A Converse Lyapunov Theorem for a Class of Dynamical Systems Which Undergo Switching

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Abstract

We investigate the stability of a system in which the dynamics at any instant in time will follow one of a fixed set of vector fields. We allow switching between members of the family of vector fields to be completely random. The focus of this paper is to prove a converse Lyapunov theorem for this class of systems.

1 Introduction

A wide variety of natural and engineering systems are inherently multi-modal in the sense that several dynamical models are required to describe their behavior with reasonable accuracy. For example, a mobile robot is programmed to respond to environmental stimuli, e.g. avoid obstacles, turn at a corner, etc.; and predominant dynamical aspects vary from mode to mode. In the aircraft industry gain scheduled controllers are still very popular. Even in the animal world insects are known to vary gaits to overcome obstacles or to avoid being

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captured by a predator. These examples have one key aspect: switching from one mode to another cannot be predicted a priori and is instead determined by environmental factors which are usually not even part of the mathematical model. Indeed, a mobile robot would ideally need to adapt to its environment and hence it is baseless to assume a predetermined switching pattern between its various modes.

Traditionally, multi-modal systems were handled by assuming that the system would remain in one mode for a long enough time for transient effects to dissipate sufficiently; this is the basis for most gain scheduling philosophies. However, modern computer controlled systems are beginning to test this assumption. They are capable of very fast servo rates, and, in principle, capable of switching between controller modes at a rate exceeding the mechanical time constants. Hence we are fast reaching the point where the transients due to modal switching needs much closer attention.

The human muscular control system provides a canonical example of a system with rapid switching. Most of the muscles of the body occur in opposing pairs and only one muscle of the pair can be actively used at any one instant. In a simple movement, such as the movement of the eye in a horizontal plane, the exterior rectus muscle controls the temporal motion and the interior rectus controls the nasal motion. To track an object that is oscillating, using only eye movements, the control switches between the internal and external rectus muscle. In more complicated motion there may be many muscle pairs involved that are switching between opposing muscles and even opposing muscle groups. One only need to watch a gymnast performing on the pommel horse to see rapid switching between muscle groups. It is reasonably clear in the pommel horse that lack of coordination during switching will result in an instability. Significant work remains to be done to analyze the stability of even the simplest aspects of the human muscular control system and we expect to find that the role of the switching pattern is very important.

The study of multi-mode systems capable of fast switching rates is indeed very challenging. Stochastic control provides a framework for modeling and analysis if some probabilistic data on switching patterns is available. Here, in this paper, we take a different path and ask the simplest possible question: If switching between modes is completely random, how can we carry out a stability analysis of the system? Basic tools to carry out such an analysis are a prerequisite to developing a successful theory of stabilization of randomly switching systems. From a heuristic viewpoint, stability is a necessary property to accompany solutions to all control paradigms, and, in this sense, we are taking the first step to develop a new control paradigm. It is our hope that in the future we will be able to answer questions such as stabilization, trajectory planning with closed loop stability, tracking, etc.

Switched and multi-mode systems have received a fair amount of attention from the control theory community. However, much of the work has focussed on finding an appropriate switching strategy in order to stabilize the system, see e.g [17], [21], [14], [16], etc. More generally variable structure control is a well-developed area of research and its aim is essentially to find a stabilizing switching strategy. On the other hand, the recent work of Branicky [5] is closely related to our work. He assumes that a class of switching paths have been enumerated and asks, "When is the system stable for all switching paths?" He has shown that if the system admits a variant of a time-varying Lyapunov function, then it is asymptotically stable. Our work aims at addressing the complimentary problem of the existence of Lyapunov functions. Another class of problems closely related to ours is that of stabilizing a family of linear systems in such a way that the closed loop systems admit a common quadratic Lyapunov function, see e.g. [2], [3], [4], [9], [15], [10], etc. The essence of this set of work is captured in [2] where it was shown that for the family,

$$\dot{x} \;=\; A_\gamma x \;+\; B_\lambda u;\; x \in \Re^n,\; \gamma \in \Gamma,\; \lambda \in \Lambda,$$

one can reduce the problem to that of finding a common quadratic Lyapunov function for a family of uncontrolled systems on a certain subspace of $\Re^n$. Our aim differs from those papers in that we ultimately would seek feedback control laws that could depend on γ and λ, but stabilize the whole family in a more general sense. Interestingly, our Example 3.1 in this paper illustrates a limitation of the stabilization problem of Barmish [2], etc., namely that asking for a quadratic Lyapunov function may be too restrictive. In general, one needs to ask for a Lyapunov function which may be of degree higher than two.

Since the term *switched system* seems to have been used predominantly to indicate that control laws are designed to switch between a family of prescribed laws, we opt to use a different terminology to describe our class. For historical reasons we will call the systems addressed here *polysystems*. In the early days of the development of the geometric theory of nonlinear control systems, it was routine to consider control inputs to be piecewise constant functions. Therefore, if one were to study controllability of a smooth nonlinear control system $\dot{x} = f(x,u)$, it is natural to consider the collection of vector fields $D = \{f(x,u) | \; u \in U\}$, where U is some admissible control set, and then study the end points of all curves obtained by concatenating integral curves of vector fields in D. Therefore, it can be thought of as a system with a multitude of modes, each corresponding to a vector field in D. Lobry [12] coined the term *dynamical polysystems* to describe such a system. We will use the compact term *polysystem* to denote the class of systems studied in this paper.

The outline of the paper is as follows. In Section 2 we give our basic definitions including those of a polysystem, stability, uniform asymptotic stability, exponential stability, etc. In Section 3 we consider the case when all vector fields in the polysystem are linear. We prove that uniform exponential stability is equivalent to the existence of a smooth positive definite function, which is a common Lyapunov function for each member of the polysystem. Surprisingly, this does not imply the existence of a common quadratic Lyapunov function. We illustrate this via an example. In Section 4 we consider the case of a general polysystem and show that exponential stability once again implies the existence of a common Lyapunov function for the member vector fields.

In this paper we will not address the general converse Lyapunov theorem, i.e., when the polysystem is uniformly asymptotically stable rather than exponentially stable. The primary reason is the technical complexity of the analysis which is quite comparable to what is found in Kurzweil's paper on converse Lyapunov theorems for continuous systems [11]. We will report the complete details elsewhere.

2 Polysystems and Stability

In this section we give basic definitions relevant to this paper.

Definition 2.1 *A **polysystem** on $\Re^n$ is a collection of smooth vector fields $D = \{X_\gamma | \gamma \in \Gamma\}$ on $\Re^n$, where Γ is some index set. A polysystem is a **finite polysystem** when Γ is finite.*

We remark here that classically the term *dynamical polysystem* was used. We use the abbreviated terminology mostly because we need to prepend adjectives such as linear and compact and the original terminology would make the text sound awkward.

The notion of an integral curve of a polysystem is due to C. Lobry [12]. In essence it is a concatenation of integral curves of member vector fields of a polysystem.

Definition 2.2 *An **integral curve** of a polysystem D is a piecewise smooth curve $\sigma : (a, b) \to \Re^n$ such that for all but finitely many $\tau \in (a, b)$ there exists $\epsilon > 0$ and $X \in D$ such that $\dot{\sigma}(t) = X(\sigma(t))$ for all $t \in (\tau - \epsilon, \tau + \epsilon)$. For $p \in \Re^n$ if $\sigma(t) = p$ for some t, σ is an **integral curve passing through** p.*

Definition 2.3 *Let $D = \{X_\gamma | \gamma \in \Gamma\}$ be a polysystem. A **switching path** is a piecewise constant function $s : [0, \infty) \to \Gamma$. We denote the set of all possible switching paths by S.*

A switching path is to be thought of as a recipe for switching between member vector fields of a polysystem. Associated with a switching path s are real numbers $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and indices $\gamma_0 < \gamma_1 < \cdots < \gamma_k < \cdots$ such that $s(t) = \gamma_k$ for all $t_k \leq t < t_{k+1}$, $k = 0, 1, \cdots$.

Definition 2.4 *Let $D = \{X_\gamma | \gamma \in \Gamma\}$ be a polysystem and let s be a switching path and let $\{t_k\}_{k=0,1,\cdots}$ and $\{\gamma_k\}_{k=0,1,\cdots}$ be such that $s(t) = \gamma_k$ for $t_k < t < t_{k+1}$. An **integral curve of** s is a curve $\sigma : [0, \infty) \to \Re^n$ such that $\dot{\sigma} = X_{\gamma_k}$ for $t_k \leq t < t_{k+1}$. If $\sigma(0) = p$, we say that σ is an integral curve of s originating from p.*

Notation 2.1 *Let s be a switching path. We denote by $(t,p) \mapsto \phi_t^s(p)$ the local flow defined by $\phi_t^s(p) = \sigma(t)$ where σ is the integral curve of s originating from p.*

It is clear that for each $p \in \Re^n$ there are infinitely many integral curves passing through p. Indeed the notion of dynamical polysystem was invented in the early nineteen seventies in order to address the controllability of nonlinear systems with piecewise constant inputs. In that context controllability is equivalent to the existence of an integral curve of the generated polysystem joining a given initial and a final state. In our context we are interested in the exact opposite of controllability; namely, that all integral curves converge to a fixed equilibrium point.

Definition 2.5 *A point $p \in \Re^n$ is an* ***equilibrium*** *of the polysystem D if $X(p) = 0$ for all $X \in D$.*

Definition 2.6 *An equilibrium point p of a polysystem D is* ***uniformly stable*** *if for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|\phi_t^s(q) - p\| < \epsilon$ for all $t \geq 0$, for all q such that $\|q - p\| < \delta$, and for all switching paths s. D is* ***uniformly bounded*** *if for all $\delta > 0$ there exists $L > 0$ such that $\|\phi_t^s(x)\| \leq L$ for all $\|x\| < \delta$, all $s \in S$ and all $t > 0$.*

Definition 2.7 *An equilibrium point p of a polysystem is* ***uniformly asymptotically stable*** *if it is stable, and if for all $\epsilon > 0$ and small enough $\delta > 0$ there exists $T > 0$ such that $\|\phi_t^s(q) - p\| < \epsilon$ for all q such that $\|q-p\| < \delta$, all $t > T$, and all switching paths s. Equilibrium p is* ***globally uniformly asymptotically stable*** *if the above hold for all $\delta > 0$ and D is uniformly bounded. An equilibrium p is* ***uniformly exponentially stable*** *if there are constants $\delta > 0$, $c > 0$ and $\lambda > 0$ such that $\|\phi_t^s(q) - p\| \leq c\, e^{-\lambda t}\|q-p\|$ for all q such that $\|q - p\| < \delta$, all s and all $t \geq 0$. An equilibrium p is* ***globally uniformly exponentially stable*** *if there are constants $c > 0$ and $\lambda > 0$ such that $\|\phi_t^s(q) - p\| \leq c\, e^{-\lambda t}\|q - p\|$ for all $q \in \Re^n$, all s and all $t \geq 0$.*

We are interested in characterizing uniform asymptotic stability and uniform exponential stability of polysystems via converse Lyapunov theorems. First, we observe that existence of a common Lyapunov function for all member fields is sufficient to guarantee uniform asymptotic stability of a polysystem if it contains either finitely many vector fields or the set of vector fields is compact in the C^1 compact open topology. This follows from the observation that the derivative of the common Lyapunov function is bounded above by a negative constant on any annulus of $\Re^n$ lying between two level sets. Our main result in this paper is that under the additional stronger hypothesis of uniform exponential stability the converse is true as well. Our converse Lyapunov theorem turns out to be true even without this additional assumption, however, the proof is very technical in nature and the technical features somewhat obscure the main ideas behind the proof. Therefore, we opted to present this version here, illustrating the main ideas, and to present the general version elsewhere.

We wish to point out that we purposefully avoided defining a notion where each switching path results in an asymptotically stable system, but the rate of decay of solutions may not be uniform with respect to the switching space. Clearly such a notion would be less demanding than ours, but it isn't clear whether it would be useful from an engineering viewpoint; for if the decay rate depends on the switching choices yet to be made, then the reliability of the overall system is questionable. In the case of linear systems we believe that if the polysystem contains finitely many vector fields (or is compact in a suitable sense) then the two notions may turn out to be equivalent. We haven't yet succeeded in proving this.

3 Linear Polysystems

In this section we consider a polysystem in which all member vector fields are linear. The main theorem proven here is that if the polysystem is compact (see below for the definition) and uniformly exponentially stable, then there is a common Lyapunov function for the family. Somewhat surprisingly this Lyapunov function may fail to be quadratic. Finding an upper bound on the degree of a candidate Lyapunov function remains unresolved.

Definition 3.1 *A polysystem D is a* ***linear polysystem*** *if all of its members fields are linear vector fields, i.e.* $D = \{A_\gamma x |\ A_\gamma \in \Re^{n\times n},\ \gamma \in \Gamma\}$.

Notation 3.1 *By abuse of notation we will denote members of a linear polysystem by coefficient matrices, i.e., we write $A \in D$ to mean that the vector field Ax is a member of the linear polysystem D.*

Definition 3.2 *A linear polysystem* $D = \{A_\gamma |\ \gamma \in \Gamma\}$ *is* ***compact*** *if* $\{A_\gamma |\ \gamma \in \Gamma\}$ *is a compact subset of* $\Re^{n\times n}$.

We will now state the main theorem of this section. Throughout the paper when we state that a certain function $V : \Re^n \to \Re$ is homogeneous of some degree p we mean that $V(ax) = a^p V(x)$ for all $a > 0$ and $x \in \Re^n$.

Theorem 3.1 *Let D be a compact linear polysystem. The following are equivalent.*

(i) The origin is a uniformly exponentially stable equilibrium.

(ii) The origin is a uniformly asymptotically stable equilibrium.

(iii) There exists a C^1 positive definite function $V : \Re^n \to \Re$, homogeneous of degree 2, such that $x \mapsto \nabla V(x)Ax$ is negative definite for all $A \in D$.

(iv) There exists a C^∞ positive definite function $V : \Re^n \to \Re$ such that $\nabla V(x)Ax$ is negative definite for all $A \in D$.

We prove this theorem via a series of lemmas. Of course it is easy to see that (i) and (ii) are equivalent, and that (iii) or (iv) implies (i) and (ii). To prove that (i) implies (iii) and (iv) the basic idea is to first construct a continuous positive definite function which will be strictly decreasing along all integral curves of the polysystem. This function will be shown to be Lipschitz continuous, and, therefore, amenable to approximation by a smooth function while preserving the negative definiteness of the derivative along member fields of the polysystem.

We will henceforth assume that D is a compact linear polysystem and that it is uniformly exponentially stable. We will first construct a positive definite continuous function which will be strictly decreasing along all integral curves. For single vector fields this construction was first proposed by Malkin [8].

Let us denote the set of switching paths by S (see Definition 2.3). For the rest of the section we fix $c > 0$ and $\lambda > 0$ such that $\|\phi_t^s(x_0)\| \leq c\, e^{-\lambda t}\|x_0\|$ for all $x_0 \in \Re^n$, $s \in S$ and all $t > 0$. Let us define $V : \Re^n \to \Re$ by,

$$V(x_0) = \sup_{s \in S} \int_0^\infty \|\phi_t^s(x_0)\|^2 dt. \tag{1}$$

Note that $t \to \phi_t^s(x_0)$ is continuous and $\|\phi_t^s(x_0\| \leq ce^{-\lambda t}\|x_0\|$ for all t. Note also that for each $s \in S$ the integral in (1) make sense and, it is bounded by $c^2/2\lambda$. Therefore, V is well defined.

Each switching path s corresponds to a linear time-varying system. Let us denote its state transition matrix by $\Phi^s(t, \tau)$. Let $Q^s = \int_0^\infty (\Phi^s(t, 0))^T \Phi^s(t, 0)dt$. Since $\|\Phi^s(t, 0)\| \leq ce^{-\lambda t}$ and $\Phi^s(t, 0)$ is nonsingular for all $t \geq 0$, Q^s exists and it is positive definite for all s.

Let $W = \{Q^s | s \in S\}$ and $K = cl(W)$ in $\Re^{n\times n}$. Using the bound $\|\phi^s(t, 0)\| \leq ce^{-\lambda t}$ we notice that $\|Q^s\| \leq c^2/(2\lambda)$ for all $s \in S$. Therefore, W is bounded as a subset of $\Re^{n\times n}$ and hence K is compact. Now $V(x) = \max\{x^T Qx| \; Q \in K\}$.

Lemma 3.1 *V is continuous.*

Proof: Since $V(ax) = a^2V(x)$ for all $a > 0$ it suffices to show that $V|_{S^{n-1}}$ is continuous, where S^{n-1} is the unit sphere in $\Re^n$. Let us consider the continuous map,

$$\psi : K \times S^{n-1} \to \Re,$$

$\psi(Q, x) = x^T Qx$. Now $V(x) = \max\{\psi(Q, x)| \; Q \in K\}$. Therefore, V is continuous. •

The next step is to prove the local Lipschitz continuity of V. This is done in an indirect way.

Lemma 3.2 *V is convex.*

Proof: Let x_0 and x_1 be in $\Re^n$ and let $x_\mu = \mu x_1 + (1-\mu)x_0$ for $\mu \in [0,1]$. Since positive definite quadratic forms are convex we have for each $Q \in W$, $x_\mu^T Q x_\mu \leq \mu x_1^T Q x_1 + (1-\mu)x_0^T Q x_0$. Therefore, by taking supremum over $Q \in W$ we obtain $V(x_\mu) \leq \mu V(x_1) + (1-\mu)V(x_0)$ for all $\mu \in [0,1]$. •

Lemma 3.3 *V is locally Lipschitz continuous.*

Proof: Continuous convex functions on $\Re^n$ are always locally Lipschitz continuous, see, e.g., [19]. •

Now we show that V is strictly decreasing along integral curves of the polysystem.

Lemma 3.4 *There exists $\delta > 0$ such that $V(x) - V(e^{tA}x) \geq (t/2)\|x\|^2$ for all $x \in \Re^n$, $A \in D$ and all $t \in [0,\delta]$.*

Proof: Let s be a switching path and $x \in \Re^n$. We may think of $t \to \phi_t^s(e^{tA}x)$ in two ways. First, it is the integral curve of s originating from $e^{tA}x_0$. Second, it is an integral curve originating from x corresponding to a switching path which starts in the direction of Ax and then follows s. This second interpretation immediately lead to:

$$V(x) \geq \int_0^t \|e^{\tau A}x\|^2 d\tau + \int_0^\infty \|\phi_\tau^s(e^{tA}x)\|^2 d\tau. \tag{2}$$

Since (2) is valid for all $s \in S$, by taking the supremum over s we obtain,

$$V(x) \geq \int_0^t \|e^{\tau A}x\|^2 d\tau + V(e^{tA}x).$$

Now by taking t small we have,

$$V(x) - V(e^{tA}x) \geq t/2\|x\|^2; \; 0 \leq t \leq \delta, \tag{3}$$

where δ is a positive constant independent of $x \in \Re^n$ and $A \in D$. $\bullet$

Lemma 3.4 almost looks like it is saying that $\nabla V(x)Ax < -\|x\|^2/2$, except that V need not be smooth. However, we have already shown that V is locally Lipschitz continuous. This allows us to approximate V by a smooth function without significantly perturbing the difference quotient. We will carry this analysis out below.

First, V is already smooth in the radial direction since $V(ax) = a^2V(x)$ for $a > 0$. So we only need to smooth it out in the spherical direction.

Let $SO(n)$ denote the group of $n \times n$ rotation matrices and let $f : SO(n) \to [0, \infty)$ be a smooth function such that its support is a small neighborhood of the identity matrix and such that $\int_{SO(n)} f(R)dR = 1$. We will specify the smallness of the support of f momentarily. Let us approximate V by the smooth function,

$$\tilde{V}(x) = \int_{SO(n)} V(Rx)f(R)dR.$$

From now on we will restrict x to be on the unit sphere in $\Re^n$. The primary purpose is to keep $\|x\|$ and $\|x\|^2$ equal in our estimates.

Since V is locally Lipschitz continuous we have for $A \in D$,

$$|V(e^{tA}x) - V(x + tAx)| \le t\|x\|^2/4, \text{ all } t \in [0, \delta] \quad \text{(we have possibly made } \delta \text{ small).}$$

Therefore, from (3), $V(x) - V(x + tAx) \ge t\|x\|^2/4, \; t \in [0, \delta]$.

Now we will create some freedom to perturb the vector fields. Since V is continuous we can find a neighborhood $\mathcal{U}$ of A in $\Re^{n\times n}$ such that, $V(x) - V(x + \delta\tilde{A}x) \ge \delta\|x\|^2/6$ for all $\tilde{A} \in \mathcal{U}$. By the convexity of V we have for $\mu \in [0, \delta]$,

$$\begin{aligned} V(x + (\mu/\delta)\delta\tilde{A}x) &\le (1 - \mu/\delta)V(x) + (\mu/\delta)V(x + \delta\tilde{A}x) \\ &\le (1 - \mu/\delta)V(x) + (\mu/\delta)[V(x) - \delta\|x\|^2/6] \\ &= V(x) - \mu\|x\|^2/6. \end{aligned}$$

Since D is a compact linear polysystem we can find common δ and $\mathcal{U}$ which work for all $A \in D$.

Now we do the first step of the smoothening process as follows. We pick our bump function f such that $R \in supp(f)$ implies that $RAR^T \in \mathcal{U}$ for all $A \in D$, and define $\tilde{V}(x) = \int_{SO(n)} V(Rx)f(R)dR$ for all $x \in \Re^n$.

Now we have,

$$\begin{aligned}
[\tilde{V}(x) - \tilde{V}(x+tAx)]/t &= \int_{SO(n)} ([V(Rx) - V(R(x+tAx)]/t)\ f(R)dR \\
&= \int_{SO(n)} ([V(Rx) - V(Rx + tRAR^T\ Rx)]/t)\ f(R)dR \\
&\geq \int_{SO(n)} \frac{1}{4}\|Rx\|^2 f(R)dR \\
&= \frac{1}{4}\|x\|^2, \quad \text{all } t \in [0,\delta].
\end{aligned}$$

Therefore, we have

$$\nabla\tilde{V}(x)Ax \ \leq \ -\frac{1}{4}\|x\|^2. \tag{4}$$

Earlier in the proof we had restricted x to be on the unit sphere. However, since V is homogeneous of degree 2, i.e. $V(ax) = a^2V(x)$ for all $a > 0$, it follows that $\tilde{V}$ is homogeneous of degree 2 as well, and (4) holds for all $x \in \Re^n$. Thus we have, $\nabla\tilde{V}(x)Ax \leq -\frac{1}{4}\|x\|^2$ for all $x \in \Re^n$ and $A \in D$.

The Lyapunov function $\tilde{V}$ we have found is C^∞ except at the origin where it is only C^1. Therefore, it satisfies (iv) of the theorem. However, it can be further smoothened if necessary to produce a C^∞ function by using the following lemma due to Kurzweil [11].

Lemma 3.5 *Let V be a positive definite function on $\Re^n$ which is continuous at 0 and C^∞ elsewhere. Then there exists a positive definite C^∞ function ψ on $\Re$ such that $d\psi(x)/dx > 0$ for all $x > 0$ and $\psi \circ V$ is a positive definite and C^∞ function on $\Re^n$.*

We refer the reader to [11] for a proof of the lemma.

Let ψ be as in the lemma. Then $\psi \circ \tilde{V}$ is a C^∞ function which is as in (iii) of Theorem 3.1. This concludes the proof of Theorem 3.1.

Knowing the existence of a Lyapunov function is useful in the design of feedback controllers for a variety of reasons: they provide a measure of robustness; they provide a way to construct stabilizing control laws using the theory of control Lyapunov functions, they play a central role in the theory of dissipative systems, etc. However, we frequently need additional information on the structure of the Lyapunov function in order to construct one explicitly. Thus it would be natural to ask whether the common Lyapunov function predicted in Theorem 3.1 could be found to be quadratic as in the case of standard linear systems. It turns out that quadratic Lyapunov functions may fail to exist in general. This can be seen via the following example.

3.1 Example: A uniformly exponential compact linear polysystem system which does not admit a common quadratic Lyapunov function

Let us consider the linear polysystem D where $D = \{A_1, A_2\}$ and

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} -1 & -a \\ 1/a & -1 \end{bmatrix},$$

where $a \in [1, \infty)$. We will show that the range of a for which the system is uniformly exponentially stable strictly contains the range of a for which both vector fields admit a common quadratic Lyapunov function.

Let us analyze the polysystem D. First observe that the phase portrait of both member fields consist of stable foci circling toward the origin in a counterclockwise direction. This makes it possible to carry out the entire stability analysis via phase plane techniques.

We will first show that the worst-case switching path corresponds to switching between members when A_1x and A_2x are parallel. Here, by worst-case, we mean if the system is asymptotically stable for this switching path, then the polysystem is uniformly exponentially stable as well. Using this it is straightforward to numerically compute the range of a for which the switching family is uniformly exponentially stable. This range turns out to *include* the interval $[1, 10]$. It is also possible to find the range of a for which the pair admit a common quadratic Lyapunov function of the form $x_1^2 + px_1x_2 + qx_2^2$; and this range is *contained* in the interval $[1, 6]$. Therefore, for $6 < a < 10$ the switching family is uniformly exponentially stable, yet it does not admit a common quadratic Lyapunov function! We will carry out this analysis below.

Let us first define l_- and l_+ as the lines on which A_1x and A_2x are parallel. Then on $l_\pm$ we have $(x_1 - x_2)/(x_1 + x_2) = (x_1/a - x_2)/(x_1 + ax_2)$. Solving this equation we obtain, $x_2 = \nu_\pm x_1$, where $\nu_\pm = \{(a+1) \pm \sqrt{a^2 + 6a + 1}\}/2a$. Let $l_\pm = \{(x_1, x_2) | \, x_2 = \nu_\pm x_1\}$. Let $\mathcal{R}_1 = \{(x_1, x_2) | \, \nu_- x_1 < x_2 < \nu_+ x_1\}$ and $\mathcal{R}_2 = \{(x_1, x_2) | \, x_2 \geq \nu_- x_1$ and $x_2 > \nu_+ x_1\}$. The lines $l_\pm$ and regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are depicted in Figure 1.

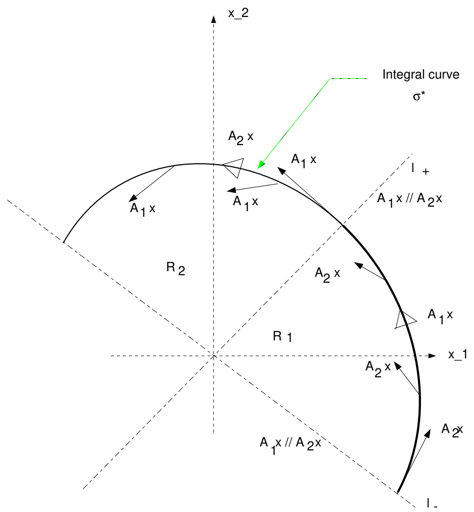

Figure 1: Worst Case Integral Curve

For the rest of this example let us define the switching path s^* to be such that the vector field governing the dynamics is A_1x on $\mathcal{R}_1 \cup -\mathcal{R}_1$ and A_2x on $\mathcal{R}_2 \cup -\mathcal{R}_2$. Let us denote an integral curve of s^* by σ^*.

First, we observe that A_2x points to the left of A_1x on $\mathcal{R}_1$ and to the right on $\mathcal{R}_2$. Therefore, intuition suggests that s^* is the worst case switching strategy. But this problem is subtle since the time parameter has not been properly accounted for in the pictorial representation in Figure 1. Indeed, it is likely that some other switching path corresponds to a slower decay rate. Therefore, we will carry out a careful analysis below that will verify our guess.

For proper comparison of all integral curves, let us first parameterize them by their polar coordinate θ, where θ is allowed to increase continually from some θ_0 to infinity. Let s be any switching path and σ its integral curve. We assume the initial state for σ and σ^* to be the same. It is clear from the foregoing discussion on the relative orientation of A_1x and A_2x on $\mathcal{R}_1$ and $\mathcal{R}_2$ that $\|\sigma(\theta)\| \le \|\sigma^*(\theta)\|$ for all θ. Furthermore, it is seen by a direct computation that on σ,

$$\frac{d\theta(t)}{dt} = \begin{cases} 1, & \text{when } \dot{\sigma}(t) = A_1\sigma(t), \\ 1/a \; cos^2\theta(t) + a\sin^2\theta(t), & \text{when } \dot{\sigma}(t) = A_2\sigma(t). \end{cases}$$

Therefore, $1/a \le d\theta(t)/dt \le a$ for all t and all switching paths. Therefore, for all t

$$\frac{t}{a} \le [\theta(t) - \theta(0)] \le at.$$

From this we conclude that, for all t

$$\frac{[\theta^*(t) - \theta(0)]}{a^2} \le [\theta(t) - \theta(0)] \le a^2[\theta^*(t) - \theta(0)],$$

where $\theta^*(t)$ is the phase angle of $\sigma^*(t)$ (Recall that by our choice $\sigma^*(0) = \sigma(0)$).

We have seen earlier that $\|\sigma(\theta)\| \le \|\sigma^*(\theta)\|$ for all $\theta > \theta(0)$. Therefore, it follows that,

$$\|\sigma(\theta(t))\| \\ \le \sup\{\|\sigma^*(\alpha)\| \mid \frac{\theta^*(t) - \theta(0)}{a^2} + \theta(0) \le \alpha \le \theta(0) + a^2(\theta^*(t) - \theta(0))\}. \quad (5)$$

From this inequality we conclude that the linear polysystem $\{A_1, A_2\}$ is asymptotically stable if and only if $\sigma^*(t)$ converges to the origin exponentially for all initial conditions. (In this case we have reduced the determination of uniform exponential stability of the polysystem to that of the stability analysis of a single linear time-varying system. It is highly unlikely that such simplification is feasible in general.)

Now, in order to determine the range of a for which $\{A_1, A_2\}$ is uniformly exponentially stable we only need to consider the system,

$$\dot{x} = \begin{cases} A_1 x, & x \in \mathcal{R}_1 \cup -\mathcal{R}_1, \\ A_2 x, & x \in \mathcal{R}_2 \cup -\mathcal{R}_2, \end{cases}$$

where, $\mathcal{R}_1$ and $\mathcal{R}_2$ are as depicted in Figure 1.

Perhaps the simplest way to determine stability of this system is to follow a trajectory originating from l_- until it gets to l_- again and evaluate its expansion or contraction in the radial direction. It is a contraction if and only if the system is asymptotically stable. Let T_1 and T_2 denote the time it takes for the trajectory to get from l_- to l_+ and from l_+ to l_-, respectively. Let $x_0 \in \ l_- \setminus \{0\}$. Then (6) is asymptotically stable if and only if $\|\exp(T_1 A_1)\exp(T_2 A_2)x_0\|_2 \ < \|x_0\|_2$, where $\|.\|_2$ denote the Euclidean norm. By a straightforward calculation it is seen that $T_1 = \tan^{-1}(\nu_+) - \tan^{-1}(\nu_-)$ and $T_2 = \pi - [\tan^{-1}(a\nu_+) - \tan^{-1}(a\nu_-)]$. (This follows from (5). Perhaps an even simpler way to see this is to note that $\dot{\theta} = 1$ when $a = 1$, and the transformation $y_1 = x_1, \ y_2 = ax_2$ transforms $\dot{x} = A_2 x$ into $\dot{y} = A_1 y$.) Now we can numerically compute the range of a for which the system is stable. It turns out that this range contains $[1, 10]$. Therefore, for $1 \le a \le 10$ the linear polysystem $\{A_1, A_2\}$ is uniformly exponentially stable. Indeed the range of all a for uniform exponential stability is considerably larger, but the range given above suffices to illustrate our point. For that matter, the motivated reader may even use a hand calculator to do the pertinent calculations when $a = 10$ in order to demonstrate the stability. Since we will show below that there is no common quadratic Lyapunov function when $a = 10$, this suffices to prove our claim.

Let us now compute the range of a for which both vector fields admit a common quadratic Lyapunov function. We start with the parametric form $V(x_1, x_2) = x_1^2/2 + px_1x_2 + qx_2^2/2$ and compute its derivative along A_1x and A_2x,

$$\begin{aligned} \nabla V(x)A_1 x &= (p-1)x_1^2 + (q-2p-1)x_1x_2 - (p+q)x_2^2, \\ \nabla V(x)A_2 x &= (p/a-1)x_1^2 + (q/a-2p-a)x_1x_2 - (pa+q)x_2^2. \end{aligned}$$

Now the quadratic form $\nabla V(x)A_1x$ is negative definite if and only if $p^2 + (q-3)^2/8 < 1$. Similarly $\nabla V(x)A_2(x)$ is negative definite if and only if $8\,p^2 + (q-3a^2)^2/a^2 < 8a^2$. Noticing that these two regions correspond to the interior of two ellipses in the $p-q$ plane with centers at $(0,3)$ and $(0,3a^2)$ respectively (see Figure 2.), we conclude that the regions have empty intersection if and only if

$$3+\sqrt{8} < (3-\sqrt{8})a^2;$$

and this is true if and only if

$$3+\sqrt{8} < a.$$

Thus for these a the intersection is empty and hence there cannot exist a common quadratic cost function.

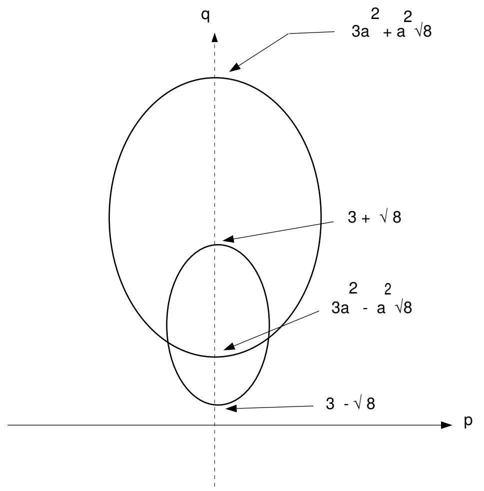

Figure 2: Admissible Regions in the Parameter Space of Lyapunov Functions

We now conclude that for $3+\sqrt{8} < a < 10$ the polysystem $\{A_1, A_2\}$ is exponentially stable and therefore member vector fields A_1x and A_2x admit a common smooth Lyapunov function, yet it cannot be found to be a positive definite quadratic form!

This somewhat surprising result naturally leads to the following question: what is the lowest degree of a common Lyapunov function of a compact linear polysystem assuming it is uniformly exponentially stable? Even if we assume that there are only finitely many vector fields in it, the question seems to be very hard; and we have not yet succeeded in finding an answer.

4 Uniform Asymptotic Stability of a Nonlinear Polysystem

In this section we study a special class of globally uniformly asymptotically stable polysystems, namely the ones which are in addition locally uniformly exponentially stable. We use the acronym **GALES** to denote this type of stability. We choose to study this class for four reasons. First, the notion of global exponential stability is heavily dependent upon the choice of a coordinate system on $\Re^n$. (This can be seen by considering the example $\dot{x} = -x$ which ceases to be globally exponentially stable under any coordinate transformation $y = \phi(x)$ such that $\phi(x) = \ln(x)$ for all $|x| > 2$.) Second, in standard dynamical systems theory, asymptotically stable—but not locally exponential stable systems—are the exception rather than the rule. Third, the heuristic reasoning often made to conclude that asymptotically stable nonlinear systems respond reliably to control commands turns out to be true only in the case of local exponential stability. Last, this class is amenable to analysis while keeping technical difficulties within reasonable bounds.

We remark that one could also study polysystems which are uniformly exponentially stable on all compact subsets of $\Re^n$. Essential ideas in this section will carry through to this case. However, if we were to remove the hypothesis of local exponential stability the analysis becomes quite complex. Most of the construction we do here will carry through up to the point of producing a continuous Lyapunov function. However, it may fail to to be Lipschitz continuous. Therefore, one needs to resort to fairly complex technical constructions to produce a smooth Lyapunov function.

Let us denote our polysystem by D. We assume that D consists of C^1 vector fields

$\{X_\gamma |\ \gamma \in \Gamma\}$ on $\Re^n$ and that the origin is an equilibrium point (recall that this means $X_\gamma(0) = 0$ for all γ). Let us denote $(\partial X_\gamma/\partial x)(0)$ by A_γ and define the linearization of D at the origin as $DL = \{A_\gamma |\ \gamma \in \Gamma\}$. We assume that D is a compact subset of the space of C^1 vector fields on $\Re^n$ in the compact open topology. (Readers who aren't familiar with this terminology may think of D as consisting of either finitely many vector fields or the convex hull of such.) It then follows that DL is a compact linear polysystem.

We now state the second main theorem of the paper.

Theorem 4.1 *The origin is a* GALES *equilibrium point of D if an only if it admits a C^1 function $V : \Re^n \to \Re$ such that the following hold:*

(i) $\nabla V(x) X_\gamma(x) < 0$ *for all* $x \neq 0$ *and* $\gamma \in \Gamma$.

(ii) There exists a neighborhood K of the origin and positive constants a_1, a_2 and λ such that

$a_1\|x\|^2 \leq V(x) \leq a_2\|x\|^2$ *and*

$\nabla V(x) X_\gamma(x) \leq -\lambda V(x)$ *for all* $x \in K$ *and* $\gamma \in \Gamma$.

(iii) $V(x) \to \infty$ *as* $\|x\| \to \infty$.

Proof: First we observe that (i)-(iii) imply that the origin is *GALES*. Indeed if V satisfies (i) and (iii) then it is easy to see (compactness of D is important here) that D is globally uniformly asymptotically stable. From (ii) it is easily seen that for any integral curve σ originating from inside a level set of V contained in K, then $\|\sigma(t)\| \leq \sqrt{a_2/a_1} e^{-\lambda t/2} \|\sigma(0)\|$. Therefore, the origin is locally uniformly exponentially stable.

Now we proceed to prove the converse. For the remainder of this section we will assume that the origin is a GALES equilibrium point of D.

It turns out that a slight modification of vector fields in D helps immensely to prove (iii) of Theorem 4.1. This consists of multiplication of vector fields by a suitable positive function in order to keep the magnitudes within reasonable bounds.

Let Y be an arbitrary smooth vector field on $\Re^n$. There is a standard construction of a strictly positive smooth function ζ on $\Re^n$ such that ζY is a complete vector field (see e.g. [6]). The basic idea is as follows. For any natural number m let U_m be the closed annulus bounded by spheres of radii m and $m+1$. Let $\eta_m = max\{\|Y(x)\| | x \in U_m\}$. Now let ζ be any positive smooth function on $\Re^n$ such that $\zeta(x) = 1$ on U_0 and $0 < \zeta(x) \leq 1/\eta_m$ for all $m \geq 2$. Thus $\|\zeta(x)Y(x)\| \leq 1$ for all $\|x\| \geq 2$. Among other things this implies that the vector field ζY is complete.

Clearly we can extend this constuction to the entire family D by exploiting its compactness in the compact open topology. Therefore, we assume that we have fixed a strictly positive smooth function ζ such that $\|\zeta(x)X_\gamma(x)\| \leq 1$ for all $\|x\| \geq 2$ and $\gamma \in \Gamma$. We further restrict ζ to be such that $\zeta(x) = 1$ for all $\|x\| \leq 1$.

Let us make three observations now.

- The family $\{\zeta X_\gamma | \gamma \in \Gamma\}$ is compact in the space of C^1 vector fields on $\Re^n$ endowed with the compact open topology. This follows since multiplication by a bounded smooth function is a continuous endomorphism on C^1 vector fields.

- Origin is a **GALES** equilibrium of $\{\zeta X_\gamma | \gamma \in \Gamma\}$. To see this recall that multiplication of a smooth vector field by a strictly positive smooth function leaves phase orbits invariant (only the time parametrization is affected). Thus, we see that all integral curves of ζX_γ remain uniformly bounded with same bounds as for X_γ and, converge to the origin. Since $\zeta(x) = 1$ for all $\|x\| \leq 1$, multiplication by ζ preserve the local exponential stability as well.

- V is a Lyapunov function which satisfies (i), (ii) and (iii) of Theorem 4.1 for the family D if and only if they are satisfied for the family $\{\zeta X_\gamma | \gamma \in \Gamma\}$. This is quite obvious since we may take K to lie inside the unit ball where $\zeta = 1$, and (i) isn't affected by multiplication of X_γ by positive functions.

In view of these three observations we will henceforth assume without any loss of generality that $\|X_\gamma\| \le 1$ whenever $\|x\| > 2$.

Let us now proceed onto demonstrating the existence of V as in Theorem 3.1. Steps involved are as follows. First, we use the uniform exponential stability of the family of linear approximations to construct V on a small compact neighborhood of the origin using Theorem 3.1. Next we use a slightly modified version of the construction of V in Theorem 3.1 to produce a Lyapunov function candidate. In the last step we smoothen it to produce the desired Lyapunov function.

Let us denote the local flows of D and DL by ϕ and $\tilde{\phi}$, respectively. We note, as in standard theory of nonlinear dynamics, that for any switching path s,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \phi_t^s(\epsilon x) \;=\; \tilde{\phi}_t^s(x).$$

Therefore, from the local exponential stability of D we conclude that

$$\begin{aligned} \|\tilde{\phi}_t^s(x)\| &\le \lim_{\epsilon \to 0} \frac{1}{\epsilon} c e^{-\lambda t} \|\epsilon x\|, \\ &= c e^{-\lambda t} \|x\|, \end{aligned}$$

where c and λ are as in the definition of local exponential stability of D.

Therefore, the linear polysystem DL is uniformly exponentially stable. Since DL is also compact, by Theorem 3.1 we can find a C^1 Lyapunov function which is homogeneous of degree two. Since X_γ is C^1 we conclude that on a small enough neighborhood of the origin V remains a Lyapunov function of D as well. Let us fix such a neighborhood K. By making K smaller if necessary, we further take K to be compact and having a level set of V as its boundary. Now on K (ii) holds.

Let us now construct an extension of V to all of $\Re^n$ as follows. Let S denote the set of switching paths.

Define,

$$\tilde{V}(x) = \begin{cases} V(x), & \text{if } x \in K, \\ \sup\{\int_0^T \|\phi_t^s(x)\|^2 dt + V(\phi_T^s(x)) \mid s \in S; & \\ \quad T \text{ is the smallest } t \text{ such that } \phi_t^s(x) \in \partial K\}, & \text{if } x \notin K. \end{cases}$$

To begin with let us observe that uniform asymptotic stability of D guarantees that $\tilde{V}$ is well defined.

We will now show that $\tilde{V}$ is locally Lipschitz continuous. This will enable us to smooth it in such a way that its derivative along solutions is negative. (Indeed, without assuming local exponential stability, this is the difficult part of the proof. Any straightforward modification of the proof leads to a $\tilde{V}$ which loses negative definiteness of the derivative upon application of the smoothing process.)

Now let $x_0 \in \Re^n \backslash K$. Let $\mathcal{W}$ be a small compact neighborhood of x_0. Choose a compact set M such that $\phi_t^s(y) \in M$ for all $y \in \mathcal{W}$, all $s \in S$ and all $t > 0$. Let $T_1 > 0$ be such that $\phi_t^s(y) \in K$ for all $y \in \mathcal{W}$, all $s \in S$, and all $t \geq T_1$. Let us consider the set of diffeomorphisms on $\mathcal{W}$,

$$Y = \{\phi_t^s : \mathcal{W} \to \Re^n | \ s \in S, \ 0 \leq t \leq T_1\},$$

and consider them as a subset of $C^0(\mathcal{W}, \Re^n)$, the space of continuous functions from $\mathcal{W}$ to $\Re^n$.

By our choice of M, $f(\mathcal{W}) \subset M$ for all $f \in Y$. Therefore, f is a bounded subset in $\|.\|_\infty$ norm topology. Furthermore, $Y \subset C^1(\mathcal{W}, \Re^n)$ (since each element is a concatenation of smooth maps). We now show that Y is a bounded subset of $C^1(\mathcal{W}, \Re^n)$ as well. To do so let us first observe that,

$$\begin{aligned} \frac{\partial}{\partial t}\frac{\partial}{\partial x}\phi_t^s(x) &= \frac{\partial}{\partial x}\frac{\partial}{\partial t}\phi_t^s(x) \\ &= \frac{\partial}{\partial x}X^{s(t)}(\phi_t^s(x)) \\ &= \frac{\partial}{\partial y}X^{s(t)}(y)|_{y=\phi_t^s(x)}\frac{\partial}{\partial x}\phi_t^s(x). \end{aligned}$$

Therefore, $\frac{\partial}{\partial x}\phi_t^s(x)$ satisfies a linear differential equation,

$$\frac{\partial}{\partial t}[\frac{\partial}{\partial x}\phi_t^s(x)] \;=\; H^s(t)[\frac{\partial}{\partial x}\phi_t^s(x)],$$

where, $H^s(t) \;=\; (\partial/\partial y)X^{s(t)}(y)|_{y \,=\, \phi_t^s(x)}$.

Let $L \;=\; \sup\{\|(\partial/\partial x)X_\gamma(x)\| \;\mid x \in M,\; \gamma \in \Gamma\}$. Note that L is finite since M is a compact subset of $\Re^n$ and D is compact in the C^1 compact open topology. It then follows that,

$$\frac{\partial}{\partial t}\|\frac{\partial}{\partial x}\phi_t^s(x)\|^2 \;\leq\; (2L)\|\frac{\partial}{\partial x}\phi_t^s(x)\|^2.$$

Therefore,

$$\|\frac{\partial}{\partial x}\phi_t^s(x)\| \;\leq\; e^{2LT_1}$$

for all $0 \;\leq\; t \;\leq T_1$, $s \;\in\; S$ and $x \;\in\; \mathcal{W}$. This shows that Y is a bounded subset of $C^1(\mathcal{W}, \Re^n)$.

Let us now define $L_1 \;=\; 2e^{2LT_1}$ and make $\mathcal{W}$ smaller if necessary, so that

$$\|\phi_t^s(x) \;-\; \phi_t^s(y)\| \;<\; L_1\|x-y\|$$

for all $s \;\in\; S$, $x, y \;\in\; \mathcal{W}$ and $0 \;\leq\; t \;\leq T_1$. Let $L_2 \;=\; \max\{\|y\| \mid y \;\in\; M\}$. It now follows that for all $x, y \;\in \mathcal{W}$, $s \;\in\; S$,

$$\int_0^{T(x)} \|\phi_t^s(x)\|^2 dt \;-\; \int_0^{T(y)} \|\phi_t^s(y)\|^2 dt \;\leq\; 2L_1L_2\int_0^{T(y)} \|x-y\| dt \;+\; \int_{T(y)}^{T(x)} \|\phi_t^s(x)\|^2 dt, \quad (6)$$

where $T(x)$ and $T(y)$ are the shortest times to get to K from x and y respectively, and without loss of generality we assumed that $T(y) \;\leq\; T(x)$.

Since $t \mapsto \phi_t^s(x)$ cuts ∂K transversally (∂K is a level set of V) and since $\|\phi_{T(y)}^s(y) \;-\; \phi_{T(y)}^s x\| \;\leq\; L_1\|x-y\|$, by taking $\mathcal{W}$ smaller if necessary, we can find L_3 such that

$$\int_{T(y)}^{T(x)} \|\phi_t^s(x)\|^2 dt \;\leq\; L_3\|x-y\|, \quad (7)$$

and L_3 is independent of the choice of $x, y \in \mathcal{W}$.

Since V is locally Lipschitz continuous on the compact set K we can find L_4 such that

$$|V(p) - V(q)| \leq L_4 \|p - q\| \text{ for all } p, q \in K. \tag{8}$$

Now from equations (6), (7) and (8) it follows that,

$$|\tilde{V}(x) - \tilde{V}(y)| \leq (2L_1 L_2 T_1 + L_3 + L_1 L_4)\|x - y\|,$$

for all x, y in $\mathcal{W}$. Thus we have shown that $\tilde{V}$ is locally Lipschitz continuous on $\Re^n$.

By our construction it is seen that $\tilde{V}(x) \to \infty$ when $\|x\| \to \infty$ (this depends heavily on the uniform boundedness of the vector fields with respect to the Euclidean norm in D). Also, just as in the linear case we see immediately that for all $s \in S$, $\gamma \in \Gamma$, $t, \tau \in [0, \infty)$,

$$\tilde{V}(\phi_\tau^{X_\gamma}(x)) - \tilde{V}(x) \leq -\int_0^\tau \|\phi_r^{X_\gamma}(x)\|^2 dr.$$

Therefore, for small enough $\tau > 0$,

$$\frac{\tilde{V}(\phi_\tau^{X_\gamma}(x)) - \tilde{V}(x)}{\tau} \leq -\frac{1}{2}\|x\|^2.$$

This almost says that $\dot{\tilde{V}}$ is negative definite along integral curves of X_γ. However, $\tilde{V}$ may fail to be C^1. We will now describe the smoothing process. It follows Kurzweil's construction in [11] very closely.

Our objective here is to construct a new Lyapunov function $\hat{V}$ in such a way that it agrees with $\tilde{V}$ on some small neighborhood of the origin, and away from the origin $\hat{V}$ and its directional derivatives in the X_γ directions will be approximately equal to $\tilde{V}$ and its difference quotients. This is done by first covering $\Re^n$ by an infinite family of compact sets, constructing a smooth approximant of $\tilde{V}$ on each set (where we can predefine the level of approximation needed) and interpolating between them using a smooth partition of unity of $\Re^n$.

In the sequel we use the notation B_r to denote the open ball of radius r around the origin of $\Re^n$ and $\bar{B}_r$ to denote its closure. First let us define a positive number ϵ_1 in such a way that $\bar{B}_{\epsilon_1} \subset K$. Now define $\epsilon_{-1} = 0,\ \epsilon_0 = \epsilon_1/2$ and $\epsilon_k = (k-1) + \epsilon_1,\ k = 2, 3, \cdots$. Let us also fix a smooth bump function $\delta : \Re^n \rightarrow \Re$ such that its support is in B_1 and its integral over $\Re^n$ is equal to one. By definition δ is a nonnegative function which is identically zero outside of the unit ball. In the sequel we will carry out the smoothing of $\tilde{V}$ on $\bar{B}_{\epsilon_k} \backslash B_{\epsilon_{k-1}}$ by convolving it with appropriately normalized dilations of δ. Thus we will be working with a sequence of positive reals $r_k,\ k = 2, 3, \cdots$ corresponding to these dilated versions of δ, namely bump functions $x \mapsto r_k^n \delta(x/r_k)$. As we go along we will impose some restrictions on the size of r_k. To begin with we assume $r_k < 1$ for $k > 2$ and $r_1 < \epsilon_1 - \epsilon_0$.

Now we define a sequence of smooth functions by $V_0 = \tilde{V}$ and,

$$V_k(x) = \int_{\Re^n} \tilde{V}(x-y) r_k^n \delta(y/r_k) dy, \quad k = 1, 2, \cdots .$$

Here, V_k is to be thought of as an approximant of $\tilde{V}$ on $B_{\epsilon_{k+1}} \setminus B_{\epsilon_k}$. On this set V_k depends only on $V|_{B_{\epsilon_{k+2}} \backslash B_{\epsilon_{k-1}}}$ by our choice of the r_k sequence.

Let us define $W_k = B_{\epsilon_{k+1}} \backslash \bar{B}_{\epsilon_{k-1}},\ k = 1, 2, \cdots$ and $W_0 = B_{\epsilon_1}$. Let us fix a smooth partition of unity $\{\lambda_k\}_{k=0,1,\cdots}$ of the open cover $\{W_k\}_{k=0,1,\cdots}$ of $\Re^n$. Our smoothened version of $\tilde{V}$ will be $\hat{V} = \sum_{k=0}^{\infty} \lambda_k V_k$. We will now explain precisely how we will choose the $\{r_k\}$ sequence such that derivatives of $\hat{V}$ along X_γ will be negative.

Of course $\hat{V} = \tilde{V} = V$ on B_{ϵ_0}. Therefore, $\hat{V}$ will satisfy (ii) of Theorem 4.1. Let us compute the derivative of $\hat{V}$ along X_γ:

$$\begin{aligned}
\nabla \hat{V}(x) X_\gamma(x) &= \sum_{k=0}^{\infty} \{\nabla \lambda_k(x) X_\gamma(x) V_k(x)\} + \{\lambda_k(x) \nabla V_k(x) X_\gamma(x)\} \\
&= \sum_{k=0}^{\infty} \{\nabla \lambda_k(x) X_\gamma(x)(V_k(x) - \tilde{V}(x))\} + \sum_{k=0}^{\infty} \{\lambda_k(x) \nabla V_k(x) X_\gamma(x)\} \quad (9) \\
&\qquad \left(\text{since } \textstyle\sum_{k=0}^{\infty} \lambda_k(x) = 1\right).
\end{aligned}$$

Let us restrict attention to $M_k = \bar{B}_{\epsilon_{k+1}} \setminus B_{\epsilon_k}$ and now examine the second summand.

On M_k only λ_k and λ_{k+1} are nonzero. Therefore, we only need to examine $\nabla V_k(x) X_\gamma(x)$ and $\nabla V_{k+1}(x) X_\gamma(x)$. Let $j \in \{k, k+1\}$. Now for $x \in M_k$,

$$
\begin{aligned}
\nabla V_j(x) X_\gamma(x) &= \frac{d}{dt}|_{t=0} V_j(\phi_t^{X_\gamma}(x)) \\
&= \lim_{t\to 0} \int_{\Re^n} \frac{[\tilde{V}\big(\phi_t^{X_\gamma}(x) - y\big) - \tilde{V}(x-y)]}{t} dy \\
&\leq \int_{\Re^n} \limsup_{t\to\infty} \frac{[\tilde{V}\big(\phi_t^{X_\gamma}(x) - y\big) - \tilde{V}(x-y)]}{t} r_k^n \delta(y/r_k) dy.
\end{aligned}
$$

Let us recall that by construction $\tilde{V}$ is locally Lipschitz continuous and that $\tilde{V}\big(\phi_t^{X_\gamma}(x)\big) - \tilde{V}(x) \leq -(t/2)\|x\|^2 + o(t)$. Let L be a Lipschitz constant of $\tilde{V}$ on $\bar{B}_{\epsilon_{k+2}} \backslash \bar{B}_{\epsilon_{k-1}}$. We observe that,

$$
\begin{aligned}
\tilde{V}\big(\phi_t^{X_\gamma}(x) - y\big) - \tilde{V}(x-y) &= [\tilde{V}\big(\phi_t^{X_\gamma}(x) - y\big) - \tilde{V}\big(\phi_t^{X_\gamma}(x-y)\big)] \; - \\
&\quad\; [\tilde{V}\big(\phi_t^{X_\gamma}(x-y)\big) - \tilde{V}(x-y)] \\
&\leq [\tilde{V}\big(x + \; tX_\gamma(x) + o(t) - y\big) \; - \\
&\quad\; \tilde{V}\big(\; (x-y) + tX_\gamma(x-y) \; + \; o(t)\big)] \\
&\quad - \; (t/2)\|x-y\|^2 \; + \; o(t) \\
&\leq L[t\big(X_\gamma(x) - X_\gamma(x-y) + o(t)\big)] \; - \\
&\quad\; (t/2)\|x\|^2 + o(t) \; + \; t < x, y > .
\end{aligned}
$$

Now since y appears only as the argument of $\delta(y/r_k)$ for $x \in M_k$ the only y that matters are those with magnitude less than r_k and r_{k+1}. Since X_γ is smooth we can now choose r_k and r_{k+1} small enough such that, $\tilde{V}\big(\phi_t^{X_\gamma}(x) - y\big) - \tilde{V}(x-y) \leq -(t/4)\|x\|^2 + o(t)$ uniformly on M_k. This choice can be made to work for all $\gamma \in \Gamma$ since D is compact. Therefore, the second summand in (9) satisfies $\sum_{l=0}^{\infty} \lambda_l(x) \nabla V_l(x) X_\gamma(x) \leq -\|x\|^2/4$. In the first summand $V_k(x)$, $V_{k+1}(x)$ and $\tilde{V}(x)$ can be made as close to each other as we want on M_k by choosing r_k and r_{k+1} small enough. Therefore, we may ensure that $\sum_{l=0}^{\infty} \nabla\lambda_l(x)[V_l(x) - \tilde{V}(x)] \leq \|x\|^2/8$.

Thus we conclude that we can choose the sequence $\{r_k\}$ such that

$$
\nabla \hat{V}(x) X_\gamma(x) \leq -\|x\|^2/8,
$$

for all $x \in \Re^n$. Thus we have shown that our smoothened version of the Lyapunov function $\hat{V}$ satisfies $(i), (ii)$ and (iii) of Theorem 4.1. This concludes the proof.

5 Concluding Remarks

The next logical step is to study the stabilization problem for polysystems. There are two ways to get started. One may seek a control Lyapunov function and seek feedback functions or one may cast the problem in a game theoretic framework where switching paths are thought to maximize a cost function and a feedback law is sought which will correspond to a Nash equilibrium. The first approach requires us to know a priori some information about the control Lyapunov function. Even in the linear case this isn't straightforward since there may not be quadratic Lyapunov functions, as we saw in Example 3.1. Therefore, the question of finding an upper bound on the degree of a homogeneous Lyapunov function for a linear polysystem would be rather interesting and important. Unfortunately this problem appears to be very difficult.

Another avenue of interest is when we know some a priori information about the switching paths. For example, switching times may be bounded below, or they could be modeled via some Markov chains, e.g., one switch assigns probabilities for the next switch, and perhaps switching times behave according to a stochastic process, e.g., a Poisson process. It has recently come to our attention that Gurvits, [7], has done some work in this direction. It would be interesting to formulate the corresponding problems and study them in depth.

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