REGULARITY PROPERTIES AND CONDITIONING IN VARIATIONAL ANALYSIS AND OPTIMIZATION[1]

A. L. Dontchev
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R. T. Rockafellar
Dept. of Mathematics, University of Washington, Seattle, WA 98195-4350

Abstract. Concepts of conditioning have long been important in numerical work on solving systems of equations, but in recent years attempts have been made to extend them to feasibility conditions, optimality conditions, complementarity conditions and variational inequalities, all of which can be posed as solving "generalized equations" for set-valued mappings. Here, the conditioning of such generalized equations is systematically organized around four key notions: metric regularity, subregularity, strong regularity and strong subregularity. Various properties and characterizations already known for metric regularity itself are extended to strong regularity and strong subregularity, but metric subregularity, although widely considered, is shown to be too fragile to support stability results such as a radius of good behavior modeled on the Eckart-Young theorem.

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1 Introduction

In numerical work, a measure of "conditioning" of a problem is typically conceived as an upper bound on the ratio of the size of solution (output) error to the size of data (input) error. The measure is linked to some sort of well-posedness property and may further serve, through its reciprocal, to describe the perturbation distance of the given problem from ill-posedness.

At its simplest, this pattern seen is when solving $Ax = y$ for x in the case of $X = Y = I\!R^n$ and a nonsingular matrix $A \in I\!R^{n\times n}$; the data input then is y and the solution output is $A^{-1}y$. In terms of errors δx in output corresponding to errors δy in input, one has the tight estimate $\|\delta x\| \leq \|A^{-1}\|\|\delta y\|$ for the absolute size of errors and, on the other hand, $(\|\delta x\|/\|x\|) \leq \|A\|\|A^{-1}\|(\|\delta y\|/\|y\|)$ for the relative size of errors. The condition number of A is traditionally defined from the second estimate, with $\operatorname{cond} A = \|A\|\|A^{-1}\|$, but the first estimate adds to the overall view of "conditioning" as well because of Eckart-Young theorem [12], which says that

$$\inf\Big\{ \|B\| \,\Big|\, A + B \text{ singular} \Big\} = \frac{1}{\|A^{-1}\|}\,.$$

This is the prototype for a theorem giving an upper bound on how far a given mapping, in this case A, may be perturbed before good numerical behavior in the problem breaks down.

In recent years perturbation results of this form have attracted the interest of researchers working in the complexity of algorithms. For such results in numerical analysis, see e.g. [6] and [5]. In optimization, the distance to ill-posedness in the sense of constraint inconsistency plays an important role in the complexity study of interior point methods; see [25], [13], [15], [23] and the references therein. There is a rapidly growing literature on conditioning of various optimization problems such as linear [4], nonlinear [31], semidefinite [22], stochastic [30], semi-infinite [2], and quadratic [32] programming problems. A major challenge is the extent to which the conditioning paradigm in the Eckart-Young result, where the radius of good behavior is the reciprocal of a constant associated with an estimate of absolute error sizes, persists in more general circumstances.

A broad framework for studies of conditioning has emerged in the methodology of variational analysis, which allows a focus on set-valued mappings (with single-valuedness as a special case). For Banach spaces X and Y, a set-valued mapping F from X to Y, indicated by $F : X \rightrightarrows Y$, is identified with a *graph* set $\operatorname{gph} F \subset X \times Y$, and has *effective domain* $\operatorname{dom} F = \{\, x \in X \mid F(x) \neq \emptyset \}$ and *effective range* $\operatorname{rge} F = \{\, y \in Y \mid \exists\, x \text{ with } F(x) \ni y \}$. Its *inverse* $F^{-1} : Y \rightrightarrows X$, obtained by reversing all pairs in the graph, has $\operatorname{dom} F^{-1} = \operatorname{rge} F$ and $\operatorname{rge} F^{-1} = \operatorname{dom} F$. Single-valuedness of F on a set D, signaled with the notation $F : D \to Y$, means that $F(x)$ reduces to exactly one element y for each $x \in D$; when F is neither single-valued nor empty-valued at a point x, it is called multi-valued at x.

In the unrestricted picture of such possibilities for F as well as for F^{-1}, the basic problem to be studied is that of solving a "generalized equation":

$$\mathcal{P}_F(y) \qquad\qquad \text{for given } y, \text{ determine } x \text{ such that } F(x) \ni y.$$

The data input is the vector $y \in Y$ and the solution output is some $x \in F^{-1}(y)$, or perhaps the entire solution set $F^{-1}(y) \subset X$. Note that nothing would really be gained by the alternative formulation of finding x solving $G(x, y) \ni 0$ for some mapping G and a parameter y, since one could immediately pass to F defined by $F(x) = \{\, y \mid G(x,y) \ni 0\}$ to turn this into the simple form proposed. Similarly, nothing new would, in principle, come out of a formulation with an additional parameter, such as that of solving $H(x,w) \ni y$ for x in terms of y and w, because that reduces to solving $F(x) \ni (w,y)$ for $F(x) = \{\, (w,y) \mid H(w,x) \ni y\}$, although both alternatives might, in some situations, provide extra information to be used in computations.

A vast array of problems is obviously covered by this generalized equation model in $\mathcal{P}_F(y)$. Beyond ordinary equation solving, there are applications to finding solutions to constraint systems (feasibility problems) and to variational inequalities, which in particular may characterize optimality or equilibrium. Everything depends ultimately on the specific structure of F and what can be done with it, but on the abstract level much can be learned by trying to see what can be said about conditioning in its very essence. Global perturbation results like those for solving $Ax = y$ must be relinquished for local results that refer to some particular $\bar{y} \in Y$ and $\bar{x} \in F^{-1}(\bar{y})$, and are derived by analyzing the graph of F around $(\bar{x}, \bar{y})$. Nonetheless, the way may be cleared for a deeper understanding of well-posedness and complexity issues in computation.

The key to this is found in "regularity" properties of F that correspond to "Lipschitz-like" properties of F^{-1}. Such properties can take on several forms, in each case associating with F and the reference pair $(\bar{x}, \bar{y})$ a constant called a regularity modulus. Finiteness of that modulus means that the generalized equation problem is, from a certain perspective, well-posed. It is possible then to formulate the question of how far the given problem is from problems that, in this same sense, are ill-posed, and to see whether that distance is given by the reciprocal of the designated regularity modulus. Such a result describes the radius of the largest ball (in perturbation space) in which good behavior is assured and thus yields what we may conveniently call a *perturbation radius theorem*.

In a previous paper [11], we went down that path with the property of *metric regularity* of F, arriving substantial extensions of the Eckart-Young theorem and articulating them further for some special problem formats. Our goal now is to follow through, as far as possible, with the related properties which we systematize as *strong metric regularity*, *metric subregularity* and *strong metric subregularity* of F (but also have other names). Whereas metric regularity of F is tied to the Aubin property of F^{-1}, the basic seed of all Lipschitz continuity behavior in the two-point sense, subregularity of F is tied to the calmness of F^{-1}, a Lipschitz-type property in the one-point sense, which localizes Robinson's "upper Lipschitz" continuity. All these concepts are extensively used, under various names, in stability analysis of variational problems; for background material, basic results, and references; see the recent monographs [1], [14], [19] and [29].

It will be demonstrated that perturbation radius theorems closely parallel to those developed for ordinary regularity in [11] are obtainable for strong regularity and strong subregularity. Metric subregularity, however, although considered in many papers (not always in clear terminological distinction from metric regularity), is an unstable property for which radius theorems in the proposed pattern are inherently unavailable.

First on our agenda will be a brief review of metric regularity and its role. Then will come an elucidation of why metric subregularity fails to lead to a parallel theory as far as conditioning and well-posedness are concerned. The main contributions of a positive nature will follow in sections devoted to strong regularity and strong subregularity. Finally, applications to variational inequalities will be described. Specializations to individual circumstances can ultimately be carried out in terms of calculus rules in variational analysis that have been built up for set-valued mappings and their graphical derivatives and coderivatives, as in [29].

2 Background in Metric Regularity

The norms in the Banach spaces X and Y will both be denoted by $\|\cdot\|$, and the closed unit balls by $I\!\!B$; the same also for the dual spaces X^* and Y^* when they come into play. More generally, $I\!\!B_r(a)$ will be the closed ball of radius r centered at a; thus $I\!\!B_r(a) = a + rI\!\!B$. (It will always be clear from the context which space is involved.) The distance in X (or any other space) between a point x and a set C will be denoted by $d(x, C)$; thus, $d(x, C) = \inf\{\, \|x - x'\| \,|\, x' \in C\}$.

Definition 2.1 (metric regularity). *A mapping $F : X \rightrightarrows Y$ is metrically regular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exists $\kappa \in [0, \infty)$ with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$d(x, F^{-1}(y)) \,\leq\, \kappa d(y, F(x)) \;\text{ for all } x \in U,\; y \in V. \tag{2.1}$$

The infimum of the set of values κ for which this holds is the modulus of metric regularity, denoted by $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$. (The absence of metric regularity is signaled by having $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \infty$.)

The inequality (2.1) has direct use in providing an estimate for how far a point x is from being a solution to the general equation problem for F and the data y; the expression $d(y, F(x))$ measures the "residual" when $F(x) \not\ni y$. Smaller values of κ correspond to more favorable behavior. Metric regularity requires a uniformity in such estimates with respect to local perturbations of both $\bar{x}$ and $\bar{y}$, the robustness of which is influenced by the size of $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$.

In the case of $F(x) = Ax + a$ for nonsingular $A \in I\!\!R^{n\times n}$ and $a \in I\!\!R^n$, one gets $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \|A^{-1}\|$. This and many other examples and applications are laid out in detail in our paper [11].

The concept of metric regularity has a long history. It already appeared indirectly in the work of Lyusternik in the 1930's and more explicitly in the work of Graves in the 1950's. It has had a valuable role of optimization since the 1960's, in particular as a constraint qualification for developing optimality conditions; for a recent survey see [17].

The estimation inequality (2.1) supports the interpretation of metric regularity as a condition of *well-posedness* of the generalized equation $F(x) \ni \bar{y}$ in reference to its solution $\bar{x}$, with $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ as the associated *condition number*. That idea is enhanced by the following fact (cf. [17], [29]); recall here that a set is said to be *locally closed* at one of its points if some neighborhood of that point has closed intersection with the set.

Theorem 2.2 (characterization by the inverse Aubin property). *For a mapping $F : X \rightrightarrows Y$, let $F(\bar{x}) \ni \bar{y}$ and suppose that $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y})$. Then F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if its inverse $F^{-1} : Y \rightrightarrows X$ has the Aubin property at $\bar{y}$ for $\bar{x}$, i.e. there exists $\kappa \in [0, \infty)$ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$F^{-1}(y') \cap U \,\subset\, F^{-1}(y) + \kappa\|y' - y\|I\!\!B \;\text{ for all } y, y' \in V. \tag{2.2}$$

Moreover, the infimum of all such κ is equals $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$.

In the notation of [29], the infimum of all κ for which (2.2) holds is $\operatorname{lip} F^{-1}(\bar{y}\,|\,\bar{x})$; in such terms, Theorem 2.2 says that

$$\operatorname{lip} F^{-1}(\bar{y}\,|\,\bar{x}) = \operatorname{reg} F(\bar{x}\,|\,\bar{y}).$$

The modulus $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ thus quantifies the extent to which Lipschitz behavior, in its most basic manifestation, is present locally in the response of solutions to data perturbations.

Lipschitz properties of single-valued mappings $G : X \to Y$ are important in ascertaining what happens to metric regularity under perturbations. For such mappings, there is no need to write the Lipschitz modulus as $G(\bar{x}\,|\,\bar{y})$ with $\bar{y} = G(\bar{x})$; we simply set

$$\operatorname{lip} G(\bar{x}) = \limsup_{\substack{x,x' \to \bar{x} \\ x \neq x'}} \frac{\|G(x) - G(x')\|}{\|x - x'\|}.$$

Having $\operatorname{lip} G(\bar{x}) < \infty$ is equivalent to having G be Lipschitz continuous around $\bar{x}$. When $G(x) = Ax + a$, for some $A \in L(X, Y)$ (the space of continuous linear mappings from X to Y), one has $\operatorname{lip} G(\bar{x}) = \|A\|$.

A generalized equation $F(x) \ni \bar{y}$ with solution $x = \bar{x}$ can be perturbed by adding to F a mapping G with $G(\bar{x}) = 0$, so as to get a generalized equation $(F + G)(x) \ni \bar{y}$ still having solution $x = \bar{x}$. What is the effect of such a perturbation on the modulus of metric regularity with respect to $\bar{x}$ and $\bar{y}$? This question is answered by a fundamental estimate.

Theorem 2.3 [11] (perturbation estimate for metric regularity)**.** *Consider a mapping* $F : X \rightrightarrows Y$ *and any* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *at which* $\operatorname{gph} F$ *is locally closed. If* $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \kappa < \infty$ *and if* $G : X \to Y$ *is a mapping such that* $G(\bar{x}) = 0$ *and* $\operatorname{lip} G(\bar{x}) < \lambda < \kappa^{-1}$*, then*

$$\operatorname{reg}(F + G)(\bar{x}\,|\,\bar{y}) \;<\; (\kappa^{-1} - \lambda)^{-1} = \frac{\kappa}{1 - \lambda\kappa}.$$

As noted in [17], it was apparently known to Milyutin already in the 60's that an estimate like this for *smooth, single-valued* F can be derived by modifying the original proof of a theorem of Lyusternik from the 30's. Closer to Theorem 2.3 than Lyusternik's theorem, however, is a result of Graves from the 50's, which he stated as a nonlinear analogue of the Banach open mapping principle. With the development of optimization theory in the 60's and 70's, this kind of result evolved into what is now called the Lyusternik-Graves theorem, which is mainly used to obtain a constraint qualification that leads to necessary conditions for optimality. More about the contributions of Lyusternik and Graves is provided in [8].

Theorem 2.3 gives the following relation between the regularity modulus of the mapping F and its perturbation $F + G$, involving the Lipschitz modulus of G.

Corollary 2.4. *Consider a mapping* $F : X \rightrightarrows Y$ *and any* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *at which* $\operatorname{gph} F$ *is locally closed and* $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \infty$*. If* $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) > 0$*, then for any* $G : X \to Y$ *such that* $G(\bar{x}) = 0$ *and* $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \cdot \operatorname{lip} G(\bar{x}) < 1$*, one has*

$$\operatorname{reg}(F + G)(\bar{x}\,|\,\bar{y}) \;\leq\; (\operatorname{reg} F(\bar{x}\,|\,\bar{y})^{-1} - \operatorname{lip} G(\bar{x}))^{-1}.$$

If, in addition, $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \cdot \operatorname{lip} G(\bar{x}) < 1/2$*, then*

$$(\operatorname{reg} F(\bar{x}\,|\,\bar{y})^{-1} + \operatorname{lip} G(\bar{x}))^{-1} \;\leq\; \operatorname{reg}(F + G)(\bar{x}\,|\,\bar{y}).$$

If $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = 0$*, then* $\operatorname{reg}(F + G)(\bar{x}\,|\,\bar{y}) = 0$ *for any* $G : X \to Y$ *with* $\operatorname{lip} G(\bar{x}) < \infty$*.*

Proof. The first inequality follows directly from Theorem 2.3. If $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \cdot \operatorname{lip} G(\bar{x}) < 1/2$, then, through the first inequality, we obtain $\operatorname{reg}(F + G)(\bar{x}\,|\,\bar{y}) \cdot \operatorname{lip} G(\bar{x}) < 1$. Hence, from Theorem

2.3 again, applied to $F = (F + G) - G$ and taking into account that $\operatorname{lip} G(\bar{x}) = \operatorname{lip}[-G](\bar{x})$, we obtain the second inequality. The last implication also follows from Theorem 2.3. $\square$

Corollary 2.4 reveals in particular that the metric regularity of a mapping F is inherited by, or can be determined from, any "strict first-order approximation" at the points in question. Recall that a mapping $F_0 : X \rightrightarrows Y$ is called a *strict first-order approximation* to F at $\bar{x}$ if

$$F = F_0 + G \text{ with } G : X \to Y \text{ satisfying } G(\bar{x}) = 0 \text{ and } \operatorname{lip} G(\bar{x}) = 0. \tag{2.3}$$

Corollary 2.5 (stability under strict first-order approximations)**.** *If a mapping $F_0 : X \rightrightarrows Y$ is a strict first-order approximation to a mapping $F : X \rightrightarrows Y$ at $\bar{x}$, where $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y})$, then F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if F_0 is metrically regular at $\bar{x}$ for $\bar{y}$, and indeed*

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \operatorname{reg} F_0(\bar{x}\,|\,\bar{y}).$$

Corollary 2.6 (partial strict linearization)**.** *Let $F = f + M$ for a continuous single-valued mapping $f : X \to Y$ and a mapping $M : X \rightrightarrows Y$ with closed graph, and let $\bar{y} \in F(\bar{x})$. Suppose f is strictly differentiable at $\bar{x}$. Then*

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \operatorname{reg}(f_0 + M)(\bar{x}\,|\,\bar{y}) \quad \textit{for the linearization } f_0(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x}).$$

Detail. The mapping $F_0 = f_0 + M$ is a strict first-order approximation to F, so Corollary 2.5 is applicable. $\square$

The classical case of a strict linearization of F itself corresponds to having M be the zero mapping in Corollary 2.6, so $F = f$ and the first-order approximation is $F_0(x) = F(\bar{x}) + DF(\bar{x})(x - \bar{x})$. Then

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \operatorname{reg} DF(\bar{x}), \text{ where } DF(\bar{x}) \in L(X, Y).$$

A consequence of Theorem 2.3 in part, but resting on additional arguments, is our main result in [11] about the perturbation distance to the failure of metric regularity.

Theorem 2.7 [11] (radius theorem for metric regularity)**.** *For a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ at which $\operatorname{gph} F$ is locally closed, one has*

$$\inf_{\substack{G:X\to Y\\ G(\bar{x})=0}} \left\{ \operatorname{lip} G(\bar{x}) \,\middle|\, F + G \textit{ not metrically regular at } \bar{x} \textit{ for } \bar{y} \right\} \geq \frac{1}{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}. \tag{2.4}$$

When $\dim X < \infty$ and $\dim Y < \infty$, one is sure actually to have

$$\inf_{\substack{G:X\to Y\\ G(\bar{x})=0}} \left\{ \operatorname{lip} G(\bar{x}) \,\middle|\, F + G \textit{ not metrically regular at } \bar{x} \textit{ for } \bar{y} \right\} = \frac{1}{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}, \tag{2.7}$$

and then moreover the infimum is unchanged if taken with respect to affine mappings G of rank 1.

The Eckart-Young theorem is the immediate corollary when $X = Y = I\!R^n$ and $F(x) = Ax$. The case of $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = 0$ is also covered, by Corollary 2.4, under the convention $\infty = 1/0$.

The general inequality in (2.4) comes directly from the estimate in Theorem 2.3, but obtaining the equality in the finite-dimensional case requires separate work. When Theorem 2.7 was proved in

[11], it seemed possible that the equation in (2.7) might hold even without the assumption of finite-dimensionality. Mordukhovich [21] subsequently showed this is true at least for a significant class of infinite-dimensional mappings, but recently Ioffe [18] settled the issue in general by exhibiting an infinite-dimensional mapping F for which the inequality in (2.4) is strict. We will be able here to add more to the topic by demonstrating that trouble spot of strict inequality in (2.7) is absent when dealing with *strong* metric regularity (cf. Theorem 4.6 below).

It was shown in [11] that the equality (2.7) can still be obtained in infinite dimensions if the mapping F acting between Banach spaces is *sublinear*, that is, $\operatorname{gph} F \subset X \times Y$ is a closed and convex cone. Specifically, we have the following fact.

Theorem 2.8 [11] (radius theorem for sublinear mappings). *For any $F : X \rightrightarrows Y$ that is sublinear,*

$$\inf_{G \in L(X,Y)} \Big\{ \|G\| \, \Big| \, F + G \text{ not metrically regular at } 0 \text{ for } 0 \Big\} = \frac{1}{\operatorname{reg} F(0\,|\,0)}. \tag{2.8}$$

Example 2.9 (feasibility mappings). *Let $F(x) = f(x) + K$ for a function $f : X \to Y$ that is strictly differentiable at $\bar{x}$ and a closed, convex cone $K \subset Y$, and let $\bar{y} = f(\bar{x}) \in F(\bar{x})$. Then, with a change of variables, we have*

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \operatorname{reg} F_0(\bar{x}\,|\,\bar{y}) = \operatorname{reg}(Df(\bar{x}) + K)(0\,|\,0) \quad \text{for } F_0(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x}) + K.$$

Since the mapping $Df(\bar{x}) + K$ is sublinear, from (2.8) we obtain

$$\inf_{G \in L(X,Y)} \Big\{ \|G\| \, \Big| \, F + G \text{ not metrically regular at } \bar{x} \text{ for } \bar{y} \Big\} = \frac{1}{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}.$$

Feasibility mappings can be studied in a much broader setting than Example 2.9. Many other results about them were presented in our paper [11].

The finite-dimensional case enjoys a characterization of metric regularity and its modulus through graphical differentiation, and that opens the way to an extensive calculus. Graphical differentiation is defined through the general notion of the tangent cone $T_G(\bar{z})$ and normal cone $N_G(\bar{z})$ to a set G at a point $\bar{z} \in G$, by taking $G \subset X \times Y$ and $\bar{z} = (\bar{x}, \bar{y})$. For $F : X \rightrightarrows Y$, the *graphical derivative* of F at $\bar{x}$ for $\bar{y}$, where $\bar{y} \in F(\bar{x})$, is the mapping $DF(\bar{x}\,|\,\bar{y}) : X \rightrightarrows Y$ defined by

$$u \in DF(\bar{x}\,|\,\bar{y})(w) \iff (w, u) \in T_{\operatorname{gph} F}(\bar{x}, \bar{y}),$$

whereas the *coderivative* is the mapping $D^*F(\bar{x}\,|\,\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$v \in D^*F(\bar{x}\,|\,\bar{y})(z) \iff (v, -z) \in N_{\operatorname{gph} F}(\bar{x}, \bar{y}).$$

The theory of tangent and normal cones in finite-dimensional spaces, which suffices for our applications here, is available in the book [29]. Infinite-dimensional extensions of these concepts can be found for example in [21], but are less potent in characterizing and quantifying regularity properties. When F is single-valued, $DF(\bar{x}\,|\,\bar{y})$ and $D^*F(\bar{x}\,|\,\bar{y})$ simplify in notation to $DF(\bar{x})$ and $D^*F(\bar{x})$. When F is continuously differentiable, these mappings reduce to the usual derivative and its adjoint.

Theorem 2.10 [29] (coderivative characterization of metric regularity). *In the case of* $\dim X < \infty$ *and* $\dim Y < \infty$, *and as long as* $\operatorname{gph} F$ *is locally closed at* $(\bar{x}, \bar{y})$, *one has* F *metrically regular at* $\bar{x}$ *for* $\bar{y}$ *if and only if* $D^*F(\bar{x}\,|\,\bar{y})^{-1}(0) = \{0\}$, *and in fact*

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \|D^*F(\bar{x}\,|\,\bar{y})^{-1}\|^+ = \sup_{v \in B} \left(\sup_{z \in D^*F(\bar{x}\,|\,\bar{y})^{-1}(v)} \|z\| \right).$$

3 Metric Subregularity

With the platform of metric regularity in place, we are ready to turn to the related concept of *sub*regularity and how it compares.

Definition 3.1 (metric subregularity). *A mapping* $F : X \rightrightarrows Y$ *is metrically subregular at* $\bar{x}$ *for* $\bar{y}$ *if* $F(\bar{x}) \ni \bar{y}$ *and there exists* $\kappa \in [0, \infty)$ *along with neighborhoods* U *of* $\bar{x}$ *and* V *of* $\bar{y}$ *such that*

$$d(x, F^{-1}(\bar{y})) \;\le\; \kappa d(\bar{y}, F(x) \cap V) \;\text{ for all } x \in U. \tag{3.1}$$

The infimum of the set of values κ *for which this holds is the modulus of metric subregularity, denoted by* $\operatorname{subreg} F(\bar{x}\,|\,\bar{y})$. *(The absence of metric subregularity is signaled by having* $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = \infty$.*)*

The difference with the metric regularity in Definition 2.1 is that the data input $\bar{y}$ is now fixed and not perturbed to a nearby y. Subregularity is thus a weaker condition, satisfied for a set of κ values that is at least as large, so we definitely have

$$\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) \;\le\; \operatorname{reg} F(\bar{x}\,|\,\bar{y}) \tag{3.2}$$

in general. The term "subregularity" is new here; this property has itself been called metric regularity in some places, but that could lead to serious misunderstandings, especially in view of its unstable behavior in contrast to that of true metric regularity, which will be brought out in Theorem 3.5. We hope that our terminological distinction will help in avoiding such misunderstandings.

Metric subregularity has extensively been studied in connection with feasibility problems and their associated mappings, often of the form

$$F(x) = \begin{cases} f(x) + K & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

with f single-valued and K a closed, convex cone (cf. Example 2.9). (Then $F^{-1}(y)$ consists of all $x \in C$ such that $y - f(x) \in K$, which can be viewed as the vector inequality $f(x) \le_K y$ in the partial ordering of Y induced by K.) The constants κ then furnish error bounds on constraint systems. More on this subject can be seen, for instance, in the conference volume [20].

Theorem 3.2 (characterization by inverse calmness). *For a mapping* $F : X \rightrightarrows Y$, *suppose that* $F(\bar{x}) \ni \bar{y}$ *and that* $F^{-1}(\bar{y})$ *is locally closed at* $\bar{x}$. *Then* F *is metrically subregular at* $\bar{x}$ *for* $\bar{y}$ *if and only if its inverse* $F^{-1} : Y \rightrightarrows X$ *is calm at* $\bar{y}$ *for* $\bar{x}$, *i.e. there exists* $\kappa \in [0, \infty)$ *along with neighborhoods* U *of* $\bar{x}$ *and* V *of* $\bar{y}$ *such that*

$$F^{-1}(y) \cap U \;\subset\; F^{-1}(\bar{y}) + \kappa\|y - \bar{y}\|\mathbb{B} \;\text{ for all } y \in V. \tag{3.3}$$

Moreover, the infimum of all such κ *equals* $\operatorname{subreg} F(\bar{x}\,|\,\bar{y})$.

Proof. To say that $x \in F^{-1}(y) \cap U$ and $y \in V$ is to say that $x \in U$ and $y \in F(x) \cap V$. For such x and y, the calmness property in (3.3) requires the ball $x + \kappa\|y - \bar{y}\|I\!\!B$ to have nonempty intersection with $F^{-1}(\bar{y})$. Then $d(x, F^{-1}(\bar{y})) \leq \kappa\|y - \bar{y}\|$ (and this is in fact equivalent to the intersection condition as long as the neighborhood U is small enough, due to our assumption that $F^{-1}(\bar{y})$ is locally closed at $\bar{x}$). In taking the infimum on the right side of this inequality with respect to $y \in F(x) \cap V$, we see that (3.1) holds. Conversely, if (3.1) holds, we replace U a smaller neighborhood if necessary, and taking advantage of local closedness, pass back to (3.3). ☐

Calmness of F^{-1} at $\bar{y}$ for $\bar{x}$ is a version, localized to $\bar{x}$, of Robinson's property of *upper Lipschitz continuity* in [27]; F^{-1} has that property at $\bar{y}$ if there exists $\kappa \in [0, \infty)$ along with a neighborhood V of $\bar{y}$ such that

$$F^{-1}(y) \subset F^{-1}(\bar{y}) + \kappa\|y - \bar{y}\|I\!\!B \text{ for all } y \in V. \tag{3.4}$$

This corresponds to taking $U = X$ in (3.3), in which case a choice of $\bar{x}$ plays no role. Particular motivation comes from the following fact, proved by Robinson in [27].

Example 3.3 [27] (piecewise polyhedral case of upper Lipschitz continuity)**.** *Suppose X and Y are finite-dimensional and* $\operatorname{gph} F$ *is the union of finitely many convex sets that are polyhedral (or equivalently, this holds for* $\operatorname{gph} F^{-1}$*). Then F^{-1} is upper Lipschitz continuous at every* $\bar{y} \in \operatorname{dom} F^{-1}$.

Robinson's result complements the classical Hoffman lemma, which says that F^{-1} is Lipschitz continuous relative to its effective domain $\operatorname{dom} F^{-1}$ when $\operatorname{gph} F$ is itself a polyhedral convex set.

As for "calmness," Clarke [3] was apparently the first to use that term in variational analysis, initially for a property of value functions in optimization in connection with multiplier rules and penalization, but in later work also for single-valued mappings more generally. Properties of calmness of single-valued and set-valued mappings are reviewed in [29].

In various situations of importance in applications, there's no need to mention a neighborhood V of $\bar{y}$ in the description of subregularity and calmness. This is clarified by our next result.

Proposition 3.4 (simplifying criterion)**.** *Let $F(\bar{x}) \ni \bar{y}$ for a mapping $F : X \rightrightarrows Y$ with $F^{-1}(\bar{y})$ such that*

$$\liminf_{\substack{x \to \bar{x} \\ x \in \operatorname{dom} F}} F(x) \ni \bar{y}. \tag{3.5}$$

Then metric subregularity of F at $\bar{x}$ for $\bar{y}$ is equivalent simply to the existence of $\kappa \in [0, \infty)$ *along with a neighborhood U of $\bar{x}$ such that*

$$d(x, F^{-1}(\bar{y})) \leq \kappa d(\bar{y}, F(x)) \quad \textit{for all } x \in U. \tag{3.6}$$

Under the additional assumption that $F^{-1}(\bar{x})$ is locally closed at $\bar{x}$, the calmness of F^{-1} at $\bar{y}$ for $\bar{x}$ can be identified with the existence of $\kappa \in [0, \infty)$ *along with a neighborhood U of $\bar{x}$ such that*

$$F^{-1}(y) \cap U \subset F^{-1}(\bar{y}) + \kappa\|y - \bar{y}\|I\!\!B \quad \textit{for all } y \in Y. \tag{3.7}$$

Proof. Obviously (3.1) is implied by (3.6). The proof of the subregularity claim can therefore be completed by demonstrating that if (3.1) holds for some U and V; then, because of the assumption in (3.5), the seemingly stronger condition in (3.6) must hold with respect to a possibly smaller substitute U' for U.

Given (3.1), we can choose within V a neighborhood of the form $V' = I\!B_\varepsilon(\bar{y})$, and next by (3.5) choose within U a neighborhood U' such that $F(x) \cap V' \neq \emptyset$ for all $x \in U'$. Then $d(\bar{y}, F(x) \cap V) = d(\bar{y}, F(x))$ when $x \in U'$. Hence from the fact that (3.1) continues to hold when U and V are replaced by the smaller sets U' and V', we obtain (3.6) for U'.

Similarly, (3.7) entails the calmness in (3.3), so attention can be concentrated on showing that we can pass from (3.3) to (3.7) under an adjustment in the size of U. We already know from Theorem 3.2 that the calmness condition in (3.3) leads to the metric subregularity condition in (3.1), and further, from the argument just given, that such subregularity in the presence of our additional assumption in (3.5) yields the condition in (3.6). But that condition can be plugged into the argument in Theorem 3.2 by taking $V = Y$, so as to get the corresponding calmness property with $V = Y$ but with U replaced by something smaller. This gives (3.7). □

The natural question to be asked about metric subregularity is whether it enjoys stability properties resembling those of metric regularity. The answer is no!

Theorem 3.5 (instability of metric subregularity). *Metric subregularity of a mapping $F : X \rightrightarrows Y$ at $\bar{x}$ for $\bar{y}$ can fail to be preserved under small Lipschitz perturbations or even under strict first-order approximations. Specifically, it is possible in some cases, even with X and Y one-dimensional and* $\operatorname{gph} F$ *closed, to have a perturbation mapping $G : X \to Y$ such that*

$$\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) < \infty \;\; \textit{but} \;\; \operatorname{subreg}(F+G)(\bar{x}\,|\,\bar{y}) = \infty, \;\; \textit{although} \;\; G(\bar{x}) = 0, \; \operatorname{lip} G(\bar{x}) = 0.$$

Proof. An example with $X = Y = I\!R$ and $\bar{x} = \bar{y} = 0$ will serve to confirm this. In $I\!R \times I\!R$, let $\operatorname{gph} F$ be the set of all (x, y) such that $x \geq 0$, $y \geq 0$, $xy = 0$. Then $F^{-1}(0) = [0, \infty) \supset F^{-1}(y)$ for all y, so F^{-1} is calm at $\bar{y} = 0$ for $\bar{x} = 0$, moreover "globally" so with $\kappa = 0$. Therefore by Theorem 3.2, F is metrically subregular at 0 for 0, in fact with $\operatorname{subreg} F(0\,|\,0) = 0$.

Consider, however, the mapping $G : I\!R \to I\!R$ defined by $G(x) = -x^2$, for which $G(0) = 0$ and $\operatorname{lip} G(0) = 0$. The perturbed mapping $F+G$ has $(F+G)^{-1}$ single-valued everywhere: $(F+G)^{-1}(y) = 0$ when $y \geq 0$, and $(F+G)^{-1}(y) = \sqrt{|y|}$ when $y \leq 0$. This mapping isn't calm at 0 for 0. Hence by Theorem 3.2 again, $F + G$ isn't metrically subregular; we have $\operatorname{subreg}(F + G)(0\,|\,0) = \infty$. □

In view of Theorem 3.5, it's *impossible* to have a perturbation radius theorem for metric subregularity in the mold of the one for metric regularity in Theorem 2.7. The radius of good behavior could well be 0! Subregularity therefore falls short in this important respect as a useful concept of conditioning for numerical purposes.

Another major drawback of metric subregularity, in contrast to metric regularity, is the lack of a "norm" characterization of $\operatorname{subreg} F(\bar{x}\,|\,\bar{y})$ along the lines of the formula in Theorem 2.10. Without such a formula, there is little hope of a viable *calculus of estimates* for this property. However, a sufficient condition for calmness, developed in [16] in terms of neighboring coderivatives, could perhaps be invoked for F^{-1} in the light of Theorem 3.2.

To conclude this section, we point out other properties which are, in a sense, derived from metric regularity but, like metric subregularity, lack stability under small Lipschitz perturbations. One of these properties relates to linear openness. It is well known that the metric regularity condition in (2.1) is equivalent to *linear openness* of F at $\bar{x}$ for $\bar{y}$ with constant κ, in the sense of having neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x+\kappa r \operatorname{int} I\!B) \supset (y + r \operatorname{int} I\!B) \cap V \quad \text{for all } x \in U,\ (x,y) \in \operatorname{gph} F,\ r > 0.$$

If we restrict this linear openness condition to the point $(\bar{x},\bar{y})$, that is, we postulate

$$F(\bar{x}+\kappa r \operatorname{int} I\!B) \ \supset\ \bar{y} + r \operatorname{int} I\!B \qquad \text{for } r > 0, \tag{3.8}$$

we get a property that fails to be preserved when F is replaced by $F+G$ for $G(\bar{x}) = 0$ and $\operatorname{lip} G(\bar{x})$ small. This is demonstrated by the following example[2]. Let $X = Y = I\!R^2$, equipped with the Euclidean norm, and define $F : X \to Y$ by taking take $F(0,0) = (0,0)$ and for, $x = (x_1,x_2) \neq (0,0)$,

$$F(x_1,x_2) = \frac{1}{\sqrt{x_1^2+x_2^2}} \begin{pmatrix} x_1^2 - x_2^2 \\ 2|x_1|x_2 \end{pmatrix}.$$

Then F satisfies (3.8) at $\bar{x} = 0$, $\bar{y} = 0$, with $\kappa = 1$, since $\|F(x)\| = \|x\|$. The function $G(x_1,x_2) = (0,x_2^3)$ has $G(0,0) = (0,0)$ and $\operatorname{lip} G(0,0) = 0$, but we have $(F+G)^{-1}(c,0) = \emptyset$ when $c < 0$.

A "metric regularity variant" of the openness property (3.8), equally failing to be preserved under small Lipschitz perturbations, as shown by this example, is the requirement that

$$d(\bar{x}, F^{-1}(y)) \le \kappa \|y - \bar{y}\| \quad \text{for} \quad y \text{ close to } \quad \bar{y}.$$

The same trouble comes up also for an "inner-semicontinuity variant": there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F^{-1}(y) \cap U \neq \emptyset \quad \text{for all } y \in V.$$

Despite these negative observations, we will later see (in Section 5) positive aspects in the stability behavior of the sharper property we call *strong* metric subregularity.

4 Strong Metric Regularity

The notion of graphical localization will now become useful. A *graphical localization* of a mapping $F : X \rightrightarrows Y$ at a pair $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is a mapping $\tilde{F} : X \rightrightarrows Y$ such that

$$\operatorname{gph} \tilde{F} \ =\ (U \times V) \cap \operatorname{gph} F \quad \text{for some neighborhood } U \times V \text{ of } (\bar{x},\bar{y}),$$

so that, in other words,

$$\tilde{F}(x) = \begin{cases} F(x) \cap V & \text{when } x \in U, \\ \emptyset & \text{when } x \notin U. \end{cases}$$

The inverse of $\tilde{F}$ then obviously furnishes a graphical localization of F^{-1} at $(\bar{y},\bar{x})$, with

$$\tilde{F}^{-1}(y) = \begin{cases} F^{-1}(y) \cap U & \text{when } y \in V, \\ \emptyset & \text{when } y \notin V. \end{cases}$$

Our primary interest in this notion will be with single-valuedness properties of such graphical localizations of F^{-1}, although convex-valuedness of graphical localizations of F will play a role at one stage as well.

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[2] This example was communicated to the first author by H. Sussmann.

Definition 4.1 (strong metric regularity)**.** *A mapping $F : X \rightrightarrows Y$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if the metric regularity condition in Definition 2.1 is satisfied by some κ, U and V such that, in addition, the graphical localization of F^{-1} with respect to U and V is nowhere multivalued, i.e.,*

$$\begin{cases} \textit{when } y \in V, \textit{ there is at most one solution} \\ x \in U \textit{ to the generalized equation } F(x) \ni y. \end{cases}$$

This definition is slightly weaker than the one we furnished in [11], where the localization in question was required to be single-valued rather than just nowhere multivalued. The two definitions are equivalent, however, in view of the characterization provided next.

Theorem 4.2 (characterization by single-valued Lipschitzian inverse)**.** *A mapping $F : X \rightrightarrows Y$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if and only if F has the single-valued Lipschitzian inverse property there, i.e., F^{-1} has a graphical localization at $(\bar{y}, \bar{x})$ that is single-valued and Lipschitz continuous on a neighborhood of $\bar{y}$: there exists $\kappa \in [0, \infty)$ with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$\begin{cases} \textit{the mapping } y \mapsto F^{-1}(y) \cap U \textit{ is single-valued on } V \\ \textit{and moreover Lipschitz continuous with constant } \kappa. \end{cases}$$

Moreover, the infimum of all such Lipschitz constants κ then equals

$\operatorname{reg} F(\bar{x}\,|\,\bar{y})$.

Proof. The sufficiency is easy to see: if F^{-1} has such a single-valued, Lipschitz continuous localization, it has the Aubin property at $\bar{y}$ for $\bar{x}$, in particular, and moreover the graph of F is locally closed at $(\bar{x}, \bar{y})$. The metric regularity of F follows then from Theorem 2.2. The necessity could almost be argued similarly through Theorem 2.2, but local closedness of $\operatorname{gph} F$ at $(\bar{x}, \bar{y})$ would then be required as an additional assumption. That can be avoided by a direct approach which utilizes instead the assumption about a localization of F^{-1} not being multivalued.

Starting from the metric regularity of F and the mentioned localization property of F^{-1}, we have, on the one hand, some $\kappa \in (0, \infty)$ and neighborhood $U_0 \times V_0$ of $(\bar{x}, \bar{y})$ such that

$$d(x, F^{-1}(y)) \;\le\; \kappa d(y, F(x)) \quad \text{when } x \in U_0,\ y \in V_0, \tag{4.1}$$

and on the other hand, a neighborhood $U_1 \times V_1$ of $(\bar{x}, \bar{y})$ such that

$$F^{-1}(y) \cap U_1 \quad \text{is a singleton or empty when } y \in V_1. \tag{4.2}$$

Choose new neighborhoods $U = I\!\!B_\delta(\bar{x}) \subset U_0 \cap U_1$ and $V = I\!\!B_\varepsilon(\bar{y}) \subset V_0 \cap V_1$, and also a $\lambda > 0$; make δ, ε and λ be small enough that $I\!\!B_{\delta+\lambda+2\kappa\varepsilon}(\bar{x}) \subset \operatorname{int} U_1$. From (4.1), which persists when U_0 and V_0 are reduced to U and V, we get

$$d(x, F^{-1}(y)) \le \kappa \|y' - y\| \quad \text{for all } x \in F^{-1}(y') \cap U \text{ when } y, y' \in V.$$

This inequality implies that $F^{-1}(y') \cap U$ meets $F^{-1}(y) + (\lambda + \kappa\|y' - y\|)I\!\!B$ when $y, y' \in V$, with the possible exception of cases where both $F^{-1}(y') \cap U$ and $F^{-1}(y)$ are empty. Of course $F^{-1}(y) \neq \emptyset$ when $y = \bar{y}$, so we must have $F^{-1}(y') \cap U \neq \emptyset$ for all $y' \in V$, and the exceptional cases are eliminated. Since (4.2) persists with U_1 and V_1 reduced to U and V, we see moreover that $F^{-1}(y') \cap U$ must be a singleton when $y' \in V$. Let the single element be denoted by $s(y')$. We have then that

$$s(y') \in U \;\text{ and }\; s(y') \in [F^{-1}(y) + (\lambda + \kappa\|y' - y\|)I\!\!B] \;\text{ when }\; y, y' \in V. \tag{4.3}$$

Recalling that $U = \mathbb{B}_\delta(\bar{x})$ and $V = \mathbb{B}_\varepsilon(\bar{y})$, which entails $\|y' - y\| \leq 2\varepsilon$ when $y, y' \in V$, we see in (4.3) that the ball $\mathbb{B}_{\delta+\lambda+2\kappa\varepsilon}(\bar{x})$ has to meet $F^{-1}(y)$. But this ball lies in the interior of U_1, which includes U. We already have determined that $F^{-1}(y) \cap U$ consists only of $s(y)$ when $y \in V$, while on the other hand $F^{-1}(y) \cap U_1$ has at most one element, so the conclusion is inescapable that unique element belonging to both $\mathbb{B}_{\delta+\lambda+2\kappa\varepsilon}(\bar{x})$ and $F^{-1}(y)$ is $s(y)$. Thus, we have deduced from (4.3) that

$$s(y') \in s(y) + (\lambda + \kappa\|y' - y\|)\mathbb{B} \text{ when } y, y' \in V.$$

Here λ could have been taken arbitrarily small, so it follows that s is Lipschitz continuous on V with constant κ. The mapping s is the localization of F^{-1} with respect to the neighborhoods U and V, and the proof of equivalence is therefore complete.

By virtue of Theorem 2.2 and its associated equation, the Lipschitz modulus at $\bar{y}$ for the single-valued local inverse mapping is equal in fact to $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$. (The Lipschitz continuity of the graphical localization of F^{-1} obviously entails $\operatorname{gph} F$ being locally closed at $(\bar{x}, \bar{y})$.) ☐

We have already seen in Theorem 2.3 that metric regularity is preserved under small Lipschitz perturbations and in particular, in Corollary 2.4, that it is unaffected by strict first-order approximations. To complement those results for the case of strong metric regularity, we develop now an estimate for what happens to the single-valuedness in Theorem 4.2 under perturbations.

Theorem 4.3 (perturbation estimate for inverse single-valuedness). *For a mapping $F : X \rightrightarrows Y$, suppose that F^{-1} has the localization property in Theorem 4.2, with Lipschitz constant $\kappa \in (0, \infty)$. Let $G : X \to Y$ have $G(\bar{x}) = 0$ and $\operatorname{lip} G(\bar{x}) < \lambda < \kappa^{-1}$. Then $(F + G)^{-1}$ likewise has the localization property in Theorem 4.2, but with Lipschitz constant $(\kappa^{-1} + \lambda)^{-1}$.*

Proof. We note first that our hypothesis implies through Theorem 4.2 that $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) \leq \kappa$ and $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y})$, hence by Theorem 2.3 we have that $\operatorname{reg}(F+G)(\bar{x}\,|\,\bar{y}) \leq (\kappa^{-1} + \lambda)^{-1}$. The desired conclusion will therefore follow, again with the help of Theorem 4.2, by showing that $(F + G)^{-1}$ has the localization property in the definition of strong metric regularity.

By assumption we have neighborhoods U and V enjoying the property described in Theorem 4.2; denote by $s(y)$ the unique element of $F^{-1}(y) \cap U$ when $y \in V$. We also have a neighborhood U' of $\bar{x}$ on which G is Lipschitz continuous with constant λ. Because $G(\bar{x}) = 0$, we can find neighborhoods $U_0 = \mathbb{B}_\delta(\bar{y}) \subset U \cap U'$ and $V_0 = \mathbb{B}_\varepsilon(\bar{y}) \subset V$ such that

$$x \in U_0,\ y \in V_0 \implies y - G(x) \in V. \tag{4.4}$$

Consider now the graphical localization of $(F + G)^{-1}$ corresponding to U_0 and V_0. Each $y \in V_0$ maps to the set $(F + G)^{-1}(y) \cap U_0$; it will be demonstrated that this set can have at most one element, and that will finish the proof.

On the contrary, assume that $y \in V$ and $x, x' \in U$, $x \neq x'$, are such that both x and x' belong to $(F+G)^{-1}(y)$. Clearly $x \in (F+G)^{-1}(y) \cap U_0$ if and only if $x \in U_0$ and $y \in F(x) + G(x)$, or equivalently $y - G(x) \in F(x)$. The latter, in turn, is the same as having $x \in F^{-1}(y - G(x)) \cap U = s(y - G(x))$, where $y - G(x) \in V$ by (4.4). Then

$$0 < \|x - x'\| = \|s(y - G(x)) - s(y - G(x'))\| \leq \kappa\|G(x) - G(x')\| \leq \kappa\lambda\|x - x'\| < \|x - x'\|,$$

which is absurd, and we are done. ☐

The above result is supplemented by Corollary 2.4, where now $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ refers to the Lipschitz constant of the single-valued graphical localization of F at $(\bar{x}, \bar{y})$.

Corollary 4.4 (stability under strict first-order approximations)**.** *If a mapping* $F_0 : X \rightrightarrows Y$ *is a strict first-order approximation to a mapping* $F : X \rightrightarrows Y$ *at* $\bar{x}$, *then* F *is strongly metrically regular at* $\bar{x}$ *for* $\bar{y}$ *if and only if* F_0 *is strongly metrically regular at* $\bar{x}$ *for* $\bar{y}$. *Indeed, the Lipschitz modulus at* $\bar{y}$ *for the graphical localization of* F^{-1} *in this property coincides with the one obtained for* F_0^{-1}.

Proof. This is immediately obtained from the case of Theorem 4.3 in which $\operatorname{lip} G(\bar{x}) = 0$. The modulus claim merely reflects the relation already established in Corollary 2.5; only the combination with graphically localized single-valuedness is new here. □

Corollary 4.5 (stability under partial strict linearizations)**.** *Let* $F = f + M$ *for a single-valued mapping* $f : X \to Y$ *and a mapping* $M : X \rightrightarrows Y$, *and let* $\bar{y} \in F(\bar{x})$. *Suppose* f *is strictly differentiable at* $\bar{x}$, *and let* $F_0 = f_0 + M$ *with* $f_0(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x})$.

Then F^{-1} *has a graphical localization around* $(\bar{y}, \bar{x})$ *that is single-valued and Lipschitz continuous on a neighborhood of* $\bar{y}$ *if and only if* F_0^{-1} *has such a localization. Moreover, the Lipschitz modulus at* $\bar{y}$ *for the localizations in question will be the same in each case.*

Proof. We specialize Corollary 4.4 to $F_0 = f_0 + M$. □

We prove next a perturbation radius theorem for strong metric regularity. Note that this result lies close to the one for metric regularity in Theorem 2.7 but, in contrast to it, guarantees an equation even when X and Y are infinite-dimensional.

Theorem 4.6 (radius theorem for strong metric regularity)**.** *For a mapping* $F : X \rightrightarrows Y$ *that is strongly metrically regular at* $\bar{x}$ *for* $\bar{y}$, *one has*

$$\inf_{\substack{G: X \to Y \\ G(\bar{x}) = 0}} \Big\{ \operatorname{lip} G(\bar{x}) \,\Big|\, F + G \text{ not strongly metrically regular at } \bar{x} \text{ for } \bar{y} \Big\} = \frac{1}{\operatorname{reg} F(\bar{x}\,|\,\bar{y})}. \tag{4.5}$$

Moreover the infimum is unchanged if taken with respect to affine mappings G *of rank 1.*

Proof. The combination of Theorem 4.3 with the general inequality for metric regularity in Theorem 2.7 reveals that "$\geq$" holds in (4.5). The opposite inequality, along with the final assertion, will be obtained by demonstrating that for any $r > 1/\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ there exists $A \in L(X, Y)$ with $\|A\| < r$ such that the mapping $F + G$ for $G(x) = A(x - \bar{x})$ fails to have a graphical localization at $(\bar{x}, \bar{y})$ exhibiting the single-valued inverse property demanded by strong metric regularity. The case of $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = 0$ is covered in this under the convention that $1/0 = \infty$, since, according to Corollary 2.4 taken together with Theorem 4.2, we have $\operatorname{reg}(F + G)(\bar{x}\,|\,\bar{y}) = 0 < \infty$ for every $G : X \to Y$ with $G(\bar{x}) = 0$ and $\operatorname{lip} G(\bar{x}) < \infty$.

Fix neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that the corresponding localization of F^{-1} reduces to a single-valued, Lipschitz continuous mapping s on V, as guaranteed through Theorem 4.2 by the strong regularity assumption on F; let the Lipschitz constant be κ. Choose $\varepsilon > 0$ small enough that $\mathbb{B}_{\varepsilon(1+r\kappa)}(\bar{y}) \subset V$, in which case $\mathbb{B}_\varepsilon \subset V$ in particular, of course.

Since $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \operatorname{lip} F^{-1}(\bar{y}\,|\,\bar{x}) = \operatorname{lip} s(\bar{y})$, we have $\operatorname{lip} s(\bar{y}) > 1/r$. Hence there exist $y_0, y_1 \in \mathbb{B}_\varepsilon(\bar{y})$, with $y_0 \neq y_1$, such that $\|x_1 - x_0\|/\|y_1 - y_0\| > 1/r$ for $x_0 = s(y_0)$ and $x_1 = s(y_1)$. Then $x_0 \neq x_1$, so there exists an x^* in the dual space X^* such that $\langle x_1 - x_0, x^* \rangle = \|x_1 - x_0\|$, $\|x^*\| = 1$. Define $A \in L(X, Y)$ by

$$A(x) = -\frac{\langle x, z \rangle}{\|x_1 - x_0\|}(y_1 - y_0),$$

noting that A is of rank 1 with $\|A\| < r$ and $A(x_1 - x_0) = y_0 - y_1$. Let $\tilde{y} = y_0 + A(x_0 - \bar{x})$; we have

$$\|\tilde{y} - \bar{y}\| = \|y_0 - \bar{y} + A(s(y_0) - s(\bar{y}))\| \leq \|y_0 - \bar{y}\| + \|A\|\|s(y_0) - s(\bar{y})\| \leq \varepsilon + r\kappa\varepsilon,$$

and consequently $\tilde{y} \in V$ by our choice of ε. Since $x_0 \in F^{-1}(y_0)$ and $x_1 \in F^{-1}(y_1)$ we have in terms of $G(x) = A(x - \bar{x})$ that

$$(F + G)(x_0) = F(x_0) + A(x_0 - \bar{x}) \ni y_0 + A(x_0 - \bar{x}) = \tilde{y},$$

and on the other hand

$$(F + G)(x_1) = F(x_1) + A(x_1 - x_0) + A(x_0 - \bar{x}) \ni y_1 + y_0 - y_1 + A(x_0 - \bar{x}) = \tilde{y}.$$

Thus, $(F + G)^{-1}(\tilde{y})$ contains both $s(y_0)$ and $s(y_1)$ and is not a singleton. By choosing arbitrarily small ε, we can make $s(y_0)$ and $s(y_1)$ arbitrarily close to $s(\bar{y}) = \bar{x}$ and $\tilde{y}$ arbitrarily close to $\bar{y}$. This shows that $(F + G)^{-1}$ fails to have a single-valued localization around $(\bar{y}, \bar{x})$. ☐

Applications of this result may be assisted by a characterization of $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ which makes use of strict graphical derivatives instead of coderivatives. Recall that, for $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $\bar{y} \in F(\bar{x})$, the *strict graphical derivative mapping* $D_*F(\bar{x}\,|\,\bar{y})$ is defined by

$$(w, z) \in \operatorname{gph} D_*F(\bar{x}\,|\,\bar{y}) \quad\Longleftrightarrow\quad \begin{cases} \exists\,(x^\nu, y^\nu) \to (\bar{x}, \bar{y}),\ \tau^\nu \searrow 0 \text{ and } (w^\nu, z^\nu) \to (w, z), \\ \text{ for which } z^\nu \in [F(x^\nu + \tau^\nu w^\nu) - y^\nu]/\tau^\nu. \end{cases}$$

In stating the characterization in question, we say that F is *inner semicontinuous around* $\bar{x}$ *for* $\bar{y}$ when there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that, for every $x \in U$ and $\varepsilon > 0$, one can find $\delta > 0$ with

$$F(x) \cap V \subset F(x') + \varepsilon\mathbb{B} \text{ and } F(x') \cap V \subset F(x) + \varepsilon\mathbb{B} \text{ when } x' \in U \cap \mathbb{B}(x, \delta).$$

(This property is a localized version of what has traditionally been called "lower semicontinuity" and corresponds to the Painlevé-Kuratowski "liminf" in set convergence, but the associated property that has traditionally been called "upper semicontinuity" does not correspond to the Painlevé-Kuratowski "limsup," except in the special context of compact spaces which excludes $\mathbb{R}^n$. For this reason, the terminology of inner and outer semicontinuity was adopted in [29] for the properties built on Painlevé-Kuratowski semi-limits, and we are conforming to that here.)

Theorem 4.7 [29] (strict derivative characterization of strong regularity). *For $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $\bar{y} \in F(\bar{x})$, suppose F is inner semicontinuous around $\bar{x}$ for $\bar{y}$, and furthermore has a graphical localization around $(\bar{x}, \bar{y})$ that is convex-valued. Then F is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $D_*F(\bar{x}\,|\,\bar{y})^{-1}(0) = 0$, in which case $m = n$ and in fact*

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \|D_*F(\bar{x}\,|\,\bar{y})^{-1}\|^+ = \sup_{v \in B} \left(\sup_{z \in D_*F(\bar{x}\,|\,\bar{y})^{-1}(v)} \|z\| \right). \tag{4.9}$$

Proof. See [29, Theorem 9.54(b)]. (This is a corrected version in the second edition of that book. The version in first edition misstated the continuity assumption and neglected the convex-valuedness assumption.) ☐

5 Strong Metric Subregularity

We are ready now to take up the property that fills the position of relating to metric subregularity in parallel in much the same way that strong metric regularity relates to metric regularity, but manages to get around the serious instability in the behavior of metric subregularity.

Definition 5.1 (strong metric subregularity). *A mapping $F : X \rightrightarrows Y$ is strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exists $\kappa \in [0, \infty)$ along with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$\|x - \bar{x}\| \;\le\; \kappa d(\bar{y}, F(x) \cap V) \;\; \text{for all } x \in U, \tag{5.1}$$

or in other words, if F is metrically subregular at $\bar{x}$ for $\bar{y}$ and, in addition, $\bar{x}$ is an isolated point of $F^{-1}(\bar{y})$ (in other words, $F^{-1}(\bar{y}) \cap U = \{\bar{x}\}$ for some neighborhood U of $\bar{y}$).

Theorem 5.2 (characterization by inverse isolated calmness). *A mapping $F : X \rightrightarrows Y$ is strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if and only if its inverse F^{-1} has a graphical localization at $(\bar{y}, \bar{x})$ that is single-valued at $\bar{y}$ itself (with value $\bar{x}$) and calm there. Specifically, this means the existence of $\kappa \in [0, \infty)$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that*

$$F^{-1}(y) \cap U \;\subset\; \bar{x} + \kappa \|y - \bar{y}\| I\!B \;\; \text{when } y \in V. \tag{5.2}$$

Proof. Suppose first that the condition in Definition 5.1 is satisfied. Consider any $y \in V$ and $x \in F^{-1}(y) \cap U$. That entails $y \in F(x) \cap V$, hence $d(\bar{y}, F(x) \cap V) \le \|y - \bar{y}\|$ and consequently $\|x - \bar{x}\| \le \kappa \|y - \bar{y}\|$ by (5.1). Thus $x \in \bar{x} + \kappa \|y - \bar{y}\| I\!B$, and we conclude that (5.2) holds.

For the converse, suppose now that (5.2) holds for certain neighborhoods U and V. Consider any $x \in U$. For arbitrary $y \in F(x) \cap V$ we have $x \in F^{-1}(x) \cap U$, and therefore $x \in \bar{x} + \kappa \|y - \bar{y}\| I\!B$ by (5.2), which means $\|x - \bar{x}\| \le \kappa \|y - \bar{y}\|$. This being true for all $y \in F(x) \cap V$, we must have $\|x - \bar{x}\| \le \kappa d(\bar{y}, F(x) \cap V)$. Thus, (5.1) holds. ◻

The property of F^{-1} in (5.2) is "calmness relative to an isolated image point," in the terminology we used in [10], although other terms and variants abound. This property in "forward mode" for a general mapping F appeared in the literature as early as 1987 in [28], where it was characterized in finite-dimensions in terms of the graphical derivatives of F. That characterization, translated to F^{-1}, yields through Theorem 5.2 the following criterion for strong metric subregularity:

Theorem 5.3 (derivative characterization of strong subregularity). *In the case of $\dim X < \infty$ and $\dim Y < \infty$, as long as $\operatorname{gph} F$ is locally closed at $(\bar{x}, \bar{y})$, one has F strongly metrically subregular at $\bar{x}$ for $\bar{y}$ if and only if $DF(\bar{x}\,|\,\bar{y})^{-1}(0) = \{0\}$, and in fact*

$$\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = \|DF(\bar{x}\,|\,\bar{y})^{-1}\|^{+} = \sup_{u \in B} \left(\sup_{w \in DF(\bar{x}\,|\,\bar{y})^{-1}(u)} \|w\| \right).$$

In [7], calmness relative to an isolated image point was called the "local upper-Lipschitz property at a point in the graph" and observed to be preserved under perturbations of order $o(x)$. By Robinson's result about upper Lipschitz continuity recalled in Example 3.3, a "polyhedral" mapping F is strongly subregular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{x}$ is an isolated point of $F^{-1}(\bar{y})$. Theorem 5.3 expresses this principle far more widely.

Strong subregularity of F at $\bar{x}$ for $\bar{y}$ does *not* imply that a localization of F^{-1} around $(\bar{y}, \bar{x})$ is nonempty-valued; there need not exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that $F^{-1}(y) \cap U \neq \emptyset$ for $y \in V$. In some situations, it could be natural to demand such nonemptiness, but we have avoided implanting it in the definition of strong subregularity itself, because that would excessively narrow the concept and remove from the scene of applications some important classes of problems, such as linear complementarity.

For positively homogeneous mappings, a "norm" formula related to the one in Theorem 2.3 is available for calculating the modulus of strong subregularity even in infinite dimensions. Recall that a mapping $F : X \rightrightarrows Y$ is *positively homogeneous* when $0 \in F(0)$ and $F(\lambda x) \supset \lambda F(x)$ for $\lambda > 0$, or equivalently, when $\operatorname{gph} F$ is a cone in $X \times Y$.

Theorem 5.4 [11, Proposition 2.5] (positively homogeneous mappings). *For a mapping $F : X \rightrightarrows Y$ that is positively homogeneous, one has*

$$\operatorname{subreg} F(0\,|\,0) = \|F^{-1}\|^{+} \;=\; \sup_{\|x\|=1} \frac{1}{d(0, F(x))}.$$

The connection between Theorems 5.4 and 5.3 is seen through the fact that, when F is positively homogeneous with closed graph, we simply have $DF(0\,|\,0) = F$. Theorem 5.3 itself fails for general F in infinite dimensions, because $DF(\bar{x}\,|\,\bar{y})$ does not then yield an "adequately uniform" local approximation to F around $(\bar{x}, \bar{y})$.

The class of positively homogeneous mappings includes linear mappings, of course, and for those we get a special conclusion about injectivity.

Example 5.5 (strong subregularity of linear mappings). *For a mapping $F \in L(X, Y)$, strong subregularity of F at $\bar{x}$ for $\bar{y} = F(\bar{x})$ implies that F is injective, i.e., $\ker F = \{0\}$, and is equivalent to injectivity when $X < \infty$ and $Y < \infty$.*

We look next at perturbations of F by a single-valued mapping G in the pattern that was followed for the other regularity properties. In line with the developments above, we introduce the *calmness* modulus

$$\operatorname{clm} G(\bar{x}) = \inf\{\, \lambda \mid \|G(x) - G(\bar{x})\| \leq \lambda \|x - \bar{x}\| \text{ for } x \text{ near } \bar{x} \,\}, \tag{5.3}$$

noting that G is calm at $\bar{x}$ for $\bar{y} = G(\bar{x})$ if and only if $\operatorname{clm} G(\bar{x}) < \infty$.

Theorem 5.6 (perturbation estimate for strong subregularity). *Consider a mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. If $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) < \kappa < \infty$ and if $G : X \to Y$ is a mapping such that $G(\bar{x}) = 0$ and $\operatorname{clm} G(\bar{x}) < \lambda < \kappa^{-1}$, then*

$$\operatorname{subreg}(F + G)(\bar{x}\,|\,\bar{y}) \;<\; (\kappa^{-1} - \lambda)^{-1} \;=\; \frac{\kappa}{1 - \lambda\kappa}.$$

Proof. Because $\operatorname{clm} G(\bar{x}) < \lambda$ and $G(\bar{x}) = 0$, there exists $a > 0$ such that

$$\|G(x)\| \leq \lambda \|x - \bar{x}\| \;\text{ when }\; x \in I\!B_a(\bar{x}). \tag{5.4}$$

On the other hand, since $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) < \kappa$, we can arrange, by taking a smaller if necessary, that

$$\|x - \bar{x}\| \leq \kappa \|y - \bar{y}\| \;\text{ when }\; (x, y) \in \operatorname{gph} F \cap (I\!B_a(\bar{x}) \times I\!B_a(\bar{y})). \tag{5.5}$$

Let $\lambda' = \max\{1, \lambda\}$ and consider any

$$z \in I\!B_{a/2}(\bar{y}), \qquad x \in (F+G)^{-1}(z) \cap I\!B_{a/2\lambda'}(\bar{x}).$$

These relations entail $z \in F(x) + G(x)$, hence $z = y + G(x)$ for some $y \in F(x)$. From (5.5) and $x \in I\!B_{a/2\lambda'}(\bar{x})$, we have $\|G(x)\| \leq a/2\lambda' \leq a/2$ (inasmuch as $\lambda' \geq 1$). Using the fact that $y - \bar{y} = z + G(x) - \bar{y}$, we get $\|y - \bar{y}\| \leq \|z - \bar{y}\| + \|G(x)\| \leq a/2 + a/2 = a$. However, because $z - G(x) \in F(x)$ we also have $x \in F^{-1}(z - g) \cap I\!B_a(\bar{x})$ and therefore from (5.5),

$$\|x - \bar{x}\| \leq \kappa \|z - \bar{y}\| \leq \kappa \|z - \bar{y}\| + \kappa \|G(x)\| \leq \kappa \|z - \bar{y}\| + \kappa\lambda \|x - \bar{x}\|,$$

hence $\|x - \bar{x}\| \leq \kappa/(1 - \lambda\kappa)\|z - \bar{y}\|$, as required. □

A corollary analogous to Corollary 2.4 can immediately be derived from Theorem 5.6:

Corollary 5.7. *Consider a mapping* $F : X \rightrightarrows Y$ *and any* $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ *with* $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) < \infty$. *If* $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) > 0$, *then for any* $G : X \to Y$ *such that* $G(\bar{x}) = 0$ *and* $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) \cdot \operatorname{clm} G(\bar{x}) < 1$, *one has*

$$\operatorname{subreg}(F+G)(\bar{x}\,|\,\bar{y}) \;\leq\; \big(\operatorname{subreg} F(\bar{x}\,|\,\bar{y})^{-1} - \operatorname{clm} G(\bar{x})\big)^{-1}.$$

If, in addition, $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) \cdot \operatorname{clm} G(\bar{x}) < 1/2$, *then*

$$\big(\operatorname{subreg} F(\bar{x}\,|\,\bar{y})^{-1} + \operatorname{clm} G(\bar{x})\big)^{-1} \;\leq\; \operatorname{subreg}(F+G)(\bar{x}\,|\,\bar{y}).$$

If $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = 0$, *then* $\operatorname{subreg}(F+G)(\bar{x}\,|\,\bar{y}) = 0$ *for any* $G : X \to Y$ *with* $\operatorname{clm} G(\bar{x}) < \infty$.

This result implies that $\operatorname{subreg}(F+G) = \operatorname{subreg} F$ for any mapping G with $\operatorname{clm} G = 0$. The latter fact can be posed in parallel to Corollary 2.5. For that purpose, recall that a mapping F_0 is called a (plain, not strict) *first-order approximation* to F at $\bar{x}$ if

$$F = F_0 + G \text{ with } G : X \to Y \text{ satisfying } G(\bar{x}) = 0 \text{ and } G(x) = o(\|x - \bar{x}\|). \tag{5.6}$$

The conditions on G correspond in the present context to having $\operatorname{clm} G(\bar{x}) = 0$ and can be compared in this way to the prescription in (2.3) for a *strict* first-order approximation, where the requirement is that $\operatorname{lip} G(\bar{x}) = 0$.

Corollary 5.8 (stability under first-order approximations)**.** *If a mapping* $F_0 : X \rightrightarrows Y$ *is a first-order approximation to a mapping* $F : X \rightrightarrows Y$ *at* $\bar{x}$, *then* F *is strongly subregular at* $\bar{x}$ *for* $\bar{y}$ *if and only if* F_0 *is strongly subregular at* $\bar{x}$ *for* $\bar{y}$, *and indeed*

$$\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = \operatorname{subreg} F_0(\bar{x}\,|\,\bar{y}).$$

Corollary 5.9 [7] (stability under partial linearizations)**.** *Let* $F = f + M$ *for mappings* $f : X \to Y$ *and* $M : X \rightrightarrows Y$, *and let* $\bar{y} \in F(\bar{x})$. *Suppose* f *is differentiable at* $\bar{x}$, *and let* $F_0 = f_0 + M$ *for* $f_0(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x})$.

Then F^{-1} *has a graphical localization around* $(\bar{y}, \bar{x})$ *that is single-valued at* $\bar{y}$ *(with value* $\bar{x}$*) and calm there, if and only if* F_0^{-1} *has such a localization. Moreover the calmness modulus for the localizations in question will be the same in both cases.*

Proof. The mapping $F_0 = f_0 + M$ fits the definition of being a first-order approximation of F at $\bar{x}$, so Corollary 5.8 is applicable. □

Corollary 5.9 covers more than just partial linearizations (and full linearizations, where M in Corollary 5.9 is the zero mapping and $F = f$). The rule described is valid even when f lacks differentiability at $\bar{x}$, but $Df(\bar{x})$, interpreted now as the graphical derivative and not required to be an element of $L(X, Y)$, is single-valued with

$$f(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x}) + o(\|x - \bar{x}\|). \tag{5.7}$$

In the finite-dimensional case, this property has been analyzed in detail in [29, Chap. 7] under the heading of "semidifferentiability." It is closely related to what has been called "B-differentiability."

Corollary 5.10 (strong subregularity from polyhedral structure). *Let X and Y be finite-dimensional and $\bar{y} \in F(\bar{x})$, where $F = f + M$ for mappings $f : X \to Y$ and $M : X \to Y$ such that f is differentiable at $\bar{x}$ and $\operatorname{gph} M$ is the union of finitely many convex sets that are polyhedral. Let $F_0 = f_0 + M$, where f_0 is the linearization of f at $\bar{x}$ as in (5.7). Then F is strongly subregular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{x}$ is an isolated point of $F_0^{-1}(\bar{y})$.*

Proof. Because F_0 is a first-order approximation of F at $\bar{x}$, the desired strong subregularity of F is equivalent to that of F_0 by Corollary 5.9. But F_0, like M, is a mapping whose graph is the union of finitely many convex sets that are polyhedral. Through Theorem 5.2 and the result of Robinson in Example 3.3, therefore, the strong subregularity of F_0 at $\bar{x}$ for $\bar{y}$ is equivalent to $\bar{x}$ being an isolated point of $F_0^{-1}(\bar{y})$. □

An important case for Corollary 5.10 is the one in which M is the subgradient mapping associated with a piecewise linear-quadratic convex function; cf. [29]. The case within that where M is the normal cone mapping associated with a polyhedral convex set will be taken up in the next section.

Theorem 5.11 (radius theorem for strong metric subregularity). *For a mapping $F : X \rightrightarrows Y$ and any $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, one has*

$$\inf_{\substack{G:X\to Y\\ G(\bar{x})=0}} \left\{ \operatorname{clm} G(\bar{x}) \,\middle|\, F + G \text{ not strongly subregular at } \bar{x} \text{ for } \bar{y} \right\} = \frac{1}{\operatorname{subreg} F(\bar{x}\,|\,\bar{y})}. \tag{5.8}$$

Moreover, the infimum is not changed if taken with respect to affine mappings G of rank 1.

Proof. We start by dealing with some trivial cases. If F is not strongly subregular then the infumum in (5.8) is 0, whereas by definition $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = \infty$, and hence the equality does hold. We can therefore suppose that $\operatorname{subreg}(F)(\bar{y}\,|\,\bar{x}) < \infty$. For another special case, if there is a neighborhood O of $(\bar{x}, \bar{y})$ such that $(x, y) \in O \cap \operatorname{gph} F$ implies $x = \bar{x}$, then $\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = 0$ and this localization property will be inherited by $F + G$ for any G with $\operatorname{clm} G(\bar{x}) < \infty$. Thus, (5.8) will again be correct under the convention that $1/0 = \infty$. If, on the other hand, there exists a neighborhood O of $(\bar{x}, \bar{y})$ such that $(x, y) \in O \cap \operatorname{gph} F$ implies $y = \bar{y}$, then from strong subregularity we have $x = \bar{x}$, the previous case.

We can henceforth suppose that, for any neighborhood O of $(\bar{x}, \bar{y})$, the set $O \cap \operatorname{gph} F$ contains points (x, y) for which both $x \neq \bar{x}$ and $y \neq \bar{y}$. Then

$$0 < \operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = \limsup \left\{ \frac{\|x - \bar{x}\|}{\|y - \bar{y}\|} \,\middle|\, (x, y) \in \operatorname{gph} F,\ (x, y) \to (\bar{x}, \bar{y}),\ y \neq \bar{y} \right\}.$$

Take $r > 1/\operatorname{subreg} F(\bar{x}\,|\,\bar{y})$. Then there exists $(\hat{x}, \hat{y}) \in \operatorname{gph} F$ with $\hat{x} \neq \bar{x}$, $\hat{y} \neq \bar{y}$, such that

$$\frac{\|\hat{x} - \bar{x}\|}{\|\hat{y} - \bar{y}\|} > \frac{1}{r}\,. \tag{5.9}$$

Here $(\hat{x}, \hat{y})$ can be chosen arbitrarily close to $(\bar{x}, \bar{y})$. Define G to be the linear mapping of rank 1 given by

$$G(x) = \|\hat{x} - \bar{x}\|^{-1}\langle x, \hat{x}^*\rangle(\bar{y} - \hat{y}),$$

where $\hat{x}^*$ is an element of the dual space X^* such that $\langle \hat{x} - \bar{x}, \hat{x}^*\rangle = \|\hat{x} - \bar{x}\|$ and $\|\hat{x}^*\| = 1$. From (5.9), we have

$$\|G\| = \frac{\|\hat{y} - \bar{y}\|}{\|\hat{x} - \bar{x}\|} < r. \tag{5.10}$$

Further, since $G(\hat{x} - \bar{x}) = \bar{y} - \hat{y}$, we have

$$F(\hat{x}) + G(\hat{x}) - \bar{y} - G(\bar{x}) \;=\; F(\hat{x}) + \bar{y} - \hat{y} - \bar{y} = F(\hat{x}) - \hat{y} \ni 0,$$

so that $F(\hat{x}) + G(\hat{x}) \ni \bar{y} + G(\bar{x})$. Because $\bar{y} \in F(\bar{x})$, this says $\bar{x} \in (F + G)^{-1}(\bar{y} + G(\bar{x}))$. Since $\hat{x}$ can be arbitrarily close to $\bar{x}$, this means that $\bar{x}$ is not an isolated point of $(F + G)^{-1}(\bar{y})$; hence $F + G$ is not strongly subregular at $\bar{x}$ for $\bar{y} + G(\bar{x})$.

But here we can take r arbitrarily close to $1/\operatorname{subreg} F(\bar{y}\,|\,\bar{x})$, and therefore, through (5.10), we are able to conclude that

$$\text{infimum in (5.8)} \;\leq 1/\operatorname{subreg} F(\bar{x}\,|\,\bar{y}).$$

To verify that equality actually holds in this relation, we can apply Theorem 5.6. Indeed, from that result the infimum in (5.8) cannot be less than $1/\operatorname{subreg} F(\bar{x}\,|\,\bar{y})$. □

In view of the final assertion in Theorem 5.11 and the fact that $\operatorname{clm} G(\bar{x}) = \operatorname{lip} G(\bar{x})$ when G is locally Lipschitz continuous around $\bar{x}$, the radius formula (5.8) can also be written validly as

$$\inf_{\substack{G: X \to Y \\ G(\bar{x}) = 0}} \Big\{ \operatorname{lip} G(\bar{x}) \,\Big|\, F + G \text{ not strongly subregular at } \bar{x} \text{ for } \bar{y} \Big\} \;=\; \frac{1}{\operatorname{subreg} F(\bar{x}\,|\,\bar{y})}. \tag{5.11}$$

The parallel with the results in Theorems 2.7 and 4.6 for metric regularity and strong metric regularity comes out then especially clearly and serves to emphasize all the more the surprising instability of metric subregularity itself, for which no such statement is possible.

Theorem 5.11 can be supplemented by a result tailored to positively homogeneous mappings. In [11, Section 2], positively homogeneous mappings with $\|F^{-1}\|^+ < \infty$ were called *nonsingular*. Nonsingularity of the inverse F^{-1} implies strong subregularity of F at 0 for 0. In finite dimensions the converse statement holds as well.

Theorem 5.12 [11, Theorem 2.6] (radius of nonsingularity). *For any mapping $F : X \rightrightarrows Y$ that is positively homogeneous, one has*

$$\inf_{G \in L(X,Y)} \Big\{ \|G\| \,\Big|\, F + G \textit{ singular} \Big\} \;=\; 1/\operatorname{subreg} F(0\,|\,0) \;=\; 1/\|F^{-1}\|^+ \;=\; \inf_{\|x\| = 1} d(0, F(x)).$$

6 Applications to Variational Inequalities

The roots of the strong regularity property lie in efforts to extend the classical inverse and implicit function theorems in ways demanded by the fact that those theorems, while of great significance for equations, say nothing directly about "generalized equations" such as those coming from "variational inequalities," which are known to be crucial in the expression of optimality conditions, equilibrium conditions, and many other applications as well. We take up that topic now in connection not only with the partial linearization result in Corollary 4.5 for strong regularity, but also with the corresponding result in Corollary 5.9 for strong subregularity.

The generalized equations on which we focus for this purpose have the particular form

$$f(x) + N_C(x) \ni y \quad \text{with } x \in X,\ y \in Y = X^* \text{ (dual space)} \tag{6.1}$$

in the case of a mapping $f : X \to X^*$, a nonempty, closed, convex set $C \subset X$ and its normal cone mapping $N_C : X \rightrightarrows X^*$ in the sense of convex analysis, given by

$$N_C(x) = \begin{cases} \{\, y \mid \langle x' - x, y \rangle \le 0,\ \forall\, x' \in C \} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases} \tag{6.2}$$

According to the definition of N_C, to say that x satisfies (6.1) is to say that

$$x \in C \ \text{ and } \ \langle f(x) - y,\, x' - x \rangle \ge 0 \ \text{ for all } \ x' \in C.$$

This relation is called the *variational inequality* for f and C with parameter y. (Other kinds of variational inequalities, with f replaced by a set-valued mapping and N_C by a more general subgradient mapping of convex analysis could be considered as well, but this is the type to which we limit our attention here.)

When $C = X$, the normal cone mapping N_C reduces to the zero mapping, so the task of solving the variational inequality reduces to that of solving the equation $f(x) = y$ for x in terms of y. The result obtained in that context by taking $M = 0$ in Corollary 4.5 can thereby be viewed as a version of the classical inverse function theorem which merely demands strict differentiability of f at $\bar{x}$, instead of smoothness (continuous differentiability) around $\bar{x}$, but in compensation relinquishes local smoothness of the solution mapping in favor of Lipschitz continuity around $\bar{y}$.

In this special case of equation solving, the weaker conclusion of Lipschitz continuity may not seem worth the advantage of the weaker assumption, and indeed applications may typically have f smooth anyway. The key fact, however, is that in going beyond equations to variational inequalities, as Corollary 4.5 does, there is little hope of concluding smoothness of the solution mapping, regardless of any assumption on f. Smoothness is lost because of the nature of N_C, but Lipschitz continuity can be retained, at least.

Historically, attempts to extend the inverse and implicit function theorems to generalized equations of variational inequality type as in (6.1) have been a prime motivation for regularity studies. In his seminal paper [26], Robinson proved that, for mappings $F = f + N_C$ with f smooth, if the "linearized" variational inequality at $\bar{x}$, with f replaced by the linearization f_0 in (4.5), has a solution mapping with a single-valued Lipschitz continuous localization at $(\bar{y}, \bar{x})$, then the same holds for the original variational inequality. In fact, he proved a broader result—an extension of the implicit function theorem in which f depends on a parameter element p—which has this as an immediate consequence. Corollary 4.5 covers Robinson's inverse function result.

In Robinson's original terminology, "strong regularity" of the variational inequality for f and C at the elements in question referred to the single-valued Lipschitz continuous localization property of the solution mapping to the *linearized* variational inequality. In view of the equivalences in Theorem 4.2 and Corollary 4.5, this property comes down to the same thing as strong metric regularity in Definition 4.1.

We now have, through the equivalences in Theorem 5.2 and Corollary 5.8, something entirely analogous for "strong subregularity" of the variational inequality for f and C. This property too can be confirmed merely by verifying that it holds with respect to a linearization of f, and the pattern of the classical inverse function theorem is thereby propagated one level further. To summarize, we combine these observations into the statement of a theorem in which, for simplicity, we take f to be smooth.

Theorem 6.1 (linearization rules for variational inequalities)**.** *In the variational inequality setting where $F = f + N_C$ with f smooth, let $\bar{x}$ be a solution for $\bar{y}$ and let $F_0 = f_0 + N_C$ for the linearization $f_0(x) = f(\bar{x}) + Df(\bar{x})(x - \bar{x})$. Then*

(a) *F is strongly regular at $\bar{x}$ for $\bar{y}$ if and only if F_0 is strongly regular at $\bar{x}$ for $\bar{y}$, where moreover one has*

$$\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \operatorname{reg} F_0(\bar{x}\,|\,\bar{y}).$$

(b) *F is strongly subregular at $\bar{x}$ for $\bar{y}$ if and only if F_0 is strongly subregular at $\bar{x}$ for $\bar{y}$, where moreover one has*

$$\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = \operatorname{subreg} F_0(\bar{x}\,|\,\bar{y}).$$

The perturbation radius formulas for strong regularity in Theorem 4.6 and for strong subregularity in Theorem 5.11 can be put together in this context with the insights in Theorem 6.1 so as to express the radius in terms of the linearized variational inequality.

Corollary 6.2 (radius formulas for variational inequalities)**.** *In the framework and the assumptions of Theorem 6.1, one has that*

(a) *if F_0 is strongly regular at $\bar{x}$ for $\bar{y}$, then*

$$\inf_{\substack{G:X\to Y\\ G(\bar{x})=0}} \left\{ \operatorname{lip} G(\bar{x}) \,\middle|\, F + G \text{ not strongly regular at } \bar{x} \text{ for } \bar{y} \right\} = \frac{1}{\operatorname{reg} F_0(\bar{x}\,|\,\bar{y})}, \tag{6.3}$$

(b) *if F_0 is strongly subregular at $\bar{x}$ for $\bar{y}$, then*

$$\inf_{\substack{G:X\to Y\\ G(\bar{x})=0}} \left\{ \operatorname{clm} G(\bar{x}) \,\middle|\, F + G \text{ not strongly subregular at } \bar{x} \text{ for } \bar{y} \right\} = \frac{1}{\operatorname{subreg} F_0(\bar{x}\,|\,\bar{y})}. \tag{6.4}$$

In both cases, the formulas would be unchanged if G were restricted to being affine, in which case $\operatorname{lip} G(\bar{x})$ and $\operatorname{clm} G(\bar{x})$ both reduce to the norm of the linear part of G.

Note that in these results in which $F = f + N_C$, the perturbation of F to $F + G$ amounts to a perturbation of f to $f + G$, since the normal cone mapping N_C stays the same.

Especially of interest for many applications is the case where $X = Y = I\!R^n$ and the convex set C is polyhedral. The two results recalled next can then come into play.

Theorem 6.3 [9] (strong regularity for polyhedral variational inequalities)**.** *For* $F = f + N_C$ *in the case of smooth* f *and polyhedral* C*, metric regularity of* F *at* $\bar{x}$ *for* $\bar{y}$ *is equivalent to strong regularity of* F *at* $\bar{x}$ *for* $\bar{y}$*. Thus,* F *is strongly regular at* $\bar{x}$ *for* $\bar{y}$ *if and only if* $\operatorname{reg} F(\bar{x}, \bar{y}) < \infty$.

Proof. Specifically, it was proved in [9] that, in the smooth and polyhedral case, metric regularity of F at $\bar{x}$ for $\bar{y}$ automatically implies localized single-valuedness of F^{-1} around $\bar{y}$, and that of course comes out as strong metric regularity. ☐

In [11] a formula for computing $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ in the circumstances of Theorem 6.3 was developed in terms of the coderivative of $D^*F(\bar{x}\,|\,\bar{y})$, utilizing the fact in Theorem 2.10. Strong regularity can in principle be confirmed, therefore, by checking whether this formula implies $\operatorname{reg} F(\bar{x}, \bar{y}) < \infty$.

We can also obtain a more detailed picture of strong metric subregularity for $F = f + N_C$ when f is smooth and C is polyhedral. Let $A = Df(\bar{x})$ and $\bar{q} = \bar{y} - f(\bar{x}) - A\bar{x}$. Then, from Theorem 6.1 and by a change of variables,

$$\operatorname{subreg}(f + N_C)(\bar{x}\,|\,\bar{y}) = \operatorname{subreg}(A + N_C)(\bar{x}\,|\,\bar{q}).$$

A further simplification of the formula for the subregularity modulus can be obtained by employing the so-called *critical cone*, defined as

$$\bar{K} = \{\, u \in T_C(\bar{x}) \,|\, u \perp \bar{q} - A\bar{x}\},$$

where $T_C(\bar{x})$ is the tangent cone to C at $\bar{x}$. By the Reduction Lemma in [9], there is a neighborhood O of the origin in $I\!R^n \times I\!R^n$ such that for $(x, v) \in O$ one has

$$\bar{q} - A\bar{x} + v \in N_C(\bar{x} + x) \quad \Longleftrightarrow \quad v \in N_{\bar{K}}(x). \tag{6.5}$$

Thus, strong subregularity of $A + N_C$ at $\bar{x}$ for $\bar{q}$ is equivalent to strong subregularity of $A + N_{\bar{K}}$ at 0 for 0. Summarizing, we can state the following result.

Theorem 6.4 (strong subregularity for polyhedral variational inequalities)**.** *For* $F = f + N_C$ *in the case of smooth* $f : I\!R^n \to I\!R^n$ *and polyhedral* $C \subset I\!R^n$*, strong subregularity of* F *at* $\bar{x}$ *for* $\bar{y}$ *is equivalent to the condition that the origin in* $I\!R^n$ *is an isolated point of* $(A + N_{\bar{K}})^{-1}(0)$*. Moreover,*

$$\operatorname{subreg} F(\bar{x}\,|\,\bar{y}) = \operatorname{subreg}(A + N_{\bar{K}})(0\,|\,0) = \sup_{\|x\|=1} \frac{1}{d(-Ax, N_{\bar{K}}(x))},$$

where

$$d(-Ax, N_{\bar{K}}(x)) = \begin{cases} \inf_{v \in \bar{K}^*, v \perp x} \|Ax + v\| & \textit{for } x \in \bar{K}, \\ \infty & \textit{otherwise.} \end{cases}$$

Proof. The first part of the claim specializes Corollary 5.10 to the case of $M = N_C$, while the second utilizes (6.5) and Theorem 5.4, inasmuch as the mapping $A + N_{\bar{K}}$ is positively homogeneous. The last equality follows from the equivalence relation that $v \in N_{\bar{K}}$ if and only if $x \in \bar{K}$, $v \in \bar{K}^*$ and $x \perp v$, where $\bar{K}^*$ is the cone that is polar to the critical cone $\bar{K}$. ☐

7 Final Remarks

The aim of this paper has been to study regularity properties of set-valued mappings from certain new perspective which addresses their "stability" and "conditioning." Our main observation has been that the regularity properties can be divided into two types. The mappings having a property of the first type form "open sets" in relation to other mappings, in the sense that the property in question is stable (persistent, robust) under small perturbations. In contrast, the set of mappings having a property of the second type contain "dense subsets" of mappings that do not possess this property, so that the property is prone to failure under arbitrarily small perturbation. It is quite remarkable that, for mappings of the first type, the "radius" of stability is, as a rule, equal to the reciprocal of the associated regularity modulus. Apparently it is not just a coincidence that, for the first class of properties there also is a rich calculus involving various kinds of derivatives, whereas no such general tools are available for the second class.

In a previous paper [11] we showed that metric regularity is of the first type. Here, we have seen that metric subregularity, a one-point alternative to the two-point property of metric regularity, is of the second type. To establish this, we gave a counterexample. This counterexample, however, does not fit with some of the common areas in which applications of subregularity have been contemplated, such as mappings associated with constraint systems. Whether such instability persists for such mappings, in relation to feasibility problems and their associated error bounds, is still an open question. The same is true for the other unstable properties, such as openness at a point and inner semicontinuity at a point, which were presented at the end of Section 3.

We have demonstrated that when the reference point of the domain space in the property of metric subregularity is an isolated solution, the resulting sharper property, which we have called strong metric subregularity, does indeed turn out to be stable in our sense. We have demonstrated further that when the localized absence of multivaluedness extends to a neighborhood, furnishing strong metric regularity, that property is stable as well; this is of particular importance in applications connected with optimization. The radius theorem we have presented for the strong regularity is quite general, but the class of perturbations in it is large and does not offer fine tuning to various special cases. For instance, this result cannot be used for evaluating the distance to nonregularity when the perturbations are restricted to subgradient mappings, such as might come from perturbing the objective function of an optimization problem. A resent result of Zolezzi [32] gives an estimate in that vein for a quadratic optimization problem. In a different direction, an extension of the radius theorem for metric regularity of sublinear mappings (Theorem 2.9 in [11]) for a specific class of perturbations is presented in the manuscript [24].

These days, moduli of regularity can be seen to have not only a role in stability and error bounds, but also in evaluations of convergence and complexity of algorithms. Finding efficient ways for estimating them is another area for future research.

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