ON THE EXISTENCE OF SOLUTIONS TO A CLASS OF MINIMUM TIME CONTROL PROBLEMS INCLUDING THE BRACHYSTOCRONE

A. CELLINA, A. FERRIERO, AND E. M. MARCHINI

ABSTRACT. We prove a theorem for the existence of solutions to minimum time control problems, under assumptions that do not require the convexity of the images. Our result applies to the minimum time problem raised by Bernoulli, the Brachystocrone.

1. Introduction

Around 1693 Jakob Bernoulli raised the following question: find the path from an initial point x^0 to a target point x^f such that a body, subject to gravity only, starting from x^0 with initial velocity zero, would reach x^f in minimum time.

In 1959 [2] A.F.Filippov proved the first general theorem on the existence of solutions to minimum time control problems of the form

$$x'(t) = f(x(t), u(t)); \quad u(t) \in U(x(t))$$

requiring that the set valued map $x \to U(x)$ be upper semicontinuous (with respect to the inclusion) and that the values $F(x) = f(x, U(x))$ be compact and convex. In Theorem 2.2 of the present paper, we prove the existence of solutions to minimum time problems, under assumptions that do not require the convexity of the images $F(x) = f(x, U(x))$. In section 3 we present a model for the problem raised by Bernoulli; the appropriate form is that of a control problem, where at each point x, the control set consists of all vectors of length one, interpreted as the derivatives, with respect to the arc length, of all possible rectifiable curves through that point. We then show that our model (a non-convex control problem) satisfies the assumptions required by our Theorem 2.2 for the existence of solutions to minimum time problems, thus proving the existence of a solution to the minimum time problem raised by Bernoulli.

2. The existence of solutions to minimum time problems

For a compact subset $A \subset \mathbb{R}^d$, set coA be its convex hull. For basic results relating to solutions to differential inclusions, measurable selections and properties of set-valued maps we refer to any standard text on the subject.

Lemma 2.1. *Let $x : [0, t^*] \to \mathbb{R}^d$ be absolutely continuous; let $X = \{x(t) : t \in [0, t^*]\}$ and let F be defined on X. Assume that, for almost every $t \in [0, t^*]$,*

$$x'(t) \in F(x(t))$$

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and that there exists $E \subset [0, t^*]$ of positive measure such that $x'(t) = 0$ on E. Then there exist $\tau^* < t^*$ and an absolutely continuous function $\tilde{x} : [0, \tau^*] \to X$, such that $\tilde{x}(0) = x(0) = x^0$, $\tilde{x}(\tau^*) = x(t^*)$ and

$$\tilde{x}'(t) \in F(\tilde{x}(t))$$

for almost every $\tau \in [0, \tau^*]$.

Proof. a) It follows from the assumptions that there exists a closed subset K of E of positive measure. The complement of K consists of at most countably many open non-overlapping intervals $(a_i, b_i), i \in I$. Since the intervals (a_i, b_i) are disjoint, we must have that $\tau^* = \sum_{i\in I}(b_i - a_i) < t^*$. For each $i \in I$, set

$$\tau(b_i) = \sum_{j\in I: b_j < b_i} (b_j - a_j).$$

If $b_l > b_m$, we have that $\tau(b_l) \geq \tau(b_m) + (b_m - a_m)$.

Consider the intervals $(\tau(b_i), \tau(b_i)+b_i-a_i)$; they are disjoint, since in case we had $(\tau(b_l), \tau(b_l) + b_l - a_l) \cap (\tau(b_m), \tau(b_m) + b_m - a_m) \neq \emptyset$ with $b_l > b_m$, we would obtain $\tau(b_m) + (b_m - a_m) > \tau(b_l)$, a contradiction. Set $T = \cup_{i\in I}(\tau(b_i), \tau(b_i) + b_i - a_i)$; since $\tau \leq \sum_{i\in I}(b_j - a_j) = \tau^*$, $\forall \tau \in T$, we have that $T \subset [0, \tau^*]$. Moreover, the measure of T equals τ^*, in fact

$$m(T) = \sum_{i\in I}(b_i - a_i) = \tau^*.$$

b) Define the absolutely continuous function $\tilde{x} : [0, \tau^*] \to \mathbb{R}^d$ by $\tilde{x}(\tau) = x^0 + \int_0^\tau \tilde{x}'(s)ds$, where

$$\tilde{x}'(s) = \begin{cases} x'(s + a_i - \tau(b_i)) & , s \in (\tau(b_i), \tau(b_i) + b_i - a_i), i \in I \\ 0 & , s \in [0, \tau^*] \setminus T \end{cases} ;$$

we have $\tilde{x}(0) = x^0$.

Fix $\tau \in T$, there exists $i \in I$ such that $\tau \in (\tau(b_i), \tau(b_i) + b_i - a_i)$. Notice that $\tau \in (\tau(b_i), \tau(b_i) + b_i - a_i)$ if and only if $\tau + a_i - \tau(b_i) \in (a_i, b_i)$. We have

$$\begin{aligned}
\tilde{x}(\tau) - x^0 &= \int_0^{\tau(b_i)} \tilde{x}'(s)ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)ds \\
&= \sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{\tau(b_j)}^{\tau(b_j)+b_j-a_j} \tilde{x}'(s)ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)ds \\
&= \sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{\tau(b_j)}^{\tau(b_j)+b_j-a_j} x'(s + a_j - \tau(b_j))ds + \int_{\tau(b_i)}^{\tau} x'(s + a_i - \tau(b_i))ds \\
&= \sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{a_j}^{b_j} x'(s)ds + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)ds.
\end{aligned}$$

Notice that

$$\sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{a_j}^{b_j} x'(s)ds = \sum_{j:b_j < b_i} \int_{a_j}^{b_j} x'(s)ds;$$

in fact, by the definition, $\tau(b_j) + b_j - a_j \leq \tau(b_i)$ if and only if $\tau(b_j) < \tau(b_i)$. Hence

$$\begin{aligned}\tilde{x}(\tau) - x^0 &= \sum_{j:b_j<b_i} \int_{a_j}^{b_j} x'(s)ds + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)ds \\ &= \int_0^{a_i} x'(s)\chi_{\{[0,t^*]\setminus K\}}(s)ds + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)ds \\ &= x(\tau + a_i - \tau(b_i)) - x_0.\end{aligned}$$

The previous equality implies that $\tilde{x}$ is a solution to the differential inclusion. In fact we have that, almost everywhere in $[0, \tau^*]$,

$$\tilde{x}'(\tau) = x'(\tau + a_i - \tau(b_i)) \in F(x(\tau + a_i - \tau(b_i)) = F(\tilde{x}(\tau)).$$

c) Set $B = \sup\{b_i\}$. Then either the supremum is attained or it is not. In the first case, for some $j_{\tilde{k}}$, $B = b_{j_{\tilde{k}}}$ and $\tau(b_{j_{\tilde{k}}}) + (b_{j_{\tilde{k}}} - a_{j_{\tilde{k}}}) = \tau^*$. From b) we have that $x(t) = \tilde{x}(t - a_{j_{\tilde{k}}} + \tau(b_{j_{\tilde{k}}}))$, $\forall t \in [a_{j_{\tilde{k}}}, b_{j_{\tilde{k}}}]$. Since $x'(t) = 0$ on $[B, t^*]$, by continuity we have that $x(t^*) = x(B) = x(b_{j_{\tilde{k}}}) = \tilde{x}(b_{j_{\tilde{k}}} - a_{j_{\tilde{k}}} + \tau(b_{j_{\tilde{k}}})) = \tilde{x}(\tau^*)$.

In the second case, let $\{b_{j_k}\}$ be an increasing sequence, converging to B. From

$$|x(t^*) - x(a_{j_k})| = \left|\int_{a_{j_k}}^{B} x'(s)ds\right| \leq \left|\sum_{\{j\in I:b_j>b_{j_{k-1}}\}} \int_{(a_j,b_j)} x'(s)ds\right|$$

it follows that $x(a_{j_k}) \to x(t^*)$, while from

$$\sum_{\{j\in I:b_j>b_{j_k}\}} b_j - a_j < B - b_{j_k} \quad \text{and} \quad \tau^* = \sum_{j\in I} b_j - a_j = \tau(b_{j_k}) + \sum_{j\in I:b_j\geq b_{j_k}} b_j - a_j$$

we obtain that $\tau(b_{j_k}) \to \tau^*$. By the previous point b) we have that $\tilde{x}(\tau(b_{j_k})) = x(a_{j_k})$ and by continuity we infer that $x(t^*) = \tilde{x}(\tau^*)$. □

In what follows we shall consider the following *minimum time problem* for solutions to a differential inclusion: X and S are closed subset of $\mathbb{R}^d$, $S \subset X$, $x^0 \in X$ and F a set valued map. Consider the problem of reaching the target set S from x^0, satisfying the constraint $x(t) \in X$, and $x(.)$ is a solution to the differential inclusion

$$x'(t) \in F(x(t)).$$

Theorem 2.2. *Let $X \subset \mathbb{R}^d$ be closed and let F be an upper semicontinuous set-valued map defined on X with compact non empty images, with the additional property that, for every $x \in X$, for every $\xi \in coF(x)$, with $\xi \neq 0$, there exists $\lambda \geq 1$ such that $\lambda\xi \in F(x)$. Assume that there exists $\tilde{t} > 0$ and a solution x to*

$$x'(t) \in coF(x(t)) \quad x(0) = x^0$$

such that $x(t) \in X$ for every $t \in [0, \tilde{t}]$ and that $x(\tilde{t}) \in S$. Then the minimum time problem for00

$$x'(t) \in F(x(t))$$

has a solution.

Proof. a) Let $t^* = \inf\{t : A_0^{co}(t) \cap S \neq \emptyset\}$. Let (t_n) be decreasing to t^* and le x_n be solutions to the differential inclusion

$$x'(t) \in coF(x(t))$$

such that $x_n(0) = 0$ and $x(t_n) \in S$, $x(t) \in X$ for $t \in [0, t_n]$. A subsequence of this sequence converges uniformly to x_*. Clearly, $x_*(0) = 0$, $x_*(t^*) \in S$ and $x_*(t) \in X$ for $t \in [0, t^*]$. It is known that, under the assumptions of the Theorem, x_* is again a solution to

$$x'(t) \in coF(x(t)).$$

Hence, x_* is a solution to the convexified problem that reaches S in minimum time, and t^* is the value of the minimum time for the convexified problem.

b) Applying Lemma 1, we infer that $x_*'(t) \neq 0$ for a.e. $t \in [0, t^*]$. In fact, otherwise, we could define a different solution to the convexified differential inclusion, defined on an interval $[0, \tau^*]$ with $\tau^* < t^*$, having the same initial and final point: hence t^* would not be the value of the minimum time for the convexified problem.

c) By the previous point and the assumption on $F(x)$, for almost every t there exists a non empty set $\Lambda(t)$ such that $\lambda \in \Lambda(t)$ implies $\lambda x_*'(t) \in F(x_*(t))$ and $\lambda \geq 1$. Reasoning as in [1], we obtain that $\Lambda(.)$ is measurable on $[0, t^*]$, hence, by standard arguments, that there exists a measurable selection $\lambda(.)$ from $\Lambda(.)$. Define the absolutely continuous map s by $s(0) = 0$ and $s'(t) = 1/\lambda(t)$: s is an increasing map and maps $[0, t^*]$ onto $[0, s^*]$, where $s^* \leq t^*$. Let $t = t(s)$ be its inverse and consider the map $\tilde{x}(s) = x_*(t(s))$. We obtain in particular that $\tilde{x}(0) = x_*(0)$ and that $\tilde{x}(s^*) = x_*(t(s^*)) = x_*(t^*)$. We also have

$$\frac{d}{ds}\tilde{x}(s) = x_*'(t(s))t'(s) = x_*'(t(s))\frac{1}{s'(t(s))}$$

$$= x_*'(t(s))\lambda_1(t(s)) \in F(x_*(t(s))) = F(\tilde{x}(s)).$$

Hence we have obtained that $\tilde{x}$ is a solution to the original differential inclusion but also a solution to the minimum time problem for I_{co}. Since every solution to the original problem is also a solution to the convexified problem, the infimum of the times needed to reach S along the solutions to the original problem cannot be lesser than the minimum time for the convexified problem. Hence $\tilde{x}$ is a solution to the minimum time problem for the original differential inclusion. □

The following result is on the existence of solutions to initial value problems for differential inclusions with non-convex right hand side.

Theorem 2.3. *Let Ω be open, $x^0 \in \Omega$ and let F be as in Theorem 2.2. Assume that there exists $t^* > 0$ such that, on $[0, t^*]$, the Cauchy Problem*

$$x'(t) \in coF(x(t)) \quad x(0) = x^0$$

admits a solution $x \not\equiv x^0$. Then the Cauchy Problem

$$x'(t) \in F(x(t)) \quad x(0) = x^0$$

admits a solution on some interval $[0, \tau^*]$

Proof. Since $x \not\equiv x^0$, there exists $t^1 \in [0, t^*]$ such that $x(t^1) \neq x^0$. Consider the minimum time problem for the convexified inclusion, with target set $S = \{x(t^1)\}$. This problem has a solution with minimum time τ^*, where $0 < \tau^* \leq t^1$. By Theorem 2.2, the original non-convexified problem has a solution on $[0, \tau^*]$. □

3. The Brachystocrone

The scheme commonly used to attack the minimum time problem raised by Bernoulli is to transform it into the problem of minimizing

$$\int \frac{\sqrt{1+(\xi_2')^2}}{\sqrt{-\xi_2}}dt$$

then to apply necessary conditions to find candidates for the solution; finally to show that the family of such candidates covers the plane, hence that there is one that passes through the final point. This approach, besides being very indirect, is somewhat unsatisfactory, mainly because the integrand is *not* defined at the initial condition $(0,0)$ and the necessary conditions are applied without a previous proof of existence of solutions. Instead, Bernoulli's problem can be stated as follows: in the plane, an initial condition (ξ_1^0, ξ_2^0) is given; consider all the possible oriented rectifiable curves passing through it. Each such curve is defined by assigning $(u_1(.), u_2(.))$, a unit vector describing the direction of its (oriented) tangent. In this way, the parameter t is the arc-length parametrization of the curve.

The system of equations

$$\begin{cases} \xi_1' = u_1\xi_3 \\ \xi_2' = u_2\xi_3 \\ \xi_3' = -gu_2 \end{cases} \qquad (B)$$

where the maps $u_1(.)$ and $u_2(.)$ are measurable and $u_1^2(t)+u_2^2(t)=1$ a.e., describes the motion of a body in the plane, defined by the coordinates $(\xi_1(t), \xi_2(t))$, with (scalar) velocity $\xi_3(t) = \sqrt{(\xi_1'(t))^2 + (\xi_2'(t))^2}$, along a curve identified assigning the direction of its tangent vector $(u_1(.), u_2(.))$, subject to the gravity g.

Hence, the problem raised by Bernoulli can be stated as the following minimum time control problem:

The Brachystocrone Minimum Time Problem: find a solution to the control system

$$\begin{cases} \xi_1' = u_1\xi_3 \\ \xi_2' = u_2\xi_3 \\ \xi_3' = -gu_2 \end{cases}$$

$(\xi_1(0), \xi_2(0), \xi_3(0)) = (\xi_1^0, \xi_2^0, 0)$, subject to the constraint $\xi_3 \geq 0$, with control set

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 = 1)\},$$

that would reach the target set $S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)$, where $\xi_2^f \leq \xi_2^0$, in minimum time.

Theorem 3.1. *The Brachystocrone Minimum Time Problem admits a solution.*

Proof. Let $x = (\xi_1, \xi_2, \xi_3)$, $X = \{\xi_3 \geq 0\}$, $S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)$; let $f(x,u)$ be the right hand side of (B) and set $F(x) = f(x,U)$. Then, as one can check, $F(x)$ satisfies the assumptions of Theorem 2.2. Hence it is enough to show that there are solutions $x(t)$ issuing from $x^0 = (0,0,0)$ with $\xi_3(t) \geq 0$ and such that, at some finite time t^*, $\xi_1(t^*) = \xi_1^f, \xi_2(t^*) = \xi_2^f$.

This is so in the case where $\xi_2^f < 0$. In fact, in this case we have that, by choosing the constant control

$$\begin{cases} u_1 = \frac{\xi_1^f}{\sqrt{(\xi_1^f)^2+(\xi_2^f)^2}} \\ u_2 = \frac{\xi_2^f}{\sqrt{(\xi_1^f)^2+(\xi_2^f)^2}} \end{cases}$$

we obtain that

$$\begin{cases} \xi_1(t) = u_1(-gu_2\frac{t^2}{2}) \\ \xi_2(t) = u_2(-gu_2\frac{t^2}{2}) \end{cases}$$

so that $\xi_1(t^f) = \xi_1^f$ and $\xi_2(t^f) = \xi_2^f$ for $t^f = \sqrt{\frac{2}{g}}\sqrt{\frac{(\xi_1^f)^2+(\xi_2^f)^2}{-\xi_2^f}}$. We also obtain that $\xi_3(t) > 0$ on $(0, t^f)$ and that $\xi_3(t^f) = \sqrt{-2g\xi_2^f}$.

Consider the case $\xi_2^f = 0$. We can assume $\xi_1^f \neq 0$, otherwise $t^* = 0$ is the solution to the minimim time problem. In the case $\xi_1^f > 0$, consider the solution with the constant control $u_1 = \frac{1}{\sqrt{2}}$,$u_2 = -\frac{1}{\sqrt{2}}$ on $[0, \sqrt{2\xi_1^f/g}]$. At time $t = \sqrt{2\xi_1^f/g}$, we have that $\xi_1(\sqrt{2\xi_1^f/g}) = \frac{\xi_1^f}{2}$, $\xi_2(\sqrt{2\xi_1^f/g}) = -\frac{\xi_1^f}{2}$ and $\xi_3(\sqrt{2\xi_1^f}) = \sqrt{g\xi_1^f}$. The solution with constant control $u_1 = \frac{1}{\sqrt{2}}$, $u_2 = \frac{1}{\sqrt{2}}$ on the interval $(\sqrt{2\xi_1^f/g}, 2\sqrt{2\xi_1^f/g})$, with initial conditions $\xi_1(\sqrt{2\xi_1^f/g}) = \frac{\xi_1^f}{2}$, $\xi_2(\sqrt{2\xi_1^f/g}) = -\frac{\xi_1^f}{2}$ and $\xi_3(\sqrt{2\xi_1^f/g}) = \sqrt{g\xi_1^f}$ is such that at $t = 2\sqrt{2\xi_1^f/g}$, $\xi_1(2\sqrt{2\xi_1^f/g}) = \xi_1^f$ and $\xi_2(2\sqrt{2\xi_1^f/g}) = 0$. Hence $t^f = 2\sqrt{2\xi_1^f/g}$ and $\xi_3(t) > 0$ on $(0, t^f)$. □

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Dipartimento di Matematica e Applicazioni, Universitá degli Studi di Milano-Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milano

E-mail address: `cellina@matapp.unimib.it`

Dipartimento di Matematica e Applicazioni, Universitá degli Studi di Milano-Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milano

E-mail address: `ferriero@matapp.unimib.it`

Dipartimento di Matematica e Applicazioni, Universitá degli Studi di Milano-Bicocca, Via Bicocca degli Arcimboldi 8, 20126 Milano

E-mail address: `elsa@matapp.unimib.it`