization. In order to gain partial insight into the positive minimality problem, we state hereafter necessary and sufficient conditions for a third order transfer functions with distinct positive real poles to have a third order (minimal) positive realization.

Theorem 16: [120] Let

$$H(z) = \frac{R_1}{z - p_1} + \frac{R_2}{z - p_2} + \frac{R_3}{z - p_3}$$

be a third order transfer function (*i.e.* R_1, R_2, $R_3 \neq 0$) with distinct positive real poles $p_1 > p_2 > p_3 > 0$. Then, $H(z)$ has a third order positive realization if and only if the following conditions hold:

1) $R_1 > 0$
2) $R_1 + R_2 + R_3 \geq 0$
3) $(p_1 - \overline{\eta})\, R_1 + (p_2 - \overline{\eta})\, R_2 + (p_3 - \overline{\eta})\, R_3 \geq 0$
4) $(p_1 - \eta)^2\, R_1 + (p_2 - \eta)^2\, R_2 + (p_3 - \eta)^2\, R_3 \geq 0$ for all η such that $\overline{\eta} \leq \eta \leq p_3$ where

$$\overline{\eta} = \max\left\{0, \frac{p_1 + p_2 + p_3 - 2\sqrt{f(p_1, p_2, p_3)}}{3}\right\}$$

with $f(p_1,p_2,p_3) = (p_2 - p_3)^2 + (p_1 - p_2)(p_1 - p_3)$. ■

It is worth noting that the *a priori* knowledge about nonnegativity of the impulse response is not required by the previous theorem so that there is no need to check such condition on h_k before applying the theorem.

Conditions for a positive realization to be minimal have been given for a special class of positive systems such as the tree–compartmental systems considered in reference [108] and the positive reachable systems in reference [121].

Moreover, as previously stated, a reformulation of the minimality of a positive realization in terms of the factorization of the Hankel matrix into two nonnegative matrices (called *positive factorization*) can be found in [114], [115], [16], [125]. However, in this context, a procedure to evaluate the minimal order of a positive realization is not available so far.

Finally, in reference [122], the connections between the McMillan degree n of a given transfer function, the size N of any minimal positive realization and the minimum of the number r of extremal vectors of cones satisfying the conditions of Theorem 2, are discussed. It is there shown that, in general $N \le r$ and examples are given, for the MIMO case, for which this inequality is strict.

VIII. The generation problem

This section deals with the generation problem, that is the problem of finding how all the minimal positive realizations of a given transfer function are related one to the other. To the best of our knowledge there are few results on this problem [110], [125]; to have a glimpse on the difficulties encountered when tackling it, hereafter we provide a very simple example showing that a change of coordinates consisting of positive linear combinations of the state variables of a positive system, may destroy the positivity of the representation in the new coordinates:

Example 10: Consider the positive system described by the triple $\{A_+, b_+, c_+\}$ *with nonnegative entries*

$$A_+ = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix},\ b_+ = \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix},\ c_+^T = \begin{pmatrix} c_1^+ \\ c_2^+ \end{pmatrix}$$

Consider then the change of coordinates $z_1 = x_1 + x_2$, $z_2 = x_2$ *so that the system representation becomes*

$$A = \begin{pmatrix} a_{12}^+ + a_{22}^+ & a_{11}^+ - a_{12}^+ + a_{21}^+ - a_{22}^+ \\ a_{12}^+ & a_{11}^+ - a_{12}^+ \end{pmatrix}$$

$$b = \begin{pmatrix} b_1^+ + b_2^+ & b_2^+ \end{pmatrix},\ c = \begin{pmatrix} c_2^+ & c_1^+ - c_2^+ \end{pmatrix}$$

It is straightforward to notice that positivity is generally lost, since it may be well the case that $a_{11}^+ < a_{12}^+$, $a_{11}^+ + a_{21}^+ < a_{12}^+ + a_{22}^+$ *or* $c_1^+ < c_2^+$. ■

In general, positivity is certainly maintained only when the change of coordinates reduces to a simple positive rescaling and reordering (*i.e.* when using a nonnegative generalized permutation matrix[6]) of the state variables, as stated by the following theorem:

[6] A nonnegative generalized permutation matrix is a permutation matrix in which the 1's are replaced by positive real numbers.

Theorem 17: Given a change of coordinates $z = Tx$ with $T \in \mathbb{R}^{n \times n}$ full rank, then every positive system $\{A_+, b_+, c_+\}$ of dimension n is transformed into a positive system $\{TA_+T^{-1},\ Tb_+,\ c_+T^{-1}\}$ if and only if T is a nonnegative generalized permutation matrix.

Sketch of proof. The if part is obvious so that we consider only the *only if* part. Since Tb_+ must be nonnegative for any nonnegative b_+, then T is a nonnegative matrix. Analogously, T^{-1} is also a nonnegative matrix when considering nonnegativity of c_+T^{-1}. Hence, from Lemma 1.1 in [89], T is a generalized permutation matrix. △

IX. Open problems and new directions

As it is clear from the issues so far discussed, there is a considerable number of open issues related to the minimality and generation problems for positive systems. For example, it is not clear what kind of mathematical "instruments" should be used to effectively tackle these problems. In fact, the geometrical approach (*i.e.* that of considered in Theorem 2) has proved to be a fundamental tool for determining the existence of a positive realization. By contrast, such approach, has led so far to the determination of necessary and sufficient conditions for the existence of a minimal positive realization for the third order case only. Nevertheless, several different promising reformulations of the positive realization problem have been proposed in the literature and are mainly related to the positive factorization approach and to the concept of *positive system rank* proposed by Picci, van den Hof and van Schuppen in references [115], [125], to the factorization of the transfer function in Z_{RC} functions as proposed by Maeda *et al.* in reference [108] or to the Tarski–Seidenberg theory as proposed by Anderson in reference [1].

An important issue related to minimality of positive systems is the study of "hidden" eigenvalues, *i.e.* of the eigenvalues which are not poles and that possibly one has to add in order to obtain a minimal positive realization. A full characterization of this property may lead to a deeper and valuable insight into the problem. For instance, in examples 7 and 8, the eigenvalue of the minimal order matrix A_+ which is not a pole of $H(z)$ is one of the dominant eigenvalues. Consequently, its value is related to the Perron–Frobenius Theorem; on the other hand the situation is in general far more complicated as Example 9 clearly has shown.

Moreover, in the case of SISO systems, it is worth investigating whether the inequality $N \le r$ between the size N of any minimal positive realization and the minimum of the number r of extremal vectors of cones satisfying the conditions of Theorem 2, reduces to $N = r$.

Further, among all possible minimal realizations, it would be very interesting to define a canonical minimal positive realization. Some preliminary results are provided in reference [120] for the case of third order transfer functions with distinct positive real poles, and by Commault in [2].

Another important issue related to minimality [1] is the determination of "tight" lower and upper bounds to the minimal order of a positive realization. Some results are presented in references [123] and [126]. Moreover, it would be very interesting [1] to determine directly from the system's parameters (say, residues and eigenvalues), the minimum number of samples of the impulse response to be checked in order to infer nonnegativity of the whole impulse response. In reference [111] an upper estimate of this number is provided.

Finally, it would be very useful for applications [1] to approximate a positive realization by a lower dimension one.

X. Acknowledgments

We wish to thank the reviewers for their suggestions which has helped us to substantially improve the quality and readability of the paper. Moreover, we would like to express our gratitude to Professor Brian D. O. Anderson for giving us support and new ideas which has made us enthusiastic and encouraged to pursue the study of positive systems.

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