An Inverse Function Theorem for Metrically Regular Mappings

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Abstract

We prove that if a mapping $F : X \rightrightarrows Y$, where X and Y are Banach spaces, is metrically regular at $\bar{x}$ for $\bar{y}$ and its inverse F^{-1} is convex and closed valued locally around $(\bar{x}, \bar{y})$, then for any function $G : X \to Y$ with $\operatorname{lip} G(\bar{x}) \cdot \operatorname{reg} F(\bar{x}\,|\,\bar{y})) < 1$, the mapping $(F + G)^{-1}$ has a continuous local selection $x(\cdot)$ around $(\bar{x}, \bar{y} + G(\bar{x}))$ which is also calm.

Key words: set-valued mapping, metric regularity, continuous selections, inverse/implicit function theorem.

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1 Introduction

The classical inverse function theorem stated for a function $f : X \to Y$, with X and Y Banach spaces, assumes that f is continuously differentiable in a neighborhood of a given reference point $\bar{x}$ and, most importantly, the Fréchet derivative $\nabla f(\bar{x})$ has a linear and bounded inverse; then the theorem claims that there exist neighborhoods U of $\bar{x}$ and V of $\bar{y} := f(\bar{x})$ such that the mapping

$$V \ni y \mapsto f^{-1}(y) \cap U \tag{1.1}$$

is single valued (a function defined on V) which is moreover continuously differentiable ($\mathcal{C}^1$) in V and whose derivative is the inverse of ∇f. It is perhaps less known that an inverse function type theorem may be obtained when the Jacobian $\nabla f(\bar{x})$ is merely surjective. Indeed, in this case the mapping (1.1), although in general set-valued, may have a local single-valued selection $x(\cdot)$, that is, a function $x(\cdot)$ exists with $x(y) \in f^{-1}(y) \cap U$ for all $y \in V$, which is continuously differentiable in V. In other words, a smooth inverse function exists but it is only a part of the inverse f^{-1} which may be set-valued. The precise result is as follows:

Theorem 1.1. *Let X and Y be Hilbert spaces and let $f : X \to Y$ be a function which is $\mathcal{C}^1$ around $\bar{x}$ and such that the derivative $B := \nabla f(\bar{x})$ is surjective. Then there exist a neighborhood V of $\bar{y} := f(\bar{x})$ and a $\mathcal{C}^1$ function $x : V \to X$ such that*

$$x(\bar{y}) = \bar{x} \qquad \text{and} \qquad f(x(y)) = y \quad \text{for every } y \in V,$$

and moreover $\nabla x(\bar{y}) = (B^*B)^{-1}B^*$.

Proof. In terms of the adjoint operator B^* consider the mapping

$$(x, u) \mapsto g(x, u) := \begin{pmatrix} x + B^*u \\ f(x) \end{pmatrix},$$

which satisfies $g(\bar{x}, 0) = (\bar{x}, \bar{y})$ and whose Jacobian is

$$J = \begin{pmatrix} I & B^* \\ B & 0 \end{pmatrix}.$$

It is well known that, in Hilbert spaces, when B is surjective than the operator J is invertible in the sense that J^{-1} is linear and bounded from $X \times Y$ into itself. Hence, by the classical inverse function theorem the mapping g^{-1}, when restricted to a neighborhood of the point $((\bar{x}, 0), (\bar{x}, \bar{y}))$ in its graph, is single-valued and continuously differentiable. In particular, for some neighborhoods U of $\bar{x}$ and V of $\bar{y}$, the function $x(y) = \xi(\bar{x}, y)$ satisfies $y = f(x(y)) \cap U$ for $y \in V$. It remains to observe that B^*B is invertible and, from the equation $B^*f(x(y)) = B^*y$, the derivative of $x(\cdot)$ with respect to y satisfies $B^*B\nabla x(\bar{y}) = B^*$. □

The implicit function theorem corresponding to Theorem 1.1 is easy to prove by using the standard passage from inverse to implicit function theorems. An interesting reading about the history and theory of various implicit function theorems is the recent book [6].

If X and Y are arbitrary Banach spaces, it is in our opinion quite unlikely that a result of the form of Theorem 1.1 holds; however, we do not know a counterexample. Still, in Banach spaces the surjectivity of the Jacobian implies the existence of a selection of (1.1) which may be not smooth but it is continuous and calm. Specifically, we have

Theorem 1.2. *Let X and Y be Banach spaces and let $f : X \to Y$ be a function which is strictly differentiable at $\bar{x}$ and such that the strict derivative $\nabla f(\bar{x})$ is surjective. Then there exist a neighborhood V of $\bar{y} := f(\bar{x})$, a continuous function $x : V \to X$ and a constant $\gamma > 0$ such that*

$$f(x(y)) = y \quad \text{and} \quad \|x(y) - \bar{x}\| \le \gamma \|y - \bar{y}\| \quad \text{for every } y \in V.$$

Theorem 2.1 was communicated to the author by H. Sussmann [9] who put it in the context of the Lyusternik theorem. A version of this theorem, without the estimate, appeared in [3].

In this paper we will obtain Theorem 1.2 as a corollary of the following more general result: Let a set-valued mapping $F : X \rightrightarrows Y$ be metrically regular at $\bar{x}$ for $\bar{y}$ and its inverse F^{-1} be convex and closed valued locally around $(\bar{x}, \bar{y})$. Then F^{-1} has a continuous local selection $x(\cdot)$ around $(\bar{x}, \bar{y})$ which is calm. Moreover, for any function $G : X \to Y$ with $\operatorname{lip} G(\bar{x}) \cdot \operatorname{reg} F(\bar{x}\,|\,\bar{y})) < 1$, the mapping $(F + G)^{-1}$ has a continuous local selection $x(\cdot)$ around $(\bar{x}, \bar{y} + G(\bar{x}))$ which is calm. Here $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$ is the modulus of strong regularity of F which is defined in further lines and $\operatorname{lip} G(\bar{x})$ is the Lipschitz modulus of G, see Section 3 for a definition.

The general paradigm behind our result is the same as in standard inverse/implicit mapping theorems: Suppose a mapping f can be represented as the sum $f = F + G$, where F is "nice" (metrically regular with locally convex and closed inverse) so that it has an "inverse" and G is "small" (small Lipschitz constant); then f has an "inverse" as well. In Theorem 1.2 F is the Jacobian mapping of the function f and G is the difference $f - F$, as in the classical inverse function theorem.

In the remaining part of this section we describe the notation and terminology we use, which is consistent with the book [8], and briefly discuss some related results. Throughout, unless stated otherwise, X and Y are real Banach spaces with norms $\|\cdot\|$ and closed unit balls $\mathbb{B}$; a ball centered at a with radius r is $\mathbb{B}_r(a)$. The distance from a point x to a set A is denoted by $d(x, A)$. The notation $F : X \rightrightarrows Y$ means that F is a *set-valued* mapping from X to the subsets of Y; if F is a function, that is, for each $x \in X$ the set of values $F(x)$ consists of no more than one element, then we write $F : X \to Y$. The *graph* of F is $\operatorname{gph} F = \{(x, y) \mid y \in F(x)\}$ and its *inverse* F^{-1} is defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$. A mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ has a *local selection around* $(\bar{x}, \bar{y})$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ and a function $s : U \to V$ such that $s(\bar{x}) = \bar{y}$ and $s(x) \in F(x) \cap V$ for all $x \in U$. A mapping F has a *continuous local selection* if it has a selection $s : U \to V$ which is continuous in U. We say that a function f is $\mathcal{C}^1$ when f is continuously differentiable (smooth). The (strict or Fréchet) derivative of f at $\bar{x}$ is denoted by $\nabla f(\bar{x})$. A function $f : X \to Y$ is *calm at* $\bar{x}$ when there exist a neighborhood V of $\bar{x}$ and a constant $\gamma > 0$ such that

$$\|f(x) - f(\bar{x})\| \le \gamma \|x - \bar{x}\| \quad \text{for every } x \in V. \tag{1.2}$$

The infimum of γ for which (1.2) holds is called *modulus of calmness* and is denoted by $\operatorname{clm} f(\bar{x})$.

A mapping $F : X \rightrightarrows Y$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ if there exists a constant $\kappa > 0$ such that

$$d(x, F^{-1}(y)) \le \kappa d(y, F(x)) \text{ for all } (x, y) \text{ close to } (\bar{x}, \bar{y}). \tag{1.3}$$

The infimum of κ for which (1.3) holds is the *modulus of metric regularity* which we denote by $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$.

The concept of metric regularity has its roots in the work of L. A. Lyusternik in the 30s and L. Graves in the 50s. It has been playing a central role in optimization for obtaining necessary conditions for extremum. In a very general setting, the metric regularity is "stable under linearization," in the line of the Lyusternik-Graves theorem and its various extensions. Further, it is "robust"; that is, if it holds for a mapping at certain point, it holds also in "neighborhoods" of the mapping and the point. As shown recently in [2], in a finite-dimensional setting, the distance from a given metrically regular mapping to the set of mappings that are not metrically regular, measured by the Lipschitz modulus of the perturbation, is equal to the reciprocal to the modulus of metric regularity. A discussion of various developments around this concept has recently been given by Ioffe [4], details are also available in [8].

In finite dimensions, more can be said about functions implicitly defined by metrically regular mappings. If $f : I\!\!R^n \to I\!\!R^n$ is continuously differentiable around $\bar{x}$, then the metric regularity of f at $\bar{x}$ simply means that the Jacobian $\nabla f(\bar{x})$ is a nonsingular matrix and then the localization of f^{-1} around the point $(f(\bar{x}), \bar{x})$ is single-valued and $\mathcal{C}^1$. The equivalence of the metric regularity with the *Lipschitz continuous* single-valued localization of f^{-1} is actually valid for more general set-valued mappings of the form $f + N_C$ where f is a smooth function and N_C is the normal cone mapping to a convex polyhedral set C. This inverse function theorem for variational inequalities was established in [1] together with a formula for the Lipschitz modulus of the localization. Here the theory of inverse function for metrically regular mappings merges with another fundamental result, due to S. Robinson [7], regarding the "stability under linearization" of the property of existence of a Lipschitz continuous single-valued localization. The existence of a single-valued and continuous localization of the inverse also holds for merely continuous functions $f : I\!\!R^n \to I\!\!R^n$ whose inverse has a continuous local selection; this follows from Brower's invariance of domain theorem, see the recent book [5] where the interested reader can also find implicit function theorems for nonsmooth mappings.

2 Aubin continuity and continuous local selections

It is well known that F is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if F^{-1} has the so-called *Aubin property* at $\bar{y}$ for $\bar{x}$: there exist $\kappa \in (0, \infty)$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F^{-1}(y') \cap U \subset F^{-1}(y) + \kappa \|y' - y\| I\!\!B \quad \text{for all } y, y' \in V; \tag{2.1}$$

moreover the constants κ in (1.3) and (2.1) agree, that is, the modulus $\operatorname{reg} F(\bar{x}, \bar{y})$ is also the infimum of all κ for which (2.1) holds.

The Aubin property (1.4) is a local property of a mapping around a point in its graph, which is preserved after a truncation of the mapping with a neighborhood of the reference point. However, such a truncation may not be lower semicontinuous, in general.

Counterexample. Consider the mapping A from $I\!\!R$ to $I\!\!R$ whose graph is the union of the graphs of the functions $x = y + 1/k$ for $k = \pm 1, \pm 2, \cdots$ and the function $x = y$. This mapping is Aubin continuous at zero for zero however, for any $\varepsilon > 0$ and $\delta > 0$, if we consider the restriction of the graph of A in the box $[-\delta, \delta] \times [-\varepsilon, \varepsilon]$, there will be points in this restriction with coordinates

(y,ε) with $y<\delta$ that cannot be approached by a sequence x_n so that $(y_n,x_n)\in \operatorname{gph} A$, $y_n\to y$ and $\delta>y_n>y$.

In the following lemma we show that if A is convex and closed valued locally around the reference point, then the mapping obtained by truncation of A with a ball centered at $\bar{x}$ with radius proportional to the distance to $\bar{y}$ is lower semicontinuous in a neighborhood of $\bar{y}$.

Lemma 2.1. *Consider a mapping $A : Y \rightrightarrows X$ and any $(\bar{y},\bar{x})\in \operatorname{gph} A$ and suppose that A is Aubin continuous at $\bar{y}$ for $\bar{x}$ with a constant κ. Let, for some $c>0$, the sets $A(y)\cap I\!B_c(\bar{x})$ be convex and closed for all $y\in I\!B_c(\bar{y})$. The for any $\alpha>\kappa$ there exists $\beta>0$ such that the mapping*

$$I\!B_\beta(\bar{y})\ni y\mapsto M_0(y):=\{x\in A(y)\mid \|x-\bar{x}\|\le\alpha\|y-\bar{y}\|\}$$

is nonempty, closed and convex valued, and lower semicontinuous.

Proof. Let $\kappa<\alpha$ and let $I\!B_a(\bar{x})$ and $I\!B_b(\bar{y})$ be the neighborhoods of $\bar{x}$ and $\bar{y}$, respectively, that are associated with the Aubin continuity of A (metric regularity of A^{-1}) with constant κ. Choose $\beta>0$ such that $\beta\le a/\kappa$ that $\max\{a,\beta\}\le c$. For such a β the mapping M_0 has nonempty closed convex values. It remains to show that M_0 is lower semicontinuous on $I\!B_\beta(\bar{y})$.

Let $(x,y)\in \operatorname{gph} M_0$ and $y_k\to y$, $y_k\in I\!B_\beta(\bar{y})$. First, let $y=\bar{y}$. Then $M_0(y)=\bar{x}$ and from the Aubin continuity of A there exists a sequence $x_k\in A(y_k)$ such that $\|x_k-\bar{x}\|\le\kappa\|y_k-\bar{y}\|$. Thus $x_k\in M_0(y_k)$, $x_k\to x$ as $k\to\infty$ and we are done in this case.

Now let $y\ne\bar{y}$. From the Aubin property of A there exists $\check{x}_k\in A(y_k)\cap I\!B_c(\bar{x})$ such that

$$\|\check{x}_k-\bar{x}\|\le\kappa\|y_k-\bar{y}\|$$

and also there exists $\tilde{x}_k\in A(y_k)\cap I\!B_c(\bar{x})$ such that

$$\|\tilde{x}_k-x\|\le\kappa\|y_k-y\|.$$

Let

$$\mu_k=\frac{\|y_k-y\|}{(\alpha-\kappa)\|y_k-\bar{y}\|+\|y_k-y\|}.$$

Then $\mu_k\to 0$ as $k\to\infty$ and hence for large k we have $0\le\mu_k<1$. Let $x_k=\mu_k\check{x}_k+(1-\mu_k)\tilde{x}_k$. Then $x_k\in A(y_k)$. We have

$$\begin{aligned}\|x_k-\bar{x}\| &\le \mu_k\|\check{x}_k-\bar{x}\|+(1-\mu_k)\|\tilde{x}_k-\bar{x}\|\\ &\le \mu_k\kappa\|y_k-\bar{y}\|+(1-\mu_k)(\|\tilde{x}_k-x\|+\|x-\bar{x}\|)\\ &\le \mu_k\kappa\|y_k-\bar{y}\|+(1-\mu_k)\kappa\|y_k-y\|+(1-\mu_k)\alpha\|y-\bar{y}\|\\ &\le \alpha\|y-\bar{y}\|-\mu_k(\alpha-\kappa)\|y_k-\bar{y}\|+(1-\mu_k)\kappa\|y_k-y\|\le\alpha\|y-\bar{y}\|,\end{aligned}$$

because of the choice of μ_k. Thus $x_k\in M_0(y_k)$ and since $x_k\to x$, the proof is complete. ◻

Lemma 2.1 allows us to apply the Michael selection theorem to the mapping M_0 obtaining, in terms of a metrically regular mapping F, the following theorem:

Theorem 2.2. *Consider a mapping $F : X\rightrightarrows Y$ and $(\bar{x},\bar{y})\in \operatorname{gph} F$ at which $\operatorname{reg} F(\bar{x}\,|\,\bar{y})<\infty$. Let, for some $c>0$, the sets $F^{-1}(y)\cap I\!B_c(\bar{x})$ be convex and closed for all $y\in I\!B_c(\bar{y})$. Then the mapping F^{-1} has a continuous local selection $x(\cdot)$ around $(\bar{x},\bar{y})$ which is calm at $\bar{y}$ with $\operatorname{clm} x(\bar{y})\le\operatorname{reg} F(\bar{x}\,|\,\bar{y})$.*

In the following section we will show that on the same assumptions, the conclusion of this theorem holds when F is perturbed by a function G with a sufficiently small Lipschitz constant.

3 The inverse mapping theorem

The metric regularity of a mapping F is preserved when F is perturbed by a function with a small Lipschitz constant. This property of the metric regularity allows one to pass from linear to nonlinear and back and is usually identified with the Lyusternik-Graves theorem. Here we refer to the following version of this result proved in [2].

Theorem 3.1. *Consider a mapping $F : X \rightrightarrows Y$ with $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ and let $\operatorname{gph} F$ has a closed intersection with a neighborhood of $(\bar{x}, \bar{y})$. Consider also a mapping $G : X \to Y$. If $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \kappa < \infty$ and $\operatorname{lip} G(\bar{x}) < \lambda < \kappa^{-1}$, then*

$$\operatorname{reg}(F+G)(\bar{x}\,|\,\bar{y} + G(\bar{x})) < (\kappa^{-1} - \lambda)^{-1}.$$

Recall that the Lipschitz modulus $\operatorname{lip} G(\bar{x})$ of a single-valued mapping G at a point $\bar{x}$ is defined as

$$\operatorname{lip} G(\bar{x}) := \limsup_{\substack{x, x' \to \bar{x} \\ x, x' \neq \bar{x}}} \frac{\|G(x') - G(x)\|}{\|x' - x\|}.$$

For a function $F : X \to Y$ which is strictly differentiable at $\bar{x}$, Theorem 3.1 implies

$$\operatorname{reg} F(\bar{x}\,|\,F(\bar{x})) = \operatorname{reg} \nabla F(\bar{x}).$$

Since $\nabla F(\bar{x})$ is linear and bounded, from the Banach open mapping theorem, F is metrically regular at $\bar{x}$ for $F(\bar{x})$ if and only if $\nabla F(\bar{x})$ is surjective.

The following theorem is the main result of this paper. We show that if a mapping F satisfies the condition of Theorem 2.2 in the previous section, and hence has a continuous local selection around the reference point, then for any function $G : X \to Y$ with $\operatorname{lip} G(\bar{x}) < \operatorname{reg} F(\bar{x}\,|\,\bar{y})$, the mapping $(F+G)^{-1}$ has a continuous and calm local selection around the reference point which is also calm. Note that we claim the existence of a continuous selection of a possibly nonconvex valued mapping. Also note that this result is an inverse function theorem for a mapping Q which can be represented as the sum $Q = F + G$ where F and G have corresponding properties. We prove this theorem using repeatedly the argument in the proof of Lemma 2.1 in a way which resembles the classical proofs of Lyusternik and Graves.

Theorem 3.2. *Consider a mapping $F : X \rightrightarrows Y$ which is metrically regular at $\bar{x}$ for $\bar{y}$. Let $\operatorname{gph} F$ have a closed intersection with a neighborhood of $(\bar{x}, \bar{y})$ and let for some $c > 0$ the mapping $I\!B_c(\bar{y}) \ni y \mapsto F^{-1} \cap I\!B_c(\bar{x})$ be convex valued. Let $G : X \to Y$ satisfy $\operatorname{lip} G(\bar{x}) \operatorname{reg} F(\bar{x}\,|\,\bar{y}) < 1$. Then the mapping $(G+F)^{-1}$ has a continuous local selection $x(\cdot)$ around $(\bar{y} + G(\bar{x}), \bar{x})$ which is calm at $\bar{y}$ with*

$$\operatorname{clm} x(\bar{y}) \leq \frac{2 \operatorname{reg} F(\bar{x}\,|\,\bar{y})}{1 - \operatorname{lip} G(\bar{x}) \operatorname{reg} F(\bar{x}\,|\,\bar{y})}. \tag{3.1}$$

Proof. Choose γ that is greater than the right hand side of (3.1) and let κ, α and λ be such that $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) < \kappa < \alpha < 1/\lambda$, $\lambda > \operatorname{lip} G(\bar{x})$ and $2\kappa/(1 - \alpha\lambda) \leq \gamma$. For simplicity, we assume that $G(\bar{x}) = 0$. Let $I\!B_a(\bar{x})$ and $I\!B_b(\bar{y})$ be the neighborhoods of $\bar{x}$ and $\bar{y}$, respectively, that are associated with the metric regularity of F at $\bar{x}$ for $\bar{y}$ with constant κ and G is Lipschitz continuous in $I\!B_a(\bar{x})$

with constant λ. Without loss of generality, let $\max\{a, b\} \le c$. Note that $F^{-1}(y) \neq \emptyset$ for any $y \in I\!\!B_b(\bar{y})$. From Theorem 2.2, if we choose b small enough, there exists a continuous function $z_0 : I\!\!B_b(\bar{x}) \to X$ such that

$$F(z_0(y)) \ni y \quad \text{ and } \quad \|z_0(y) - \bar{x}\| \le \kappa\|y - \bar{y}\|$$

for all $y \in I\!\!B_b(\bar{y})$. Choose a positive τ such that

$$\tau \le (1 - \alpha\lambda) \min\Big\{\frac{a}{2\kappa}, \frac{b}{1 + \kappa\lambda}\Big\}. \tag{3.2}$$

Consider the mapping

$$I\!\!B_\tau(\bar{y}) \ni y \mapsto M_1(y) := \{\, x \in F^{-1}(y - G(z_0(y))) \mid \|x - z_0(y)\| \le \kappa(1 + \kappa\lambda)\|y - \bar{y}\|\}.$$

The mapping M_1 is closed and convex valued and $(\bar{y}, \bar{x}) \in \operatorname{gph} M_1$. For $y \in I\!\!B_\tau(\bar{y})$ we have

$$\|y - G(z_0(y)) - \bar{y}\| \le \tau + \lambda\|z_0(y) - \bar{x}\| \le \tau + \lambda\kappa\tau \le b,$$

therefore the mapping M_1 is also nonempty valued. We will show that this mapping is lower semicontinuous in $I\!\!B_\tau(\bar{y})$.

Let $y \in I\!\!B_\tau(\bar{y})$ and $x \in M_1(y)$, and let $y_k \in I\!\!B_\tau(\bar{y})$, $y_k \to y$ as $k \to \infty$. First, assume that $y = \bar{y}$. Then $M_1(y) = \bar{x} = z_0(\bar{y}) = z_0(y) \in F^{-1}(\bar{y} - G(\bar{x}))$ and from the Aubin property of F^{-1} there exists $x_k \in F^{-1}(y_k - G(z_0(y_k)))$ such that

$$\|x_k - z_0(\bar{y})\| \le \kappa(\|y_k - \bar{y}\| + \|G(z_0(y_k)) - G(\bar{x})\|).$$

Then, using the calmness of z_0 we obtain

$$\|x_k - z_0(y)\| \le \kappa(\|y_k - \bar{y}\| + \lambda\|z_0(y_k) - \bar{x}\|) \le \kappa(1 + \kappa\lambda)\|y_k - \bar{y}\|.$$

Thus $x_k \in M_1(y_k)$ and $x_k \to \bar{x}$.

Now suppose that $y \neq \bar{y}$. Using the calmness of z_0 and (3.2) we have

$$\|y_k - G(z_0(y_k)) - \bar{y}\| \le \|y_k - \bar{y}\| + \|G(z_0(y_k)) - G(\bar{x})\| \le (1 + \kappa\lambda)\tau < b \tag{3.3}$$

and

$$\|z_0(y_k) - \bar{x}\| \le \kappa\tau \le a. \tag{3.4}$$

Since $z_0(y_k) \in F^{-1}(y_k - G(\bar{x}))) \cap I\!\!B_a(\bar{x})$, from the Aubin continuity of F^{-1} there exists $\check{x}_k \in F^{-1}(y_k - G(z_0(y_k)))$ such that

$$\|\check{x}_k - z_0(y_k)\| \le \kappa\|G(z_0(y_k)) - G(\bar{x})\| \le \kappa\lambda\|z_0(y_k) - \bar{x}\| \le \kappa^2\lambda\|y_k - \bar{y}\|. \tag{3.5}$$

Similarly, since $x \in F^{-1}(y - G(z_0(y))) \cap I\!\!B_a(\bar{x})$, from the estimations (3.3) and (3.4) with y_k replaced by y and from the Aubin continuity of F^{-1} there exists $\tilde{x}_k \in F^{-1}(y_k - G(z_0(y_k)))$ such that

$$\|\tilde{x}_k - x\| \le \kappa\|y_k - y\| + \kappa\lambda\|z_0(y_k) - z_0(y)\|, \tag{3.6}$$

thus $\tilde{x}_k \to x$ as $k \to \infty$. Remembering that $y \neq \bar{y}$, for k sufficiently large we have $\|y_k - \bar{y}\| > \|y - \bar{y}\|/2 > 0$. For these large k, denote

$$\varepsilon_k := \frac{\|\tilde{x}_k - x\| + \|z_0(y_k) - z_0(y)\|}{\kappa\|y_k - \bar{y}\| - \kappa^2\lambda\|y_k - y\|}.$$

From (3.6) and the continuity of z_0 we obtain that $\varepsilon_k \to 0$. Let β_k be a convergent to zero sequence such that $\beta_k \geq \varepsilon_k$. From the local convexity of the values of F^{-1}, the point

$$x_k = \beta_k \check{x}_k + (1 - \beta_k)\tilde{x}_k$$

is an element of $F^{-1}(y_k - G(z_0(y_k)))$ and also $x_k \to x$ as $k \to \infty$. Then (3.5), (3.6) and the choice of ε_k and β_k yield

$$\begin{aligned}
\|x_k - z_0(y_k)\| &\leq \beta_k\|\check{x}_k - z_0(y_k)\| + (1 - \beta_k)\|\tilde{x}_k - z_0(y_k)\| \\
&\leq \beta_k\kappa^2\lambda\|y_k - y\| + \beta_k\kappa^2\lambda\|y - \bar{y}\| \\
&\quad +(1 - \beta_k)(\|\tilde{x}_k - x\| + \|x - z_0(y)\| + \|z_0(y) - z_0(y_k)\|) \\
&\leq \beta_k\kappa^2\lambda\|y_k - y\| + \beta_k\kappa^2\lambda\|y - \bar{y}\| \\
&\quad +(1 - \beta_k)(\|\tilde{x}_k - x\| + \kappa(1 + \kappa\lambda)\|y - \bar{y}\| + \|z_0(y) - z_0(y_k)\|) \\
&\leq \kappa(1 + \kappa\lambda)\|y - \bar{y}\| - \beta_k\kappa\|y - \bar{y}\| + \beta_k\kappa^2\lambda\|y_k - y\| + \|\tilde{x}_k - x\| + \|z_0(y) - z_0(y_k)\| \\
&= \kappa(1 + \kappa\lambda)\|y - \bar{y}\| - (\beta_k - \varepsilon_k)(\kappa\|y_k - \bar{y}\| - \kappa^2\lambda\|y_k - y\|) \\
&\leq \kappa(1 + \kappa\lambda)\|y - \bar{y}\|
\end{aligned}$$

for k sufficiently large. We obtain that $x_k \in M_1(y_k)$ for such k. Hence M_1 is lower semicontinuous in $\mathbb{B}_\tau(\bar{y})$.

The mapping M_1 is nonempty, closed and convex valued and lower semicontinuous in $\mathbb{B}_\tau(\bar{y})$, hence it has a continuous selection $z_1(\cdot) : \mathbb{B}_\tau(\bar{y}) \to X$ which by definition satisfies

$$z_1(y) \in F^{-1}(y - G(z_0(y))) \quad \text{and} \quad \|z_1(y) - z_0(y)\| \leq \kappa(1 + \lambda\kappa)\|y - \bar{y}\|. \tag{3.7}$$

Hence for all $y \in \mathbb{B}_\tau(\bar{y})$,

$$\|z_1(y) - \bar{x}\| \leq \|z_1(y) - z_0(y)\| + \|z_0(y) - \bar{x}\| \leq \kappa(2 + \kappa\lambda)\|y - \bar{y}\|. \tag{3.8}$$

In particular, from (3.2)

$$\|z_1(y) - \bar{x}\| \leq a.$$

and also

$$\|y - G(z_1(y)) - \bar{y}\| \leq \|y - \bar{y}\| + \|G(z_1(y)) - G(\bar{x})\| \leq b. \tag{3.9}$$

Now consider the mapping

$$\mathbb{B}_\tau(\bar{y}) \ni y \mapsto M_2(y) := \{\, x \in F^{-1}(y - G(z_1(y))) \mid \|x - z_1(y)\| \leq \alpha\lambda\|z_1(y) - z_0(y)\|\}.$$

Again, we will apply the Michael selection theorem to M_2 after proving that it is lower semicontinuous in $\mathbb{B}_\beta(\bar{y})$. Of course, M_2 is nonempty, closed and convex valued and $(\bar{y}, \bar{x}) \in \operatorname{gph} M_2$.

Let $y \in \mathbb{B}_\tau(\bar{y})$ and $x \in M_2(y)$, and let $y_k \in \mathbb{B}_\tau(\bar{y})$, $y_k \to y$ as $k \to \infty$. If $z_0(y) = z_1(y)$ then $M_2(y) = \{z_1(y)\}$ and therefore $x = z_1(y)$, and since

$$z_1(y_k) \in F^{-1}(y_k - G(z_0(y_k))) \cap B_a(\bar{x}) \qquad \text{and} \qquad y_k - G(z_0(y_k)) \in \mathbb{B}_b(\bar{y})$$

the Aubin continuity of F^{-1} implies that there exists $x_k \in F^{-1}(y_k - G(z_1(y_k)))$ such that

$$\|x_k - z_1(y_k)\| \le \kappa \|G(z_1(y_k)) - G(z_0(y_k))\| \le \alpha\lambda \|z_1(y_k) - z_0(y_k)\|.$$

Hence $x_k \in M_2(y_k)$ and from the continuity of the functions z_0 and z_1 we obtain that $x_k \to z_1(y) = x$, thus M_2 is lower semicontinuous.

Now let $z_0(y) \ne z_1(y)$. The estimations (3.8) and (3.9) clearly hold for y replaced by y_k, and since $z_1(y_k) \in F^{-1}(y_k - G(z_0(y_k))) \cap \mathbb{B}_a(\bar{x})$, the Aubin continuity of F^{-1} implies the existence of $\check{x}_k \in F^{-1}(y_k - G(z_1(y_k)))$ such that

$$\|\check{x}_k - z_1(y_k)\| \le \kappa \|G(z_1(y_k)) - G(z_0(y_k))\| \le \kappa\lambda \|z_1(y_k) - z_0(y_k)\|. \tag{3.10}$$

Also, taking into account (3.8) and (3.9) and the inclusion $x \in F^{-1}(y - G(z_1(y))) \cap \mathbb{B}_a(\bar{x})$, the Aubin continuity of F^{-1} yields that there exists $\tilde{x}_k \in F^{-1}(y_k - G(z_1(y_k)))$ such that

$$\|\tilde{x}_k - x\| \le \kappa(\|y_k - y\| + \lambda \|z_1(y_k) - z_1(y)\|) \to 0 \quad \text{as} \quad k \to \infty. \tag{3.11}$$

Let β_k be an arbitrary sequence of positive numbers that is convergent to zero as $k \to \infty$ and let

$$\varepsilon_k := \frac{\beta_k \kappa\lambda \|z_1(y_k) - z_0(y_k)\| + (1 - \beta_k)(\|\tilde{x}_k - x\| + \|z_1(y_k) - z_1(y)\|)}{\lambda \|z_1(y_k) - z_0(y_k)\|}.$$

Note that $\|z_1(y_k) - z_0(y_k)\| \ge \|z_1(y) - z_0(y)\|/2 > 0$ for all large k and therefore $\varepsilon_k \to 0$ as $k \to \infty$. Let

$$x_k = \beta_k \check{x}_k + (1 - \beta_k)\tilde{x}_k.$$

Since F^{-1} is locally convex valued, we have $x_k \in F^{-1}(y_k - G(z_1(y_k)))$ and, since $\tilde{x}_k \to x$ and $\beta_k \to 0$, we obtain $x_k \to x$ as $k \to \infty$. From (3.10), (3.11) and the choice of ε_k and β_k we have

$$\begin{aligned}
\|x_k - z_1(y_k)\| &\le \beta_k \|\check{x}_k - z_1(y_k)\| + (1 - \beta_k)\|\tilde{x}_k - z_1(y_k)\| \\
&\le \beta_k \kappa\lambda \|z_1(y_k) - z_0(y_k)\| + (1 - \beta_k)(\|\tilde{x}_k - x\| + \|x - z_1(y)\| + \|z_1(y) - z_1(y_k)\|) \\
&\le \beta_k \kappa\lambda \|z_1(y_k) - z_0(y_k)\| \\
&\quad + (1 - \beta_k)(\|\tilde{x}_k - x\| + \kappa\lambda \|z_0(y) - z_1(y)\| + \|z_1(y) - z_1(y_k)\|) \\
&\le \alpha\lambda \|z_0(y) - z_1(y)\| - (\alpha - \kappa)\lambda \|z_0(y) - z_1(y)\| \\
&\quad + \beta_k \kappa\lambda \|z_1(y_k) - z_1(y)\| + (1 - \beta_k)(\|z_1(y_k) - z_1(y)\| + \|\tilde{x}_k - x\|) \\
&\le \alpha\lambda \|z_0(y) - z_1(y)\| - (\alpha - \kappa - \varepsilon_k)\lambda \|z_0(y) - z_1(y)\| \\
&\le \alpha\lambda \|z_0(y) - z_1(y)\|
\end{aligned}$$

for sufficiently large k. We obtain that $x_k \in M_2(y_k)$ for all large k and since $x_k \to x$, the mapping M_2 is lower semicontinuous in $\mathbb{B}_\tau(\bar{y})$. Hence it has a continuous selection $z_2(\cdot) : \mathbb{B}_\tau(\bar{y}) \to X$; by definition it satisfies

$$z_2(y) \in F^{-1}(y - G(z_1(y))) \quad \text{and} \quad \|z_2(y) - z_1(y)\| \le \alpha\lambda \|z_1(y) - z_0(y)\| \text{ for all } y \in \mathbb{B}_\tau(\bar{y}).$$

The induction step is analogous. Let z_0, z_1 and z_2 be as above and suppose we have also found functions $z_3, z_4, \cdots, z_n$, such that any z_j, $j = 3, 4, \cdots, n$, is a continuous local selection of the mapping

$$I\!B_\tau(\bar{y}) \ni y \mapsto M_j(y) := \{\, x \in F^{-1}(y - G(z_{j-1}(y))) \mid \|x - z_{j-1}(y)\| \leq \alpha\lambda \|z_{j-1}(y) - z_{j-2}(y)\|\}$$

Then for $y \in I\!B_\tau(\bar{y})$ we obtain

$$\|z_j(y) - z_{j-1}(y)\| \leq (\alpha\lambda)^{j-1}\|z_1(y) - z_0(y)\|$$

and therefore, using Theorem 2.2 and (3.7),

$$\|z_j(y) - \bar{x}\| \leq \sum_{i=1}^{j}(\alpha\lambda)^{i-1}\|z_1(y) - z_0(y)\| + \|z_0(y) - \bar{x}\| \leq \frac{2\kappa}{1-\alpha\lambda}\|y - \bar{y}\|.$$

Hence, from (3.2), for $j = 3, 4 \cdots, n$,

$$\|z_j(y) - \bar{x}\| \leq \frac{2\kappa\tau}{1-\alpha\lambda} \leq a \tag{3.12}$$

and

$$\|y - G(z_j(y)) - \bar{y}\| \leq \tau + \|z_j(y) - \bar{x}\| \leq \tau + \frac{2\kappa\lambda\tau}{1-\alpha\lambda} \leq b. \tag{3.13}$$

Consider the mapping

$$I\!B_\tau(\bar{y}) \ni y \mapsto M_{n+1}(y) := \{\, x \in F^{-1}(y - G(z_n(y))) \mid \|x - z_n(y)\| \leq \alpha\lambda\|z_n(y) - z_{n-1}(y)\|\}.$$

which is nonempty, closed and convex valued and $(\bar{y}, \bar{x})$ is in its graph. Let $y \in I\!B_\tau(\bar{y})$ and $x \in M_{n+1}(y)$, and let $y_k \in I\!B_\tau(\bar{y})$, $y_k \to y$ as $k \to \infty$. If $z_{n-1}(y) = z_n(y)$ then $M_{n+1}(y) = \{z_n(y)\}$ and hence $x = z_n(y)$, and from $z_n(y_k) \in F^{-1}(y_k - G(z_{n-1}(y_k))) \cap B_a(\bar{x})$ and $y_k - G(z_{n-1}(y_k)) \in I\!B_b(\bar{y})$, and using the Aubin property of F^{-1}, we obtain that there exists $x_k \in F^{-1}(y_k - G(z_n(y_k)))$ such that

$$\|x_k - z_n(y_k)\| \leq \kappa\|G(z_n(y_k)) - G(z_{n-1}(y_k))\| \leq \alpha\lambda\|z_n(y_k) - z_{n-1}(y_k)\|.$$

Therefore $x_k \in M_{n+1}(y_k)$, $x_k \to z_1(y) = x$ as $k \to \infty$, and hence M_2 is lower semicontinuous.

Let $z_n(y) \neq z_{n-1}(y)$. From (3.12) and (3.13) for $y = y_k$, since $z_n(y_k) \in F^{-1}(y_k - G(z_{n-1}(y_k))) \cap I\!B_a(\bar{x})$, the Aubin continuity of F^{-1} implies the existence of $\check{x}_k \in F^{-1}(y_k - G(z_n(y_k)))$ such that

$$\|\check{x}_k - z_1(y_k)\| \leq \kappa\|G(z_n(y_k)) - G(z_{n-1}(y_k))\| \leq \kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\|.$$

Similarly, the estimations (3.12) and (3.13) and the Aubin continuity of F^{-1} yield that, since $x \in F^{-1}(y - G(z_n(y))) \cap I\!B_a(\bar{x})$, there exists $\tilde{x}_k \in F^{-1}(y_k - G(z_n(y_k)))$ such that

$$\|\tilde{x}_k - x\| \leq \kappa(\|y_k - y\| + \lambda\|z_n(y_k) - z_n(y)\|) \leq \kappa(\|y_k - y\| + \|G(z_n(y_k)) - G(z_n(y))\|) \to 0 \quad \text{as} \quad k \to \infty.$$

Choose an arbitrary sequence $\beta_k \to 0$ which is convergent to zero sequence and let

$$\varepsilon_k := \frac{\beta_k\kappa\lambda\|z_n(y_k) - z_{n-1}(y_k)\| + (1-\beta_k)(\|\tilde{x}_k - x\| + \|z_n(y_k) - z_n(y)\|)}{\lambda\|z_n(y_k) - z_{n-1}(y_k)\|}.$$

Then $\varepsilon_k \to 0$ as $k \to \infty$. Taking

$$x_k = \beta_k \check{x}_k + (1-\beta_k)\tilde{x}_k,$$

we estimate the distance $\|x_k - z_n(y_k)\|$ in the same way as in the first step, by just replacing z_1 by z_n and z_0 by z_{n-1}; then we conclude that $x_k \in M_{n+1}(y_k)$ for all large k. Since $x_k \to x$ as $k \to \infty$, the mapping M_{n+1} is lower semicontinuous in $I\!B_\tau(\bar{y})$. Hence M_{n+1} has a continuous selection $z_{n+1}(\cdot) : I\!B_\tau(\bar{y}) \to X$ which then satisfies

$$z_{n+1}(y) \in F^{-1}(y - G(z_n(y))) \quad \text{and} \quad \|z_{n+1}(y) - z_n(y)\| \le \alpha\lambda \|z_n(y) - z_{n-1}(y)\|.$$

Putting all together we have

$$\|z_{n+1}(y) - z_n(y)\| \le (\alpha\lambda)^n \|z_1(y) - z_0(y)\| \tag{3.14}$$

and

$$\|z_{n+1}(y) - \bar{x}\| \le \sum_{i=1}^{n+1} (\alpha\lambda)^{i-1} \|z_1(y) - z_0(y)\| + \|z_0(y) - \bar{x}\| \le \frac{2\kappa}{1-\alpha\lambda}\|y - \bar{y}\|,$$

and the induction step is complete.

Thus, we obtain an infinite sequence of functions $z_0, z_1, \cdots, z_n, \cdots$ for which (3.7) and (3.14) yield

$$\sup_{y \in I\!B_\tau(\bar{y})} \|z_{n+1}(y) - z_n(y)\| \le (\alpha\lambda)^n \kappa(1 + \lambda\kappa)\tau,$$

therefore, since $\alpha\lambda < 1$, $\{z_n\}$ is a Cauchy sequence in the space of functions that are continuous on $I\!B_\tau(\bar{y})$ equipped with the supremum norm. Then this sequence has a limit $x(\cdot)$ which is a continuous function in $I\!B_\tau(\bar{y})$ and satisfies

$$x(y) \in F^{-1}(y - G(x(y))) \text{ and } \|x(y) - \bar{x}\| \le \frac{2\kappa}{1-\alpha\lambda}\|y - \bar{y}\| \le \gamma\|y - \bar{y}\|$$

for all $y \in I\!B_\tau(\bar{y})$. This completes the proof. □

Proof of Theorem 1.2. Apply Theorem 3.2 with $F(x) = \nabla f(\bar{x})(x - \bar{x})$ and $G(x) = f(x) - \nabla f(\bar{x})(x - \bar{x})$. Metric regularity of F is equivalent to the surjectivity of $\nabla f(\bar{x})$ and, since $\nabla f(\bar{x})$ is linear and continuous, the mapping F^{-1} is convex and closed valued. The mapping G has $\operatorname{lip} G(\bar{x}) = 0$ and finally $F + G = f$. □

In this case a formula for the modulus of metric regularity of the linear and bounded mapping $\nabla f(\bar{x})$ is available from [2], Example 1.1, and $\operatorname{lip} G(\bar{x}) = 0$; then the upper bound for the modulus of calmness has the form

$$\operatorname{clm} x(\bar{y}) \le 2 \sup\{d(0, \nabla f(\bar{x})^{-1}(y)) \mid y \in I\!B\}.$$

4 Applications

Theorems 3.2 can be also stated in a corresponding “implicit function” form as follows:

Theorem 4.1. *Let X, Y and Z be Banach spaces. Consider a mapping $F : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ which satisfies the conditions in Theorem 3.2. Consider also a mapping $G : X \times Z \to Y$ which is continuous in a neighborhood of $(\bar{x}, \bar{p})$ and with $\operatorname{lip}_x G(\bar{x}, \bar{p}) < \operatorname{reg} F(\bar{x}|\bar{y})^{-1}$ (here the Lipschitz modulus of $G(x, p)$ is with respect to x where* lim sup *is also with respect to $p \to \bar{p}$). Then there exists a neighborhood U of $\bar{x}$ and P of $\bar{p}$, a continuous function $x(\cdot) : P \to U$, and a constant γ such that*

$$\bar{y} \in G(x(p), p) + F(x(p)) \qquad \text{and} \qquad \|x(p) - \bar{x}\| = \gamma \|G(\bar{x}, p) - G(\bar{x}, \bar{p})\| \quad \text{for every} \quad p \in P.$$

Sketch of proof. The proof is parallel to the proof of Theorem 3.2. First we choose κ, α and λ such that $\operatorname{reg} F(\bar{x}|\bar{y}) < \kappa < \alpha < 1/\lambda$ and $\lambda > \operatorname{lip}_x G(\bar{x}, \bar{p})$ and neighborhoods of $\bar{x}$, $\bar{y}$ and $\bar{p}$ that are associated with the metric regularity of F at $\bar{x}$ for $\bar{y}$ with constant κ and G is Lipschitz continuous with respect to x with constant λ uniformly in p. For simplicity, let $G(\bar{x}, \bar{p}) = 0$ and $\bar{y} = 0$. By appropriately choosing a sufficiently small radius τ of a ball around $\bar{p}$, we construct an infinite sequence of continuous functions $z_j : I\!\!B_\tau(\bar{p}) \to X$, $j = 0, 1, \cdots$, that is uniformly in $I\!\!B_\tau(\bar{p})$ convergent to a function $x(\cdot)$ that satisfies the conclusion of the theorem. The first z_0 is obtained with the help of Theorem 2.2 and satisfies

$$z_0(p) \in F^{-1}(-G(\bar{x}, p)) \qquad \text{and} \qquad \|z_0(p) - \bar{x}\| \le \kappa \|G(\bar{x}, p)\|.$$

The function z_1 is a continuous selection of the mapping

$$I\!\!B_\tau(\bar{p}) \ni y \mapsto M_1(p) := \{\, x \in F^{-1}(-G(z_0(p), p)) \mid \|x - z_0(p)\| \le \kappa(1 + \lambda\kappa)\|G(\bar{x}, p)\| \}$$

while, analogously, z_j is a continuous selection of

$$I\!\!B_\tau(\bar{p}) \ni y \mapsto M_j(p) := \{\, x \in F^{-1}(-G(z_{j-1}(p), p)) \mid \|x - z_{j-1}(p)\| \le \alpha\lambda \|z_{j-1}(p) - z_{j-2}(p)\| \}.$$

Then for $p \in I\!\!B_\tau(\bar{p})$ we obtain

$$\|y$$

$$_k)\| \le \kappa \|G(z_n(y_k)) - G(z_{n-1}(y_k))\| \le \kappa\lambda \|z_n(y_k) - z_{n-1}(y_k)\|.$$

Similarly, the estimations (3.12) and (3.13) and the Aubin continuity of F^{-1} yield that, since $x \in F^{-1}(y - G(z_n(y))) \cap I\!\!B_a(\bar{x})$, there exists $\tilde{x}_k \in F^{-1}(y_k - G(z_n(y_k)))$ such that

$$\|\tilde{x}_k - x\| \le \kappa(\|y_k - y\| + \lambda \|z_n(y_k) - z_n(y)\|) \le \kappa(\|y_k - y\| + \|G(z_n(y_k)) - G(z_n(y))\|) \to 0 \quad \text{as} \quad k \to \infty.$$

Choose an arbitrary sequence $\beta_k \to 0$ which is convergent to zero sequence and let

$$\varepsilon_k := \frac{\beta_k \kappa\lambda \|z_n(y_k) - z_{n-1}(y_k)\| + (1 - \beta_k)(\|\tilde{x}_k - x\| + \|z_n(y_k) - z_n(y)\|)}{\lambda \|z_n(y_k) - z_{n-1}(y_k)\|}.$$

Then $\varepsilon_k \to 0$ as $k \to \infty$. Taking

$$x_k = \beta_k \check{x}_k + (1-\beta_k)\tilde{x}_k,$$

we estimate the distance $\|x_k - z_n(y_k)\|$ in the same way as in the first step, by just replacing z_1 by z_n and z_0 by z_{n-1}; then we conclude that $x_k \in M_{n+1}(y_k)$ for all large k. Since $x_k \to x$ as $k \to \infty$, the mapping M_{n+1} is lower semicontinuous in $I\!B_\tau(\bar{y})$. Hence M_{n+1} has a continuous selection $z_{n+1}(\cdot) : I\!B_\tau(\bar{y}) \to X$ which then satisfies

$$z_{n+1}(y) \in F^{-1}(y - G(z_n(y))) \qquad \text{and} \qquad \|z_{n+1}(y) - z_n(y)\| \le \alpha\lambda \|z_n(y) - z_{n-1}(y)\|.$$

Putting all together we have

$$\|z_{n+1}(y) - z_n(y)\| \le (\alpha\lambda)^n \|z_1(y) - z_0(y)\| \tag{3.14}$$

and

$$\|z_{n+1}(y) - \bar{x}\| \le \sum_{i=1}^{n+1} (\alpha\lambda)^{i-1} \|z_1(y) - z_0(y)\| + \|z_0(y) - \bar{x}\| \le \frac{2\kappa}{1-\alpha\lambda}\|y - \bar{y}\|,$$

and the induction step is complete.

Thus, we obtain an infinite sequence of functions $z_0, z_1, \cdots, z_n, \cdots$ for which (3.7) and (3.14) yield

$$\sup_{y \in B_\tau(\bar{y})} \|z_{n+1}(y) - z_n(y)\| \le (\alpha\lambda)^n \kappa(1 + \lambda\kappa)\tau,$$

therefore, since $\alpha\lambda < 1$, $\{z_n\}$ is a Cauchy sequence