A sample of methods for the study of the controllability of nonlinear partial differential equations

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Chapter 1

Introduction

For finite dimensional systems, one knows many powerful sufficient conditions for local controllability of a nonlinear control system (see chapter 2 and the references therein). This is not the case in infinite dimension, where, roughly speaking, the only known general result is that, if the linearized control system at an equilibrium is controllable, then the nonlinear control system is locally controllable at the equilibrium. We present here three methods to get in some cases controllability results for some control systems modelled by partial differential equations even if the linearized control system around the equilibrium is not controllable. These methods are

1. the return method,
2. quasi-static deformations,
3. power series expansion.

Return method. The idea of the return method goes as follows: If one can find a trajectory of the nonlinear control system such that

- it starts and ends at the equilibrium,
- the linearized control system around this trajectory is controllable,

then, in general, the implicit function theorem implies that one can go from every state close to the equilibrium to every other state close to the equilibrium. In this paper, we sketch some results in flow control which have been obtained by that method, namely

- global controllability results for the Euler equations of incompressible fluids,
- global controllability results for the Navier-Stokes equations of incompressible fluids,
- local controllability of a 1-D tank containing a fluid modelled by the shallow water equations,
- null global asymptotic stabilizability by means of explicit boundary feedback laws for the 2-D inviscid incompressible fluids on simply connected domains.

Quasi-static deformations. The quasi-static deformation method allows one to prove in some cases that one can move from every equilibrium to every other equilibrium if these two equilibriums are connected in the set of equilibriums. The idea is just to move very slowly the control (quasi-static deformation) so that at each time the state is close to the curve of equilibriums connecting the two given equilibriums. If some of these equilibriums are instable, one uses also suitable feedback laws in order to stabilize them: without these feedback laws the quasi-static deformation would not work. We present an application to a semilinear heat equation.

Power series expansion. In this method one does some power series expansion in order to see if the nonlinearity allows to move in the directions which are not controllable for the linearized control system around the equilibrium. We present an application to the Korteweg-de Vries equation.

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Chapter 2

Lie brackets and local controllability in finite dimension

Throughout out this chapter, by (C) we denote the nonlinear control system

$$\dot{x} = f(x, u), \tag{C}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control. We assume that

$$f(0, 0) = 0, \tag{2.1}$$

and, unless otherwise specified, $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^n)$.

Let us first give the definition we use in these notes for small time locally controllable –we should in fact say small time locally controllable with small controls at 0–

Definition 1 *Control system (C) is small time locally controllable if, for every real number $\epsilon > 0$, there exists a real number $\eta > 0$ such that, for every $x_0 \in B_\eta := \{x \in \mathbb{R}^n; |x| < \eta\}$ and for every $x_1 \in B_\eta$, there exists a measurable function $u : [0, \epsilon] \to \mathbb{R}^m$ such that*

$$|u(t)| < \epsilon, \forall t \in [0, T],$$
$$(\dot{x} = f(x, u(t)), x(0) = x_0) \Rightarrow (x(\varepsilon) = x_1).$$

One does not know any interesting necessary and sufficient condition for small time local controllability, even for analytic control systems. However, one knows powerful necessary conditions and powerful sufficient conditions. Let us recall some of these conditions. Both rely on Lie brackets. Before giving these conditions let us mention that it is quite natural that Lie brackets appear for controllability conditions:

- for necessary conditions of controllability because of the Frobenius theorem,

- for sufficient conditions because of the following observation. Let us assume that $m = 2$ and that $f(x, u) = u_1 f_1(x) + u_2 f_2(x)$. Then one knows how to move in the direction of $\pm f_1(0)$ even with small control: just take $u_1 = \eta \neq 0$ and $u_2 = 0$. Similarly, one knows how to move in the directions of $\pm f_2(0)$. Let us explain how to move in the direction of the Lie bracket $[f_1, f_2](0)$, defined by

$$[f_1, f_2]^j(0) = \left(\sum_{k=1}^{n} f_2^k \frac{\partial f_1^j}{\partial x_k} - f_1^k \frac{\partial f_2^j}{\partial x_k} \right)(0), \ \forall j \in \{1, \cdots, n\}.$$

Let $\Psi_i : \mathbb{R} \times \mathbb{R}^n, (t, x) \to \Psi_i^t(x)$ be the flow associated to f_i (for $i \in \{1, 2\}$), that is

$$\frac{\partial \Psi_i}{\partial t} = f_i(\Psi_i),\ \Psi_i^0(x) = x.$$

Then

$$\lim_{\delta \to 0} \frac{(\Psi_2^{-\delta}(\Psi_1^{-\delta}(\Psi_2^{\delta}(\Psi_1^{\delta}(0)))))}{\delta^2} = [f_1, f_2](0). \tag{2.2}$$

Hence, if one starts at 0, that is if $x(0) = 0$, and if one takes

- $(u_1(t), u_2(t)) = (\eta_1, 0)$, for $t \in (0, \epsilon)$,
- $(u_1(t), u_2(t)) = (0, \eta_2)$, for $t \in (\epsilon, 2\epsilon)$,
- $(u_1(t), u_2(t)) = (-\eta_1, 0)$, for $t \in (2\epsilon, 3\epsilon)$,
- $(u_1(t), u_2(t)) = (0, -\eta_2)$, for $t \in (3\epsilon, 4\epsilon)$,

with $(\eta_1, \eta_2) \in \mathbb{R}^2$ then, as $\epsilon \to 0$,

$$x(4\epsilon) = \eta_1 \eta_2 \epsilon^2 [f_1, f_2](0) + o(\epsilon^2). \tag{2.3}$$

Hence we have indeed succeeded in moving in the directions of $\pm[f_1, f_2](0)$.

Even in the case with a drift term, Lie brackets appear also naturally. Let us consider for example the case where $m = 1$ and

$$f(x, u) = f_0(x) + u f_1(x). \tag{2.4}$$

Clearly one can move in the directions of $\pm f_1(0)$ -recall that, by (2.1), $f_0(0) = 0$-. Let $\eta \neq 0$ be fixed and let us consider, for $\epsilon > 0$, the control defined on $[0, 2\epsilon]$ by

$$u(t) = \eta \text{ for } \in (0, \epsilon),$$
$$u(t) = -\eta \text{ for } \in (\epsilon, 2\epsilon).$$

Let $x : [0, 2\epsilon] \to \mathbb{R}^n$ be the solution of

$$\dot{x} = f_0(x) + u(t) f_1(x),\ x(0) = 0.$$

Then, straightforward computations show that, as $\epsilon \to 0$,

$$x(2(\epsilon)) = -2\epsilon^2 \eta [f_0, f_1](0) + o(\epsilon^2). \tag{2.5}$$

Hence we have succeeded to move in the directions of $\pm[f_0, f_1](0)$. Of course the case of a system with a drift term is more complicated than the case of a driftless. For example, in the previous driftless case, one can keep going and for example move in the direction of $[[f_1, [f_1, f_2]](0)$. However, in the case with the drift term, one cannot always move in the direction of $[[f_1, [f_1, f_0]](0)$.

In order to first give a necessary condition, let us provide new definitions.

Definition 2 *[19]. The strong jet accessibility subspace of (C) at $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ is the subspace of $\mathbb{R}^n$, denoted by $a(\bar{x}, \bar{u})$, spanned by*

$$\{g(\bar{x}); g \in \{\partial^{|\alpha|} f / \partial u^\alpha(\cdot, \bar{u}),\ \alpha \in \mathcal{N}^m,\ |\alpha| \geq 1\} \cup Br_2(f, \bar{u}), \tag{2.6}$$

where $Br_2(f, \bar{u})$ denotes the set of iterated Lie brackets of length at least 2 of vector fields in $\{\partial^{|\alpha|} f / \partial u^\alpha(\cdot, \bar{u}); \alpha \in \mathcal{N}^m\}$.

Remark 3 *One easily checks that the usual strong accessibility subspace of* (C) *at* $\bar{x}$ *(i.e. the space denoted* $\mathcal{F}_0 D(\bar{x})$ *in [110, p. 109]) contains* $a(\bar{x},\bar{u})$ *for all* $\bar{u}$ *in* $\mathbb{R}^m$ *and that, if* f *is analytic with respect to* x *and* u *or is a polynomial with respect to* u*, these inclusions are all equalities.*

Definition 4 *The control system* (C) *satisfies the strong Lie algebra rank condition at* $(\bar{x},\bar{u})$ *if*

$$a(\bar{x},\bar{u}) = \mathbb{R}^n. \tag{2.7}$$

Remark 5 *It follows from Remark 3 that, if* (C) *satisfies the strong Lie algebra rank condition at* $(\bar{x},\bar{u})$ *(2.7), then it satisfies the usual strong accessibility rank condition at* $\bar{x}$ *(i.e. dim* $\mathcal{F}_0 D(\bar{x}) = n$ *with the notations of [110]) and the converse holds if* f *is analytic with respect to* x *and* u *or is a polynomial with respect to* u*.*

With these definitions, one has the following well-known necessary condition for small time local controllability of analytic control system due to Hermann [55] and Nagano [83] (see also Sussmann [104])

Theorem 6 *Assume that the control system* (C) *is locally controllable and that* f *is analytic. Then the control system* (C) *satisfies the strong Lie algebra rank condition at* $(0,0)$*.*

This necessary condition is sufficient for important control systems such as, for example,

- Linear control system $\dot{x} = Ax + Bu$. This follows from the Kalman condition [100, Theorem 2 p. 88] and the fact that
$$a(0,0) = \text{ Span } \{A^i Bu;\, i \in \{0,\cdots,n-1\},\, u \in \mathbb{R}^m\}.$$
- Driftless control system $\dot{x} = \sum_{i=1}^m u_i f_i(x)$. This is the classical Rashevski-Chow theorem [87, 14]. (For an explicit method to produce trajectories joining two given points for driftless systems satisfying the strong Lie algebra condition at every (x,u), see the paper [108] by Sussmann and Liu.)

However, this necessary condition is not, in general, sufficient, as the two following simple control systems show

$$(n=1, m=1),\ \dot{x} = u^2,$$
$$(n=2, m=1),\ \dot{x}_1 = x_2^2, \dot{x}_2 = u.$$

Let us now give sufficient conditions for small time local controllability. Let us assume, for the time being, that

$$f(x,u) = f_0(x) + \sum_{i=1}^m u_i f_i(x).$$

Let $L(f_0,\dots,f_m)$ be the free Lie algebra generated by $f_0,\dots,f_m$ and let us denote by $\text{Br}(f) \subset L(f_0,\dots,f_m)$ the set of *formal iterated* Lie brackets of $\{f_0, f_1, ..., f_m\}$; see [107] for more details and precise definitions. For example

$$h = [[[f_0,[f_1,f_0]],f_1],f_0] \in \text{ Br}(f). \tag{2.8}$$

For $h \in L(f_0,\dots,f_m)$, let $h(0) \in \mathbb{R}^n$ be the "evaluation" of h at 0. For h in $\text{Br}(f)$ and $i \in [0,m]$, let $\delta_i(h)$ be the number of times that f_i appears in h. For example, with h given by (2.8), $\delta_0(h) = 3$,

$\delta_1(h) = 2$ and $\delta_i(h) = 0$, for any $i \in [2, m]$. Let S_m be the group of permutations of $\{1, \ldots, m\}$. For π in S_m, let $\tilde{\pi}$ the automorphism of $L(f_0, \ldots, f_m)$ which sends f_0 to f_0 and f_i to $f_{\pi(i)}$ for $i \in [1, m]$. For $h \in \mathrm{Br}(f)$, we let

$$\sigma(h) = \sum_{\pi \in S_m} \tilde{\pi}(h) \in L(f_0, \ldots, f_m).$$

For example, if h is given by (2.8) and $m = 2$, one has

$$\sigma(h) = [[[f_0, [f_1, f_0]], f_1], f_0] + [[[f_0, [f_2, f_0]], f_2], f_0].$$

For $\theta \in [0, +\infty]$, $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ satisfies the Sussmann condition $S(\theta)$ [107, Section 7] if it satisfies the strong Lie algebra rank condition (2.7) at $(0, 0)$ and, if, for every $h \in \mathrm{Br}(f)$ with $\delta_0(h)$ odd and $\delta_i(h)$ even for all i in $[1, m]$, $\sigma(h)(0)$ is in the span of the $g(0)$'s where the g's are in $\mathrm{Br}(f)$ and satisfy

$$\theta \delta_0(g) + \sum_{i=1}^m \delta_i(g) < \theta \delta_0(h) + \sum_{i=1}^m \delta_i(h). \tag{2.9}$$

with the convention that, when $\theta = +\infty$, (2.9) is replaced by $\delta_0(g) < \delta_0(h)$. H. Sussmann has proved in [107]

Theorem 7 *[107, Thm. 7.3] If, for some θ in [0,1], $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ satisfies the condition $S(\theta)$, then the control system (C) is small time locally controllable.*

Let us notice that one can easily check that

Proposition 8 *Let θ be in [0,1]. Then $\dot{x} = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ satisfies the condition $S(\theta)$ if and only if $\dot{x} = f_0(x) + \sum_{i=1}^m y_i f_i(x)$, $\dot{y} = u$, where the state is $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and the control is $u \in \mathbb{R}^m$, satisfies $S(\theta/(1-\theta))$ (with the convention $1/0 = +\infty$).*

This proposition allows us to extend the condition $S(\theta)$ to $\dot{x} = f(x, u)$ in the following way

Definition 9 *Let $\theta \in [0, 1]$. The control system $\dot{x} = f(x, u)$ satisfies the condition $S(\theta)$ if the control system $\dot{x} = f(x, y)$, $\dot{y} = u$ satisfies $S(\theta/(1-\theta))$.*

What is called the Hermes condition is $S(0)$ ([59] and [106]). It follows from [107] that

Theorem 10 *If, for some θ in $[0, 1]$, the control system $\dot{x} = f(x, u)$ satisfies the condition $S(\theta)$ then it is small time locally controllable.*

Proof: Apply [107] to $\dot{x} = f(x, y)$, $\dot{y} = u$ with the constraint $\int_0^t |u(s)|\, ds \leqslant r$ (instead of $|u| \leqslant 1$).

One can find other sufficient conditions for small time local controllability in Agrachev [3], Bianchini-Stefani [8], Kawski [66], Tret'yak [111] and the references therein.

Example 11 *If $f(x, u) = \sum_{i=1}^m u_i f_i(x)$ or if $f(x, u) = Ax + Bu$ (i.e. for driftless control systems and linear control systems), the control system (C) satisfies the Hermes condition if and only if it satisfies the strong Lie algebra rank condition at $(0, 0)$. Hence, by Sussmann' Theorem 7, one recovers Rashevski-Chow's theorem (i.e. that for driftless control systems, small time local controllability is implied by the strong Lie algebra rank condition at $(0, 0)$) and that the Kalman condition*

$$\mathit{Span}\,\{A^i B u;\, i \in [0, n-1],\, u \in \mathbb{R}^m\} = \mathbb{R}^n$$

implies the controllability of the linear control system $\dot{x} = Ax + Bu$).

Example 12 *We take* $n = 2$, $m = 1$ *and*

$$\dot{x}_1 = x_2^3, \ \dot{x}_2 = u.$$

In other words, our system is

$$\dot{x} = f_0(x) + uf_1(x),$$

with

$$f_0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2^3 \\ 0 \end{pmatrix}, \ f_0 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then

$$f_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ [f_1, [f_1, [f_1, f_0]]] \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

span $\mathbb{R}^2$. *Hence our system satisfies the strong Lie algebra rank condition at*$(0, 0)$. *Moreover, if* $h \in Br(f)$ *with* $\delta_0(h)$ *even and* $\delta_1(h)$ *odd, then* $h(0) = 0$. *Hence our system satisfies the Hermes condition at* 0 *and so is small time locally controllable.*

Example 13 *Let us consider the following classical model for rigid spacecraft with control torques provided by thruster jets. Let* $\eta = (\phi, \theta, \psi)$ *be the Euler angles of a frame attached to the spacecraft representing rotations with respect to a fixed reference frame. Let* $\omega = (\omega_1, \omega_2, \omega_3)$ *be the angular velocity of the frame attached to the spacecraft with respect to the reference frame, expressed in the frame attached to the spacecraft and let* J *be the inertia matrix of the satellite. The evolution of the spacecraft is governed by the equations*

$$J\dot{\omega} = S(\omega)J\omega + \sum_{i=1}^{m} u_i b_i, \quad \dot{\eta} = A(\eta)\omega, \tag{2.10}$$

where the $u_i \in \mathbb{R}$, $1 \leqslant i \leqslant m$, *are the controls* ($u_i b_i \in \mathbb{R}^3$, $1 \leqslant i \leqslant m$ *are the torques applied to the spacecraft),* $S(\omega)$ *is the matrix representation of the wedge-product, i.e.*

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}, \tag{2.11}$$

and

$$A(\eta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ \sin\theta \tan\phi & 1 & -\cos\theta \tan\phi \\ -\sin\theta / \cos\phi & 0 & \cos\theta / \cos\phi \end{pmatrix}. \tag{2.12}$$

Without loss of generality, we assume that the vectors $b_1, \ldots, b_m$ *are linearly independent. Then the following results hold.*

- *If* $m = 3$, *the control system (2.10) is small time locally controllable and globally controllable in large time (that is given two states, there exists a time* $T > 0$ *and an open loop control* $u \in L^\infty(0, T)$ *steering the first state to the second one). This result is due to Bonnard [11] (see also [33]).*

- *If $m = 2$, the control system (2.10) satisfies the strong Lie algebra rank condition at $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$ if and only if (see [11, 33])*

$$\textit{Span}\ \{b_1, b_2, S(\omega)J^{-1}\omega;\ \omega \in\ \textit{Span}\ \{b_1, b_2\}\} = \mathbb{R}^3. \tag{2.13}$$

  Moreover, if (2.13) holds,

  - *The control system (2.10) satisfies Sussmann's condition $S(1)$, and so is small time locally controllable; see Keraï [68],*
  - *The control system (2.10) is globally controllable in large time; this result is due to Bonnard [11], see also [33].*

- *If $m = 1$, the control system (2.10) satisfies the strong Lie algebra rank condition at $(0,0) \in \mathbb{R}^6 \times \mathbb{R}$ if and only if (see [11, 33])*

$$\textit{Span}\ \{b_1, S(b_1)J^{-1}b_1 S(\omega)J^{-1}\omega;\ \omega \in\ \textit{Span}\ \{b_1, S(b_1)J^{-1}b_1\}\} = \mathbb{R}^3. \tag{2.14}$$

  Moreover

  - *The control system (2.10) does not satisfy a necessary condition for small time local controllability due to Hermes [57] and Sussmann [106] (see the next theorem) and so is not small time locally controllable; see [68].*
  - *If (2.14) holds, the control system (2.10) is globally controllable in large time; this result is due to Bonnard [11], see also [33].*

With the notations we have introduced we can state also an interesting necessary condition for small time local controllability due to Hermes [57] and Sussmann [106].

Theorem 14 *If the control system (2.4) is small time locally controllable then*

$$[f_1, [f_1, f_2]](0) \in \textit{Span}\ \{h \in \textit{Br}(f); \delta_1(h) \leq 1\}. \tag{2.15}$$

One can find other necessary conditions in [57, 58, 66, 106] and the references therein.

Chapter 3

Return method

3.1 Description of the method

In order to explain this method, let us first consider the problem of local controllability of a control system in finite dimension. So we consider the control system

$$\dot{x} = f(x, u),$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control ; we assume that f is of class $\mathcal{C}^\infty$ and satisfies

$$f(0, 0) = 0.$$

Note that all the controllability conditions given in the previous chapter rely on a Lie bracket formulation. Unfortunately this geometric tool does not seem to give good results for distributed control systems - in this case x belongs to an infinite dimensional space -. The main reason is that, for many interesting distributed control systems, (2.3) and (2.5) do not hold. More precisely, the left hand sides of (2.3) and (2.5) divided by ϵ^2 have no limit as $\epsilon \to 0$ in many interesting cases. On the other hand for *linear* distributed control systems there are powerful methods to prove controllability - e.g. the H.U.M. method due to J.-L. Lions, see [77]. The return method consists in reducing the local controllability of a nonlinear control system to the existence of - suitable - periodic (or "almost periodic" -see below the cases of the Navier-Stokes control system and of the shallow water equations) trajectories and to the controllability of *linear* systems. The idea is the following one: assume that, for every positive real number T and every positive real number ϵ, there exists a measurable bounded function $\bar{u} : [0, T] \to \mathbb{R}^m$ with $\|\bar{u}\|_{L^\infty(0,T)} \leq \epsilon$ such that, if we denote by $\bar{x}$ the (maximal) solution of $\dot{\bar{x}} = f(\bar{x}, \bar{u}(t))$, $\bar{x}(0) = 0$, then

$$\bar{x}(T) = 0, \tag{3.1}$$

$$\text{the linearized control system around } (\bar{x}, \bar{u}) \text{ is controllable on } [0, T]. \tag{3.2}$$

Then it follows easily from the inverse mapping theorem - see e.g. [100, Theorem 7 p. 126] - that $\dot{x} = f(x, u)$ is small time locally controllable. Let us recall that the linearized control system around $(\bar{x}, \bar{u})$ is the time-varying control system

$$\dot{y} = A(t)y + B(t)v, \tag{3.3}$$

where the state is $y \in \mathbb{R}^n$, the control is $v \in \mathbb{R}^m$ and $A(t) = (\partial f/\partial x)(\bar{x}(t), \bar{u}(t))$, $B(t) = (\partial f/\partial u)(\bar{x}(t), \bar{u}(t))$.

For the linear control system (3.3), controllability on $[0,T]$ means, by definition, that for every y_0 and y_1 in $\mathbb{R}^n$, there exists a bounded measurable function $v : [0,T] \to \mathbb{R}^m$ such that if $\dot{y} = A(t)y + B(t)v$ and $y(0) = y_0$, then $y(T) = y_1$. There is a well-known Kalman-type sufficient condition for the controllability of (3.3) due to Silverman and Meadows [97] -see also [100, Prop. 3.5.16]-.

Proposition 15 *Assume that, for some $\bar{t}$ in $[0,T]$,*

$$\operatorname{Span}\left\{\left(\frac{d}{dt} - A(t)\right)^i B(t)_{|t=\bar{t}}\, v;\ v \in \mathbb{R}^m,\ \ i \geq 0\right\} = \mathbb{R}^n, \tag{3.4}$$

then the linear control system (3.3) is controllable on $[0,T]$. Moreover if A and B are analytic on $[0,T]$ and if the linear control system (3.3) is controllable on $[0,T]$, then (3.4) holds for all $\bar{t}$ in $[0,T]$.

Note that if one takes $\bar{u} \equiv 0$, then the above method just gives the well-known fact that, if the time-invariant linear control system

$$\dot{y} = \frac{\partial f}{\partial x}(0,0)y + \frac{\partial f}{\partial u}(0,0)v,$$

is controllable, then the nonlinear control system $\dot{x} = f(x,u)$ is small time locally controllable. However, it may happen that (3.2) does not hold for $\bar{u} \equiv 0$, but holds for other choices of $\bar{u}$. Let us give simple examples.

Example 16 We take $n = 2$, $m = 1$ and again consider (see Example 12 above) the control system

$$\dot{x}_1 = x_2^3,\ \dot{x}_2 = u. \tag{3.5}$$

For $\bar{u} \equiv 0$, $\bar{x} \equiv 0$ and the linearized control system around $(\bar{x}, \bar{u})$ is

$$\dot{y}_1 = 0,\ \dot{y}_2 = v,$$

which is clearly not controllable (one cannot change y_1). Let us now take $\bar{u} \in \mathcal{C}^\infty([0,T];\mathbb{R})$ such that

$$\int_0^{T/2} \bar{u}(t)dt = 0,$$
$$\bar{u}(T-t) = \bar{u}(t),\ \forall\, t \in [0,T].$$

Then one easily checks that

$$\bar{x}_2(T/2) = 0,$$
$$\bar{x}_2(T-t) = -\bar{x}_2(t),\ \forall t \in [0,T],$$
$$\bar{x}_1(T-t) = \bar{x}_1(t),\ \forall t \in [0,T].$$

In particular, we have

$$\bar{x}_1(T) = 0\,,\ \bar{x}_2(T) = 0.$$

The linearized control system around $(\bar{x}, \bar{u})$ is

$$\dot{y}_1 = 3\bar{x}_2^2(t)y_2\,,\ \dot{y}_2 = v.$$

Hence

$$A(t) = \begin{pmatrix} 0 & 3\bar{x}_2(t)^2 \\ 0 & 0 \end{pmatrix}, \ B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and one easily sees that (3.4) holds if and only if

$$\exists\, i \in \mathbb{N} \text{ such that } \frac{d^i \bar{x}_2}{dt^i}(\bar{t}) \neq 0. \tag{3.6}$$

Note that (3.6) holds for at least a $\bar{t}$ in $[0,T]$ if (and only if) $\bar{u} \not\equiv 0$. So (3.2) holds if (and only if) $\bar{u} \not\equiv 0$. We recover that the control system (3.5) is small time locally controllable.

Example 17 *We take $n = 3$, $m = 2$ and the control system is*

$$\dot{x}_1 = u_1, \ \dot{x}_2 = u_2, \ \dot{x}_3 = x_1 u_2 - x_2 u_1. \tag{3.7}$$

Again one can check that the linearized control system around $(\bar{x}, \bar{u})$ is controllable on $[0,T]$ if and only if $\bar{u} \not\equiv 0$. Note that, for the control system (3.7), it is easy to achieve the "return condition" (3.1). Indeed, if

$$\bar{u}(T - t) = -\bar{u}(t), \forall\, t \in [0,T],$$

then

$$\bar{x}(T - t) = \bar{x}(t), \forall\, t \in [0,T]$$

and, in particular,

$$\bar{x}(T) = \bar{x}(0) = 0.$$

So the control system (3.7) is again small time locally controllable (which follows also directly from the Rashevski-Chow theorem [87, 14]).

Example 18 In this example $n = 4$ and $m = 2$ and the control system we consider is given by

$$\dot{x}_1 = x_2, \, \dot{x}_2 = -x_1 + u, \, \dot{x}_3 = x_4, \, \dot{x}_4 = -x_3 + 2x_1 x_2, \tag{3.8}$$

where the state is $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and the control is $u \in \mathbb{R}$. One easily checks that this system satisfies the strong Lie algebra condition at $(0,0)$. Here we want to see that the return method can be used to get large time local controllability. Let us first check that this control system is not small time locally controllable. Indeed if $(x,u) : [0,T] \to \mathbb{R}^4 \times \mathbb{R}$ is a trajectory of the control system (3.8) such that $x(0) = 0$ then

$$x_3(T) = \int_0^T x_1^2(t) \cos(T - t) dt, \tag{3.9}$$

$$x_4(T) = x_1^2(T) - \int_0^T x_1^2(t) \sin(T - t) dt. \tag{3.10}$$

In particular, if $x_1(T) = 0$ and $T \leqslant \pi$, then $x_4(T) \leqslant 0$ with equality if and only if $x \equiv 0$. So, if for $T > 0$ we denote by $\mathcal{P}(T)$ the following controllability property

$\mathcal{P}(T)$ There exists $\varepsilon > 0$ such that, for every $x_0 \in \mathbb{R}^4$ and for every $x_1 \in \mathbb{R}^4$ both of norm less than ε, there exists a bounded measurable function $u : [0,T] \to \mathbb{R}$ such that, if x is the (maximal) solution of (3.8) which satisfies $x(0) = x_0$, then $x(T) = x_1$, then, for every $T \in (0, \pi]$, $\mathcal{P}(T)$ is false. Let us show how the return method can be used to prove that

$$\mathcal{P}(T) \text{ holds for every } T \in (\pi, +\infty). \tag{3.11}$$

Let $T > \pi$. Let

$$\eta = \frac{1}{2}\text{Min } (T - \pi, \pi).$$

Let $\bar{x}_1 : [0, T] \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ such that

$$\bar{x}_1(t) = 0\, \forall t \in [\eta, \pi] \cap [\pi + \eta, T], \tag{3.12}$$
$$\bar{x}_1(t + \pi) = \bar{x}_1(t)\, \forall t \in [0, \eta]. \tag{3.13}$$

Let $\bar{x}_2 : [0, T] \to \mathbb{R}$ and $\bar{u} : [0, T] \to \mathbb{R}$ be such that

$$\bar{x}_2 = \dot{\bar{x}}_1,\ \bar{u} = \dot{\bar{x}}_2 + \bar{x}_1.$$

In particular

$$\bar{x}_2(t) = 0\, \forall t \in [\eta, \pi] \cap [\pi + \eta, T], \tag{3.14}$$
$$\bar{x}_2(t + \pi) = x_2(t)\, \forall t \in [0, \eta]. \tag{3.15}$$

Let $\bar{x}_3 : [0, T] \to \mathbb{R}$ and $\bar{x}_4 : [0, T] \to \mathbb{R}$ be defined by

$$\dot{\bar{x}}_3 = \bar{x}_4,\ \dot{\bar{x}}_4 = -\bar{x}_3 + 2\bar{x}_1\bar{x}_2, \tag{3.16}$$
$$\bar{x}_3(0) = 0,\ \bar{x}_4(0) = 0. \tag{3.17}$$

So $(\bar{x}, \bar{u})$ is a trajectory of the control system (3.8). Then, using (3.9), (3.10), (3.12), (3.14), (3.13), (3.15), one sees that

$$\bar{x}(T) = 0.$$

If $\bar{x}_1 \equiv 0$, $(\bar{x}, \bar{u}) \equiv 0$ and then the linearized control system around $(\bar{x}, \bar{u})$ is not controllable. However, as one easily checks by using the Kalman-type sufficient condition for the controllability of linear time-varying control system due to Silverman and Meadows (Proposition 15), if $\bar{x}_1 \not\equiv 0$ then the linearized control system around $(\bar{x}, \bar{u})$ is controllable. This establishes (3.11).

One may wonder if the local controllability of $\dot{x} = f(x, u)$ implies the existence of u in $\mathcal{C}^\infty([0, T]; \mathbb{R}^m)$ such that (3.1) and (3.2) hold. This is indeed the case as proved by Sontag in [99]. Let us also remark that the above examples suggest that for many choices of $\bar{u}$ then (3.2) holds. This in fact holds in general. More precisely let us assume that

$$\left\{ h(0); h \in \text{ Lie}\left\{ \frac{\partial f}{\partial u^\alpha}(\cdot, 0),\ \alpha \in \mathbb{N}^m \right\} \right\} = \mathbb{R}^n, \tag{3.18}$$

where Lie $\mathcal{F}$ denotes the Lie algebra generated by the vector fields in $\mathcal{F}$; then for generic u in $\mathcal{C}^\infty([0, T]; \mathbb{R}^m)$ (3.2) holds; this is proved in [19], and in [102] if f is analytic. Let us recall that by Theorem 6 due to Hermann and Nagano, (3.18) is a necessary condition for local controllability if f is analytic.

The return method does not seem to give any new interesting controllability result if x lies in a finite dimensional space. In particular

- the small time local controllability in Example 16 follows from the Hermes condition [59, 107],
- the small time local controllability in Example 17 follows from Rashevski-Chow's theorem [87, 14],
- the large time local controllability (more precisely (3.11)) follows from a general result obtained by R. Bianchini about unilateral variations in [10] (one considers the trajectory $(x, u) \equiv (0, 0)$).

However, it gives some new results

- for the stabilization of driftless system (see section 3.2),
- for the controllability of distributed control systems (see sections 3.3 and 3.4).

3.2 Stabilization of driftless systems in finite dimension

In this section we assume that

$$f(x, u) = \sum_{i=1}^{m} u_i f_i(x).$$

Let us first remark that in this case, as pointed out by Brockett in [12], if $m < n$ such systems are in general not asymptotically stabilizable by means of continuous (stationary) feedback laws even if they are in general controllable. But, using the return method, we are going to see that most of the driftless control systems can be globally asymptotically by means of time-varying feedback laws.

Let us denote by $\mathrm{Lie}\{f_1, \ldots, f_m\} \subset C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ the Lie sub-algebra generated by the vector fields $f_1, \ldots, f_m$. Then one has

Theorem 19 *Assume that, for all $x \in \mathbb{R}^n \setminus \{0\}$,*

$$\{g(x);\ g \in Lie\{f_1, \ldots, f_m\}\} = \mathbb{R}^n. \tag{3.19}$$

Then, for all $T > 0$, there exists u in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ such that

$$u(0, t) = 0,\ \forall t \in \mathbb{R}, \tag{3.20}$$

$$u(x, t + T) = u(x, t),\ \forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R}, \tag{3.21}$$

and 0 is globally asymptotically stable for

$$\dot{x} = f(x, u(x, t)) = \sum_{i=1}^{m} u_i(x, t) f_i(x). \tag{3.22}$$

Remark 20 *By Rashevski-Chow's theorem [87, 14], property (3.19) implies (and, by Theorem 6, is necessary if the f_i, $1 \leqslant i \leqslant m$, are analytic) the global controllability of the driftless control system $\dot{x} = \sum_{i=1}^{m} u_i f_i(x)$ in $\mathbb{R}^n \setminus \{0\}$, i.e., for every $x_0 \in \mathbb{R}^n \setminus \{0\}$, every $x_1 \in \mathbb{R}^n \setminus \{0\}$ and every $T > 0$, there exists $u \in L^\infty((0, T); \mathbb{R}^m)$ such that, if $\dot{x} = \sum_{i=1}^{m} u_i(t) f_i(x)$ and $x(0) = x_0$, then $x(T) = 0$.*

This theorem is proved in [17]. Let us just briefly describe the idea of the proof: assume that, for any positive real number T, there exists $\bar{u}$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ satisfying (3.20) and (3.21) such that, if $\dot{x} = f(x, \bar{u}(x,t))$, then

$$x(T) = x(0), \tag{3.23}$$

$$\text{if } x(0) \neq 0, \text{ the linearized control system around } (x, \bar{u}) \text{ is controllable on } [0,T]. \tag{3.24}$$

Using (3.23) and (3.24), one can construct a "small" feedback v in $C^\infty(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^m)$ satisfying (3.20) and (3.21) such that, if

$$\dot{x} = f(x, (\bar{u} + v)(x,t)), \tag{3.25}$$

and $x(0) \neq 0$, then

$$|x(T)| < |x(0)|, \tag{3.26}$$

which implies that 0 is globally asymptotically stable for (3.22) with $u = \bar{u} + v$.

So it only remains to construct $\bar{u}$. In order to get (3.23), one just imposes on $\bar{u}$ that

$$\bar{u}(x,t) = -\bar{u}(x, T-t), \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{3.27}$$

which implies that $x(t) = x(T-t)$, $\forall t \in [0,T]$ and therefore gives (3.23). Finally, one proves that (3.24) holds for "many" $\bar{u}$.

Remark 21 *The first example showing that controllability does not imply asymptotic stabilizability by means of continuous feedback laws is due to Sussmann [105]. See also [103] by Sontag and Sussmann. Two alternatives have been then proposed for the asymptotic stabilization*

- *stabilization by means of time-varying feedback law. See e.g. [5, 17, 20, 21, 95, 102] and the references therein,*
- *stabilization by means of discontinuous feedback law. See e.g. [15, 29, 56, 86, 89, 91, 103, 105] and the references therein.*

3.3 Controllability of the Euler and Navier-Stokes equations

In this section we show how the return method can be used to give controllability results for the Euler and Navier-Stokes equations. Let us introduce some notations. Let $l \in \{2,3\}$ and let Ω be a bounded nonempty connected open subset of $\mathbb{R}^l$ of class $\mathcal{C}^\infty$. Let Γ_0 be an open subset of $\Gamma := \partial\Omega$ and let Ω_0 be an open subset of Ω. We assume that

$$\Gamma_0 \cup \Omega_0 \neq \emptyset. \tag{3.1}$$

The set Γ_0 is the part of the boundary Γ and Ω_0 is the part of the domain Ω on which the control acts. The fluid that we consider is incompressible so that the velocity field y satisfies

$$\text{div } y = 0.$$

On the part of the boundary $\Gamma \backslash \Gamma_0$ where there is no control, the fluid does not cross the boundary: it satisfies

$$y.n = 0 \text{ on } \Gamma \backslash \Gamma_0, \tag{3.2}$$

where n denotes the outward unit normal vector field on Γ. When the fluid is viscous it satisfies on $\Gamma \backslash \Gamma_0$, besides (3.2), some extra conditions which will be specified later on. For the moment, let us just call by **BC** all the boundary conditions (*including* (3.2)) satisfied by the fluid on $\Gamma \backslash \Gamma_0$.

Let us introduce the following definition.

Definition 22 *A trajectory of the Navier-Stokes control system (resp. Euler control system) on the time interval* $[0,T]$ *is a map* $y : [0,T] \times \overline{\Omega} \to \mathbb{R}^l$ *of class* $\mathcal{C}^\infty$ *such that, for some function* $p : [0,T] \times \overline{\Omega} \to \mathbb{R}$ *of class* $\mathcal{C}^\infty$*,*

$$\frac{\partial y}{\partial t} - \nu \Delta y + (y \cdot \nabla) y + \nabla p = 0 \ \textit{in} \ [0,T] \times (\overline{\Omega} \backslash \Omega_0), \tag{3.3}$$

$$(\textit{resp.} \ \frac{\partial y}{\partial t} + (y \cdot \nabla) y + \nabla p = 0 \ \textit{in} \ [0,T] \times (\overline{\Omega} \backslash \Omega_0)) \tag{3.4}$$

$$\textit{div} \ y = 0 \ \textit{in} \ [0,T] \times \overline{\Omega}, \tag{3.5}$$

$$y(t, \cdot) \ \textit{satisfies the boundary conditions BC on} \ \Gamma \backslash \Gamma_0, \ \forall t \in [0,T]. \tag{3.6}$$

The real number $\nu > 0$ appearing in (3.3) is the viscosity. J.–L. Lions' problem of controllability is the following one: let $T > 0$, let y_0 and y_1 in $\mathcal{C}^\infty(\overline{\Omega}; \mathbb{R}^l)$ be such that

$$\text{div } y_0 = 0 \text{ in } \overline{\Omega}, \tag{3.7}$$

$$\text{div } y_1 = 0 \text{ in } \overline{\Omega}, \tag{3.8}$$

$$y_0 \text{ satisfies the boundary conditions BC on } \Gamma \backslash \Gamma_0, \tag{3.9}$$

$$y_1 \text{ satisfies the boundary conditions BC on } \Gamma \backslash \Gamma_0, \tag{3.10}$$

does there exist a trajectory y of the Navier-Stokes or the Euler control system such that

$$y(0, \cdot) = y_0 \text{ in } \overline{\Omega}, \tag{3.11}$$

and, for an appropriate topology –see [78, 79]–,

$$y(T, \cdot) \text{ is "close" to } y_1 \text{ in } \overline{\Omega}? \tag{3.12}$$

That is to say, starting with the initial data y_0 for the velocity field, we ask whether there are trajectories of the control system considered (Navier-Stokes if $\nu > 0$, Euler if $\nu = 0$) which, at a fixed time T, are arbitrarily close to the given velocity field y_1. If this problem has always a solution one says that the control system considered is approximately controllable.

Note that (3.3), (3.5), (3.6) and (3.11) have many solutions. In order to have uniqueness one needs to add extra conditions. These extra conditions are the controls.

In the case of the Euler control system one can even require instead of (3.12) the stronger condition

$$y(T, \cdot) = y_1 \text{ in } \overline{\Omega}. \tag{3.13}$$

If y still exists with this stronger condition, one says that the Euler control system is exactly controllable. Of course, due to the smoothing of the Navier-Stokes equations, one cannot expect to have (3.13) instead of (3.12) for general y_1. We will see in subsection 3.3.2 a way to replace (3.13) in order to recover a natural definition of (exact) controllability of the Navier-Stokes equation.

This section is organized as follows

- in subsection 3.3.1 we consider the case of the Euler control system,
- in subsection 3.3.2 we consider the case of the Navier-Stokes control system.

3.3.1 Controllability of the Euler equations

In this subsection the boundary conditions BC in (3.6), (3.9), and (3.10) are respectively

$$y(t,x).n(x) = 0, \forall (t,x) \in [0,T] \times (\Gamma \backslash \Gamma_0), \tag{3.14}$$

$$y_0(x).n(x) = 0, \quad \forall x \in \Gamma \backslash \Gamma_0, \tag{3.15}$$

$$y_1(x).n(x) = 0, \quad \forall x \in \Gamma \backslash \Gamma_0. \tag{3.16}$$

For simplicity we assume that

$$\Omega_0 = \emptyset,$$

i.e. we study the case of boundary control (see [22] when $\Omega_0 \neq \emptyset$ and $l = 2$). In that case a control is given by $y.n$ on Γ_0 with $\int_{\Gamma_0} y.n = 0$ and by curl y if $l = 2$ and (curl $y).n$ if $l = 3$ at the points of $[0,T] \times \Gamma_0$ where $y.n < 0$: these boundary conditions, (3.14), and the initial condition (3.11) imply the uniqueness of the solution to the Euler equations (3.4) -up to an arbitrary function of t which may be added to p-; see also [67] for the existence of a solution.

Let us first point out that in order to have (exact) controllability one needs that

$$\Gamma_0 \text{ intersects every connected component of } \Gamma. \tag{3.17}$$

Indeed, let $\mathcal{C}$ be a connected component of Γ which does not intersect Γ_0 and assume that, for some smooth Jordan curve γ_0 on $\mathcal{C}$ (if $l = 2$ one takes $\gamma_0 = \mathcal{C}$),

$$\int_{\gamma_0} y_0.ds \neq 0, \tag{3.18}$$

but that

$$y_1(x) = 0, \forall x \in \mathcal{C}. \tag{3.19}$$

Then there is no solution to our problem, that is there is no $y \in \mathcal{C}^\infty([0,T] \times \overline{\Omega}; \mathbb{R}^2)$ and $p \in \mathcal{C}^\infty([0,T] \times \overline{\Omega}; \mathbb{R})$ such that (3.5), (3.4), (3.11), (3.13), and (3.14) hold. Indeed, if such a solution (y,p) exists, then, by Kelvin's law,

$$\int_{\gamma(t)} y(t,\cdot).ds = \int_{\gamma_0} y_0.ds (\in \mathbb{R}), \tag{3.20}$$

where $\gamma(t)$ is the Jordan curve obtained, at time t, from the points of the fluids which at time 0 where on γ_0; in other words $\gamma(t)$ is the image of γ_0 by the flow map associated to the time-varying vector field y. But (3.13), (3.18), (3.19) and (3.20) are in contradiction.

Conversely, if (3.17) holds, then the Euler control system is exactly controllable:

Theorem 23 *Assume that Γ_0 intersects every connected component of $\partial\Omega$. Then the Euler control system is exactly controllable.*

Theorem 23 has been proved in

- [18] when Ω is simply-connected and $l = 2$,
- [22] when Ω is multi-connected and $l = 2$,
- [50] when Ω is contractible and $l = 3$,
- [51] when Ω is not contractible and $l = 3$.

The strategy of the proof of Theorem 23 relies on the "return method". Applied to the controllability of the Euler control system the return method consists in looking for $(\bar{y}, \bar{p})$ such that (3.5), (3.4), (3.11), (3.13) hold, with $y = \bar{y}, p = \bar{p},\ y_0 = y_1 = 0$ and such that the linearized control system around the trajectory $\bar{y}$ is controllable under the assumptions of Theorem 23. With such a $(\bar{y}, \bar{p})$ one may hope that there exists (y, p) -close to $(\bar{y}, \bar{p})$- satisfying the required conditions, at least if y_0 and y_1 are "small". Finally, by using some scaling arguments, one can deduce from the existence of (y, p) when y_0 and y_1 are "small" the existence of (y, p) even if y_0 and y_1 are not "small".

Let us emphasize that one cannot take $(\bar{y}, \bar{p}) = (0, 0)$. Indeed, with such a choice of $(\bar{y}, \bar{p})$, (3.5), (3.4), (3.11), (3.13) hold, with $y = \bar{y}, p = \bar{p},\ y_0 = y_1 = 0$, but the linearized control system around $\bar{y} = 0$ is not at all controllable. Indeed the linearized control system around $\bar{y} = 0$ is

$$\text{div } z = 0 \text{ in } \bar{[}0, T] \times \Omega, \tag{3.21}$$

$$\frac{\partial z}{\partial t} + \nabla \pi = 0 \text{ in } [0, T] \times \overline{\Omega}, \tag{3.22}$$

$$z(t, x).n(x) = 0,\ \forall (t, x) \in [0, T] \times (\Gamma \backslash \Gamma_0).$$

Taking the curl of (3.22), one gets

$$\frac{\partial \text{curl } z}{\partial t} = 0,$$

which clearly shows that the linearized control system is not controllable (one cannot modify the curl with the control). So one needs to consider other $(\bar{y}, \bar{p})$. Let us briefly explain how one constructs "good" $(\bar{y}, \bar{p})$ when $l = 2$ and Ω is simply connected. In such a case one easily checks the existence of a harmonic function θ in $\mathcal{C}^\infty(\overline{\Omega})$ such that

$$\nabla \theta(x) \neq 0,\ \forall x \in \overline{\Omega},$$

$$\frac{\partial \theta}{\partial n} = 0 \text{ on } \Gamma \backslash \Gamma_0.$$

Let $\alpha \in \mathcal{C}^\infty(0, T)$ vanishing 0 and T. Let

$$(\bar{y}, \bar{p})(t, x) = (\alpha(t) \nabla \theta(x), -\alpha'(t)\theta(x) - \frac{1}{2}\alpha^2(t) \mid \nabla \theta(x) \mid^2).$$

Then (3.5), (3.4), (3.11), (3.13) hold, with $y = \bar{y}, p = \bar{p},\ y_0 = y_1 = 0$. Moreover, using arguments relying on an extension method analogous to the one introduced by Russell in [93], one can see that the linearized control system around $\bar{y}$ is controllable.

When Γ_0 does not intersect all the connected components of Γ_0 one can get, if $l = 2$, approximate controllability and even exact controllability outside every arbitrarily small neighborhood of the union $\Gamma^\star$ of the connected components of Γ which does not intersect Γ_0. More precisely, one has

Theorem 24 *[22]. Assume that $l = 2$. There exists a constant c_0 depending only on Ω such that, for every Γ_0 as above, every $T > 0$, every $\varepsilon > 0$, and every y_0, y_1 in $\mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying (3.7), (3.8), (3.15) and (3.16), there exists a trajectory y of the Euler control system on $[0,T]$ satisfying (3.11) such that*

$$y(T,x) = y_1(x),\ \forall x \in \overline{\Omega} \text{ such that } dist(x,\Gamma^\star) \geqslant \varepsilon, \tag{3.23}$$

$$|y(T,\cdot)|_{L^\infty} \leqslant c_0(|y_0|_{L^2} + |y_1|_{L^2} + |curl y_0|_{L^\infty} + |curl y_1|_{L^\infty}). \tag{3.24}$$

In (3.23), $\text{dist}(x,\Gamma^\star)$ denotes the distance of x to $\Gamma^\star$, i.e.

$$\text{dist } (x,\Gamma^\star) = \text{ Min } \{|x - x^\star|; x^\star \in \Gamma^\star\}. \tag{3.25}$$

We use the convention dist $(x,\emptyset) = +\infty$ and so Theorem 24 implies Theorem 23. In (3.24), $|\ |_{L^r}$, for $r \in [1,+\infty]$, denotes the L^r-norm on Ω. Let us point out that, y_0, y_1, and T as in Theorem 24 being given, it follows from (3.23) and (3.24) that, for every r in $[1,+\infty)$,

$$\lim_{\varepsilon\to 0^+} |y(T,\cdot) - y_1|_{L^r} = 0; \tag{3.26}$$

that is Theorem 24 implies approximate controllability in the L^r-space for every r in $[1,+\infty)$. Let us notice that, if $\Gamma^\star \neq \emptyset$, then, again by Kelvin's law, approximate controllability for the L^∞-norm does not hold. More precisely let us consider the case $l = 2$ and let us denote by $\Gamma_1^\star, \dots, \Gamma_k^\star$ the connected components of Γ which does not meet Γ_0. Let y_0, y_1 in $\mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying (3.7), (3.8), (3.15) and (3.16). Assume that, for some $i \in \{1,\dots,k\}$,

$$\int_{\Gamma_i^\star} y_0.ds \neq \int_{\Gamma_i^\star} y_1.ds.$$

Then, for $\varepsilon > 0$ small enough, there is no trajectory y of the Euler control system on $[0,T]$ satisfying (3.11) such that

$$|y(T,\cdot) - y_1|_{L^\infty} \leq \varepsilon. \tag{3.27}$$

One may wonder if, on the contrary, one assumes that

$$\int_{\Gamma_i^\star} y_0.ds = \int_{\Gamma_i^\star} y_1.ds,\ \forall i \in \{1,\dots,k\}. \tag{3.28}$$

Then Glass has proved that one has approximate controllability in L^∞ and even in the Sobolev spaces $W^{1,p}$ for every $p \in [1,+\infty)$. Indeed he has proved in [52]

Theorem 25 *Assume that $l = 2$. For every $T > 0$, and every y_0, y_1 in $\mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying (3.7), (3.8), (3.15), (3.16) and (3.28). there exists a sequence $(y^k)_{k\in\mathbb{N}}$ of trajectories of the Euler control system on $[0,T]$ satisfying (3.11) such that*

$$y^k(T,x) = y_1(x), \forall x \in \overline{\Omega} \text{ such that } dist(x,\Gamma^\star) \geqslant 1/k, \forall k \in \mathbb{N}, \tag{3.29}$$

$$y^k(T,\cdot) \to y_1 \text{ in } W^{1,p}(\Omega) \text{ as } k \to +\infty, \forall p \in [1,+\infty). \tag{3.30}$$

Again the convergence in (3.30) is optimal: since the vorticity curl y is conserved along the trajectories of the vector field y one cannot have the convergence in $W^{1,\infty}$. In order to have convergence in $W^{1,\infty}$ one needs to add a relation between curl y_0 and curl y_1 on the Γ_i for $i \in \{1,\dots,l\}$. In that direction, Glass has proved in [52]

Theorem 26 *Assume that $l = 2$. Let $T > 0$, and let y_0, y_1 in $\mathcal{C}^\infty(\overline{\Omega}; \mathbb{R}^2)$ be such that (3.7), (3.8), (3.15). (3.16) and (3.28) hold. Assume that, for every $i \in \{1, \ldots, l\}$, there exits a diffeomorphism D_i of $\Gamma_i^\star$ preserving the orientation such that*

$$curl\ y_1 = (curl\ y_0) \circ D_i.$$

Then there exists a sequence $(y^n)_n$ of trajectories of the Euler control system on $[0, T]$ satisfying (3.11), (3.29) and

$$y^k(T, \cdot) \to y_1 \text{ in } W^{2,p}(\Omega) \text{ as } k \to +\infty, \forall p \in [1, +\infty). \tag{3.31}$$

Again, one cannot expect a convergence in $W^{2,\infty}$ without an extra assumption on y_0 and y_1 -see [52]-.

Let us end this subsection by mentioning that, for the Vlasov-Poisson system (a system related to the Euler equations), Glass, using the return method, has obtained beautiful controllability results in [54].

3.3.2 Controllability of the Navier-Stokes equations

In this subsection, $\nu > 0$. We now need to specify the boundary conditions BC. Three types of conditions are considered:

- Stokes boundary condition,
- Navier boundary condition,
- curl condition.

The Stokes boundary condition is the well-known no-slip boundary condition

$$y = 0 \text{ on } \Gamma \backslash \Gamma_0, \tag{3.32}$$

which of course implies (3.2).

The Navier boundary condition [84] imposes, condition (3.2), which is always assumed, and

$$\overline{\sigma} y.\tau + (1 - \overline{\sigma}) n^i \left(\frac{\partial y^i}{\partial x^j} + \frac{\partial y^j}{\partial x^i} \right) \tau^j = 0 \text{ on } \Gamma \backslash \Gamma_0, \tag{3.33}$$

where $\overline{\sigma}$ is a constant in $[0, 1)$, $n = (n^1, \ldots, n^l)$ and $\tau = (\tau^1, \ldots, \tau^l)$ is any tangent vector field on the boundary Γ. In (3.33) we also have used the usual summation convention. Note that the Stokes boundary condition (3.32) corresponds to the case $\overline{\sigma} = 1$, which we will not include in the Navier boundary condition considered here. The boundary condition (3.33) with $\overline{\sigma} = 0$ corresponds to the case where there the fluid slips on the wall without friction. It is the appropriate physical model for some flow problems; see [49] for example. The case $\overline{\sigma} \in (0, 1)$ corresponds to a case where there the fluid slips on the wall with friction; it is also used in models of turbulence with rough walls; see, e.g., [74]. Note that in [16] F. Coron has derived rigorously the Navier boundary condition (3.33) from the boundary condition at the kinetic level (Boltzmann equation) for compressible fluids. Let us also recall that Bardos, Golse, and Levermore have derived in [6] the incompressible Navier-Stokes equations from a Boltzmann equation.

Let us point out that, using (3.2), one sees that, if $l = 2$ and if τ is the unit tangent vector field on $\partial\Omega$ such that (τ, n) is a direct basis of $\mathbb{R}^2$, (3.33) is equivalent to

$$\sigma y.\tau + \text{curl } y = 0 \text{ on } \Gamma \backslash \Gamma_0,$$

with $\sigma \in \mathcal{C}^\infty(\Gamma;\mathbb{R})$ defined by

$$\sigma(x) = \frac{2(1-\overline{\sigma})\kappa(x) - \overline{\sigma}}{1-\overline{\sigma}}, \forall x \in \Gamma, \tag{3.34}$$

where κ is the curvature of Γ defined through the relation $\frac{\partial n}{\partial \tau} = \kappa\tau$. In fact we will not use this particular character of (3.34) in our considerations; Theorem 30 below holds for every $\sigma \in \mathcal{C}^\infty(\Gamma;\mathbb{R})$.

Finally the curl condition is considered in dimension 2 ($l = 2$). This condition is, condition (3.2) which is always assumed, and

$$\text{curl } y = 0 \text{ on } \Gamma \backslash \Gamma_0. \tag{3.35}$$

It corresponds to the case $\sigma = 0$ in (3.34).

As mentioned in the introduction, due to smoothing property of the Navier-Stokes equation, one cannot expect to get (3.13), at least for general y_1. For these equations, the good notion for exact controllability is not passing from a given state y_0 to another given state y_1. As proposed by A. Fursikov and O. Yu Imanuvilov in [44, 45], the good definition for exact controllability is passing from a given state y_0 to a given *trajectory* $\hat{y}_1$. This leads to the following, still open, problem of exact controllability of the Navier-Stokes equation with the Stokes, or Navier, or curl condition.

Open Problem 27 *Let $T > 0$. Let $\hat{y}_1$ be a trajectory of the Navier-Stokes control system on $[0,T]$. Let $y_0 \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^l)$ satisfying (3.7) and (3.9). Does there exist a trajectory y of the Navier-Stokes control system on $[0,T]$ such that*

$$y(0,x) = y_0(x), \forall x \in \overline{\Omega}, \tag{3.36}$$

$$y(T,x) = \hat{y}_1(T,x), \forall x \in \overline{\Omega}? \tag{3.37}$$

Let us point out that the (global) approximate controllability of the Navier-Stokes control system is also an open problem. Related to the open problem 27 one knows two types of results

- local results,
- global results,

which we briefly describe in the next subsections

Local results

These results do not rely on the return method, but on the HUM and difficult Carleman's inequalities. Let us introduce the following definition.

Definition 28 *The Navier-Stokes control system is locally (for the Sobolev H^1-norm) exactly controllable along the trajectory $\hat{y}_1$ on $[0,T]$ of the Navier-Stokes control system if there exists $\epsilon > 0$ such that, for every $y_0 \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R}^l)$ satisfying (3.7), (3.9) and*

$$| y_0 - \hat{y}_1(0,\cdot) |_{H^1(\Omega)} < \epsilon,$$

there exists a trajectory y of the Navier-Stokes control system on $[0,T]$ satisfying (3.36) and (3.37).

Then one has the following results.

Theorem 29 *The Navier-Stokes control system is locally exactly controllable* *(i) along every trajectory for the curl condition or the Navier boundary condition in dimension 2 (l=2),*

(ii) along every trajectory if $\Gamma_0 = \Gamma$,

(iii) along every stationary trajectory with compact support for the Stokes condition.

Case (i) has been obtained by A.V. Fursikov and O. Yu Imanuvilov in [45, 46]. Case (ii) has been obtained by A.V. Fursikov in [43]. Case (iii) has been obtained by O. Yu Imanuvilov in [64] and [65].

Global results

Let $d \in \mathcal{C}^0(\overline{\Omega};\mathbb{R})$ be defined by

$$d(x) = \text{ dist } (x,\Gamma) = \text{ Min } \{|x-x'|; x' \in \Gamma\}.$$

In [23] the following theorem is proved

Theorem 30 *Let $T>0$, let y_0 and y_1 in $\mathcal{C}^\infty(\overline{\Omega},\mathbb{R}^2)$ be such that (3.7) and (3.8) hold. Let us also assume that y_0 and y_1 satisfy the Navier boundary condition (3.33). Then there exists a sequence $(y^k; k \in \mathbb{N})$ of trajectories of the Navier-Stokes control system on $[0,T]$ with the Navier boundary condition (3.33) such that, as $k \to +\infty$,*

$$\int_\Omega d^\mu |y^k(T,\cdot) - y_1| \to 0, \ \forall \mu > 0, \tag{3.38}$$

$$|y^k(T,\cdot) - y_1|_{W^{-1,\infty}(\Omega)} \to 0, \tag{3.39}$$

and, for all compact K included in $\Omega \cup \Gamma_0$,

$$|y^k(T,\cdot) - y_1|_{L^\infty(K)} + |curl\ y^k(T,\cdot) - curl\ y_1|_{L^\infty(K)} \to 0. \tag{3.40}$$

In this theorem, $W^{-1,\infty}(\Omega)$ denotes the usual Sobolev space of first derivatives of functions in $L^\infty(\Omega)$ and $|\ |_{W^{-1,\infty}(\Omega)}$ one of it's usual norms, for example the norm given in [2, Section 3.10].

As in the proof of the controllability of the 2-D Euler equations of incompressible inviscid fluids (see subsection 3.3.1), one uses the return method. Let us recall that it consists in looking for a trajectory of the Navier-Stokes control system $\bar{y}$ such that

$$\bar{y}(0,\cdot) = \bar{y}(T,\cdot) = 0 \text{ in } \overline{\Omega}, \tag{3.41}$$

and such that the linearized control system around the trajectory $\bar{y}$ has a controllability in a "good" sense. With such a $\bar{y}$ one may hope that there exists y – close to $\bar{y}$ – satisfying the required conditions, at least if y_0 and y_1 are "small". Note that the linearized control system around $\bar{y}$ is

$$\frac{\partial z}{\partial t} - \nu\Delta z + (\bar{y}\cdot\nabla)z + (z\cdot\nabla)\bar{y} + \nabla\pi = 0 \text{ in } [0,T]\times(\overline{\Omega}\backslash\Omega_0), \tag{3.42}$$

$$\text{div} z = 0 \text{ in } [0,T]\times\overline{\Omega}, \tag{3.43}$$

$$z.n = 0 \text{ on } [0,T]\times(\Gamma\backslash\Gamma_0), \tag{3.44}$$

$$\sigma z.\tau + \text{curl } z = 0 \text{ on } [0,T] \times (\Gamma \backslash \Gamma_0). \tag{3.45}$$

In [45, 46] A. Fursikov and O. Immanuvilov have proved that this linear control system is controllable (see also [76] for the approximate controllability). Of course it is tempting to consider the case $\bar{y} = 0$. Unfortunately, it is not clear how to deduce from the controllability of the linear system (3.42) with $\bar{y} = 0$, the existence of a trajectory y of the Navier-Stokes control system (with the Navier boundary condition) satisfying (3.11) and (3.12) if y_0 and y_1 are *not small*. For this reason, one does not use $\bar{y} = 0$, but $\bar{y}$ similar to the one constructed in [22] to prove the controllability of the 2-D Euler equations of incompressible inviscid fluids; these $\bar{y}$ are chosen to be "large" so that, in some sense, "Δ" is small compared to "$(\bar{y} \cdot \nabla) + (\ \cdot \nabla)\bar{y}$".

Remark 31 *In fact, with the $\bar{y}$ we use, one does not have (3.41): we have only the weaker property*

$$\bar{y}(0,\cdot) = 0,\ \bar{y}(T,\cdot) \textit{ is "close" to } 0 \textit{ in } \overline{\Omega}. \tag{3.46}$$

However the controllability of the linearized control system around $\bar{y}$ is strong enough to take care of the fact that $\bar{y}(\cdot,T)$ is not equal to 0 but only close to 0. A similar observation can be found, in a different setting, in [109]. With the notation introduced by Sussmann in [109], $\bar{y}$ plays the role of ξ_* in the proof of [109, Theorem12] and the Euler control system plays for the Navier-Stokes control system the role played by $\mathcal{G}$ for $\mathcal{F}$.

Note that (3.38), (3.39), and (3.40) are not strong enough to imply

$$|y^k(T,\cdot) - y_1|_{L^2(\Omega)} \to 0, \tag{3.47}$$

i.e. to get the approximate controllability in L^2 of the Navier-Stokes control system. But, in the special case where $\Gamma_0 = \Gamma$, (3.38), (3.39), and (3.40) are strong enough to imply (3.47). Moreover, gluing together the proofs of Theorem 29 and of (ii) of Theorem 30, one gets

Theorem 32 *[28] The open problem 27 has a positive answer when $\Gamma_0 = \Gamma$ and $l = 2$.*

This result has been recently generalized by Fursikov and Immanuvilov in [48] to the case $l = 3$ Let us also mention that, in [38], Fabre has obtained, in every dimension, an approximate controllability result of two natural "cut off" Navier-Stokes equations. Her proof relies on a general method introduced by Zuazua in [113] to prove approximate controllability of semilinear wave equations. This general method is based on H.U.M. (Hilbert Uniqueness Method, due to J.–L. Lions [77]) and on a fixed point technique; see also [39] where Fabre, Puel and Zuazua use this method to prove approximate controllability of semilinear heat equations.

Remark 33 *It is usually accepted that the viscous Burgers equation provides a realistic simplification of the Navier-Stokes system in fluid Mechanics. But J.I. Diaz has proved in [35] that the viscous Burgers equation is not approximately controllable; see also [44]. For the nonviscous Burgers equation, results have been obtained by Ancona and Marson in [4] and by Horsin in [62].*

3.4 Local controllability of a 1-D tank containing a fluid modelled by the Saint-Venant equations

In this section, we consider a 1-D tank containing an inviscid incompressible irrotational fluid. The tank is subject to one-dimensional horizontal moves. We assume that the horizontal acceleration of the tank is small compared to the gravity constant and that the height of the fluid is small compared to the length of the tank. These physical considerations motivate the use of the Saint-Venant equations [94] (also called shallow water equations) to describe the motion of the fluid; see e.g. [34, Sec. 4.2]. Hence the dynamics equations considered are -see [37]-

$$H_t\,(t,x) + (Hv)_x\,(t,x) = 0, \tag{3.1}$$

$$v_t\,(t,x) + \left(gH + \frac{v^2}{2}\right)_x (t,x) = -u\,(t)\,, \tag{3.2}$$

$$v(t,0) = v(t,L) = 0, \tag{3.3}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}\,(t) = u\,(t)\,, \tag{3.4}$$

$$\frac{\mathrm{d}D}{\mathrm{d}t}\,(t) = s\,(t)\,, \tag{3.5}$$

where (see figure 3.1),

- L is the length of the 1-D tank,
- $H\,(t,x)$ is the height of the fluid at time t and for $x \in [0,L]$,
- $v\,(t,x)$ is the horizontal water velocity of the fluid *in a referential attached to the tank* at time t and for $x \in [0,L]$ (in the shallow water model, all the points on the same vertical have the same horizontal velocity),
- $u\,(t)$ is the horizontal acceleration of the tank in the absolute referential,
- g is the gravity constant,
- s is the horizontal velocity of the tank,
- D is the horizontal displacement of the tank.

This is a control system, denoted Σ, where

- the state is $Y = (H, v, s, D)$,
- the control is $u \in \mathbb{R}$.

Our goal is to study the local controllability of this control system Σ around the equilibrium point

$$(Y_e, u_e) := ((H_e, 0, 0, 0), 0).$$

This problem has been raised by Dubois, Petit and Rouchon in [37].

Of course, the total mass of the fluid is conserved so that, for every solution of (3.1) to (3.3),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L H\,(t,x)\,\mathrm{d}x = 0. \tag{3.6}$$

(One gets (3.6) by integrating (3.1) on $[0,L]$ and by using (3.3) together with an integration by parts.) Moreover, if H and v are of class $\mathcal{C}^1$, it follows from (3.2) and (3.3) that

$$H_x(t,0) = H_x(t,L) \quad (= -u\,(t)\,/g). \tag{3.7}$$

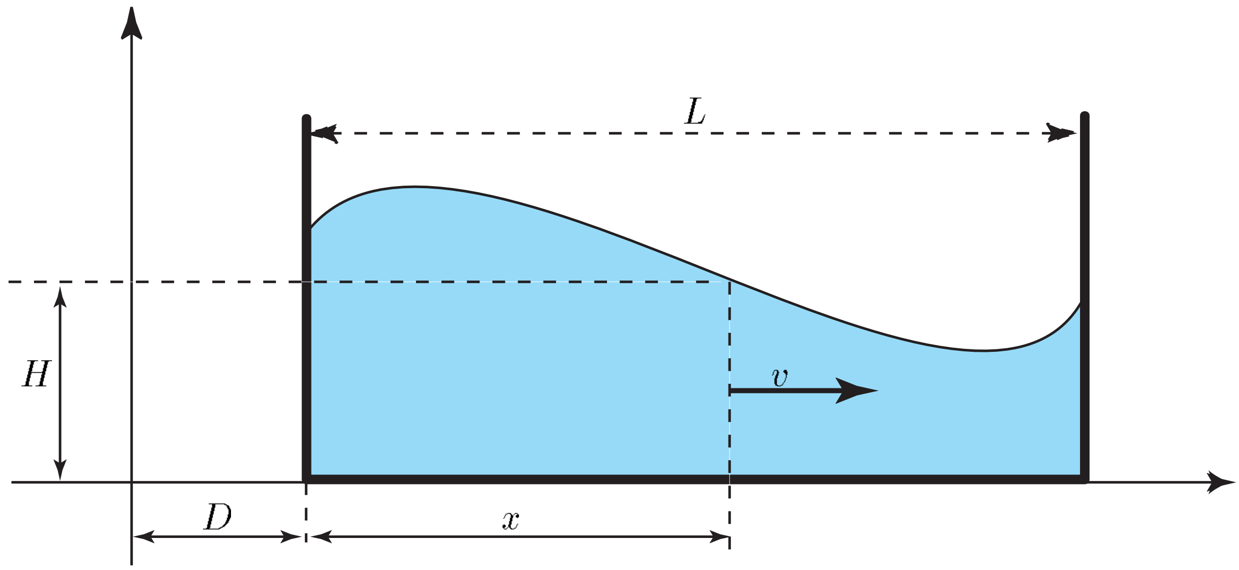

Figure 3.1: Fluid in the 1-D tank

Therefore we introduce the vectorial space E of functions $Y = (H, v, s, D) \in \mathcal{C}^1([0, L]) \times \mathcal{C}^1([0, L]) \times \mathbb{R} \times \mathbb{R}$ such that

$$H_x(0) = H_x(L), \tag{3.8}$$
$$v(0) = v(L) = 0, \tag{3.9}$$

and consider the affine subspace $\mathcal{Y} \subset E$ of $Y = (H, v, s, D) \in E$ satisfying

$$\int_0^L H(x)\mathrm{d}x = LH_e. \tag{3.10}$$

The vectorial space E is equipped with the usual norm

$$|Y| := |H|_1 + |v|_1 + |s| + |D|,$$

where, for $w \in \mathcal{C}^1([0, L])$,

$$|w|_1 := \mathrm{Max}\{|w(x)| + |w_x(x)|;\ x \in [0, L]\}.$$

With these notations, we can define a trajectory of the control system Σ.

Definition 34 *Let T_1 and T_2 be two real numbers satisfying $T_1 \leqslant T_2$. A function $(Y, u) = ((H, v, s, d), u) : [T_1, T_2] \to \mathcal{Y} \times \mathbb{R}$ is a trajectory of the control system Σ if*

(i) the functions H and v are of class $\mathcal{C}^1$ on $[T_1, T_2] \times [0, L]$,

(ii) the functions s and D are of class $\mathcal{C}^1$ on $[T_1, T_2]$ and the function u is continuous on $[0, T]$,

(iii) the equations (3.1) to (3.5) hold for every $(t, x) \in [T_1, T_2] \times [0, L]$.

Our main result states that the control system Σ is locally controllable around the equilibrium point (Y_e, u_e). More precisely, one has the following theorem.

Theorem 35 *([26]) There exist $T > 0$, $C_0 > 0$ and $\eta > 0$ such that, for every $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}$, and for every $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}$ such that*

$$|H_0 - H_e|_1 + |v_0|_1 < \eta, \ |H_1 - H_e|_1 + |v_1|_1 < \eta, \ |s_1 - s_0| + |D_1 - s_0 T - D_0| < \eta,$$

there exists a trajectory $(Y, u) : [0, T] \ \to \mathcal{Y} \times \mathbb{R}, \ t \mapsto ((H(t), v(t), s(t), D(t)), u(t))$ of the control system Σ such that

$$Y(0) = Y_0 \text{ and } Y(T) = Y_1, \tag{3.11}$$

and, for every $t \in [0, T]$,

$$\begin{aligned} &|H(t) - H_e|_1 + |v(t)|_1 + |u(t)| \leqslant \\ &\quad C_0 \left(\sqrt{|H_0 - H_e|_1 + |v_0|_1 + |H_1 - H_e|_1 + |v_1|_1} + |s_1 - s_0| + |D_1 - s_0 T - D_0| \right). \end{aligned} \tag{3.12}$$

As a corollary of this theorem, any steady state $Y_1 = (H_e, 0, 0, D_1)$ can be reached from any other steady state $Y_0 = (H_e, 0, 0, D_0)$. More precisely, one has the following corollary.

Corollary 36 *([26]) Let T, C_0 and η be as in Theorem 35. Let D_0 and D_1 be two real numbers and let $\eta_1 \in (0, \eta]$. Then, there exists a trajectory $(Y, u) : [0, T(|D_1 - D_0| + \eta_1)/\eta_1] \ \to \mathcal{Y} \times \mathbb{R}, \ t \mapsto ((H(t), v(t), s(t), D(t)), u(t))$ of the control system Σ such that*

$$Y(0) = (H_e, 0, 0, D_0) \text{ and } Y(T(|D_1 - D_0| + \eta_1)/\eta_1) = (H_e, 0, 0, D_1), \tag{3.13}$$

$$|H(t) - H_e|_1 + |v(t)|_1 + |u(t)| \leqslant C_0 \eta_1 \ \forall t \in [0, T(|D_1 - D_0| + \eta_1)/\eta_1]. \tag{3.14}$$

Let us explain on a toy model in finite dimension what are the main ideas to prove Theorem 35. Let us write

$$\begin{gathered} H_0 = H_e + \epsilon h_0^1 + \epsilon^2 h_0^2, \ v_0 = \epsilon v_0^1 + \epsilon^2 v_0^2, \ s_0 = \epsilon s_0^1, \ D_0 = \epsilon d_0^1, \\ H_1 = H_e + \epsilon h_1^1 + \epsilon^2 h_1^2, \ v_1 = \epsilon v_1^1 + \epsilon^2 v_1^2, \ s_1 = s_1^1, \ D_1 = \epsilon d_0^1, \ u = \epsilon u^1, \end{gathered}$$

where $\epsilon > 0$ is small. (In fact we should write also the terms in ϵ^2 for u, s, D but they turn out to be useless.) Let us expand the solution (H, v, s, D) of

$$\begin{cases} H_t(t, x) + (Hv)_x(t, x) = 0, \\ v_t(t, x) + \left(gH + \frac{v^2}{2} \right)_x (t, x) = -u(t), \\ v(t, 0) = v(t, L) = 0, \\ \frac{\mathrm{d}s}{\mathrm{d}t}(t) = u(t), \\ \frac{\mathrm{d}D}{\mathrm{d}t}(t) = s(t), \\ H(0, x) = H_0(x), \ v(x, 0) = v_0(x), \ s(0) = s_0, \ D(0) = D_0, \end{cases}$$

in terms of ϵ:

$$H = H_e + \epsilon h_1 + \epsilon^2 h_2 + \ldots, \ v = \epsilon v_1 + \epsilon^2 v_2 + \ldots, \ s = \epsilon s_1, \ D = \epsilon D_1, u = \epsilon u_1.$$

One has

$$\begin{cases} h_{1t}\,(t,x) + H_e v_{1x}\,(t,x) = 0, \\ v_{1t}\,(t,x) + g h_{1x} = -u_1\,(t)\,, \\ v_1(t,0) = v_1(t,L) = 0, \\ \frac{\mathrm{d}s_1}{\mathrm{d}t}\,(t) = u_1\,(t)\,, \\ \frac{\mathrm{d}D_1}{\mathrm{d}t}\,(t) = s_1\,(t)\,, \end{cases} \tag{3.15}$$

$$\begin{cases} h_{2t}\,(t,x) + H_e v_{2x}\,(t,x) = -(h_1 v_1)_x, \\ v_{2t}\,(t,x) + g h_{2x} = -v_1 v_{1x}, \\ v_1(t,0) = v_1(t,L) = 0, \end{cases} \tag{3.16}$$

together with the initial conditions

$$h_1(0,x) = h_0^1(x),\ v_1(0,x) = v_0^1(x),\ s_1(0) = s_1,\ D_1(0) = D_1,\ h_2(0,x) = h_0^2(x),\ v_2(0,x) = v_0^2(x).$$

We consider (3.15) as a control system where the control is u_1 and where the state $(h_1, v_1, s_1, D_1) \in C^1([0,L]) \times C^1([0,L]) \times \mathbb{R} \times \mathbb{R}$ satisfies

$$h_1(L-x) = -h_1(x),\ v_1(L-x) = v_1(x),\ v_1(0) = v_1(L) = 0. \tag{3.17}$$

Note that, by [37], this control system is controllable in any time $T > L/\sqrt{H_e g}$. Similarly we consider (3.16) as a control system where the state $(h_2, v_2) : [0,L] \to \mathbb{R}^2$ satisfies

$$h_2(L-x) = h_2(x),\ h_{2x}(0) = h_{2x}(L) = 0,\ \int_0^L h_2(x)\mathrm{d}x = 0 \tag{3.18}$$

$$v_2(L-x) = -v_2(x),\ v_2(0) = v_2(L) = 0. \tag{3.19}$$

The control in this system (i.e. $(h_1 v_1)_x$ and $v_1 v_{1x}$) depends only on the state of the control system (3.15) and is quadratic with respect to this state. The first two equations of (3.15) and (3.16) are the usual wave equations. A natural analogous of the wave equation in finite dimension is the oscillator equation. Hence a natural analogous to our control system (3.15) and (3.16) is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u, \tag{3.20}$$

$$\dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1 x_2, \tag{3.21}$$

$$\dot{s} = u,\ \dot{D} = s \tag{3.22}$$

where the state is $(x_1, x_2, x_3, x_4, s, D) \in \mathbb{R}^6$ and the control $u \in \mathbb{R}$. This control system is our toy model. We call it $\mathcal{T}$.

The linearized control system of $\mathcal{T}$ around $(0,0) \in \mathbb{R}^6 \times \mathbb{R}$ is

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3,\ \dot{s} = u,\ \dot{D} = s. \tag{3.23}$$

This linear control sytem (3.23) is not controllable. However, as one easily checks, (3.23) is steady-state controllable for arbitrary time T, that is, for every $(D_0, D_1) \in \mathbb{R}^2$ and for every $T > 0$, there exists a trajectory $((x_1, x_2, x_3, x_4, s, D), u) : [0,T] \to \mathbb{R}^6 \times \mathbb{R}$ of the linear control system (3.23) such that

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = s(0) = 0,\ D(0) = D_0,$$
$$x_1(T) = x_2(T) = x_3(T) = x_4(T) = s(T) = 0,\ D(T) = D_1.$$

But the same does not hold for the nonlinear control system $\mathcal{T}$. Indeed it follows from Example 18 that if $((x_1,x_2,x_3,x_4,s,D),u):[0,T]\to\mathbb{R}^6\times\mathbb{R}$ is a trajectory of the control system $\mathcal{T}$such that

$$x_1(0)=x_2(0)=x_3(0)=x_4(0)=s(0)=0,$$
$$x_1(T)=x_2(T)=x_3(T)=x_4(T)=s(T)=0,\ D(T)\neq D(0),$$

then $T>\pi$. This minimal time for steady state controllability is optimal. This can be seen by using the return method as in Example 18. Indeed let $T>\pi$ and let $(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4,\bar{u})$ be as in example 18 except that one requires also

$$\int_0^\eta \bar{x}_1(t)\mathrm{d}t=0,\ \int_0^\eta\int_0^t \bar{x}_1(s)\mathrm{d}s\mathrm{d}t=0. \tag{3.24}$$

Let us define $(\bar{s},\bar{D}):[0,T]\to\mathbb{R}^2$ by

$$\dot{\bar{s}}=\bar{u},\ \bar{s}(0)=0,\ \dot{\bar{D}}=\bar{s},\ \bar{D}(0)=0.$$

Then, using in particular (3.24), one easily sees that

$$\bar{s}(T)=0,\ \bar{D}(T)=0.$$

Moreover, straightforward computations show that the linearized control system around the trajectory $((\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4,\bar{s},\bar{D}),\bar{u})$ is controllable if $\bar{x}_1$ is not identically equal to 0. Hence the return method gives the local controllability of $\mathcal{T}$ in time $T>\pi$.

Unfortunately this proof to check the local controllability relies heavily on explicit computations, which we have not been able to perform in the case of the control system Σ. Let us now present a more flexible use of the return method to prove the following large time local controllability of $\mathcal{T}$.

Proposition 37 *There exists $T>0$ and $\delta>0$ such that, for every $a\in\mathbb{R}^6$ and every $b\in\mathbb{R}^6$ with $|a|<\delta$ and $|b|<\delta$, there exists $u\in L^\infty(0,T)$ such that, if $x=(x_1,x_2,x_3,x_4,s,D):[0,T]\to\mathbb{R}^6$ is the solution of the Cauchy problem*

$$\begin{gathered}\dot{x}_1=x_2,\ \dot{x}_2=-x_1+u,\\ \dot{x}_3=x_4,\ \dot{x}_4=-x_3+2x_1x_2,\\ \dot{s}=u,\ \dot{D}=s\\ x(0)=a,\end{gathered}$$

then $x(T)=b$.

Let us prove this proposition by using the return method. In order to use this method one needs, at least, to know trajectories of the control system $\mathcal{T}$such that the linearized control systems around these trajectories are controllable. The simplest trajectories one can consider are the trajectories

$$((x_1^\gamma,x_2^\gamma,x_3^\gamma,x_4^\gamma,s^\gamma,D^\gamma),u^\gamma)=((\gamma,0,0,0,\gamma t,\gamma t^2/2),\gamma), \tag{3.25}$$

where $\tau_1>0$ is fixed, γ is any real number different from 0 and $t\in[0,\tau_1]$. The linearized control system around the trajectory $(x^\gamma,u^\gamma):=((x_1^\gamma,x_2^\gamma,x_3^\gamma,x_4^\gamma,s^\gamma,D^\gamma),u^\gamma)$ is the linear control system

$$\left\{\begin{array}{l}\dot{x}_1=x_2,\ \dot{x}_2=-x_1+u,\\ \dot{x}_3=x_4,\ \dot{x}_4=-x_3+2\gamma x_2,\\ \dot{s}=u,\ \dot{D}=s.\end{array}\right. \tag{3.26}$$

Using Proposition 15 -in that case just the usual Kalman's condition-, one easily checks that this linear control system (3.26) is controllable if and only if $\gamma \neq 0$. Let us now choose $\gamma \neq 0$. Then, since the linearized control system around (x^γ, u^γ) is controllable, there exists $\delta_1 > 0$ such that, for every $a \in B(x^\gamma(0), \delta_1) := \{x \in \mathbb{R}^6; |x - x^\gamma(0)| < \delta_1\}$ and for every b in $B(x^\gamma(\tau_1), \delta_1) := \{x \in \mathbb{R}^6; |x - x^\gamma(\tau_1)| < \delta_1\}$, there exists $u \in L^\infty([0, \tau_1]; \mathbb{R})$ such that

$$\left(\left\{ \begin{array}{l} \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u, \\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2, \dot{s} = D,\ \dot{D} = u,\ x(0) = a \end{array} \right. \right) \Rightarrow (x(T_0) = b)\ .$$

Hence in order to prove Proposition 37 it suffices to check that

(i) there exists $\tau_2 > 0$ and a trajectory $(\tilde{x}, \tilde{u}) : [0, \tau_2] \to \mathbb{R}^6 \times \mathbb{R}$ of the control system $\mathcal{T}$such that $\tilde{x}(0) = 0$ and $|\tilde{x}(\tau_2) - x^\gamma(0)| < \delta_1$.

(ii) there exists $\tau_3 > 0$ and a trajectory $(\hat{x}, \hat{u}) : [0, \tau_3] \to \mathbb{R}^6 \times \mathbb{R}$ of the control system $\mathcal{T}$such that $\hat{x}(\tau_3) = 0$ and $|\hat{x}(0) - x^\gamma(\tau_1)| < \delta_1$.

Indeed, by the continuity of the solutions of the Cauchy problem with respect to the initial condition, there exists $\delta > 0$ such that

$$\left(\left\{ \begin{array}{l} \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \tilde{u}, \\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2\dot{s} = D,\ \dot{D} = \tilde{u},\ |x(0)| \leqslant \delta \end{array} \right. \right) \Rightarrow (|x(\tau_2) - x^\gamma(0)| < \delta_1)\ ,$$

and

$$\left(\left\{ \begin{array}{l} (\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \tilde{u}, \\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2, \\ \dot{s} = D,\ \dot{D} = \hat{u}(t - \tau_2 - \tau_1)|x(\tau_2 + \tau_1 + \tau_3)| \leqslant \delta \end{array} \right. \right) \Rightarrow (|x(\tau_2 + \tau_1) - x^\gamma(\tau_1)| < \delta_1\ .$$

Then, it suffises to take $T = \tau_2 + \tau_1 + \tau_3$.

Let us first consider (i). In order to prove (i) we consider quasi-static deformations. Let $g \in \mathcal{C}^2([0, 1]; \mathbb{R})$ be such that

$$g(0) = 0,\ g(1) = 1. \tag{3.27}$$

Let $\tilde{u} : [0, 1/\epsilon] \to \mathbb{R}$ be defined by

$$\tilde{u}(t) = \gamma g(\epsilon t). \tag{3.28}$$

Let $\tilde{x} := (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{s}, \tilde{D}) : [0, 1/\epsilon] \to \mathbb{R}^6$ be defined by

$$\dot{\tilde{x}}_1 = \tilde{x}_2,\ \dot{\tilde{x}}_2 = -\tilde{x}_1 + \tilde{u},$$
$$\dot{\tilde{x}}_3 = \tilde{x}_4,\ \dot{\tilde{x}}_4 = -\tilde{x}_3 + 2\tilde{x}_1\tilde{x}_2,$$
$$\dot{\tilde{s}} = \tilde{u},\ \dot{\tilde{D}} = \tilde{s},\ \tilde{x}(0) = 0.$$

One has

$$\tilde{x}_1 + i\tilde{x}_2 = i\gamma \int_0^t e^{-i(t-s)} g(\epsilon s) \mathrm{d}s, \tag{3.29}$$

$$\tilde{x}_3 + i\tilde{x}_4 = 2i \int_0^t e^{-i(t-s)} \tilde{x}_1(s)\tilde{x}_2(s) \mathrm{d}s. \tag{3.30}$$

From (3.29) and two integrations by parts, one gets

$$\tilde{x}_1 + i\tilde{x}_2 = \gamma g(\epsilon t) - \gamma g(0)e^{-it} + i\epsilon\gamma g'(\epsilon t) - i\epsilon\gamma g'(0)e^{-it} - i\epsilon^2\gamma \int_0^t e^{i(s-t)} g''(\epsilon s)\mathrm{d}s, \tag{3.31}$$

which, with (3.27), gives

$$(\tilde{x}_1(1/\epsilon), \tilde{x}_2(1/\epsilon)) \to (\gamma, 0) \text{ as } \epsilon \to 0. \tag{3.32}$$

Similarly, using (3.27), (3.30), (3.31) and integration by parts one gets

$$(\tilde{x}_3(1/\epsilon), \tilde{x}_4(1/\epsilon)) \to 0 \text{ as } \epsilon \to 0. \tag{3.33}$$

Finally

$$\tilde{s}(1/\epsilon) = (\gamma/\epsilon)\int_0^1 g(z)dz, \tag{3.34}$$

$$\tilde{D}(1/\epsilon) = (\gamma/(\epsilon^2))\int_0^1\int_0^{z_1} g(z_2)\mathrm{d}z_2\mathrm{d}z_1. \tag{3.35}$$

Hence, if

$$\int_0^1 g(z)\mathrm{d}z = 0, \int_0^1\int_0^{z_1} g(z_2)\mathrm{d}z_2\mathrm{d}z_1 = 0, \tag{3.36}$$

one gets, using (3.32), (3.33), (3.34) and (3.35),

$$\tilde{x}(1/\epsilon) \to (\gamma, 0, 0, 0, 0, 0) \text{ as } \epsilon \to 0,$$

which ends the proof of (i).

In order to get (ii) one needs to modify a little bit the above construction. Let now $g \in \mathcal{C}^2([0,1];\mathbb{R})$ be such that $g(0) = 1$ and $g(1) = 0$. Let $g_1 \in \mathcal{C}^1([0,1];\mathbb{R})$ be such that

$$\int_0^1 g_1(z)\mathrm{d}z = \tau_1, \int_0^1\int_{z_1}^1 g_1(z_2)\mathrm{d}z_2\mathrm{d}z_1 = 0. \tag{3.37}$$

Let also $g_2 \in \mathcal{C}^0([0,1];\mathbb{R})$ be such that

$$\int_0^1\int_{z_1}^1 g_2(z_2)\mathrm{d}z_2\mathrm{d}z_1 = \tau_1^2/2. \tag{3.38}$$

Finally let $\hat{u} : [0, 1/\epsilon] \to \mathbb{R}$ be defined by

$$\hat{u}(t) = g(\epsilon t) - \epsilon g_1(\epsilon t) + \epsilon^2 g_2(\epsilon t). \tag{3.39}$$

Then, similar computations as for $\tilde{x}$ show that

$$|\hat{x}(0) - (\gamma, 0, 0, 0, \gamma\tau_1, \gamma\tau_1^2/2)| \to \text{ as } \epsilon \to 0,$$

where $\hat{x} := (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{s}, \hat{D}) : [0, 1/\epsilon] \to \mathbb{R}^6$ is defined by

$$\dot{\hat{x}}_1 = \hat{x}_2,\ \dot{\hat{x}}_2 = -\hat{x}_1 + \hat{u},$$
$$\dot{\hat{x}}_3 = \hat{x}_4,\ \dot{\hat{x}}_4 = -\hat{x}_3 + 2\hat{x}_1\hat{x}_2,$$
$$\dot{\hat{s}} = \hat{u},\ \dot{\hat{D}} = \hat{s},$$
$$\hat{x}(1/\epsilon) = 0,$$

which ends the proof of (ii).

Note that the analog of Theorem 35 for the control system $\mathcal{T}$ is the following proposition.

Proposition 38 *There exists $T > 0$ such that, for every $\epsilon > 0$, there exists $\eta > 0$, for every $a = (a_1, a_2, a_3, a_4, s_0, D_0) \in \mathbb{R}^6$, and for every $b = (b_1, b_2, b_3, b_4, s_1, D_1) \in \mathbb{R}^6$ such that*

$$|a_1| + |a_2| + |a_3| + |a_4| < \eta,$$
$$|b_1| + |b_2| + |b_3| + |b_4| < \eta,$$
$$|s_0| + |s_1| + |D_0| + |D_1| < \eta,$$

there exists a trajectory

$$\begin{aligned}(x, u) : [0, T] &\to \mathbb{R}^6 \times \mathbb{R},\\ t &\mapsto (x_1(t), x_2(t), x_3(t), x_4(t), s(t), D(t), u(t))\end{aligned}$$

of the control system $\mathcal{T}$ such that

$$x(0) = a \text{ and } x(T) = b,$$

and, for every $t \in [0, T]$,

$$|x_1(t)| + |x_2(t)| + |x_3(t)| + |x_4(t)| + |u(t)| \leqslant \epsilon.$$

This proposition is stronger than Proposition 37. The proof of Proposition 38 requires some new estimates. In particular one needs an estimate on δ_1 compared to γ. See [26] for these estimates for the control system Σ.

Let us now sketch how one adapts for our control system Σ what we have said for the control system $\mathcal{T}$. Let us first point out that by scaling arguments one can assume without loss of generality that

$$L = g = H_e = 1. \tag{3.40}$$

Indeed, if we let Indeed, if we let

$$H^*(t, x) := \frac{1}{H_e} H\left(\frac{Lt}{\sqrt{H_e g}}, Lx\right), \ v^*(t, x) := \frac{1}{\sqrt{H_e g}} v\left(\frac{Lt}{\sqrt{H_e g}}, Lx\right),$$
$$u^*(t) := \frac{L}{H_e g} u\left(\frac{Lt}{\sqrt{H_e g}}\right), \ s^*(t) := \frac{1}{\sqrt{H_e g}} s\left(\frac{Lt}{\sqrt{H_e g}}\right), \ D^*(t) := \frac{1}{L} D\left(\frac{Lt}{\sqrt{H_e g}}\right),$$

with $x \in [0, 1]$, then equations (3.1) to (3.5) are equivalent to

$$\begin{aligned}
&H_t^*(t, x) + (H^* v^*)_x(t, x) = 0,\\
&v_t^*(t, x) + \left(H^* + \frac{v^{*2}}{2}\right)_x (t, x) = -u^*(t),\\
&v^*(t, 0) = v^*(t, 1) = 0,\\
&\frac{\mathrm{d}s^*}{\mathrm{d}t}(t) = u^*(t),\\
&\frac{\mathrm{d}D^*}{\mathrm{d}t}(t) = s^*(t).
\end{aligned}$$

From now on, we always assume that we have (3.40). Since $(Y, u) = ((H, v, s, D), u)$ is a trajectory of the control system Σ if and only if $((H, v, s - a, D - at - b), u)$ is a trajectory of the control system Σ, we may assume without loss of generality that $s_0 = D_0 = 0$.

The proof of Theorem 35 relies again on the return method. So one looks for a trajectory $(\bar{Y}, \bar{u}) : [0, T] \to \mathcal{Y} \times \mathbb{R}$ of the control system Σ satisfying

$$\bar{Y}(0) = \bar{Y}(T) = Y_e, \tag{3.41}$$

$$\text{the linearized control system around } (\bar{Y}, \bar{u}) \text{ is controllable.} \tag{3.42}$$

Let us point out that, as already noticed by Dubois, Petit and Rouchon in [37] (see also [85]), property (3.42) does not hold for the natural trajectory $(\bar{Y}, \bar{u}) = (Y_e, u_e)$. Indeed the linearized control system around (Y_e, u_e) is

$$\Sigma_0 \begin{cases} h_t + v_x = 0, \\ v_t + h_x = -u\,(t)\,, \\ v(t, 0) = v(t, 1) = 0, \\ \frac{\mathrm{d}s}{\mathrm{d}t}\,(t) = u\,(t)\,, \\ \frac{\mathrm{d}D}{\mathrm{d}t}\,(t) = s\,(t)\,, \end{cases} \tag{3.43}$$

where the state is $(h, v, s, D) \in \mathcal{Y}_0$, with

$$\mathcal{Y}_0 := \left\{ (h, v, s, D) \in E;\ \int_0^L h \mathrm{d}x = 0 \right\},$$

and the control is $u \in \mathbb{R}$. But (3.43) implies that, if

$$h(0, 1 - x) = -h(0, x) \text{ and } v(0, 1 - x) = v(0, x)\ \forall x \in [0, 1],$$

then

$$h(t, 1 - x) = -h(t, x) \text{ and } v(t, 1 - x) = v(t, x)\ \forall x \in [0, 1],\ \forall t.$$

Remark 39 *Even if the control system (3.43) is not controllable, one can move, as it is proved in [37], from every steady state* $(h_0, v_0, s_0, D_0) := (0, 0, 0, D_0)$ *to every steady state* $(h_1, v_1, s_1, D_1) := (0, 0, 0, D_1)$ *for this control system (see also [85] when the tank has a non-straight bottom). This does not imply that the related property (move from* $(1, 0, 0, D_0)$ *to* $(1, 0, 0, D_1)$*) also holds for the nonlinear control system* Σ*, but it follows from Corollary 36, that this property indeed also holds for the nonlinear control system* Σ*. Moreover the fact that, for the control system (3.43), it is possible –[37]–, to move from every steady state* $(h_0, v_0, s_0, D_0) := (0, 0, s_0, D_0)$ *to every steady state* $(h_1, v_1, s_1, D_1) := (0, 0, s_1, D_1)$ *explains why in the right hand side of (3.12) one has* $|s_1 - s_0| + |D_1 - s_0 T - D_0|$ *and not* $(|s_1 - s_0| + |D_1 - s_0 T - D_0|)^{1/2}$.

As in [17, 18, 22, 28, 48, 50, 51, 101] one has to look for more complicated trajectories $(\bar{Y}, \bar{u})$ in order to have (3.42). In fact, as in [23] and our toy model $\mathcal{T}$, one can require instead of (3.41), the weaker property

$$\bar{Y}(0) = Y_e \text{ and } \bar{Y}(T) \text{ is close to } Y_e, \tag{3.44}$$

and hopes that, as it happens for the Navier-Stokes control system -see above and [23]-, the controllability around $(\bar{Y}, \bar{u})$ will be strong enough to tackle the problem that $\bar{Y}(T)$ is not Y_e but only close to Y_e. Moreover, since as it is proved in [37], one can move for the linear control system Σ_0, from $Y_e = (0, 0, 0, 0)$ to $(0, 0, s_1, D_1)$, it is natural to try not to "return" to Y_e, but requires instead (3.44) the property

$$\bar{Y}(0) = Y_e \text{ and } \bar{Y}(T) \text{ is close to } (1, 0, s_1, D_1)\,. \tag{3.45}$$

In order to use this method, one first needs to have trajectories of the control system Σ such that the linearized control system around these trajectories are controllable. Let us give an example of a family of such trajectories. Let us fix a positive real number T^* in $(2, +\infty)$. For $\gamma \in (0, 1]$ and $(a, b) \in \mathbb{R}^2$, let us define $(Y^{\gamma,a,b}, u^\gamma) : \ [0, T^*] \to \mathcal{Y} \times \mathbb{R}$ by

$$Y^{\gamma,a,b}(t,x) := \left(1 + \gamma\left(\frac{1}{2} - x\right), 0, \gamma t + a, \gamma\frac{t^2}{2} + at + b\right) \ \forall t \in [0, T^*], \ \forall x \in [0, 1], \tag{3.46}$$

$$u^\gamma(t) := \gamma \ \forall t \in [0, T^*]. \tag{3.47}$$

Then, $(Y^{\gamma,a,b}, u^\gamma)$ is a trajectory of the control system Σ. The linearized control system around this trajectory is the following control system

$$\Sigma_\gamma \begin{cases} h_t + \left(\left(1 + \gamma\left(\frac{1}{2} - x\right)\right) v\right)_x = 0, \\ v_t + h_x = -u(t), \\ v(t,0) = v(t,1) = 0, \\ \frac{\mathrm{d}s}{\mathrm{d}t}(t) = u(t), \\ \frac{\mathrm{d}D}{\mathrm{d}t}(t) = s(t) \end{cases} \tag{3.48}$$

where the state is $(h, v, s, D) \in \mathcal{Y}_0$ and the control is $u \in \mathbb{R}$. This linear control system Σ_γ is controllable if $\gamma > 0$ is small enough (see [26] for a proof). Unfortunately the controllability of Σ_γ does not seem to imply directly the local controllability of the control system Σ around the trajectory $(Y^{\gamma,a,b}, u^\gamma)$. Indeed the map from $\mathcal{Y} \times \mathcal{C}^0([0, T])$ into $\mathcal{Y}$ which associates to any initial data $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}$ and to any $u \in \mathcal{C}^0([0, T])$ such that

$$H_{0x}(0) = H_{0x}(1) = -u(0)$$

the state $Y(T) \in \mathcal{Y}$, where $Y = (H, v, s, D) : \ [0, T] \to \mathcal{Y}$ satisfies (3.1) to (3.5) and $Y(0) = Y_0$ is well-defined and continuous on a small open neighborhood of $(Y_e, 0)$ (see e.g. [75]) but is not of class $\mathcal{C}^1$ on this neighborhood. So one cannot use the classical inverse function theorem to get the desired local controllability. To take care of this problem, one adapts the usual iterative scheme used to prove the existence of solutions to hyperbolic systems (see e.g. [31, p. 476-478], [61, p. 54-55], [75, p. 96-107], [80, p. 35-43] or [96, p. 106-116] -see also [18, 22, 28, 48, 50, 51] for the Euler and the Navier control system for incompressible fluids): one uses the following inductive procedure $(h^n, v^n, s^n, D^n, u^n) \mapsto \left(h^{n+1}, v^{n+1}, s^{n+1}, D^{n+1}, u^{n+1}\right)$ so that

$$h_t^{n+1} + v^n h_x^{n+1} + \left(1 + \gamma\left(\frac{1}{2} - x\right) + h^n\right) v_x^{n+1} - \gamma v^{n+1} = 0, \tag{3.49}$$

$$v_t^{n+1} + h_x^{n+1} + v^n v_x^{n+1} = -u^{n+1}(t), \tag{3.50}$$

$$v^{n+1}(t, 0) = v^{n+1}(t, L) = 0, \tag{3.51}$$

$$\frac{\mathrm{d}s^{n+1}}{\mathrm{d}t}(t) = u^{n+1}(t), \tag{3.52}$$

$$\frac{\mathrm{d}D^{n+1}}{\mathrm{d}t}(t) = s^{n+1}(t), \tag{3.53}$$

and $\left(h^{n+1}, v^{n+1}, s^{n+1}, D^{n+1}, u^{n+1}\right)$ has the required value for $t = 0$ and for $t = T^*$. Unfortunately we have only been able to prove that the control system (3.49)-(3.53), where the state is $\left(h^{n+1}, v^{n+1}, s^{n+1}, D^{n+1}\right)$ and the control is u^{n+1}, is controllable under a special assumption on (h^n, v^n), see [26]. Hence one has to insure that, at each iterative step, (h^n, v^n) satisfies this condition, which turns out to be possible. So one gets the following proposition, which is proved in [26].

Proposition 40 *There exist $C_1 > 0$,$\mu > 0$ and $\gamma_0 \in (0,1]$ such that, for every $\gamma \in [0,\gamma_0]$, for every $(a,b) \in \mathbb{R}^2$ and for every $(Y_0, Y_1) \in \mathcal{Y}^2$ satisfying*

$$|Y_0 - Y^{\gamma,a,b}(0)| \leqslant \mu\gamma^2 \text{ and } |Y_1 - Y^{\gamma,a,b}(T^*)| \leqslant \mu\gamma^2,$$

there exists a trajectory $(Y,u) : [0,T^*] \to \mathcal{Y} \times \mathbb{R}$ of the control system Σ such that

$$Y(0) = Y_0 \text{ and } Y(T^*) = Y_1,$$
$$\left| Y(t) - Y^{\gamma,a,b}(t) \right| + |u(t)| \leqslant C_1\gamma \ \forall t \in [0,T^*]. \tag{3.54}$$

One now needs to construct, for every given $\gamma > 0$ small enough, trajectories $(Y,u) : [0,T^0] \to \mathcal{Y}\times\mathbb{R}$ of the control system Σ satisfying

$$Y(0) = (1,0,0,0) \text{ and } |Y(T^0) - Y^{\gamma,a,b}(0)| \leqslant \mu\gamma^2, \tag{3.55}$$

and trajectories $(Y,u) : [T^0 + T^*, T^0 + T^* + T^1] \to \mathcal{Y} \times \mathbb{R}$ of the control system Σ such that

$$Y\left(T^0 + T^1 + T^*\right) = (1,0,s_1,D_1) \text{ and } \left| Y\left(T^0 + T^*\right) - Y^{\gamma,a,b}(T^*) \right| \leqslant \mu\gamma^2, \tag{3.56}$$

for suitable choice of $(a,b) \in \mathbb{R}^2$, $T^0 > 0$, $T^1 > 0$. Let us first point out that it follows from [37] that one knows explicit trajectories $(Y^l, u^l) : [0,T^0] \to \mathcal{Y} \times \mathbb{R}$ of *the linearized control system around* $(0,0)$ (i.e. the control system Σ_0) satisfying $Y^l(0) = 0$ and $Y^l(T^0) = Y^{\gamma,a,b}(0)$. (In fact Dubois, Petit and Rouchon have proved in [37] that the linear control system Σ_0 is flat -a notion introduced by Fliess, Lévine, Martin and Rouchon in [42]-. They have given a complete explicit parametrization of the trajectories of Σ_0 by means of an arbitrary function and a 1-periodic function.) Then, the idea is that, if one moves "slowly", the same control u^l gives a trajectory $(Y,u) : [0,T^0] \to \mathcal{Y} \times \mathbb{R}$ of the control system Σ such that (3.55) holds. More precisely, let $f_0 \in \mathcal{C}^4([0,4])$ be such that

$$f_0 = 0 \text{ in } [0,1/2] \cup [3,4], \tag{3.57}$$
$$f_0(t) = s/2 \ \forall t \in [1,3/2], \tag{3.58}$$
$$\int_0^4 f_0(t_1)\mathrm{d}t_1 = 0. \tag{3.59}$$

Similarly, let $f_1 \in \mathcal{C}^4([0,4])$ and $f_2 \in \mathcal{C}^4([0,4])$ be such that

$$f_1 = 0 \text{ in } [0,1/2] \cup [1,3/2] \text{ and } f_1 = 1/2 \text{ in } [3,4], \tag{3.60}$$
$$\int_0^3 f_1(t_1)\mathrm{d}t_1 = 0, \tag{3.61}$$
$$f_2 = 0 \text{ in } [0,1/2] \cup [1,3/2] \cup [3,4], \tag{3.62}$$
$$\int_0^4 f_2(t_1)\mathrm{d}t_1 = 1/2. \tag{3.63}$$

Let

$$\mathbb{D} := \left\{ \left(\bar{s}, \bar{D}\right) \in \mathbb{R}^2;\ |\bar{s}| \leqslant 1,\ \left|\bar{D}\right| \leqslant 1 \right\}.$$

For $\left(\bar{s}, \bar{D}\right) \in \mathbb{D}$, let $f_{\bar{s},\bar{D}} \in \mathcal{C}^4([0,4])$ be defined by

$$f_{\bar{s},\bar{D}} := f_0 + \bar{s}f_1 + \bar{D}f_2. \tag{3.64}$$

For $\epsilon \in (0,1/2]$ and for $\gamma \in \mathbb{R}$, let $u^{\epsilon,\gamma}_{\bar{s},\bar{D}} : [0,3/\epsilon] \to \mathbb{R}$ be defined by

$$u^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) := \gamma f'_{\bar{s},\bar{D}}(\epsilon t) + \gamma f'_{\bar{s},\bar{D}}(\epsilon(t+1)). \tag{3.65}$$

Let $\left(h^{\epsilon,\gamma}_{\bar{s},\bar{D}}, v^{\epsilon,\gamma}_{\bar{s},\bar{D}}, s^{\epsilon,\gamma}_{\bar{s},\bar{D}}, D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right) : [0,3/\epsilon] \to \mathcal{C}^1([0,1]) \times \mathcal{C}^1([0,1]) \times \mathbb{R} \times \mathbb{R}$ be such that (3.43) holds for $(h,v,s,D) = \left(h^{\epsilon,\gamma}_{\bar{s},\bar{D}}, v^{\epsilon,\gamma}_{\bar{s},\bar{D}}, s^{\epsilon,\gamma}_{\bar{s},\bar{D}}, D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right)$, $u = u^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ and

$$\left(h^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0,\cdot), v^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0,\cdot), s^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0), D^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0)\right) = (0,0,0,0).$$

From [37] one gets that

$$h^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t,x) = -\frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+x)) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1-x)), \tag{3.66}$$

$$v^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t,x) = \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+x)) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1-x)) - \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon t) - \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1)), \tag{3.67}$$

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) = \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon t) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(\epsilon(t+1)), \tag{3.68}$$

$$D^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) = \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(\epsilon t) + \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(\epsilon(t+1)), \tag{3.69}$$

with

$$F_{\bar{s},\bar{D}}(t) := \int_0^t f_{\bar{s},\bar{D}}(t_1)\mathrm{d}t_1.$$

In particular, using also (3.57) to (3.63), one gets

$$h^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t,x\right) = \gamma\left(\frac{1}{2}-x\right) \text{ and } v^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t,x\right) = 0 \ \forall t \in \left[0,\frac{1-2\epsilon}{2\epsilon}\right], \ \forall x \in [0,1], \tag{3.70}$$

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t\right) = \frac{\gamma}{\epsilon} + \frac{\gamma}{2} + \gamma t \ \forall t \in \left[0,\frac{1-2\epsilon}{2\epsilon}\right], \tag{3.71}$$

$$D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+t\right) = D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) + \left(\frac{\gamma}{\epsilon}+\frac{\gamma}{2}\right)t + \frac{\gamma}{2}t^2 \ \forall t \in \left[0,\frac{1-2\epsilon}{2\epsilon}\right], \tag{3.72}$$

$$h^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon},x\right) = 0 \text{ and } v^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon},x\right) = 0 \ \forall x \in [0,1], \tag{3.73}$$

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right) = \frac{\gamma}{\epsilon}\bar{s} \text{ and } D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right) = \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}. \tag{3.74}$$

Let $H^{\epsilon,\gamma}_{\bar{s},\bar{D}} = 1 + h^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ and $Y^{\epsilon,\gamma}_{\bar{s},\bar{D}} = \left(H^{\epsilon,\gamma}_{\bar{s},\bar{D}}, v^{\epsilon,\gamma}_{\bar{s},\bar{D}}, s^{\epsilon,\gamma}_{\bar{s},\bar{D}}, D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right)$. Consider

$$a_{\epsilon,\gamma} := \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(1) + \frac{\gamma}{\epsilon} f_{\bar{s},\bar{D}}(1+\epsilon) = \frac{\gamma}{\epsilon} + \frac{\gamma}{2}, \ b^{\bar{s},\bar{D}}_{\epsilon,\gamma} := \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(1) + \frac{\gamma}{\epsilon^2} F_{\bar{s},\bar{D}}(1+\epsilon) = D^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right).$$

Using (3.46), (3.70), (3.71) and (3.72), one has

$$Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) = Y^{\gamma,a_{\epsilon,\gamma},b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(0,\cdot), \tag{3.75}$$

and, if $\epsilon \in (0,1/(2(T^*+1))]$,

$$Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}+T^*\right) = Y^{\gamma,a_{\epsilon,\gamma},b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(T^*). \tag{3.76}$$

The next proposition, which is proved in [26], shows that one can achieve (3.55) with $u = u^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ for suitable choices of T^0, ϵ and γ.

Proposition 41 *There exists a constant $C_2 > 2$ such that, for every $\epsilon \in (0, 1/C_2]$, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$ and for every $\gamma \in [0, \epsilon/C_2]$, there exists one and only one map $\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}} : [0, 1/\epsilon] \to \mathcal{Y}$ satisfying the two following conditions*

$$(\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}, u^{\epsilon,\gamma}_{\bar{s},\bar{D}}) \text{ is a trajectory of the control system } \Sigma \text{ (on } [0, 1/\epsilon]),$$
$$\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(0) = (1, 0, 0, 0)\,,$$

and this unique map $\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ verifies

$$\left|\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) - Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_2\epsilon\gamma^2 \ \forall t \in [0, 1/\epsilon]. \tag{3.77}$$

In particular, by (3.70),

$$\left|\tilde{v}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right)\right|_1 + \left|\tilde{h}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) - \gamma\left(\frac{1}{2} - x\right)\right|_1 \leqslant C_2\epsilon\gamma^2. \tag{3.78}$$

Similarly, one has the following proposition, which shows that (3.56) is achieved with $u = u^{\epsilon,\gamma}_{\bar{s},\bar{D}}$ for suitable choices of T^1, ϵ and γ.

Proposition 42 *There exists a constant $C_3 > 2(T^*+1)$ such that, for every $\epsilon \in (0, 1/C_3]$, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$, and for every $\gamma \in [0, \epsilon/C_3]$, there exists one and only one map $\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}} : [(1/\epsilon)+T^*, 3/\epsilon] \to \mathcal{Y}$ satisfying the two following conditions*

$$\left(\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}, u^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right) \text{ is a trajectory of the control system } \Sigma \text{ (on } [(1/\epsilon)+T^*, 3/\epsilon]),$$
$$\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right) = \left(1, 0, \frac{\gamma}{\epsilon}\bar{s}, \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}\right) = Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{3}{\epsilon}\right),$$

and this unique map $\hat{Y}^{\epsilon,\gamma}$ verifies

$$\left|\hat{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t) - Y^{\epsilon,\gamma}_{\bar{s},\bar{D}}(t)\right| \leqslant C_3\epsilon\gamma^2 \ \forall t \in [(1/\epsilon)+T^*, 3/\epsilon]. \tag{3.79}$$

In particular, by (3.70),

$$\left|\hat{v}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right)\right|_1 + \left|\hat{h}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) - \gamma\left(\frac{1}{2} - x\right)\right|_1 \leqslant C_2\epsilon\gamma^2. \tag{3.80}$$

Let us choose

$$\epsilon := \operatorname{Min}\left(\frac{1}{C_2}, \frac{1}{C_3}, \frac{\mu}{2C_2}, \frac{\mu}{2C_3}\right) \leqslant \frac{1}{2}. \tag{3.81}$$

Let us point out that there exists $C_4 > 0$ such that, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$ and for every $\gamma \in [-\epsilon, \epsilon]$,

$$\left|H^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right|_{C^2([0,3/\epsilon]\times[0,1])} + \left|v^{\epsilon,\gamma}_{\bar{s},\bar{D}}\right|_{C^2([0,3/\epsilon]\times[0,1])} \leqslant C_4, \tag{3.82}$$

which, with straightforward estimates, leads to the next proposition, whose proof is omitted.

Proposition 43 *There exists $C_5 > 0$ such that, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$, for every $Y_0 = (H_0, v_0, s_0, D_0) \in \mathcal{Y}$ with*

$$|Y_0 - Y_e| \leqslant \frac{1}{C_5}, \; s_0 = 0, \; D_0 = 0$$

and for every $\gamma \in [0, \epsilon/C_2]$, there exists one and only one $Y : \; [0, 1/\epsilon] \rightarrow \mathcal{Y}$ such that

$$(Y, u^{\varepsilon,\gamma}_{\bar{s},\bar{D}} - H_{0x}(0)) \text{ is a trajectory of the control system } \Sigma,$$
$$Y(0) = Y_0,$$

and this unique map Y satisfies

$$\left| Y(t) - \tilde{Y}^{\varepsilon,\gamma}_{\bar{s},\bar{D}}(t) \right| \leqslant C_5 |Y_0 - Y_e|, \; \forall t \in [0, 1/\epsilon].$$

Similarly, (3.82) leads to the following proposition.

Proposition 44 *There exists $C_6 > 0$ such that, for every $(\bar{s}, \bar{D}) \in \mathbb{D}$, for every $\gamma \in [0, \epsilon/C_3]$, and for every $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}$ such that*

$$\left| Y_1 - \left(1, 0, \frac{\gamma}{\epsilon}\bar{s}, \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}\right) \right| \leqslant \frac{1}{C_6}, \; s_1 = \frac{\gamma}{\epsilon}\bar{s}, \; D_1 = \frac{\gamma}{2\epsilon}\bar{s} + \frac{\gamma}{\epsilon^2}\bar{D}$$

there exists one and only one $Y : \; [(1/\epsilon) + T^*, 3/\epsilon] \rightarrow \mathcal{Y}$ such that

$$(Y, u^{\varepsilon,\gamma}_{\bar{s},\bar{D}} - H_{1x}(0)) \text{ is a trajectory of the control system } \Sigma$$
$$Y(3/\epsilon) = Y_1,$$

and this unique map Y satisfies

$$\left| Y(t) - \hat{Y}^{\varepsilon,\gamma}_{\bar{s},\bar{D}}(t) \right| \leqslant C_6 \left| Y_1 - Y^{\varepsilon,\gamma}_{\bar{s},\bar{D}}(3/\epsilon) \right|, \; \forall t \in [(1/\epsilon) + T^*, 3/\epsilon].$$

Finally define

$$T := \frac{3}{\epsilon}, \tag{3.83}$$

$$\eta := \operatorname{Min}\left(\frac{\epsilon^2 \mu}{2C_5(C_3^2 + C_2^2)}, \frac{\epsilon^2 \mu}{2C_6(C_3^2 + C_2^2)}, \frac{\epsilon}{C_2}, \frac{\epsilon}{C_3}, \frac{1}{C_5}, \frac{1}{C_6}, \frac{\gamma_0^2 \mu}{2C_5}, \frac{\gamma_0^2 \mu}{2C_6}, \gamma_0 \right). \tag{3.84}$$

We want to check that Theorem 35 holds with these constants for a large enough C_0. Let $Y_0 = (H_0, v_0, 0, 0) \in \mathcal{Y}$ and $Y_1 = (H_1, v_1, s_1, D_1) \in \mathcal{Y}$ be such that

$$|H_0 - 1|_1 + |v_0|_1 \leqslant \eta, \; |H_1 - 1|_1 + |v_1|_1 \leqslant \eta, \; |s_1| + |D_1| \leqslant \eta. \tag{3.85}$$

Let

$$\gamma := \operatorname{Max}\left(\sqrt{\frac{2C_5}{\mu}} \sqrt{|H_0 - 1|_1 + |v_0|_1}, \sqrt{\frac{2C_6}{\mu}} \sqrt{|H_1 - 1|_1 + |v_1|_1}, |s_1| + |D_1| \right), \tag{3.86}$$

$$\bar{s} := \frac{\epsilon}{\gamma} s_1, \; \bar{D} := \frac{\epsilon^2}{\gamma}\left(D_1 - \frac{s_1}{2}\right), \tag{3.87}$$

so that, thanks to (3.74),

$$s^{\epsilon,\gamma}_{\bar{s},\bar{D}} = s_1,\ D^{\epsilon,\gamma}_{\bar{s},\bar{D}} = D_1. \tag{3.88}$$

Note that, by (3.81), (3.85), (3.86) and (3.87),

$$\left(\bar{s}, \bar{D}\right) \in \mathbb{D}. \tag{3.89}$$

By (3.84), (3.85) and (3.86), we obtain that

$$\gamma \in \left[0, \mathrm{Min}\ \left(\frac{\epsilon}{C_2}, \frac{\epsilon}{C_3}\right)\right]. \tag{3.90}$$

Then, by Proposition 43, (3.84), (3.85) and (3.90), there exists a function $Y^0 = (H^0, v^0, s^0, D^0) : [0, 1/\epsilon] \to \mathcal{Y}$ such that

$$(Y^0, u^{\epsilon,\gamma}_{\bar{s},\bar{D}} - H_{0x}(0)) \text{ is a trajectory of the control system } \Sigma \text{ on } [0, 1/\epsilon], \tag{3.91}$$

$$Y^0(0) = Y_0, \tag{3.92}$$

$$\left|Y^0\left(t\right) - \tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(t\right)\right| \leqslant C_5|Y_0 - Y_e|\ \forall t \in [0, 1/\epsilon]. \tag{3.93}$$

By (3.86) and (3.93),

$$\left|Y^0\left(\frac{1}{\epsilon}\right) - \tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right)\right| \leqslant \frac{\mu\gamma^2}{2}. \tag{3.94}$$

By Proposition 41, (3.81) and (3.90),

$$\left|\tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(\frac{1}{\epsilon}\right) - Y^{\gamma,a_{\epsilon,\gamma},b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(0)\right| \leqslant C_2\epsilon\gamma^2 \leqslant \frac{\mu\gamma^2}{2},$$

which, with (3.94), gives

$$\left|Y^0\left(\frac{1}{\epsilon}\right) - Y^{\gamma,a_{\epsilon,\gamma},b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}\right| \leqslant \mu\gamma^2. \tag{3.95}$$

Similarly, by Propositions 42 and 44, (3.81), (3.83), (3.84), (3.85), (3.86), (3.88) and (3.90), there exists $Y^1 = (H^1, v^1, s^1, D^1) :\ [(1/\epsilon) + T^*, T] \to \mathcal{Y}$ such that

$$(Y^1, u^{\epsilon,\gamma}_{\bar{s},\bar{D}} - H_{1x}(0)) \text{ is a trajectory of the control system } \Sigma \text{ on } [(1/\epsilon) + T^*, T], \tag{3.96}$$

$$Y^1(T) = Y_1, \tag{3.97}$$

$$\left|Y^1\left(t\right) - \tilde{Y}^{\epsilon,\gamma}_{\bar{s},\bar{D}}\left(t\right)\right| \leqslant C_6\left|Y_1 - (1, 0, s_1, D_1)\right|\ \ \forall t \in \left[(1/\epsilon) + T^*, T\right], \tag{3.98}$$

$$\left|Y^1\left(\frac{1}{\epsilon} + T^*\right) - Y^{\gamma,a_{\epsilon,\gamma},b^{\bar{s},\bar{D}}_{\epsilon,\gamma}}(T^*)\right| \leqslant \mu\gamma^2. \tag{3.99}$$

By (3.84), (3.85) and (3.86),

$$\gamma \leqslant \gamma_0. \tag{3.100}$$

From Proposition 40, (3.95), (3.99) and (3.100), there exists a trajectory $(Y^*, u^*) : [0, T^*] \to \mathcal{Y}$ of the control system Σ satisfying

$$Y^*(0) = Y^0\left(\frac{1}{\epsilon}\right), \tag{3.101}$$

$$\left|Y^*(t) - Y^{\gamma, a_{\epsilon,\gamma}, b_{\epsilon,\gamma}^{\bar{s},\bar{D}}}(t)\right| \leqslant C_1\mu\gamma \ \forall t \in [0, T^*], \tag{3.102}$$

$$Y^*(T^*) = Y^1\left(\frac{1}{\epsilon} + T^*\right). \tag{3.103}$$

The map $(Y, u) : [0, T] \to \mathcal{Y}$ defined by

$$(Y(t), u(t)) = (Y^0(t), u_{\bar{s},\bar{D}}^{\epsilon,\gamma}(t) - H_{0x}(0)) \ \forall t \in [0, 1/\epsilon],$$
$$(Y(t), u(t)) = (Y^*(t - (1/\epsilon)), u^*(t - (1/\epsilon))) \ \forall t \in [1/\epsilon, (1/\epsilon) + T^*],$$
$$(Y(t), u(t)) = (Y^1(t), u_{\bar{s},\bar{D}}^{\epsilon,\gamma}(t) - H_{1x}(0)) \ \forall t \in [(1/\epsilon) + T^*, T],$$

is a trajectory of the control system Σ which, by (3.92) and (3.97), satisfies (3.11). Finally the existence of $C_0 > 0$ such that (3.12) holds follows from the construction of (Y, u), (3.7), (3.54), (3.66) to (3.69), (3.77), (3.79), (3.85), (3.86), (3.93), (3.98) and (3.102).

Remark 45 *Let us notice that, since $T^* > 2$, our method requires, in Theorem 35, $T > 2L/\sqrt{gH_e}$. We conjecture that*

- *Theorem 35 holds for every $T > 2L/\sqrt{gH_e}$*
- *For every $T \leqslant 2L/\sqrt{gH_e}$, Theorem 35 does not hold and the steady state motion problem (move from $(0, 0, 0, D_0)$ to $(0, 0, 0, D_1)$) is not possible with the interval of time $[0, T]$ if $|u|_0$ and $|D_1 - D_0|$ are small with $D_1 \neq D_0$. (However, let us recall that, for the linearized control system Σ_0 $T > L/\sqrt{gH_e}$ is enough for the steady-state controllability (see [37]).)*

3.5 Null asymptotic stabilizability of the 2-D Euler control system

In this section we address a problem of stabilization of the Euler equation. In section 3.3.1 we have considered the problem of the controllability of the Euler control system of an incompressible inviscid fluid in a bounded domain. In particular we have seen that, if the controls act on an arbitrarily small open subset of the boundary which meets every connected component of that boundary, then the Euler equation is exactly controllable.

For linear control systems, the exact controllability implies the asymptotic stabilizability by means of feedback laws. This is well known for linear control systems of finite dimension and, by M. Slemrod [98], J.–L. Lions [77], I. Lasiecka-R. Triggiani [73] and V. Komornik [71], it also holds in infinite dimension in very general cases. But, as pointed out by H.J. Sussmann in [105], by E.D. Sontag and H.J. Sussmann in [103], and by R.W. Brockett in [12], this is no longer true for *nonlinear* control systems, even of finite dimension. For example (see [12]) the nonlinear control system (3.7) is globally controllable but $0 \in \mathbb{R}^3$ cannot be, even locally, asymptotically stabilized by means of feedback laws. Let us also notice that, as in this counter-example, the linearized control system of the Euler equation around the origin is not controllable.

Therefore it is natural to ask what is the situation for the asymptotic stabilizability of the origin for the 2-D Euler equation of an incompressible inviscid fluid in a bounded domain when the controls act on an arbitrarily small open subset of the boundary which meets every connected component of this boundary. In this section we are going to prove that the null global asymptotic stabilizability by means of feedback laws holds if the domain is simply connected.

Let Ω be a nonempty bounded connected and simply connected subset of $\mathbb{R}^2$ of class $\mathcal{C}^\infty$ and let Γ_0 be a non empty open subset of the boundary $\partial\Omega$ of Ω. This set Γ_0 is the location of the control. Let y be the velocity field of the inviscid fluid contained in Ω. We assume that the fluid is incompressible, so that

$$\text{div } y = 0. \tag{3.1}$$

Since Ω is simply connected, y is completely characterized by $\omega := \text{curl } y$ and $y.n$ on $\partial\Omega$, where n denotes the unit outward normal to $\partial\Omega$. For the problem of controllability, one does not really need to specify the control and the state: one considers the "Euler control system" as an under-determined system by requiring $y.n = 0$ on $\partial\Omega\setminus\Gamma_0$ instead of $y.n = 0$ on $\partial\Omega$ as for the uncontrolled usual Euler equation. For the stabilization problem, one needs to specify more precisely the control and the state. In this paper the state is ω. For the control there are at least two natural possibilities

(a) The control is $y.n$ on Γ_0 and the time derivative $\partial\omega/\partial t$ of the vorticity at the points of Γ_0 where $y.n < 0$, i.e. at the points where the fluid enters into the domain Ω,

(b) The control is $y.n$ on Γ_0 and the vorticity ω at the points where $y.n < 0$.

Let us point out that, by (3.1), in both cases $y.n$ has to satisfy $\int_{\partial\Omega} y.n = 0$. In this paper we study only case (a); for case (b), see [24].

Let us give stabilizing feedback laws. Let $g \in \mathcal{C}^\infty(\partial\Omega)$ be such that

$$\text{Support } g \subset \Gamma_0, \tag{3.2}$$
$$\Gamma_0^+ := \{g > 0\} \text{ and } \Gamma_0^- := \{g < 0\} \text{ are connected}, \tag{3.3}$$
$$g \neq 0, \tag{3.4}$$
$$\overline{\Gamma_0^+} \cap \overline{\Gamma_0^-} = \emptyset, \tag{3.5}$$
$$\int_{\partial\Omega} g = 0. \tag{3.6}$$

For every $f \in \mathcal{C}^0(\overline{\Omega})$, we denote

$$|f|_0 = \text{Max } \{|f(x)|\,; x \in \overline{\Omega}\}.$$

Our stabilizing feedback laws are

$$y.n = M\,|\omega|_0\, g \text{ on } \Gamma_0,$$
$$\frac{\partial\omega}{\partial t} = -M\,|\omega|_0\,\omega \text{ on } \Gamma_0^- \text{ if } |\omega|_0 \neq 0,$$

where $M > 0$ is large enough. With these feedback laws, a function $\omega : I \times \overline{\Omega} \to \mathbb{R}$, where I is a time interval, is a solution of the closed loop system Σ if

$$\frac{\partial \omega}{\partial t} + \text{ div } (\omega y) = 0 \text{ in } \overset{\circ}{I} \times \Omega, \tag{3.7}$$

$$\text{div } y = 0 \text{ in } \overset{\circ}{I} \times \Omega, \tag{3.8}$$

$$\text{curl } y = \omega \text{ in } \overset{\circ}{I} \times \Omega, \tag{3.9}$$

$$y(t).n = M \left|\omega(t)\right|_0 g \text{ on } \partial\Omega, \forall t \in I, \tag{3.10}$$

$$\frac{\partial \omega}{\partial t} = -M \left|\omega(t)\right|_0 \omega \text{ on } \{t;\, \omega(t) \neq 0\} \times \Gamma_0^-. \tag{3.11}$$

where, for $t \in \Omega$, $\omega(t) : \overline{\Omega} \to \mathbb{R}$ and $y(t) : \overline{\Omega} \to \mathbb{R}^2$ are defined by requiring $\omega(t)(x) = \omega(t,x)$ and $y(t)(x) = y(t,x), \forall x \in \overline{\Omega}$. More precisely, the definition of a solution of system Σ is

Definition 46 *Let I be an interval. A function $\omega : I \to \mathcal{C}^0(\overline{\Omega})$ is a solution of system Σ if*

(i) $\omega \in \mathcal{C}^0(I; \mathcal{C}^0(\overline{\Omega}))(\cong \mathcal{C}^0(I \times \overline{\Omega}))$,

(ii) For $y \in \mathcal{C}^0(I \times \overline{\Omega}; \mathbb{R}^2)$ defined by requiring (3.8) and (3.9) in the sense of distributions and (3.10), one has (3.7) in the sense of distributions,

(iii) In the sense of distributions on the open manifold $\{t \in \overset{\circ}{I};\, \omega(t) \neq 0\} \times \Gamma_0^-$ one has $\partial\omega/\partial t = -M \left|\omega(t)\right|_0 \omega$.

Our first theorem says that, for M large enough, the Cauchy problem for system Σ has at least one solution defined on $[0, +\infty)$ for every initial data in $\mathcal{C}^0(\overline{\Omega})$. More precisely one has

Theorem 47 *There exists $M_0 > 0$ such that, for every $M \geqslant M_0$, the following two properties hold*

(i) For every $\omega_0 \in \mathcal{C}^0(\overline{\Omega})$, there exists a solution of system Σ defined on $[0, +\infty)$ such that $\omega(0) = \omega_0$,

(ii) Any maximal solution of system Σ defined at time 0 is defined on $[0, +\infty)$ (at least).

Remark 48 *a. In this theorem, property (i) is in fact implied by property (ii) and Zorn's lemma. We state (i) in order to emphasize the existence of a solution to the Cauchy problem for system Σ.*
b. We do not know if the solution to the Cauchy problem is unique for positive time. (For negative time, one does not have uniqueness since there are solutions ω of system Σ defined on $[0, +\infty)$ such that $\omega(0) \neq 0$ and $\omega(T) = 0$ for $T \in [0, +\infty)$ large enough.) However, let us emphasize that, already for control systems in finite dimension, one considers feedback laws which are merely continuous; with these feedback laws, the Cauchy problem for the closed loop system may have many solutions. It turns out that this lack of uniqueness is not a real problem. Indeed, in finite dimension at least, if a point is asymptotically stable for a continuous vector field, then there exists, as in the case of regular vector fields, a (smooth) strict Lyapounov function. This result is due to Kurzweil [72].It is tempting to conjecture that a similar result holds in infinite dimension under reasonable assumptions. The existence of this Lyapounov function insures some robustness to perturbations. This is precisely this robustness which makes the interest of feedback laws compared to open loop controls. We will see that, for our feedback laws, there exists also a strict Lyapounov –see Proposition 52 below– and therefore our feedback laws provide some kind of robustness.

Our next theorem shows that, at least for M large enough, our feedback laws globally and strongly asymptotically stabilize the origin in $\mathcal{C}^0(\overline{\Omega})$ for system Σ.

Theorem 49 *There exists a positive constant $M_1 \geqslant M_0$ such that, for every $\varepsilon \in (0,1]$, every $M \geqslant M_1/\varepsilon$ and every maximal solution ω of system Σ defined at time 0,*

$$|\omega(t)|_0 \leqslant \; Min \, \{|\omega(0)|_0, \frac{\varepsilon}{t}\}, \, \forall t > 0. \tag{3.12}$$

Remark 50 *Due to the term $|\omega(t)|_0$ appearing in (3.10) and in (3.11) our feedback laws do not depend only on the value of ω on Γ_0. Let us point out that there is no asymptotically stabilizing feedback law depending only on the value of ω on Γ_0 such that the origin is asymptotically stable for the closed loop system. In fact, given a nonempty open subset Ω_0 of Ω, there is no feedback law which does not depend on the values of ω on Ω_0. This phenomenon is due to the existence of "phantom vortices": there are smooth stationary solutions $\bar{y} : \overline{\Omega} \to \mathbb{R}^2$ of the 2-D Euler equations such that Support $\bar{y} \subset \Omega_0$ and $\bar{\omega} :=$ curl $\bar{y} \neq 0$; see, e.g., [81]. Then $\omega(t) = \bar{\omega}$ is a solution of the closed loop system if the feedback law does not depend on the values of ω on Ω_0 –and vanishes for $\omega = 0$.*

Remark 51 *Let us emphasize that (3.12) implies that*

$$|\omega(t)|_0 \leqslant \varepsilon, \, \forall t \geqslant 1, \tag{3.13}$$

for <u>every</u> maximal solution ω of system Σ defined at time 0 (<u>whatever</u> is $\omega(0)$). It would be interesting to know if one could have a similar result for the 2-D Navier-Stokes equations of viscous incompressible flows, that is if, given $\varepsilon > 0$, does there exist a feedback law such that (3.13) holds for every solution of the closed loop Navier-Stokes control system? Note that $y = 0$ on Γ_0 is a feedback which leads to asymptotic stabilization of the null solution of the Navier-Stokes control system. However, this feedback does not have the required property. One may ask a similar question for the Burgers control system; for the null asymptotic stabilization of this control system, see the paper [70] by M. Krstić and the references therein.

The detailed proofs of Theorem 47 and of Theorem 49 are given in [24]. Let us just mention that Theorem 49 is proved by giving an explicit Lyapounov function $V : \mathcal{C}^0(\overline{\Omega}) \to [0, +\infty)$. That function V is defined by

$$V(\omega) = |\omega \exp(-\theta)|_0,$$

where $\theta \in \mathcal{C}^\infty(\overline{\Omega})$ satisfies

$$\Delta\theta = 0 \text{ in } \overline{\Omega}, \tag{3.14}$$

$$\frac{\partial\theta}{\partial n} = g \text{ on } \partial\Omega. \tag{3.15}$$

(Let us point out that the existence of θ follows from (3.6).) Theorem 49 is an easy consequence of the following proposition.

Proposition 52 *There exists $M_2 \geqslant M_0$ and $\mu > 0$ such that, for every $M \geqslant M_2$ and every solution $\omega : [0, +\infty) \to \mathcal{C}^0(\overline{\Omega})$ of system Σ, one has, for every $t \in [0, +\infty)$,*

$$[-\infty, 0] \ni \dot{V}(t) := \frac{d}{dt^+} V(\omega(t)) \leqslant -\mu M V^2(\omega(t)), \tag{3.16}$$

where $d/dt^+ V(\omega(t)) := \lim_{\varepsilon \to 0^+} (V(\omega(t+\varepsilon)) - V(\omega(t)))/\varepsilon$.

Let us end this section by some comments for the case where Ω is not simply connected In this case, in order to define the state, one adds to ω the real numbers $\lambda_1, \cdots, \lambda_g$ defined by

$$\lambda_i = \int y.\nabla^{\perp}\tau_i,$$

where, if one denotes by $\mathcal{C}_0, \mathcal{C}_1, \cdots, \mathcal{C}_g$ the connected components of Γ, the functions $\tau_i \in \mathcal{C}^{\infty}(\overline{\Omega}), i \in \{1, \cdots, g\}$ are defined by

$$\begin{aligned} \Delta\tau_i &= 0 \ , \\ \tau_i &= 0 \text{ on } \partial\Omega \setminus \mathcal{C}_i, \\ \tau_i &= 1 \text{ on } \mathcal{C}_i, \end{aligned}$$

and where $\nabla^{\perp}\tau_i$ denotes $\nabla\tau_i$ rotated by $\pi/2$. One has the following open problem

Open Problem 53 *Assume that $g \geq 1$ and that Γ_0 meets every connected component of Γ. Does there exist always a feedback law such that $0 \in \mathcal{C}^0(\overline{\Omega}) \times \mathbb{R}^g$ is globally asymptotically stable for the closed loop system?*

Brockett's necessary condition [12] for the existence of asymptotically stabilizing feedback laws cannot be directly applied to our situation since our control system is of infinite dimension. However, it leads to the following question.
Question *Assume that Γ_0 meets every connected component of Γ. Let $f \in \mathcal{C}^{\infty}(\overline{\Omega})$. Does there exit $y \in \mathcal{C}^{\infty}(\overline{\Omega}; \mathbb{R}^2)$ and $p \in \mathcal{C}^{\infty}(\overline{\Omega})$ such that*

$$(y.\nabla)y + \nabla p = f \ \textit{in } \overline{\Omega} \ , \tag{3.17}$$

$$\text{div } y = 0 \ \textit{in } \overline{\Omega} \ , \tag{3.18}$$

$$y.n = 0 \ \textit{on } \Gamma \setminus \Gamma_0? \tag{3.19}$$

Let us point out that, by scaling arguments, one does not have to assume that f is "small" in this question. It turns out that the answer to this question is indeed positive. This has been proved in [25] if Ω is simply connected and by O. Glass in [53] for the general case. For many results on feedback in flow control, see the book [1].

Chapter 4

Quasi-static deformations

The quasi-static deformation method has in fact been already used for the controllability of the 1-D tank (see section 3.4). Note that, in connection with our toy model $\mathcal{T}$ of the 1-D tank, the linearized system of

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + \gamma,\ \dot{x}_3 = x_4,\ \dot{x}_4 = -x_3 + 2x_1x_2$$

at equilibrium $(\gamma, 0, 0, 0)$ has i and $-i$ for eigenvalues. This is why the quasi-static deformation is so easy to perform. If this linearized would have eigenvalues with strictly positive real part, it is still possible to perform in some cases the quasi-static deformation by stabilizing the equilibriums by suitable feedbacks, as we are going to see in this chapter.

4.1 Description of the method

For sake of simplicity we explain the method in finite dimension. Let us consider in $\mathbb{R}^n$ a general control system of the form

$$\dot{y}(t) = g(y(t), u(t)), \tag{4.1}$$

where $g\ :\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^1, $u(t) \in \mathcal{U}$, and $\mathcal{U}$ denotes the set of measurable essentially bounded admissible controls. Let $y_0, y_1 \in \mathbb{R}^n$ be two equilibrium points of system (4.1), that is

$$g(y_i, u_i) = 0,\ i = 1, 2,$$

for some $u_0, u_1 \in \mathbb{R}^m$. We assume that (y_0, u_0) and (y_1, u_1) belong to a same C^1-path connected component of the set of zeros of g in $\mathbb{R}^n \times \mathbb{R}^m$. Our aim is to steer the system from y_0 to y_1 in some (large) time $T > 0$. The method splits into four steps:

First step: construction of an almost trajectory. Construct a C^1-path $(\bar{y}(\tau), \bar{u}(\tau))$, with $\tau \in [0, 1]$, connecting (y_0, u_0) to (y_1, u_1) and such that

$$\forall \tau \in [0, 1] \quad g(\bar{y}(\tau), \bar{u}(\tau)) = 0.$$

Of course this path is not in general solution of system (4.1), but if $\varepsilon > 0$ is small enough then the C^1-path $(y^\varepsilon, u^\varepsilon)$

$$\begin{array}{rcl} [0, 1/\varepsilon] & \to & \mathbb{R}^n \times \mathbb{R}^m \\ t & \mapsto & (y^\varepsilon(t), u^\varepsilon(t)) = (\bar{y}(\varepsilon t), \bar{u}(\varepsilon t)) \end{array}$$

is "almost" a solution of system (4.1). Indeed

$$\|\dot{y}^\varepsilon - f(y^\varepsilon, u^\varepsilon)\| = \mathrm{O}(\varepsilon).$$

Second step: stabilization procedure. This quasi-static trajectory is not in general stable, and thus has to be stabilized. To this aim, introduce the following change of variable:

$$z(t) = y(t) - y^\varepsilon(t),$$
$$v(t) = u(t) - u^\varepsilon(t),$$

where $t \in [0, 1/\varepsilon]$. In the new variables z, v, the control system writes, at least if $\|z(t)\| + \|v(t)\|$ is small enough,

$$\dot{z}(t) = A(\varepsilon t)z(t) + B(\varepsilon t)v(t) + \mathrm{O}(\|z(t)\|^2 + \|v(t)\|^2 + \varepsilon), \tag{4.2}$$

where $t \in [0, 1/\varepsilon]$, and where

$$A(\tau) = \frac{\partial g}{\partial y}(\bar{y}(\tau), \bar{u}(\tau)) \quad \text{and} \quad B(\tau) = \frac{\partial g}{\partial u}(\bar{y}(\tau), \bar{u}(\tau)),$$

with $\tau = \varepsilon t \in [0, 1]$. Therefore we have to stabilize near the origin a *slowly-varying in time* linear control system. We refer to [69] for this classical theory. In particular it is often sufficient to stabilize, for every $\tau \in [0, 1]$ the control system $\dot{\tilde{z}} = A(\tau)\tilde{z} + B(\tau)v$ by some feedback $v_\tau \tilde{z}$ and then define $v = v_{\epsilon t} z$ to stabilize the control system (4.2).
Third step: local controllability at the end. If the system (4.1) is *locally controllable* near the point y_1, we conclude that it is possible to steer the system in finite time from the point $y(1/\varepsilon)$ to the desired target y_1. Usually such a local controllability result is achieved by using an implicit function argument, after proving that the linearized system is controllable.

4.2 Application to a semilinear heat equation

Let us now explain how to apply this method on a specific control system modelled by a semilinear heat equation.

Let $L > 0$ fixed and $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. Let us consider the boundary control system

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t, 0) = 0, \ y(t, L) = u(t), \end{cases} \tag{4.3}$$

where the state is $y(t, .) : [0, L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$.

Our goal is to see that one can move from any given steady-state to any other one belonging to the same connected component of the set of steady-states. More precisely let us define the notion of steady-state.

Definition 54 *A function $y \in C^2([0, L])$ is a steady-state of the control system (4.3) if*

$$\frac{d^2 y}{dx^2} + f(y) = 0, \ y(0) = 0.$$

We denote by $\mathcal{S}$ the set of steady-states, endowed with the C^2 topology.

Let us also introduce the Banach space

$$Y_T = \Big\{ y(t,x), (t,x) \in (0,T) \times (0,L) \ / \ y \in L^2(0,T,W^{2,2}(0,L)) \\ \text{and } \frac{\partial y}{\partial t} \in L^2((0,T)\times(0,L)) \Big\} \tag{4.4}$$

endowed with the norm

$$\|y\|_{Y_T} = \|y\|_{L^2(0,T,W^{2,2}(0,L))} + \left\| \frac{\partial y}{\partial t} \right\|_{L^2((0,T)\times(0,L))}.$$

Notice that Y_T is continuously imbedded in $L^\infty((0,T)\times(0,L))$.

The main result of this section is the following.

Theorem 55 *([30].) Let y_0 and y_1 be two steady-states belonging to a same connected component of $\mathcal{S}$. There exist a time $T>0$ and a control function $u \in L^2(0,T)$ such that the solution $y(t,x)$ in Y_T of*

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t,0) = 0, \ y(t,L) = u(t), \\ y(0,x) = y_0(x), \end{cases} \tag{4.5}$$

satisfies $y(T,.) = y_1(.)$.

Remark 56 *In [30] we prove that, if y_0 and y_1 belong to distinct connected components of $\mathcal{S}$, then it is actually impossible to move either from y_0 to y_1 or from y_1 to y_0, whatever the time and the control are.*

Remark 57 *For any $T>0$ and $u \in L^2(0,T)$ there is at most one solution of (4.5) in the Banach space Y_T.*

Remark 58 *Concerning the global controllability problem, one of the main results [39] asserts that if f is globally lipschitzian then this control system is approximately globally controllable, see also [63] for exact controllability. When f is superlinear the situation is still widely open, in particular because of possible blowing up. Indeed it is well known that if $yf(y)>0$ as $y \neq 0$ then blow-up phenomena may occur for the Cauchy problem*

$$\begin{cases} \dfrac{\partial y}{\partial t} = \dfrac{\partial^2 y}{\partial x^2} + f(y), \\ y(t,0) = 0, \ y(t,L) = 0, \\ y(0,x) = y_0(x). \end{cases} \tag{4.6}$$

For example if $f(y) = y^3$ then for numerous initial data there exists $T>0$ such that the unique solution to the previous Cauchy problem is well defined on $[0,T)\times[0,L]$ and satisfies

$$\lim_{t\to T} \|y(t,.)\|_{L^\infty(0,L)} = +\infty,$$

see for instance [7, 13, 82, 112] and references therein.

One may ask if, acting on the boundary of $[0, L]$*, one could avoid the blow-up phenomenon. Actually the answer to this question is negative in general, see [41]: for some nonlinear functions* f *satisfying*

$$|f(y)| \sim |y| \log^p(1+|y|) \quad \text{as } |y| \to +\infty,$$

and for any time $T > 0$*, there exist initial data which lead to blow-up before time* T*, whatever the control function* u *is. Notice however that if*

$$|f(y)| = o\Big(|y| \log^{3/2}(1+|y|)\Big) \quad \text{as } |y| \to +\infty,$$

then the blow-up (which could occur in the absence of control) can be avoided by means of boundary control, see [41]. But one still does not know what is the situation for $f(y) = -y^3$*.*

Remark 59 *This is a (partial) global exact controllability result. The time needed in our proof is large, but on the other hand there are indeed cases where the time* T *of controllability cannot be taken arbitrarily small. For instance in the case where* $f(y) = -y^3$*, any solution of (4.5) starting from* 0 *satisfies the inequality*

$$\int_0^L (L-x)^4 y(T,x)^2 dx \leq 8LT,$$

and hence if $y_0 = 0$ *a minimal time is needed to reach a given* $y_1 \neq 0$*. This result is due to Bamberger [60], see also [44, Lemma 2.1].*

In order to prove Theorem 55, it suffices to follow the steps described previously.

4.2.1 Construction of an almost trajectory

The following lemma is obvious.

Lemma 60 *Let* $\phi_0, \phi_1 \in \mathcal{S}$*. Then* ϕ_0 *and* ϕ_1 *belong to the same connected component of* $\mathcal{S}$ *if and only if for any real number* α *between* $\phi_0'(0)$ *and* $\phi_1'(0)$ *the maximal solution of*

$$\frac{d^2y}{dx^2} + f(y) = 0,\ y(0) = 0, y'(0) = \alpha,$$

denoted by $y^\alpha(.)$*, is defined on* $[0, L]$*.*

Let now y_0 and y_1 in the same connected component of $\mathcal{S}$. Let us construct in $\mathcal{S}$ a C^1 path $(\bar{y}(\tau, .), \bar{u}(\tau))$, $0 \leq \tau \leq 1$, joining y_0 to y_1. For each $i = 0, 1$ set

$$\alpha_i = y_i'(0).$$

Then with our previous notations: $y_i(.) = y^{\alpha_i}(.), i = 0, 1$. Now set

$$\bar{y}(\tau, x) = y^{(1-\tau)\alpha_0 + \tau\alpha_1}(x) \ \text{ and } \ \bar{u}(\tau) = \bar{y}(\tau, L),$$

where $\tau \in [0, 1]$ and $x \in [0, L]$. By construction we have

$$\bar{y}(0, .) = y_0(.),\ \bar{y}(1, .) = y_1(.) \ \text{ and } \ \bar{u}(0) = \bar{u}(1) = 0,$$

and thus $(\bar{y}(\tau, .), \bar{u}(\tau))$ is a C^1 path in $\mathcal{S}$ connecting y_0 to y_1.

4.2.2 Stabilization procedure

This is the most technical part of the work. Let $\varepsilon > 0$. We set, for any $t \in [0, 1/\varepsilon]$ and any $x \in [0, L]$,

$$\begin{aligned} z(t,x) &= y(t,x) - \bar{y}(\varepsilon t, x), \\ v(t) &= u(t) - \bar{u}(\varepsilon t). \end{aligned} \tag{4.7}$$

Then from the definition of $(\bar{y}, \bar{u})$ we infer that z satisfies the initial-boundary problem

$$\begin{cases} z_t = z_{xx} + f'(\bar{y})z + z^2 \displaystyle\int_0^1 (1-s)f''(\bar{y}+sz)ds - \varepsilon \bar{y}_\tau, \\ z(t,0) = 0, \ z(t,L) = v(t), \\ z(0,x) = 0. \end{cases} \tag{4.8}$$

Now, in order to deal rather with a Dirichlet-type problem, we set

$$w(t,x) = z(t,x) - \frac{x}{L} v(t), \tag{4.9}$$

and we suppose that the control v is derivable. This leads to the following equation:

$$\begin{cases} w_t = w_{xx} + f'(\bar{y})w + \dfrac{x}{L} f'(\bar{y})v - \dfrac{x}{L} v' + r(\varepsilon, t, x), \\ w(t,0) = w(t,L) = 0, \\ w(0,x) = -\dfrac{x}{L} v(0), \end{cases} \tag{4.10}$$

where

$$r(\varepsilon,t,x) = -\varepsilon \bar{y}_\tau + \Big(w + \frac{x}{L} v\Big)^2 \int_0^1 (1-s) f'' \Big(\bar{y} + s(w + \frac{x}{L} v)\Big)\, ds, \tag{4.11}$$

and the next step is to prove that there exist ε small enough and a pair (v, w) solution of (4.10) such that $w(1/\varepsilon, .)$ belongs to some arbitrary neighborhood of 0 in H_0^1-topology. To achieve this one constructs an appropriate control function and a Lyapunov functional which stabilizes system (4.10) to 0. In fact as we shall see the control will be chosen in $H^1(0, 1/\varepsilon)$ and such that $v(0) = 0$.

In order to motivate what follows, let us first notice that if the residual term r and the control v were equal to zero then (4.10) would reduce to

$$\begin{aligned} w_t &= w_{xx} + f'(\bar{y})w, \\ w(t,0) &= w(t,L) = 0. \end{aligned}$$

This suggests to introduce the *one-parameter family of linear operators*

$$A(\tau) = \Delta + f'(\bar{y}(\tau, .))Id, \ \tau \in [0,1], \tag{4.12}$$

defined on $H^2(0,L) \cap H_0^1(0,L)$. Let $(e_j(\tau,.))_{j\geq 1}$ be an Hilbertian basis of $L^2(0,L)$ of eigenfunctions of $A(\tau)$, such that for each $j \geq 1$ and each $\tau \in [0,1]$,

$$e_j(\tau,.) \in H_0^1(0,L) \cap C^2([0,L]),$$

and let $(\lambda_j(\tau))_{j\geq 1}$ denote the corresponding eigenvalues. A standard application of the minimax principle (see for instance [88]) shows that these eigenfunctions and eigenvalues are C^1 functions of τ. Moreover, for each $\tau \in [0,1]$,

$$-\infty < \cdots < \lambda_n(\tau) < \cdots < \lambda_1(\tau) \quad \text{and} \quad \lambda_n(\tau) \underset{n \to -\infty}{\longrightarrow} -\infty.$$

From the continuity of the eigenvalues on $[0,1]$, we can define n as the maximal number of eigenvalues taking at least a nonnegative value as $\tau \in [0,1]$, i.e. there exists $\eta > 0$ such that

$$\forall t \in [0, 1/\varepsilon] \quad \forall k > n \quad \lambda_k(\varepsilon t) < -\eta < 0. \tag{4.13}$$

Remark 61 *Note that the integer n can be arbitrarily large. For example if $f(y) = y^3$ and if $y_1'(0) \to +\infty$ then $n \to +\infty$.*

We also set, for any $\tau \in [0,1]$ and $x \in [0,L]$,

$$a(\tau, x) = \frac{x}{L} f'(\bar{y}(\tau, x)) \ \text{ and } \ b(x) = -\frac{x}{L}.$$

In these notations system (4.10) leads to

$$w_t(t,.) = A(\varepsilon t) w(t,.) + a(\varepsilon t,.) v(t) + b(.) v'(t) + r(\varepsilon, t, .). \tag{4.14}$$

Any solution $w(t,.) \in H^2(0,L) \cap H_0^1(0,L)$ of (4.14) can be expanded as series in the eigenfunctions $e_j(\varepsilon t,.)$, convergent in $H_0^1(0,L)$,

$$w(t,.) = \sum_{j=1}^{\infty} w_j(t) e_j(\varepsilon t,.).$$

In fact the w_j's depend on ε and should be called, for example, w_j^ε. For simplicity we omit the index ε, and we shall also omit the index ε for other functions.

In what follows we are going to move, by means of an appropriate *feedback control*, the n first eigenvalues of the operator A, without moving the others, in order to make all eigenvalues negative. This pole-shifting process is the first part of the stabilization procedure, see [92, p. 711].

For any $\tau \in [0,1]$ let $\pi_1(\tau)$ denote the orthogonal projection onto the subspace of $L^2(0,L)$ spanned by $e_1(\tau,.), \ldots, e_n(\tau,.)$, and let

$$w^1(t) = \pi_1(\varepsilon t) w(t,.) = \sum_{j=1}^{n} w_j(t) e_j(\varepsilon t,.). \tag{4.15}$$

It is clear that for any τ the operators $\pi_1(\tau)$ and $A(\tau)$ commute, and moreover for any $y \in L^2(0,L)$ we have

$$\pi_1'(\tau) y = \sum_{j=1}^{n} \langle y, e_j(\tau,.) \rangle_{L^2(0,L)} \frac{\partial e_j}{\partial \tau}(\tau,.) + \sum_{j=1}^{n} \left\langle y, \frac{\partial e_j}{\partial \tau}(\tau,.) \right\rangle_{L^2(0,L)} e_j(\tau,.).$$

Hence, derivating (4.15) with respect to t, we get

$$\sum_{j=1}^{n} w_j'(t) e_j(\varepsilon t,.) = \pi_1(\varepsilon t) w_t(t,.) + \varepsilon \sum_{j=1}^{n} \left\langle w(t,.), \frac{\partial e_j}{\partial \tau}(\varepsilon t,.) \right\rangle_{L^2(0,L)} e_j(\varepsilon t,.).$$

On the other part

$$A(\varepsilon t) w^1(t) = \sum_{j=1}^{n} \lambda_j(\varepsilon t) w_j(t) e_j(\varepsilon t,.),$$

and thus (4.14) yields

$$\begin{aligned}\sum_{j=1}^{n} w_j'(t)e_j(\varepsilon t,.) =& \sum_{j=1}^{n} \lambda_j(\varepsilon t)w_j(t)e_j(\varepsilon t,.) + \pi_1(\varepsilon t)a(\varepsilon t,.)v(t) \\ &+ \pi_1(\varepsilon t)b(.)v'(t) + r^1(\varepsilon,t,.),\end{aligned} \tag{4.16}$$

where

$$r^1(\varepsilon,t,.) = \pi_1(\varepsilon t)r(\varepsilon,t,.) + \varepsilon\sum_{j=1}^{n}\left\langle w, \frac{\partial e_j}{\partial\tau}(\varepsilon t,.)\right\rangle_{L^2(0,L)} e_j(\varepsilon t,.). \tag{4.17}$$

Let us set an upper bound to the residual term r^1. First, it is not difficult to check that there exists a constant C such that if $|v(t)|$ and $\|w(t,.)\|_{L^\infty(0,L)}$ are less than 1 then the inequality

$$\|r(\varepsilon,t,.)\|_{L^\infty(0,L)} \leq C(\varepsilon + v(t)^2 + \|w(t,.)\|^2_{L^\infty(0,L)})$$

holds, where r is defined by (4.11). Therefore we get easily

$$\|r^1(\varepsilon,t,.)\|_{L^\infty(0,L)} \leq C_1(\varepsilon + v(t)^2 + \|w(t,.)\|^2_{L^\infty(0,L)}).$$

Moreover since $H^1(0,L)$ is continuously imbedded in $C^0([0,L])$, we can assert that there exists a constant C_2 such that if $|v(t)|$ and $\|w(t,.)\|_{L^\infty(0,L)}$ are less than 1 then

$$\|r^1(\varepsilon,t,.)\|_{L^\infty(0,L)} \leq C_2(\varepsilon + v(t)^2 + \|w(t,.)\|^2_{H^1_0(0,L)}). \tag{4.18}$$

Now projecting (4.16) on each $e_i, i = 1\ldots n$, one comes to

$$w_i'(t) = \lambda_i(\varepsilon t)w_i(t) + a_i(\varepsilon t)v(t) + b_i(\varepsilon t)v'(t) + r_i^1(\varepsilon,t), \quad i = 1\ldots n, \tag{4.19}$$

where

$$\begin{aligned} r_i^1(\varepsilon,t) &= \langle r^1(\varepsilon,t,.), e_i(\varepsilon t,.)\rangle_{L^2(0,L)}, \\ a_i(\varepsilon t) &= \langle a(\varepsilon t,.), e_i(\varepsilon t,.)\rangle_{L^2(0,L)} = \frac{1}{L}\int_0^L xf'(\bar{y}(\varepsilon t,x))e_i(\varepsilon t,x)dx, \\ b_i(\varepsilon t) &= \langle b(.), e_i(\varepsilon t,.)\rangle_{L^2(0,L)} = -\frac{1}{L}\int_0^L xe_i(\varepsilon t,x)dx. \end{aligned} \tag{4.20}$$

The n equations (4.19) form a differential system controlled by v, v'. Set

$$\alpha(t) = v'(t), \tag{4.21}$$

and consider now $v(t)$ as a state and $\alpha(t)$ as a control. Then the former finite dimensional system may be rewritten as

$$\begin{cases} v' = \alpha, \\ w_1' = \lambda_1 w_1 + a_1 v + b_1\alpha + r_1^1, \\ \quad\vdots \\ w_n' = \lambda_n w_n + a_n v + b_n\alpha + r_n^1. \end{cases} \tag{4.22}$$

If we introduce the matrix notations

$$X_1(t) = \begin{pmatrix} v(t) \\ w_1(t) \\ \vdots \\ w_n(t)\end{pmatrix},\ R_1(\varepsilon,t) = \begin{pmatrix} 0 \\ r_1^1(\varepsilon,t) \\ \vdots \\ r_n^1(\varepsilon,t)\end{pmatrix},$$

$$A_1(\tau) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1(\tau) & \lambda_1(\tau) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n(\tau) & 0 & \cdots & \lambda_n(\tau) \end{pmatrix}, \quad B_1(\tau) = \begin{pmatrix} 1 \\ b_1(\tau) \\ \vdots \\ b_n(\tau) \end{pmatrix},$$

then equations (4.22) yield the *finite dimensional linear control system*

$$X_1'(t) = A_1(\varepsilon t)X_1(t) + B_1(\varepsilon t)\alpha(t) + R_1(\varepsilon, t). \tag{4.23}$$

Let us now prove the following lemma.

Lemma 62 *For each $\tau \in [0,1]$ the pair $(A_1(\tau), B_1(\tau))$ satisfies the Kalman condition, i.e.*

$$\textit{rank}\left(B_1(\tau), A_1(\tau)B_1(\tau), \dots, A_1(\tau)^{n-1}B_1(\tau)\right) = n. \tag{4.24}$$

Let $\tau \in [0,1]$ be fixed. We compute directly

$$\det\left(B_1, A_1B_1, \dots, A_1^{n-1}B_1\right) = \prod_{j=1}^{n}(a_j + \lambda_j b_j)\ \mathrm{VdM}(\lambda_1, \dots, \lambda_n), \tag{4.25}$$

where $\mathrm{VdM}(\lambda_1, \dots, \lambda_n)$ is a Van der Monde determinant, and thus is never equal to zero since the $\lambda_i(\tau), i = 1 \dots n$, are distinct, for any $\tau \in [0,1]$. On the other part, using the fact that each $e_j(\tau, .)$ is an eigenfunction of $A(\tau)$ and belongs to $H_0^1(0,L)$, we compute

$$\begin{aligned} a_j(\tau) + \lambda_j(\tau)b_j(\tau) &= \frac{1}{L}\int_0^L x\left(f'(\bar{y}(\tau,x))e_j(\tau,x) - \lambda_j(\tau)e_j(\tau,x)\right)dx \\ &= -\frac{1}{L}\int_0^L x\frac{\partial^2 e_j}{\partial x^2}(\tau,x)dx \\ &= -\frac{\partial e_j}{\partial x}(\tau, L). \end{aligned}$$

But this quantity is never equal to zero since $e_j(\tau, L) = 0$ and $e_j(\tau, .)$ is a nontrivial solution of a linear second-order scalar differential equation. Therefore the determinant (4.25) is never equal to zero and we are done.

It is a standard fact that the Kalman condition (4.24) implies a *pole shifting* result, and we get the following corollary, see [69].

Corollary 63 *For each $\tau \in [0,1]$ there exist scalars $k_0(\tau), \dots, k_n(\tau)$ such that, if we denote*

$$K_1(\tau) = (k_0(\tau), \dots, k_n(\tau)),$$

then the matrix $A_1(\tau) + B_1(\tau)K_1(\tau)$ admits -1 as an eigenvalue with order $n+1$.

Moreover there exists a C^1 application $\tau \mapsto P(\tau)$ on $[0,1]$, where $P(\tau)$ is a $(n+1) \times (n+1)$ symmetric positive definite matrix, such that the identity

$$P(\tau)\left(A_1(\tau) + B_1(\tau)K_1(\tau)\right) + {}^{tr}\left(A_1(\tau) + B_1(\tau)K_1(\tau)\right)P(\tau) = -I \tag{4.26}$$

holds for any $\tau \in [0,1]$.

We are now able to construct a control Lyapunov functional in order to stabilize system (4.14). Let $c > 0$ to be chosen later. For any $t \in [0, 1/\varepsilon]$, $v \in \mathbb{R}$ and $w \in H^2(0, L) \cap H_0^1(0, L)$ we set

$$V(t, v, w) = c\, X_1(t) P(\varepsilon t) X_1(t) - \frac{1}{2} \langle w, A(\varepsilon t) w \rangle_{L^2(0,L)}, \tag{4.27}$$

where $X_1(t)$ denotes the matrix vector in $\mathbb{R}^{n+1}$

$$X_1(t) = \begin{pmatrix} v \\ w_1(t) \\ \vdots \\ w_n(t) \end{pmatrix}$$

and

$$w_i(t) = \langle w, e_i(\varepsilon t, .) \rangle_{L^2(0,L)}.$$

In particular we have

$$V(t, v, w) = c\, X_1(t) P(\varepsilon t) X_1(t) - \frac{1}{2} \sum_{j=1}^{\infty} \lambda_j(\varepsilon t) w_j(t)^2. \tag{4.28}$$

After some straightforward estimates (see [30]), one gets the existence of $\sigma > 0$ and of $\rho \in (0, \sigma]$ such that, for every $\varepsilon \in (0, 1]$ and for every $t \in [0, 1/\varepsilon]$ such that $V_1(t) \leq \rho$,

$$V_1'(t) \leq \sigma \varepsilon^2.$$

Hence, since $V_1(0) = 0$, we get, if $\varepsilon \in (0, \rho/\sigma]$, that

$$V_1(t) \leq \sigma \varepsilon, \quad \forall t \in [0, 1/\varepsilon],$$

and in particular

$$V_1 \left(\frac{1}{\varepsilon} \right) \leq \sigma \varepsilon.$$

Coming back to definitions (4.7) and (4.9), we have proved

$$\left\| y \left(\frac{1}{\varepsilon}, . \right) - y_1(.) \right\|_{H^1(0,L)} \leq \gamma \varepsilon, \tag{4.29}$$

where $y_1(.) = \bar{y}(1, .)$ is the final target and γ is a positive constant which does not depend on $\varepsilon \in (0, \rho/\sigma]$. This concludes the second step.

4.2.3 Local controllability at the end-point

The last step in the proof of Theorem 55 consists in solving a local exact controllability result: from the previous subsection $y(\frac{1}{\varepsilon}, .)$ belongs to an arbitrarily small neighborhood of $y_1(.)$ in the H^1-topology if ϵ is small enough, and our aim is now to construct a trajectory $q(t, x)$ solution of the control system steering $y(\frac{1}{\epsilon}, .)$ to $y_1(.)$ in some time $T > 0$ (for instance $T = 1$), i.e.

$$\begin{cases} q_t = q_{xx} + f(q), \\ q(t, 0) = 0, \ q(t, L) = u(t), \\ q(0, x) = y \left(\dfrac{1}{\epsilon}, x \right), \ q(T, x) = y_1(x). \end{cases}$$

Existence of such a solution q is given by [63, Theorem 3.3], with $\tilde{y}(t,x) = y_1(x)$. Actually in [63] the function f is assumed to be globally lipschitzian, but the local result one needs here readily follows from the proofs and the estimates contained in [63].

Remark 64 *One could also deduce Theorem 55 from [63, Theorem 3.3] and a straightforward compactness argument. But the method presented here leads to a control which can be easily implemented. See [30] for an example.*

Chapter 5

Power series expansion

5.1 Description of the method

We again explain first the method on a control system of finite dimension,

$$\dot{x} = f(x, u), \tag{5.1}$$

with $x \in \mathbb{R}^m$ and $u \in \mathbb{R}^m$ and $f(0,0) = 0$. Let $H := \text{Span}\,\{A^i Bu;\, u \in \mathbb{R}^m,\, i \in \{0, \cdots, n-1\}$ with

$$A = \frac{\partial f}{\partial x}(0,0),\ B = \frac{\partial f}{\partial x}(0,0).$$

If $H = \mathbb{R}^n$, the linearized control system around $(0,0)$ is controllable and therefore the nonlinear control system is small time locally controllable (at 0). Let us look to the case where the dimension of H is $n-1$. Let us make a power expansion of the control system (5.1). We write

$$x = y^1 + y^2 + \cdots,\ u = v^1 + v^2 + \cdots.$$

The order 1 is given by (y^1, v^1); the order 2 is given by (y^2, v^2) and so on. The dynamics of these different orders are given by

$$\dot{y}^1 = \frac{\partial f}{\partial x}(0,0)y^1 + \frac{\partial f}{\partial u}(0,0)v^1, \tag{5.2}$$

$$\dot{y}^2 = \frac{\partial f}{\partial x}(0,0)y^2 + \frac{\partial f}{\partial u}(0,0)v^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(y^1, y^1) + \frac{\partial^2 f}{\partial x \partial u}(y^1, u^1) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(u^1, u^1) \tag{5.3}$$

and so on. Let $e_1 \in H^\perp$. Let $T > 0$. Let us assume that there are controls $v^1_\pm$ and $v^2_\pm$ both in $L^\infty(0,T)^m$ such that, if $y^1_\pm$ and $y^2_\pm$ are solutions of

$$\begin{gathered}
\dot{y}^1_\pm = \frac{\partial f}{\partial x}(0,0)y^1_\pm + \frac{\partial f}{\partial u}(0,0)v^1_\pm,\\
y^1_\pm(0) = 0,\\
\dot{y}^2_\pm = \frac{\partial f}{\partial x}(0,0)y^2_\pm + \frac{\partial f}{\partial u}(0,0)v^2_\pm + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(y^1_\pm, y^1_\pm) + \frac{\partial^2 f}{\partial x \partial u}(y^1_\pm, u^1_\pm) + \frac{1}{2}\frac{\partial^2 f}{\partial u^2}(u^1_\pm, u^1_\pm),\\
y^2_\pm = 0,
\end{gathered}$$

then

$$\begin{gathered}
y^1_\pm(T) = 0,\\
y^2_\pm(T) = \pm e_1.
\end{gathered}$$

Let $(e_i)_{i\in\{2,\cdots n\}}$ be a basis of H. By the definition of H and classical result about the controllable part of a linear system (see e.g. [100, Section 3.3]), there are $(u_i)_{i=2,\cdots,n}$ all in $L^\infty(0,T)^m$ such that, if $(x_i)_{i=2,\cdots,n}$ are the solutions of

$$\dot{x}_i = \frac{\partial f}{\partial x}(0,0)x_i + \frac{\partial f}{\partial x}(0,0)u_i,$$
$$x_i(0) = 0,$$

then, for every $i \in \{2, \cdots, n\}$,

$$x_i(0) = e_i.$$

Let now

$$b = \sum_{i=1}^{n} b_i e_i$$

be a point in $\mathbb{R}^n$. Let

- if $b_1 \geq 0$, $v^1 := v^1_+$ and $v^2 := v^2_+$,
- if $b_1 < 0$, $v^1 := v^1_-$ and $v^2 := v^2_-$.

Then, let $u : (0,T) \to \mathbb{R}^m$ be defined by

$$u(t) := |b_1|^{1/2} v^1(t) + |b_1| v^2(t) + \sum_{i=2}^{n} b_i u_i(t).$$

Let $x : [0,T] \to \mathbb{R}^n$ be the solution of

$$\dot{x} = f(x, u(t)),\ x(0) = 0.$$

Then one has, as $b \to 0$,

$$x(T) = b + o(b).$$

Hence using the Brouwer fixed point theorem (and standard estimates on ordinary differential equations) one gets the the local controllability of $\dot{x} = f(x,u)$ (around 0) in time T. This method is classical to study the local controllability of control system in finite dimension. In fact more subtle tools have been introduced: for example different scalings on the components of the control and of the state as well as scaling on time, and the case where the codimension of H is strictly larger than 1 has been considered. See e.g. [3, 8, 9, 10, 57, 59, 66, 106, 107, 111] and the references therein. In the next section we apply this method to a partial differential equation, namely the Korteweg-de Vries equation. In fact, for that equation, an expansion to the order 2 is not sufficient: we obtain the local controllability by means of an expansion to the order 3.

5.2 Application to the Korteweg-de Vries equation

Let us consider the following Korteweg-de Vries control system

$$(\mathrm{KdV}) \begin{cases} y_t + y_x + y_{xxx} + y y_x = 0, \\ y(t,0) = y(t,L) = 0. \end{cases}$$

For this control system, $L > 0$ is given, the state is $y(.) : [0, L] \longrightarrow \mathbb{R}$, and for the control one can take, for example, $u(.) = y_x(., L) \in \mathbb{R}$. We are interested in the local controllability of (KdV) around 0. Rosier has proved in [90] that the control system (KdV) is locally controllable around 0 provided that the length of the spatial domain is not critical.

Theorem 65 *([90].) Let $T > 0$, and let us assume that*

$$L \notin \mathcal{N} := \left\{ 2\pi\sqrt{\frac{j^2 + l^2 + jl}{3}};\ j, l \in \mathbb{N}^* \right\}. \tag{5.4}$$

Then there exists $r_0 > 0$ such that for any $y_0, y_1 \in L^2(0, L)$, with $\|y_0\|_{L^2(0,L)} < r_0$ and $\|y_1\|_{L^2(0,L)} < r_0$, there exists $y \in C([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$ satisfying (KdV) such that $y(0, .) = y_0$ and $y(T, .) = y_1$.

The aim of this section is to study the local exact controllability around 0 of the nonlinear KdV equation when $L = 2k\pi$ in $\mathcal{N}$ (take $j = l = k$ in (5.4)). The main theorem is the following one.

Theorem 66 *([27].) Let k be a a positive integer and let $T > 0$. There exist $r_1 > 0$ such that for any $y_0, y_1 \in L^2(0, 2k\pi)$, with $\|y_0\|_{L^2(0,2k\pi)} < r_1$ and $\|y_1\|_{L^2(0,2k\pi)} < r_1$, there exist $y \in C([0, T], L^2(0, 2k\pi)) \cap L^2(0, T, H^1(0, 2k\pi))$ satisfying (KdV) with $L = 2k\pi$ such that $y(0, .) = y_0$ and $y(T, .) = y_1$.*

When $L = 2k\pi$ the linearized control system of (KdV) around 0 is

$$\text{(KdVL)} \left\{ \begin{array}{l} y_t + y_x + y_{xxx} = 0, \\ y(t, 0) = y(t, 2k\pi) = 0. \end{array} \right.$$

It has been shown by Rosier in [90] that this linear control system is not controllable. To prove that the nonlinear term yy_x gives the local controllability, a first idea could be to use the exact controllability of the nonlinear equation around nontrivial stationary solutions proved in [32] and to apply the method introduced we have used to get the local controllability of the 1-D tank (that is, use the return method together with quasi-static deformations; see section 3.4). However, with this method, we can only obtain the local exact controllability for *large* time. For proving Theorem 66 we use a different strategy that we have briefly described in the above section.

We first point out that, in this theorem, we may assume that $y_0 = 0$: this follows easily from the invariance of the control system (KdV) by the change of variables $\tau = T - t$, $\xi = 2k\pi - x$. Then we use the following result for the linear control system.

Theorem 67 *([90].) Let $T > 0$, $H = \left\{ y \in L^2(0, 2k\pi), \int_0^{2k\pi} y(1 - \cos(x))dx = 0 \right\}$. For every $(y_0, y_1) \in H \times H$, there exists $y \in C([0, T], L^2(0, 2k\pi)) \cap L^2(0, T, H^1(0, 2k\pi))$ satisfying $(KdVL)$ such that $y(0, .) = y_0$ and $y(T, .) = y_1$.*

Next, as we are going to see, the nonlinear term yy_x allows us to "move" in the two directions $\pm(1 - \cos(x))$ which are missed by the linearized control system $(KdVL)$. Finally in we briefly sketch how to derive Theorem 66 from these motions by means of a fixed point theorem.

Remark 68 *For the other critical lengths, we believe that the same result holds. But the situation is more complicate in these other cases : there are now four noncontrollable directions of the linearized control system around* 0 *(instead of two, namely* $1 - \cos(x)$ *and* $\cos(x) - 1$*).*

Let us now explain how the nonlinear term yy_x allows us to "move" in the two directions $\pm(1-\cos(x))$. Let $L>0$. We first recall some properties proved by Rosier in [90] about the following linear KdV Cauchy problem

$$y_t + y_x + y_{xxx} = f, \tag{5.5}$$
$$y(t,0) = y(t,L) = 0, \tag{5.6}$$
$$y_x(t,L) = h(t), \tag{5.7}$$
$$y(T_0,x) = y_0(x). \tag{5.8}$$

We adopt notations of [90]. Let A denote the operator $Aw = -w''' - w'$ defined on $\mathcal{D}(A) := \{w \in H^3(0,L);\, w(0) = w(L) = w_x(L) = 0\}$ and let $(S(t)_{t\geq 0})$ denote the semi-group of contractions associated with A (see [90, Proposition 3.1]). For $T_0 < T_1$, let

$$B_{T_0,T_1} := C([T_0,T_1], L^2(0,L)) \bigcap L^2(T_0,T_1,H^1(0,L))$$

endowed with the norm

$$\|y\|_{B_{T_0,T_1}} = \operatorname{Max}\left\{\|y(t,.)\|_{L^2(0,L)};\, t\in[T_0,T_1]\right\} + \left(\int_{T_0}^{T_1} \|y(t,.)\|^2_{H^1(0,L)}dt\right)^{\frac{1}{2}}.$$

Rosier has proved the following proposition.

Proposition 69 *([90, (Proofs of) Propositions 3.2 and 3.7].) Let $T_0 < T_1$. There exist unique linear maps Ψ_{T_0,T_1} and δ_{T_0,T_1}*

$$\begin{aligned} \Psi_{T_0,T_1} :\ L^2(0,L) \times L^2(T_0,T_1) \times L^1(T_0,T_1,L^2(0,L)) &\longrightarrow B_{T_0,T_1} \\ (y_0,h,f) &\longmapsto \Psi_{T_0,T_1}(y_0,h,f) \end{aligned}$$

$$\begin{aligned} \delta_{T_0,T_1} :\ L^2(0,L) \times L^2(T_0,T_1) \times L^1(T_0,T_1,L^2(0,L)) &\longrightarrow L^2(T_0,T_1) \\ (y_0,h,f) &\longmapsto \delta_{T_0,T_1}(y_0,h,f) \end{aligned}$$

such that, for $y_0 \in \mathcal{D}(A)$, $h \in C^2([T_0,T_1])$ with $h(T_0) = 0$ and $f \in C^1([T_0,T_1], L^2(0,L))$,

$\Psi_{T_0,T_1}(y_0,h,f)$ is the unique classical solution of (5.5)-(5.6)-(5.8),
$$\delta_{T_0,T_1}(y_0,h,f)(t) = (\Psi_{T_0,T_1}(y_0,h,f))_x(t,0).$$

The function $\Psi_{T_0,T_1}(y_0,h,f)$ is called the mild solution of (5.5) to (5.8). For simplicity, we write B for B_{T_0,T_1} and Ψ for Ψ_{T_0,T_1}. Note that the existence of the continuous linear map δ_{T_0,T_1} shows that, with $y = \Psi_{T_0,T_1}(y_0,h,f)$, "$y_x(t,0)$" makes sense in $L^2(T_0,T_1)$. For simplicity we shall write $y_x(t,0)$ instead of $\delta_{T_0,T_1}(y_0,h,f)(t)$. Let $f \in L^2(T_0,T_1,L^2(0,L))$. We say that $y : [T_0,T_1]\times[0,L] \to \mathbb{R}$ is a mild solution of

$$y_t + y_x + y_{xxx} = f,$$
$$y(t,0) = y(t,L) = 0,$$

if there exists $h \in L^2(T_0,T_1)$ such that y is the mild solution of (5.5) to (5.8) with $y_0(x) := y(T_0,x)$. Note that it follows from the proof of Theorem 67 given in [90] that this theorem holds for mild solutions of (KdVL).

In [27] we prove the following proposition.

Proposition 70 *Let $T > 0$ and let $L = 2k\pi$ with $k \in \mathbb{N} \setminus \{0\}$. Then there exists $(u_\pm, v_\pm, w_\pm)$ in $L^2(0,T)^3$ such that, if $\alpha_\pm$, $\beta_\pm$, $\gamma_\pm$ are the mild solutions of*

$$\alpha_{\pm t} + \alpha_{\pm x} + \alpha_{\pm xxx} = 0, \tag{5.9}$$
$$\alpha_\pm(t,0) = \alpha_\pm(t,L) = 0, \tag{5.10}$$
$$\alpha_{\pm x}(t,L) = u_\pm(t), \tag{5.11}$$
$$\alpha_\pm(0,x) = 0, \tag{5.12}$$
$$\tag{5.13}$$

of

$$\beta_{\pm t} + \beta_{\pm x} + \beta_{\pm xxx} = -\alpha_\pm \alpha_{\pm x}, \tag{5.14}$$
$$\beta_\pm(t,0) = \beta_\pm(t,L) = 0, \tag{5.15}$$
$$\beta_{\pm x}(t,L) = v_\pm(t), \tag{5.16}$$
$$\beta_\pm(0,x) = 0, \tag{5.17}$$

and of

$$\gamma_{\pm t} + \gamma_{\pm x} + \gamma_{\pm xxx} = -(\alpha_\pm \beta_\pm)_x, \tag{5.18}$$
$$\gamma_\pm(t,0) = \gamma_\pm(t,L) = 0, \tag{5.19}$$
$$\gamma_{\pm x}(t,L) = w_\pm(t), \tag{5.20}$$
$$\gamma_\pm(0,x) = 0, \tag{5.21}$$

then

$$\alpha_\pm(T,x) = 0,\ \beta_\pm(T,x) = 0 \text{ and } \gamma_\pm(T,x) = \pm(1 - \cos(x)). \tag{5.22}$$

Remark 71 *It would have been quite natural to look for the existence $(u_\pm, v_\pm)$ in $L^2(0,T)^2$ such that, if $\alpha_\pm$, $\beta_\pm$ are the mild solutions of*

$$\alpha_{\pm t} + \alpha_{\pm x} + \alpha_{\pm xxx} = 0,$$
$$\alpha_\pm(t,0) = \alpha_\pm(t,L) = 0,$$
$$\alpha_{\pm x}(t,L) = u_\pm(t),$$
$$\alpha_\pm(0,x) = 0,$$

and of

$$\beta_{\pm t} + \beta_{\pm x} + \beta_{\pm xxx} = -\alpha_\pm \alpha_{\pm x},$$
$$\beta_\pm(t,0) = \beta_\pm(t,L) = 0,$$
$$\beta_{\pm x}(t,L) = v_\pm(t),$$
$$\beta_\pm(0,x) = 0,$$

then

$$\alpha_\pm(T,x) = 0 \text{ and } \beta_\pm(T,x) = \pm(1 - \cos(x)).$$

The existence of such $(u_\pm, v_\pm)$ would have also implied Theorem 66. Unfortunately, as it is proved in [27], such $(u_\pm, v_\pm)$ do not exist. Roughly speaking, an expansion to the order 2 is not sufficient. This is the the expansion to the order 3 which gives the local controllability.

Let us now explain how to deduce Theorem 66 from Proposition 70. As pointed out above, the invariance of the control system (KdV) by the change of variables $\tau = T - t$, $\xi = L - x$ allows us to prove only that, for every $T > 0$, there exists $r_1' > 0$ such that, for every $y_1 \in L^2(0,L)$ with $\|y_1\|_{L^2(0,L)} \leq r_1'$, there exists $u \in L^2(0,T)$ such that the mild solution y of

$$y_t + y_x + y_{xxx} + yy_x = 0, \tag{5.23}$$
$$y(t,0) = y(t,L) = 0, \tag{5.24}$$
$$y_x(t,L) = u(t), \tag{5.25}$$
$$y(0,x) = 0, \tag{5.26}$$

satisfies $y(T,.) = y_1$. Of course, by "y is a mild solution of of (5.23) to (5.26)", we mean that y is in B and is the mild solution of

$$\begin{aligned} y_t + y_x + y_{xxx} &= f, \\ y(t,0) = y(t,L) &= 0, \\ y_x(t,L) &= u(t), \\ y(0,x) &= 0, \end{aligned}$$

with $f := -yy_x$ (note that, if y is in B, then $yy_x \in L^1(0,T,L^2(0,L))$. It is proved in [27] that, for given $u \in L^2(0,T)$, there exists at most one mild solution of (5.23) to (5.26) and that such a solution exists if $\|u\|_{L^2(0,T)}$ is small enough (the smallness depending on T and k).

By (the proof of) Theorem 67 (see [90, Remark 3.10]), there exists a continuous linear map Γ

$$\Gamma : h \in H \subset L^2(0,L) \longmapsto \Gamma(h) \in L^2(0,T) \tag{5.27}$$

such that the mild solution of the linear system

$$\begin{aligned} y_t + y_x + y_{xxx} &= 0, \\ y(t,0) = y(t,L) &= 0, \\ y_x(t,L) &= \Gamma(h)(t), \\ y(0,x) &= 0, \end{aligned}$$

satisfies $y(T,x) = h(x)$. (One can take for Γ the control obtained by means of HUM; see [90, Remark 3.10]).

Let $y_1 \in L^2(0,L)$ be such that $\|y_1\|_{L^2(0,L)} \leq r$, $r > 0$ small enough so that the maps defined below are well defined in a neighborhood of 0. Let $\mathcal{T}_{y_1}$ denote the map,

$$\begin{aligned} \mathcal{T}_{y_1} : L^2(0,L) &\longrightarrow L^2(0,L) \\ z &\longmapsto z + y_1 - F(G(z)), \end{aligned}$$

with

$$\begin{aligned} F : L^2(0,T) &\longrightarrow L^2(0,L) \\ u &\longmapsto y(T,.), \end{aligned}$$

where y is the mild solution of (5.23) to (5.26) and $G : L^2(0,L) \longrightarrow L^2(0,T)$ is defined as follows. We write $z = \mathcal{P}_H(z) + \rho(z)(1-\cos(x))$ with $\int_0^L \mathcal{P}_H(z)(1-\cos(x))dx = 0$. Then

1. If $\rho(z) \geq 0$, $G(z) := \Gamma(\mathcal{P}_H(z)) + \rho^{1/3}(z)u_+ + \rho^{2/3}(z)v_+ + \rho(z)w_+$,
2. If $\rho(z) \leq 0$, $G(z) := \Gamma(\mathcal{P}_H(z)) + |\rho(z)|^{1/3}u_- + |\rho(z)|^{2/3}v_- + |\rho(z)|w_-$.

(The functions $u_\pm$, $v_\pm$ and $w_\pm$ have already been defined in Proposition 70.)

Clearly, each fixed point z^* of $\mathcal{T}_{y_1}$ satisfies $F(G(z^*)) = y_1$, and the control $u = G(z^*)$ is a solution to our problem. In order to prove the existence of a fixed point to $\mathcal{T}_{y_1}$, at least if $\|y_1\|_{L^2(0,T)}$ is small enough, one first proves the following estimate (see [27] for the proof): There exist $K > 0$ and $R > 0$ such that

$$\|\mathcal{T}_0 z\|_{L^2(0,L)} \leq K\|z\|^{4/3}_{L^2(0,L)} \text{ for every } z \in L^2(0,L) \text{ with } \|z\|_{L^2(0,L)} \leq R.$$

Then, using the continuity of $\mathcal{T}_{y_1}(z)$ with respect to y_1 and z, we could conclude the existence of a fixed point to $\mathcal{T}_{y_1}$ at least if $\|y_1\|_{L^2(0,T)}$ is small enough and if we were in finite dimension or have, at least, a suitable compactness property. Unfortunately, we do not have this compactness property. So we proceed in a different manner. We use the Banach fixed point theorem for the part in H and the intermediate value theorem in $H^\perp$. (If $H^\perp$ would have dimension in $\mathbb{N} \setminus \{0, 1\}$, we would have used the Brouwer fixed point theorem instead of the intermediate value theorem: in dimension 1 the Brouwer fixed point theorem reduces to the intermediate value theorem.) See [27] for the detailed proofs.

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