NEEDLE VARIATIONS THAT CANNOT BE SUMMED*

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Abstract. This article analyzes sets of higher-order tangent vectors to reachable sets of analytic control systems (affine in the control). Both small-time local *output* controllability and small-time local controllability about a nonstationary reference trajectory are considered. In a series of purposefully constructed examples it is shown that the *cones* generated by *needle variations* may fail to be convex. The examples demonstrate that the usual technical conditions that *needle variations* must be *moveable* are essential in order to guarantee desirable convexity properties. Moreover, new doubts are cast on the structural stability of controllability properties, as apparently higher order perturbations can reverse the (lack of) controllability of *lower order* nilpotent approximating systems, thereby providing new insights about the ultimate question whether controllability is finitely determined.

Key words. Nonlinear controllability, optimality, control variations, tangent cones.

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1. Introduction. A fundamental dichotomy for a control system together with a reference trajectory is whether the trajectory is *optimal*, or whether the system is *controllable* about this trajectory. Topologically, this translates into the reference trajectory lying either on the boundary, or in the interior of the *funnel of reachable sets*. Derivatives of the *endpoint map* with respect to the controls serve as the main analytic tool: If the derivative has full rank, then the trajectory lies in the interior and the system is *controllable*. Conversely, a necessary condition for *optimality* is that the derivative does not have full rank, which leads to the Pontryagin Maximum Principle [19] (which is a *first derivative* test). See [23] for the current state of the art unifying nonsmooth with differential geometric approaches.

Since commonly the first derivative does not have full rank (at the reference control), there have been many efforts to obtain higher order conditions for controllability and optimality, with Krener's high-order maximal principle [17], Stefani's high order conditions for optimality [20], and Sussmann's general theorem for controllability [22] being a few prominent classical ones. See [23] and the references therein for the current state of the art. As opposed to purely theoretical statements in terms of high-order Frechet derivatives, the main interest is in *effectively computable* conditions, leading e.g. to algebraic rank conditions in terms of iterated Lie brackets of the system vector fields evaluated at a point as e.g. in [20, 22]. Underlying such conditions are notions of higher order tangent vectors together with open mapping theorems. The higher order tangent vectors are basically high-order directional derivatives of the endpoint map, obtained from *families of control variations* (i.e. curves in the space of admissible controls). Open mapping theorems guarantee that if a *cone of tangent vectors* is the whole tangent space, then the reference trajectory lies in the interior. Of course, depending on the precise technical notion of tangent object, one needs a dedicated open mapping theorems – compare e.g. Lee-Markus [18] for classical such statements.

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A notable effort in the 1980s by Frankowska employed *nonsmooth analysis* to obtain a general purpose open mapping theorem [8, 9], relying on Clarke's contingent cones as tangent objects.

The basic trade-off is between narrowly defined tangent objects (with straightforward open mapping theorems), and *large* tangent objects which may lack nice convexity and approximation properties. Recall that classical calculus of variations primarily employed families of variations that are parameterized by their amplitude. In optimal control theory several alternative approaches have been analyzed. These include both curves such that $\|u_s - u^*\|_\infty \searrow 0$ (where $u_0 = u^*$ is the reference control) and curves such that $\|u_s - u^*\|_1 \searrow 0$ as $s \searrow 0$

(compare [7]), and the so-called *needle variations.* Loosely speaking, these are such that the variation u_s agrees with the reference control u^* everywhere except on a (finite number of) interval(s) whose (combined) length goes to zero as $s \searrow 0$. The main attraction of such needle variations is that they are conceptually easy to combine for the purpose of generating convex combinations of tangent vectors, resulting in tangent cones with nice convexity properties. Of course, such schemes hinge on the intervals on which the variations differ from the reference control, to eventually (for sufficiently small $s > 0$) be disjoint. Alternatively, one might require that every needle variation be *moveable*, i.e. one must be able to *move* the variation by a small time-amount and still be able to generate the same tangent vector. Precise technical specification of such conditions can be quite involved.

This article is not overly concerned about specific ways of combining needle variations. The main results are independent of such technical intricacies as on one side we show that certain directions can be generated by the most basic needle variations, while certain other points cannot be reached by any control variations at all. Of course, this implies that one cannot reach those points, or generate those tangent vectors, by any kind of composition of needle variations either, no matter how these are defined.

For analytic, affine control systems with a *stationary* reference trajectory no such conditions are needed and a wealth of necessary and sufficient conditions for controllability and optimality have been found in recent decades, compare e.g. [22]. On the other hand, a distinctive feature of the existing literature for nonstationary reference trajectories are much more careful and more delicate notions of variations, tangent objects and open-mapping theorems, compare e.g. [1, 4, 5, 6, 16, 23]. A major open question has been whether such technical conditions are truly essential, or whether they are merely artifacts of a still imperfect knowledge level (e.g. future work might show that they are automatically satisfied for reasonable classes of systems). This article provides a definitive answer: Loosely speaking, even for the most benign classes of nonlinear systems (polynomial cascades, affine in the control, convex compact control values), the tangent *cones* generated by needle variations can be nonconvex. Moreover, even if their convex hull is the entire space the reference trajectory may lie on the boundary of the reachable sets. Both may happen even for arbitrary small positive times.

This article is organized as follows: After this introduction, we give and review technical definitions of controllability, needle variations, tangent cones, and open mapping theorems. The subsequent three sections analyze a sequence of increasingly more delicate examples. First we consider the *output controllability* of a simple polynomial cascade system about an equilibrium (stationary reference trajectory). Most of the innovative constructions and critical inequalities may be found her. The next chapter slightly modifies the system so that it falls into the class of standard small-time local controllability about a nonstationary reference trajectory. Finally, we add some terms which at first sight appear to be higher-order perturbations – but we show that these actually destroy the delicate controllability properties. The final section reflects and speculates on possible implications of such lack of robustness of the notion of controllability.

The constructions in the proofs fall into two categories: On one side we provide very explicit constructions of *needle variations* and demonstrate that they generate certain tangent vectors. Similarly, in the absence of applicable open mapping theorems, we explicitly construct controls that steer the system to any given target point. On the other side, we show that certain (regions of) points cannot be reached by any control whatsoever, and thus not by any needle variations either. Such arguments naturally involve delicate arguments with integral inequalities.

2. Control variations and approximating cones. The main thrust of this article are *counterexamples*. Thus for positive controllability results we use constructions that rely on small sets of tools, e.g. very narrow notions of (needle) variations. Such results clearly hold a-fortiori if the sets of admissible variations etc. are broadened. Conversely, negative results (lack of convexity or controllability) are usually stated in the form that even for very large sets of controls something is impossible.

The systems under consideration are finite-dimensional, deterministic, analytic systems (possibly including an analytic *output* function) that are affine in the control.

$$\dot{x} = f_0(x) + \sum_{i=1}^{\ell} u_i(t) f_i(x), \qquad x(0) = 0, \qquad y = \varphi(x). \tag{2.1}$$

Here $x \in \mathbb{R}^n$, $f_i\colon \mathbb{R}^n \mapsto T\mathbb{R}^n$ are analytic vector fields, the controls $u\colon [0,T] \mapsto U \subseteq \mathbb{R}^\ell$ are measurable functions defined on finite intervals and take values in a compact convex set U, usually $U = [-1,1]^\ell$. The output map $\varphi\colon \mathbb{R}^n \mapsto \mathbb{R}^p$ is analytic. In this article the vector fields are actually always of a polynomial cascade form, the controls in all constructions are piecewise constant, and the output map is either the identity, or a projection onto a subspace. The solution curves of (2.1) corresponding to a control u, starting from $x(0) = 0$ are denoted by $x(t,u)$, or when no confusion arises, by $x(t)$. Their images under the output map are denoted $y(t,u) = \varphi(x(t,u))$ or simply $y(t)$. Throughout this article we also conveniently identify the tangent spaces $T_p\mathbf{R}^n$ with $\mathbf{R}^n$.

The reachable sets $\mathcal{R}(t)$ consist of all points $x(t,u)$ reached in time t by trajectories of (2.1) from $x(0) = 0$ by means of admissible controls (measurable, and with values in U almost everywhere). Given a reference trajectory $x^*(t) = x(t,u^*)$, controllability is defined as:

Definition 2.1. *The system* (2.1) *is small-time locally controllable* (STLC) *about* x^* *if* $x^*(t) \in \operatorname{int}\mathcal{R}(t)$ *for all* $t > 0$. *The system* (2.1) *is small-time locally output controllable* (STLOC) *about* $y^* = \varphi(x^*)$ *if* $y^*(t) \in \operatorname{int}(\varphi(\mathcal{R}(t)))$ *for all* $t > 0$.

The latter notion is related to Kaskosz' g-controllability [12], but is not yet standard. In general one needs to distinguish STLOC about a reference output y^* with a fixed initial condition $x(0) = 0$ from a notion that only requires $x(0) \in \varphi^{-1}(y_0)$. In this article this distinction will not play a role.

Throughout this article we will use the norm $\|x\|_\infty \stackrel{\text{def}}{=} \max_{k=1}^n |x_k|$ for $x \in \mathbb{R}^n$. In particular, the open ball of *radius* $r \geq 0$ about $p \in \mathbb{R}^n$ is $B_p^\infty(r) = \{x \in \mathbb{R}^k\colon \|x -$

$p\|_\infty < r\}$. Since $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty$, this does not change the quality of any statements, but allows us to dispend with lots of additional factors of $\sqrt{n}$ throughout our constructions and statements of results.

DEFINITION 2.2. *A one-parameter family of control variations of a reference control* $u^*: [0,T] \mapsto U$ *is a curve* $s \mapsto u_s \in \mathcal{U} = \{u: [0,T] \mapsto U$ *measurable* $\}$ *defined for* $s \in [0, s_0]$ *for some* $s_0 > 0$*, and with* $u_0 = u^*$*. Such a family is called a family of control variations* at zero *if* $u_s(t) = u^*(t)$ *for all* $t \in (s,T]$*.*

In the latter case it is convenient to identity u_s with its restriction to $[0,s]$. Conversely we generally do not distinguish between a control $u_s: [0,s] \mapsto U$ and its extension to $[0,T]$ defined by setting $u_s(t) = u^*(t)$ if $t \in (s,T]$.

Using either kind of curves one may define derivatives to the family of reachable sets. Denote by $\Phi: (t,p) \mapsto \Phi_t(p) = x(t,u^*;p)$ the flow corresponding to the reference control u^*. In all systems analyzed in this article, this flow will be analytic, and hence there are no subtleties about transporting tangent vectors via the tangent maps Φ_{T*} from one tangent space to another. To facilitate comparisons, we move all tangent vectors back to the tangent space at the initial point. If $v \in \mathcal{K}^m$ is a tangent vector, we will also refer to its positive multiples $\lambda v \in \overline{\mathcal{K}}^m$, $\lambda \geq 0$, as tangent vectors. In the following definitions let $u^*: [0,T] \mapsto U$ be a admissible control generating a (reference) trajectory $x^*: [0,T] \mapsto \mathbf{R}^n$.

DEFINITION 2.3. *A vector* $v \in \mathbb{R}^n$ *is an* m*-th order tangent vector to the family of reachable sets* $\{\mathcal{R}(T)\}_{T \geq 0}$ *at (time) zero written* $v \in \mathcal{K}_0^m$*, if there exists a family of control variations at zero* $u_s: [0,s] \mapsto U$ *of* u^* *such that*

$$x(s,u_s) = x(s,u^*) + s^m \Phi_{s*,0}(v) + o(s^m) \tag{2.2}$$

DEFINITION 2.4. *A vector* $v \in \mathbb{R}^p$ *is called an* m*-th order tangent vector to the output-reachable sets of the system* (2.1) *with stationary reference trajectory* $x^* \equiv 0$*, written* $v \in \mathcal{K}_\varphi^m$*, if there exists a family of control variations* $u_s: [0,s] \mapsto U$ *such that*

$$\varphi(x(s,u_s)) = \varphi(x(0)) + s^m v + o(s^m) \tag{2.3}$$

Most of the statements in this article involve these approximating sets to the *family* of reachable sets at the initial time. However, in a few places (e.g. system (2.7) and corollary 5.8) it is convenient for precise statements to use the following notion of tangent vectors to a fixed reachable set (but transported back to the initial time for easy comparisons). In order to allow rudimentary comparisons with the above notions of higher order tangent vectors, the following definition uses the L^1-norm of the differences $(u_s - u^*)$.

DEFINITION 2.5. *A vector* $v \in \mathbb{R}^n$ *is an* m*-th order tangent vector to the reachable set* $\mathcal{R}(T)$ *at* $x^*(T)$*, written* $v \in \mathcal{K}_T^m$*, if there exists a family of control variations* $u_s: [0,T] \mapsto U$ *of* u^* *such that*

$$x(T,u_s) = x(T,u^*) + \|u_s - u^*\|_1^m \Phi_{T*,0}(v) + o(\|u_s - u^*\|_1^m). \tag{2.4}$$

If $\{u_s\}_s > 0$ is a family of control variations at zero (and thus $u_s(t) = u^*(t)$ for $s < t \leq T$) it is clear that $\|u_s - u^*\|_1^m \leq c^m s^m$ where $c = \max_{w \in U} |w|$ is finite (due to compactness of U). However, this is not enough to be able to fully compare $\mathcal{K}_T^m$ to $\mathcal{K}_0^m$ as a reverse inequality need not hold (e.g. $\|u_s - u^*\|_1^m \searrow 0$ might go to zero faster than s^m as $s \searrow 0$). Nonetheless, for typical families of control variations at zero, as constructed in the sequel, one often can readily conclude that they generate corresponding tangent vectors of the same order m, lying in $\mathcal{K}_0^m$ and $\mathcal{K}_T^m$. Write $\overline{\mathcal{K}}_T$, $\overline{\mathcal{K}}_0^m$, and $\overline{\mathcal{K}}_\varphi^m$ for the cones $\overline{\mathcal{K}}_0^m = \{\lambda v : v \in \mathcal{K}_0^m, \ \lambda \geq 0\}$ etc. generated by $\mathcal{K}_T$, $\mathcal{K}_0^m$, and $\mathcal{K}_\varphi^m$, respectively.

In the case of a stationary reference trajectory $x^* \equiv x(0)$ together with full-state output $\varphi \equiv \mathrm{id}_{\mathbf{R}^n}$ the definitions of $\mathcal{K}_0^m$ and $\mathcal{K}_\varphi^m$ agree, and they are closely related to the contingent cones of nonsmooth analysis, and have nice convexity and approximation properties. Note, that in this setting no regularity whatsoever is assumed or required for the maps $s \mapsto u_s$.

A key step to render the sets of tangent vectors useful for obtaining optimality results is to establish their convexity. For a wide variety of different settings results similar to the following may be found compare e.g. [4, 5, 6, 7, 8, 9, 13, 18])

Proposition 2.1. *For systems of form* (2.1) *with a stationary reference trajectory the following hold:*
If $\lambda \in [0,1]$*, and* $v \in \mathcal{K}_0^m$ *then* $\lambda v \in \mathcal{K}_0^m$
If $m \leq m'$ *then* $\mathcal{K}_0^m \subseteq \mathcal{K}_0^{m'}$.
If $v_1, v_2 \in \mathcal{K}_0^m$ *and* $\lambda \in [0,1]$ *then* $\lambda^m v_1 + (1-\lambda)^m v_2 \in \mathcal{K}_0^m$.

Theorem 2.2. ([8, 9, 13]) *Suppose* $x^* \equiv 0$ *is the stationary reference trajectory of a system of form* (2.1). *Then for every closed convex cone* K *that is strictly contained in* $\{0\} \cup \mathrm{int}\overline{\mathcal{K}}_0^m$ *there exist* $C > 0$ *and* $T > 0$ *such that*

$$K \cap B_0^\infty(Ct^m) \subseteq \mathcal{R}(t) \ \text{for all} \ 0 \leq t \leq T. \tag{2.5}$$

Moreover, if $\overline{\mathcal{K}}_0^m = \mathbb{R}^n$*, then the system* (2.1) *is STLC about* $x^* = 0$*, and the minimum time function* $V(q) = \min\{t \geq 0 : \exists u \quad \text{such that} \quad x(t,u) = q\}$ *is Hölder continuous of order* $\frac{1}{m}$ *at* $x = 0$.

If the reference trajectory is nonstationary, then the cones $\overline{\mathcal{K}}_0^m$ need not have similarly nice convexity and approximation properties. Thus much work in recent decades has focused on developing more refined notions of tangent objects for that setting. The classical open mapping theorems, compare e.g. [18, 19], use topological arguments that rely on continuity of the multi-parameter families of control variations that naturally generalize the (single-parameter) curves introduced above. Typically, one might have constructed n curves $s \mapsto u_s^{(i)} \in \mathcal{U}$, which generate the tangent vectors $v^{(i)} \in \mathcal{K}_0^m$, $i = 1, \dots n$. The critical next step is to somehow combine these variations in order to show that the convex combinations $(c_1 v_1 + \ldots + c_n v_n)$ with $c_1 + \ldots + c_n = 1$ are m-th order tangent vectors, i.e., are contained in $\overline{\mathcal{K}}_0^m$, too. A natural first try is to consider convex combinations of suitable reparameterizations of the original curves

$$s \mapsto u_{s,c_1,\ldots c_n} = c_1 u^{(1)}_{\alpha_1(c,s)} + c_2 u^{(2)}_{\alpha_2(c,s)} + \ldots + c_n u^{(n)}_{\alpha_n(c,s)} \tag{2.6}$$

and analyze the corresponding curves $s \mapsto x(s, u_{s,c})$ of endpoints for each value c. Variations that lend themselves especially well to generating such convex combinations are needle variations. (Alternatively compare e.g. [7] for variations such that $\|u_s - u^*\|_1 \searrow 0$). Basically, each such control u_s agrees with u^* on all of $[0,T]$ except on an interval whose length is of order s for $s \searrow 0$. Two such families of variations are easily combined unless the intervals on which they disagree from u^* overlap for all small $s > 0$. In that case, one might be tempted to simply *move* the interval in one of the families. The continuous dependence of solution curves on the data suggests that after such small (or vanishing, as $s \searrow 0$) moves, the combined variations might still generate the desired tangent vectors.

Before giving a technical (broad) definition of needle variations we discuss a very simple example that illustrates the problem that some (narrowly defined) *needle variations* might *not be able to be moved*:

$$\left\{\begin{array}{lcl} \dot{x}_0 & = & 1 \\ \dot{x}_1 & = & u \\ \dot{x}_2 & = & (x_0-1)x_1^2 \\ \dot{x}_3 & = & x_1^7 \end{array}\right. \qquad \begin{array}{rcl} |u(\cdot)| & \leq & 1 \\ x(0) & = & 0 \\ x^*(t) & = & (t,0,0,0) \\ t & \in & [0,2]=[0,T] \end{array} \tag{2.7}$$

In this case the family of *needle variations* $u_s^{\pm}\colon[0,2]\mapsto[-1,1]$ defined by $u_s^{\pm}(t)=\pm 1$ if $1-\frac{1}{2}s\leq t<1$, $u_s^{\pm}(t)=\mp 1$ if $1\leq t<1+\frac{1}{2}s$, and $u_s^{\pm}(t)=0$ else, generate the curves $x(2,u_s^{\pm})=(2,0,0,\pm 2^{-10}s^8)\in\mathcal{R}_0(2)$, and thus $v=(0,0,0,\pm 1)\in\overline{\mathcal{K}}_2^8$ as 8-th order tangent vectors at $T=2$ (note that $\|u_s-u^*\|_1=s$).
In general, suppose $u_s\colon[0,2]\mapsto[-1,1]$ is any family of control variations that agrees with $u_s(t)=u^*(t)=0$ for all $t\in[0,T]\setminus I_s$ where $I_s\subseteq[0,T]$ is an interval of any of the forms $[a,a+s]$, $[a-s,a]$ or $[a-\frac{s}{2},a+\frac{s}{2}]$ for some fixed $a\in[0,2]$. Then $x(T,u_s)=(2,0,0,Cs^8)+o(s^8)$ is only possible if I_s is of the third form, and in addition $a=1$. This means that the only *single* family of *needle variations* that can generate $v=(0,0,0,\pm 1)$ as tangent vector *"cannot be moved"*.
In this simple example it is clear that either v can easily be generated by control variations whose support (technically the support of (u_s-u^*)) consists of two disjoint intervals, e.g.

$$u_s(t)=\left\{\begin{array}{rl} 1 & \text{if } \pm\frac{1}{2}-\frac{1}{4}s\leq t<\pm\frac{1}{2}, \\ -1 & \text{if } \pm\frac{1}{2}\leq t<\pm\frac{1}{2}+\frac{1}{4}s, \\ 0 & \text{else.} \end{array}\right. \tag{2.8}$$

This demonstrates that if one only considers control variations that are supported in a single interval shrinking to a point, then it may be possible that certain tangent vectors can *only* be generated at a specific point t_0 that cannot be moved.
The example appears contrived, and the notion of needle variation unnecessarily constrained. It is customary to broaden the notion of needle variations to allow the support of (u_s-u^*) to be a finite union of intervals of total length of order s.

Definition 2.6. *A family of control variations $s\mapsto u_s\colon[0,T]\mapsto U$ defined for $s\in[0,s_0]$ with $s_0>0$ is a (family of)* needle variations *of the reference control $u^*=u_0$ if there exist a constant $C>0$ and a finite number of pairs of increasing functions $s\mapsto a_s^{(k)}$ and decreasing functions $s\mapsto b_s^{(k)}$ defining intervals $[a_s^{(k)},b_s^{(k)}]\subseteq[0,2]$, $k=1,\dots N$, such that*

$$\operatorname{supp}(u_s-u^*)\subseteq\bigcup_{k=1}^{N}[a_s^{(k)},b_s^{(k)}], \quad \textit{and} \quad \sum_{k=1}^{N}(b_s^{(k)}-a_s^{(k)})\leq Cs \ \textit{ for all } s\leq s_0 \tag{2.9}$$

Note that a family of variations *at zero* as defined earlier automatically qualifies as a family of needle variations as one may take $N=1$ and intervals defined by $a_s^{(1)}=0$ and $b_s^{(1)}=s$. Definition 2.6 also includes other common constructions with $N=1$ and $[a_s^{(1)},b_s^{(1)}]=[\alpha,\alpha+s]$ (*right* variations at $t=\alpha$), or $[a_s^{(1)},b_s^{(1)}]=[\beta-s,\beta]$ (*left* variations at $t=\beta$). The constructions in this article use mainly (right) needle variations at zero, combined with some (left) needle variations at the final time T. But the main claim of *"needle variations that cannot be summed"* holds even if the combinations (or sums) are allowed to lie in a more general class as in definition 2.6.

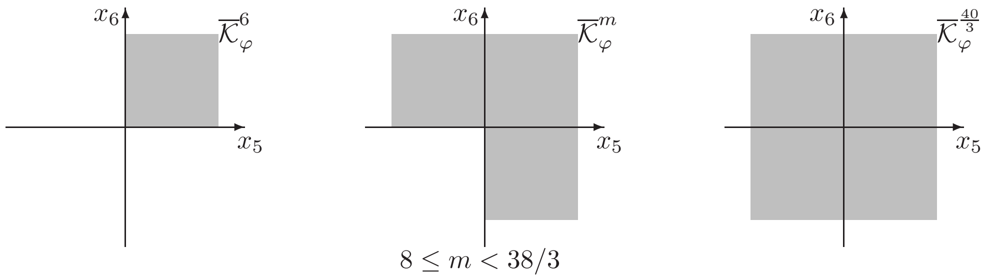

FIG. 3.1. *Cross-sections of approximating cones of tangent vectors for system (3.1)*

3. The main construction. This section analyzes a custom-designed polynomial cascade system (affine in the control) with output, which forms the heart of the promised counterexample in section 5. We demonstrate that the growth rates of its reachable sets are very sensitive to reflections about the origin. More specifically, the approximating cones of tangent vectors are described not only by intersections, but also by unions of half spaces. The basic system with *output* φ is

$$(3.1)\quad \left\{ \begin{array}{lcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_2 \\ \dot{x}_3 & = & x_1^2 \\ \dot{x}_4 & = & x_2^2 \\ \dot{x}_5 & = & x_4 x_1^2 - x_1^7 \\ \dot{x}_6 & = & x_3 x_2^2 - x_2^7 \end{array} \right. \qquad \begin{array}{rcl} |u_1(\cdot)| & \leq & 1 \\ |u_2(\cdot)| & \leq & 1 \\ x(0) & = & 0 \\ \varphi(x) & = & (x_1, x_2, x_5, x_6) \end{array}$$

Figure 1 pictorially summarizes key properties of this system, showing cross-sections of approximating cones of the image $\varphi(\mathcal{R}(t))$ of the reachable set under the output map. The remainder of this section proves technical statements that make this illustrations precise.

THEOREM 3.1. *For all $T > 0$ (sufficiently small)*

$$(3.2)\quad \varphi(R_0(T)) \supseteq B_0^{\infty}(2^{-18}T^8) \cap \{y \in \mathbb{R}^4 \colon (y_3 \geq 0 \textbf{ or } \; y_4 \geq 0 \,) \textbf{ and } y_1 = y_2 = 0\}$$

The proof shows in particular that points of the forms $x = (0, 0, *, *, -2^{-18}s^8, 0)$ and $x = (0, 0, *, *, 0, -2^{-18}s^8)$ can be reached in time s, i.e. by control variations at zero. Thus the tangent vectors $(0, 0, -1, 0), (0, 0, 0, -1) \in \overline{\mathcal{K}}_{\varphi}^{8}$ are generated by *needle-variations* of the zero reference control.

Proof. Suppose $T > 0$ and $\eta \in \mathbb{R}^4$ are given, with $\|\eta\|_\infty < 2^{-18}T^8$, $\eta_3 < 0$, and $\eta_4 \geq 0$. (The case of $\eta_4 < 0$, and $\eta_3 \geq 0$ is analogous.) Set $t_1 = (-4\eta_3)^{1/8} > 0$. Define the control $u \colon [0, T] \mapsto [-1, 1]^2$ (with $t_2 \geq 0$ to be determined later) by

$$(3.3)\qquad u(t) = \left\{ \begin{array}{lllcl} (\;\;1, & \;\;0) & \text{if} & 0 & \leq t \leq & t_1, \\ (-1, & \;\;0) & \text{if} & t_1 & \leq t \leq & 2t_1, \\ (\;\;0, & -1) & \text{if} & 2t_1 & \leq t \leq & 2t_1 + t_2, \\ (\;\;0, & \;\;1) & \text{if} & 2t_1 + t_2 & \leq t \leq & 2t_1 + 2t_2, \\ (\;\;0, & \;\;0) & \text{else .} & & & \end{array} \right.$$

From $u_2(t) = 0$ for $t \leq 2t_1$ it is easily seen that $x(2t_1, u)) = (0, 0, \frac{2}{3}(-4\eta_3)^{3/8}, 0, \eta_3, 0)$ and thus $\varphi(x(2t_1, u)) = (0, 0, \eta_3, 0)$. Then $\varphi(x(2t_1 + 2t_2, u)) = (0, 0, \eta_3, \xi(t_2))$ where

$$(3.4)\; \xi \colon t_2 \mapsto x_6(2t_1 + 2t_2) = 2\int_0^{t_2} \left(\tfrac{2}{3}(-4\eta_3)^{3/8} t^2 + t^7\right) dt = \tfrac{4}{9}(-4\eta_3)^{3/8} t_2^3 + \tfrac{1}{4} t_2^8$$

is a strictly increasing function. Thus there exists a unique $0 \leq t_2^* \leq (4\eta_4)^{1/8}$ such that $\xi(t_2^*) = \eta_4$. One easily verifies that the total time satisfies

$$(3.5)\quad (2t_1 + 2t_2^*) \leq 2(-4\eta_3)^{1/8} + 2(4\eta_4)^{1/8} \leq 4 \cdot \|4\eta\|_\infty^{1/8} = \left(2^{18}\|4\eta\|_\infty\right)^{1/8} < T.$$

Points $\eta = (0, 0, \eta_3, \eta_4)$ with $\eta_3 > 0$ and $\eta_4 > 0$ are even easier to reach using again piecewise constant controls, but now both u_1 and u_2 start with value $+1$, followed by -1 and zero. Note that in the directions of this positive quadrant the reachable set grown even faster, of order T^6. ■

Several stronger statements appear possible, e.g. characterizing the image $\varphi(\mathcal{R}_0(T)$ of the reachable set without the condition $y_1 = y_2 = 0$. However, in the sequel we have no need for this, and it is quite cumbersome to find the precise, entire boundary of the reachable sets.

One consequence of the next statement is that there does not exist a constant $C > 0$ such that one can reach points $x = (0, 0, *, *, -CT^8, -CT^8)$ in the fourth quadrant for all sufficiently small times $T > 0$, using *any* controls. Thus, a fortiori, one cannot reach such points by any combination of needle variations in time $T > 0$, and $(0, 0, -1, -1) \notin \overline{\mathcal{K}}_\varphi^8$.

Theorem 3.2. *Suppose $0 \leq T < 1$, $x_1(T, u) = x_2(T, u) = 0$. Let $C = 3^{1/5} \cdot 2^{-35/3} > 0$. If $x_5(T, u) < 0$ then $x_6(T, u) > -CT^{38/3}$, and if $x_6(T, u) < 0$ then $x_5(T, u) > -CT^{38/3}$.*

The following useful restatements and consequences are immediate corollaries from the theorem (or from its proof given below).

Corollary 3.3. *If $C, m > 0$ are such that the images of reachable set of system (3.1) contain the open balls $B_0^\infty(CT^m) \subseteq \varphi(\mathcal{R}_0(T))$ for all $T > 0$ (sufficiently small), then $m \geq \frac{38}{3}$.*

Corollary 3.4. *For the system (3.1), $(0, 0, a, b) \in \overline{\mathcal{K}}_\varphi^8$ if and only if $a \geq 0$ or $b \geq 0$.*

Proof. The *"if"* part is clear from theorem (3.1). For the *"only if"* part, suppose $(0, 0, a, b) \in \mathcal{K}_\varphi^8$ with $a < 0$ for some $b \in \mathbb{R}$, i.e. there exists a curve (not necessarily continuous) $s \mapsto u_s \in \mathcal{U}$ with corresponding endpoints $x(s, u_s) = (0, 0, *, *, as^8 + o(s^8), bs^8 + o(s^8)) \in \mathcal{R}(s)$. By theorem 3.2 this implies that $b \geq 0$. ■

Corollary 3.5. *For $8 < m < 38/3$ the approximating cones $\overline{\mathcal{K}}_\varphi^m$ of the output-reachable set are not convex.*

To streamline the proof of theorem 3.2 we first establish two technical lemmata. Consider the following simple system about its equilibrium point $x = 0$.

$$(3.6)\qquad \left\{ \begin{array}{rcl} \dot{x}_1 & = & u \\ \dot{x}_2 & = & cx_1^2 - x_1^7 \end{array} \right. \qquad \begin{array}{rcl} |u(\cdot)| & \leq & 1 \\ x(0) & = & 0 \end{array}$$

Lemma 3.6. *If $c \neq 0$ then the system (3.6) is not STLC about $x = 0$. If $T > 2\left(\frac{8}{3}|c|\right)^{\frac{1}{5}}$ then $0 \in \mathrm{int}\mathcal{R}(T)$.*

Proof. Without loss of generality assume $c > 0$ (else consider the coordinate change $(x_1, x_2) \mapsto (x_1, -x_2)$.) The first assertion follows from $x_2(T) = \int_0^T x_1^2(t)(c - x_1^5(t))\, dt \geq 0$ since $|x_1^5(t)| \leq t^5 \leq T^5 \leq c$ if $T \leq c^{\frac{1}{5}}$.

Conversely, suppose $T > 2b_0$ where $b_0 = (\frac{8}{3}c)^{\frac{1}{5}}$ is the first positive zero of $t \mapsto \frac{1}{3}ct^3 - \frac{1}{8}t^8$. Consider the 2-parameter family of controls

$$u_{a,b}(t) = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq T - 2b_0 - 2b - a \\ 1 & \text{if} \quad T - 2b_0 - 2b - a < t \leq T - b_0 - b - a \\ -1 & \text{if} \quad T - b_0 - b - a < t \leq T \end{cases} \tag{3.7}$$

Since $T - 2b_0 > 0$, these controls are defined for (a, b) in some neighborhood of $(0, 0)$. (To be precise, defined for $-b_0 \leq b$, $-b_0 \leq a + b$, and $a + 2b \leq T - 2b_0$.) It is straightforward [15] to verify that $x(T, u_{0,0}) = 0$ and that the Jacobian (for some polynomial p)

$$\frac{\partial x(T, u_{a,b})}{\partial(a, b)} = \begin{pmatrix} -1 & 0 \\ ca^2 + a^7 & -\frac{20}{9}\left(54c^7\right)^{\frac{1}{5}} + b \cdot p(b, c^{\frac{1}{5}}) \end{pmatrix} \tag{3.8}$$

of the map $(a, b) \mapsto x(T, u_{a,b})$ has full rank at $(a, b) = (0, 0)$ when $(c \neq 0)$. By the inverse mapping theorem, $0 \in \operatorname{int} \mathcal{R}(T)$, and the system (3.6) is STLC. ■

Lemma 3.7. *Suppose $f\colon [0, T] \mapsto \mathbb{R}$ is absolutely continuous with $|f'(t)| \leq \varepsilon$ a.e. and $f(0) = f(T) = 0$. Then for any even integer $m = 2k \geq 2$*

$$\int_0^T f^m(t)\, dt \geq \tfrac{2}{(m+1)\varepsilon} \|f\|_\infty^{m+1} \tag{3.9}$$

Proof. Since f is continuous there exists $t_0 \in [0, T]$ such that $f(t_0) = \|f(t)\|_\infty$. Without loss of generality assume $f(t_0) > 0$. Since $|f'(t)| \leq \varepsilon$ a.e., use the lower bound $f(t_0 \pm \tau) \geq f(t_0) - \varepsilon\tau$ for $0 \leq \tau \leq \tau_0 = f(t_0)/\varepsilon$ to obtain the estimate

$$\int_0^T f^{2k}(t)\, dt \geq \int_{t_0-\tau_0}^{t_0+\tau_0} f^{2k}(t) dt \geq 2 \cdot \int_0^{\tau_0} (\varepsilon t)^{2k}\ dt = \frac{2\varepsilon^{2k}\tau_0^{2k+1}}{2k+1} = \frac{2\,\|f\|_\infty^{2k+1}}{(2k+1)\varepsilon}. \tag{3.10}$$ ■

Proof (of theorem 3.2)**.** Before going into technical details we lay out some heuristic ideas that motivate the strategy.

The presence of the definite terms in system (3.1) suggests to use controls u_1 and u_2 with disjoint support. If one first keeps $u_2 \equiv 0$, thus also $x_4 \equiv 0$, and one easily reaches a point of the form $(0, 0, q_3, 0, -q_5, 0)$ with $q_3, q_5 > 0$ using a simple bang-bang control u_1. Next one holds $u_1 \equiv 0$ and tries to reach a point $(0, 0, q_3, q_4, -q_5, -q_6)$ with all $q_i > 0$ using a suitable control u_2. But according to lemma 3.6 this second step requires that the *available time* is sufficiently large (as determined by q_3). The subsequent estimates quantify this delicate trade-off, with the conclusion that, roughly speaking, if $x_6(T) < 0$ then $|x_5(T)| < CT^{38/3}$ for all T sufficiently small. The proof shows that this is true for all controls, not only for whatever clever combinations of the needle variations used in the proof of theorem 3.1. One of the key obstacles to making this heuristics into a general argument is that in such setting of *all possible* controls, there does not exist a clear notion of which of the two directions is obtained first. Thus the first objective is to identify an abstraction of *"which of the directions x_5 and x_6 is generated first."* Since

$$\int_0^T \left(\int_0^t x_1^2(s)\, ds\right) x_2^2(t)\, dt + \int_0^T x_1^2(t) \left(\int_0^t x_2^2(s)\, ds\right) dt = \left(\int_0^T x_1^2(t)\, dt\right) \cdot \left(\int_0^T x_2^2(t)\, dt\right)$$

we may assume without loss of generality that

$$\int_0^T \left(\int_0^t x_1^2(s)\,ds\right) x_2^2(t)\,dt \geq \tfrac{1}{2}\left(\int_0^T x_1^2(t)\,dt\right)\left(\int_0^T x_2^2(t)\,dt\right) \tag{3.11}$$

which may be rewritten in the most useful form

$$\int_0^T \left(\int_0^t x_1^2(s)\,ds\right) x_2^2(t)\,dt \geq \tfrac{1}{2}\int_0^T \left(\int_0^T x_1^2(s)\,ds\right) x_2^2(t)\,dt \tag{3.12}$$

The key advantage of this form is that the weight against which x_2^2 is integrated is now a constant, so that lemma 3.6 applies. If $x_6(T) \geq 0$ nothing needs to be shown. Thus assume $x_6(T) < 0$, then

$$0 > x_6(T) = \int_0^T \left(x_3^2(t)x_2^2(t) - x_2^7(t)\right)dt \geq \int_0^T x_2^2(t)\ \left(\tfrac{1}{2}\int_0^T x_1^2(s)\,ds - x_2^5(t)\right)dt \tag{3.13}$$

and thus

$$\xi_2 \stackrel{\text{def}}{=} \max_{0\leq t\leq T} x_2(t) > \left(\tfrac{1}{2}\int_0^T x_1^2(s)\,ds\right)^{\frac{1}{5}}. \tag{3.14}$$

The term on the right hand side may be considered a *cost*, or *energy*, that is required to move $x_6(0) = 0$ to $x_6(T) < 0$. The next step is to find a lower bound for this energy in terms of the displacement. Using lemma 3.7 with $\varepsilon = 1$ and $f = x_1$ gives the crude estimate

$$\int_0^T x_1^2(s)\,ds \geq 2\cdot\left(\tfrac{1}{3}\right)\cdot\xi_1^3 \ \stackrel{\text{def}}{=}\ \tfrac{2}{3}\max_{0\leq t\leq T} x_1(t) \tag{3.15}$$

On the other hand, using Minkowski's inequality (backwards)

$$-x_5(T) = \int_0^T (x_1^7(t) - x_4(t)x_1^2(t))dt \leq \int_0^T |x_1^7(t)|dt \leq T\cdot\xi_1^7 \tag{3.16}$$

Combining (3.14), (3.15), and (3.16) yields

$$\xi_2 > \left(\tfrac{1}{2}\int_0^T x_1^2(s)\,ds\right)^{\frac{1}{5}} \geq \left(\tfrac{1}{2}\cdot\tfrac{2}{3}\,\xi_1^3\right)^{\frac{1}{5}} \geq \left(\tfrac{1}{3}\left(\frac{-x_5(T)}{T}\right)^{\frac{3}{7}}\right)^{\frac{1}{5}} \tag{3.17}$$

Using that $\xi_2 \leq \frac{1}{2}T$ (from $u_2(\cdot)| \leq 1$ together with $x_2(0) = x_2(T) = 0$) this implies

$$-x_5(T) \leq 3^{\frac{1}{5}}\cdot\xi_2^{\frac{35}{3}}\cdot T \leq 3^{\frac{1}{5}}\cdot 2^{-\frac{35}{3}}\cdot T^{\frac{38}{3}}. \tag{3.18}$$

Note that, in particular, if $x(T) = (0, 0, *, *, -\rho, -\rho) \in \mathcal{R}_0(T)$ with $\rho > 0$, then $\rho < CT^{38/3}$ (with $C = 3^{1/5}\cdot 2^{-35/3}$). ∎

THEOREM 3.8. *There exist a constant $C > 0$ such that for all sufficiently small $T > 0$ the image $\varphi(\mathcal{R}(T))$ of the reachable set of system* (3.1) *contains the open ball $B_0^\infty(CT^{40/3})$.*

Corollary 3.9. *The system* (3.1) *is small-time locally output controllable* (STLOC) *about* $y = 0$.

Corollary 3.10. *The approximating cone* $\overline{\mathcal{K}}_\varphi^{\frac{40}{3}}$ *is the whole tangent space* $T_0\mathbb{R}^4$

Main elements of the construction. These statements all follow directly from theorem 4.3 and we refer to its proof for the full technical details. At this place we illustrate the critical elements in the construction showing a slightly weaker result. The general proof given in section 4 requires no major innovation, but mainly involves keeping track of many more quantities.

The objective in this construction is to suitably combine the families of control variations

$$(3.19)\quad u_s^{(1)}(t) = \begin{cases} (+1,0) & \text{if } 0 \le t < \frac{s}{2} \\ (-1,0) & \text{if } \frac{s}{2} \le t < s \end{cases} \quad \text{and } u_s^{(2)}(t) = \begin{cases} (0,+1) & \text{if } 0 \le t < \frac{s}{2} \\ (0,-1) & \text{if } \frac{s}{2} \le t < s \end{cases}$$

which steer to points $y(s,u_s^{(1)}) = (0,0,-2^{-7}s^8,0)$ and $y(s,u_s^{(2)}) = (0,0,0,-2^{-7}s^8)$ on the negative y_3 and y_4 axes, to obtain families of control variations that steer to points in the third quadrant $y_3 < 0$ and $y_4 < 0$ in the plane $y_1 = y_2 = 0$. Thus consider the control

$$(3.20)\qquad u(t) = \begin{cases} (+1,0) & \text{if } \;\; 0 \;\le t < \;\; t_1 = c\varepsilon^r \\ (-1,0) & \text{if } \;\; t_1 \le t < \;\; t_2 = 2c\varepsilon^r \\ (0,+1) & \text{if } \;\; t_2 \le t < \;\; t_3 = 2c\varepsilon^r + \varepsilon + \mu\varepsilon^r \\ (0,-1) & \text{if } \;\; t_3 \le t < \;\; T = 2c\varepsilon^r + 2\varepsilon + 2\mu\varepsilon^m \end{cases}$$

with $r, m \ge 1$ and constants c, μ to be determined (first think of $r = 1$ and $\mu = 0$). Verify [15] that at selected switching times the trajectory passes through

$$(3.21)\quad \begin{array}{l} x(t_1) = (c\varepsilon^r, 0, \frac{1}{3}c^3\varepsilon^{3r}, 0, -\frac{1}{8}c^8\varepsilon^{8r}, 0) \\ x(t_2) = (\;0\;, 0, \frac{2}{3}c^3\varepsilon^{3r}, 0, -\frac{1}{4}c^8\varepsilon^{8r}, 0) \\ x(t_3) = (0, \varepsilon+\mu\varepsilon^m, \frac{2}{3}c^3\varepsilon^{3r}, \frac{1}{3}(\varepsilon+\mu\varepsilon^m)^3, -\frac{1}{4}c^8\varepsilon^{8r}\, \frac{2}{9}c^3\varepsilon^{3+3r} - \frac{1}{8}\varepsilon^8 + p_{36}\cdot\mu\varepsilon^m) \\ x(T) = (\;0\;,\;0\;, \frac{2}{3}c^3\varepsilon^{3r}, \frac{2}{3}(\varepsilon+\mu\varepsilon^m)^3, -\frac{1}{4}c^8\varepsilon^{8r}, \frac{4}{9}c^3\varepsilon^{3+3r} - \frac{1}{4}\varepsilon^8 + p_{46}\cdot\mu\varepsilon^m). \end{array}$$

Here p_{36} and p_{46} are polynomial expressions in $(c, \varepsilon, \varepsilon^r, \mu\varepsilon^m)$. In particular,

$$(3.22)\qquad p_{46} = \left(\tfrac{4}{3}c^3\varepsilon^{3r+2} - 2\varepsilon^7\right) + q_{46}\cdot\mu\varepsilon^m$$

where q_{46} is a polynomial expression in $(c, \varepsilon, \varepsilon^r, \mu\varepsilon^m)$. First consider the case with $\mu = 0$ (no *perturbation*). If $r = 1$ then $x_6(2c\varepsilon^r + 2\varepsilon) > 0$ for all sufficiently small $\varepsilon > 0$. However, (still with $\mu = 0$) if one chooses $r \ge \frac{5}{3}$, then one may obtain $y_4(T < 0$ for suitable choices of c. Choosing the critical value $r = \frac{5}{3}$, yields equality of the exponents in ε^8 and ε^{3+3r}, and thus by varying the parameter c one reaches the points $y(T) = \varepsilon^{\frac{40}{3}}(0, 0, -\frac{1}{4}c^8, \frac{4}{9}c^3 - \frac{1}{4})$ in the third quadrant of the plane $y_1 = y_2 = 0$. Alternatively, fix the value $c^* = (9/16)^{1/3}$, which yields $y_4(T) = 0$ when $\mu = 0$. To reach a given point $\eta = (0, 0, \eta_3, \eta_4)$ with $\eta_3, \eta_4 < 0$ firstt choose $\varepsilon = (16/9)^{\frac{1}{5}} \cdot (-4\eta_3)^{\frac{3}{40}}$. Then, using that $p_{46}|_{\mu=0} \ne 0$, the implicit function theorem guarantees that (for $-\eta_4 > 0$ sufficiently small) one can solve $\eta_4 = \mu\varepsilon^m \cdot p_{46}$ (with $m = \frac{40}{3}$) for $\mu > 0$. Note that the total time $T = 2\varepsilon + 2c\varepsilon^{\frac{5}{3}} + 2\mu\varepsilon^{\frac{40}{3}} \overset{!}{=} s$ is of order ε (for $\varepsilon < 1$), and it is straightforward to reparameterize the controls choosing a constant multiple $s = a\varepsilon$ so that they qualify as a family of control variations at zero. Such linear reparameterization clearly does not affect the exponents such as $\frac{40}{3}$, thus yielding the statements in theorem 3.8 and its corollaries. The construction of controls steering to any point $\eta \in \mathbb{R}^4$ with η_1, η_2 not necessarily zero requires no major innovation, just much bookkeeping, and it is completely analogous to the proof of theorem 4.3 given below. ■

REMARK 3.11. Recall, that one commonly *linearly* reparameterizes needle variations when constructing convex combinations of tangent vectors. What is new here is that one must consider nonlinear reparameterizations (here basically replace s by a power s^r with r strictly larger than one). As a consequence one may here still obtain the convex combinations of the original tangent vectors, but not as tangent vectors of the same order (here the combination of eighth-order tangent vectors yields at best tangent vectors of order $(40/3)$ as shown above). Further, and much more severe implications for controllability properties in general are analyzed in section 5.

4. A nonstationary reference trajectory. A slight modification of this system with output yields a corresponding result for controllability about (and optimality of) a *nonstationary reference trajectory*. It might well be possible to construct a similar example with fewer controls. We chose this implementation with four controls for its structural simplicity, and because it allows one to easily build on the results of the previous section.

$$\left\{ \begin{array}{lcllcl} \dot{x}_1 & = & u_1 & |u_1(\cdot)| & \leq & 1 \\ \dot{x}_2 & = & u_2 & |u_2(\cdot)| & \leq & 1 \\ \dot{x}_3 & = & x_1^2 + (1+u_{01}) & |u_{01}(\cdot)| & \leq & 1 \\ \dot{x}_4 & = & x_2^2 + (1+u_{02}) & |u_{02}(\cdot)| & \leq & 1 \\ \dot{x}_5 & = & x_4 x_1^2 - x_1^7 & x(0) & = & 0 \\ \dot{x}_6 & = & x_3 x_2^2 - x_2^7 & x^*(t) & = & (0,0,t,t,0,0) \end{array} \right. \tag{4.1}$$

The key in this construction is that the *lower order* (i.e. apparently dominant) definite components $\int_0^T x_1^2(t)dt$ and $\int_0^T x_2^2(t)dt$ are aligned with the direction of the nonstationary reference trajectory, and that the *zero speed* $\|\dot{x}\| = 0$ is on the *boundary* of the set of admissible *velocities* (on the hyperplane $x_1 = x_2 = 0$ which includes the reference trajectory). With the fixed boundary velocity $u_{01} = u_{02} = -1$ the system may be considered a two-input system (with controls (u_1, u_2)) about the *stationary reference trajectory* $x \equiv 0$ (basically system (3.1)). That system is not STLC as obviously the (x_3, x_4)-directions are uncontrollable (it is impossible to reach points x with $x_3 < 0$ or $x_4 < 0$ from $x = 0$). But in (4.1) these uncontrollable directions are aligned with (as opposed to transversal to) the nonstationary reference trajectory, and thus controllability involves comparison with $\dot{x}_3 = \dot{x}_4 \equiv 1$ (as opposed to $).\dot{x}_3 = \dot{x}_4 \equiv 0$).

With this design in mind, it is natural to combine standard techniques and results about STLC and convexity of approximating cones that apply to systems about an equilibrium point (i.e. first holding $u_{01} = u_{02} = -1$ fixed), with constructions unique to nonstationary reference trajectories. In particular, after having *generated* desired tangent vectors working in the vicinity of the *equilibrium point*, it is straightforward to *catch up* with the prescribed nonstationary reference trajectory by using $u_{01} > 0$ and $u_{02} > 0$.

THEOREM 4.1. *For* $8 \leq m < \frac{38}{3}$ *the approximating cones* $\overline{\mathcal{K}}_0^m$ *of system* (4.1) *are not convex as they have* inward corners *in the sense that*

$$(0,0,0,0,v_5,v_6) \in \overline{\mathcal{K}}_0^m \iff (\ v_5 \geq 0 \ \textbf{ or } \ v_6 \geq 0\ ). \tag{4.2}$$

Proof First note that if $\xi(\cdot, u)$ and $x(\cdot, u)$ denote the solution curves of systems (3.1) and (4.1), respectively, (for the same control), then clearly $x_5(t,u) \geq \xi_5(t,u)$ and $x_6(t,u) \geq \xi_6(t,u)$ for all $t \geq 0$. Thus, it is a direct consequence of theorem 3.2 that if $m < \frac{38}{3}$, $v_5 < 0$ **and** $v_6 < 0$, then $(0,0,0,0,v_5,v_6) \notin \mathcal{K}_0^m$.
In the other direction, we show that every $v = (0,0,v_3,v_4,v_5,v_6) \in \mathbb{R}^6$ with $v_5 \geq 0$ or $v_6 \geq 0$ is contained in the cones $\overline{\mathcal{K}}_0^m$ for $m \geq 8$. (More general conclusions allowing $v_1, v_2 \neq 0$ may be obtained using theorem 4.3.) Suppose $0 < s < 1$ is given and $q = (0,0,q_3,q_4,q_5,q_6) \in R^6$ is such that $\|q\|_\infty < (1/26)^8$ and w.l.o.g. $q_6 \geq 0$. We exhibit a family of control variations at zero $u_s = u_{s,q}\colon [0,s] \mapsto [-1,1]^4$, $0 \leq s \leq 1$, such that $x(s,u_s) = x^*(s) + s^8 q$. Write $t_1 = s\,(4|q_5|)^{\frac{1}{8}}$, and let $t_2 \geq 0$ be the unique nonnegative number such that $s^8 q_6 = \frac{2}{9} t_1^3 t_2^3 + \frac{1}{8} t_2^8$. Moreover, let $t_3 = 2t_1 + 2t_2 - \frac{2}{3} t_1^3$, and $t_4 = 2t_1 + 2t_2 - \frac{2}{3} t_2^3$. Write $\tau_4 = 2t_1 + 2t_2 + t_3 + t_4$ and $\tau_5 = s - s^8(|q_3| + |q_4|)$.

It is critical that $\tau_4 \leq \tau_5$. To see this, note that $t_i \leq 2s\|q\|_\infty^{1/8}$ for both $i = 1,2$. Next $t_i \leq 8s\|q\|_\infty^{1/8}$ for both $i = 3,4$, and thus $\tau_4 \leq 24s\|q\|_\infty^{1/8}$. On the other hand, $\tau_5 \geq s - 2s\|q\|_\infty^{1/8}$ (since both $s < 1$ and $\|q\|_\infty < 1$). Thus $\tau_5 - \tau_4 \geq s(1 - 26\|q\|_\infty^{1/8}) \geq 0$ follows from the assumption that $\|q\|_\infty < (1/26)^8$. Thus the following is a well-defined one-parameter family of control variations of $u^* \equiv (0,0,0,0)$.

$$u(t) = \begin{cases} (\mathrm{sign}(q_5), 0, -1, -1) & \text{if} \quad 0 \leq t < t_1 \\ (-\mathrm{sign}(q_5), 0, -1, -1) & \text{if} \quad t_1 \leq t < 2t_1 \\ (0, 1, -1, -1) & \text{if} \quad 2t_1 \leq t < 2t_1 + t_2 \\ (0, -1, -1, -1) & \text{if} \quad 2t_1 + t_2 \leq t < 2t_1 + 2t_2 \\ (0,0,1,0) & \text{if} \quad 2t_1 + 2t_2 \leq t < 2t_1 + 2t_2 + t_3 \\ (0,0,0,1) & \text{if} \quad 2t_1 + 2t_2 + t_3 \leq t < \tau_4 \\ (0,0,0,0) & \text{if} \quad \tau_4 \leq t < \tau_5 \\ (0,0,\mathrm{sign}(q_3),0) & \text{if} \quad \tau_5 \leq t < s - s^8|q_4| \\ (0,0,0,\mathrm{sign}(q_4)) & \text{if} \quad s - s^8|q_4| \leq t < s \end{cases} \tag{4.3}$$

Note that as a family of control variations *at zero* this is a family of needle variations. One easily verifies [15] that at selected switching times the trajectory passes through

$$\begin{array}{lcl} x(2t_1) & = & (0,0,\frac{2}{3}t_1^3, 0, s^8 q_5, 0) \\ x(2t_1 + 2t_2) & = & (0,0,\frac{2}{3}t_1^3, \frac{2}{3}t_2^3, s^8 q_5, s^8 q_6) \\ x(\tau_4) & = & (0,0,\tau_4,\tau_4, s^8 q_5, s^8 q_6) \\ x(\tau_5) & = & (0,0,\tau_5,\tau_5, s^8 q_5, s^8 q_6) \\ x(s) & = & (0,0,s,s,0,0) + s^8 q \end{array} \tag{4.4}$$

The flow Φ corresponding to the reference control $u^* \equiv 0$ satisfies $\Phi_{-s}(0,0,x_3,x_4,x_5,x_6) = (0,0,x_3 - s, x_4 - s, x_5, x_6)$, and thus $x(s,u_s) = x(s,u^*) + s^8 \Phi_s(0,0,q_3,q_4,q_5,q_6)$. Consequently $(0,0,\lambda q_3, \lambda q_4, \lambda q_5, \lambda q_6) \in \overline{\mathcal{K}}_0^8$ for all $\lambda \geq 0$. ■

Note, the proof shows, in particular, that the vectors $(0,0,0,0,-1,0)$ and $(0,0,0,0,0,-1)$ are members of $\overline{\mathcal{K}}_0^8$ that are generated by *needle variations*. In contrast, according to theorem 3.2, the convex combination $\frac{1}{2}(0,0,0,0,-1,-1)$ cannot at all be generated as an eighth order tangent vector. In particular, it cannot be generated by any (whatever clever) combination of needle variations. Nonetheless, in analogy to corollary 3.10, in this particular system, this convex combination can still be generated as an even higher order tangent vector as shown below. First we establish a basic estimate needed later.

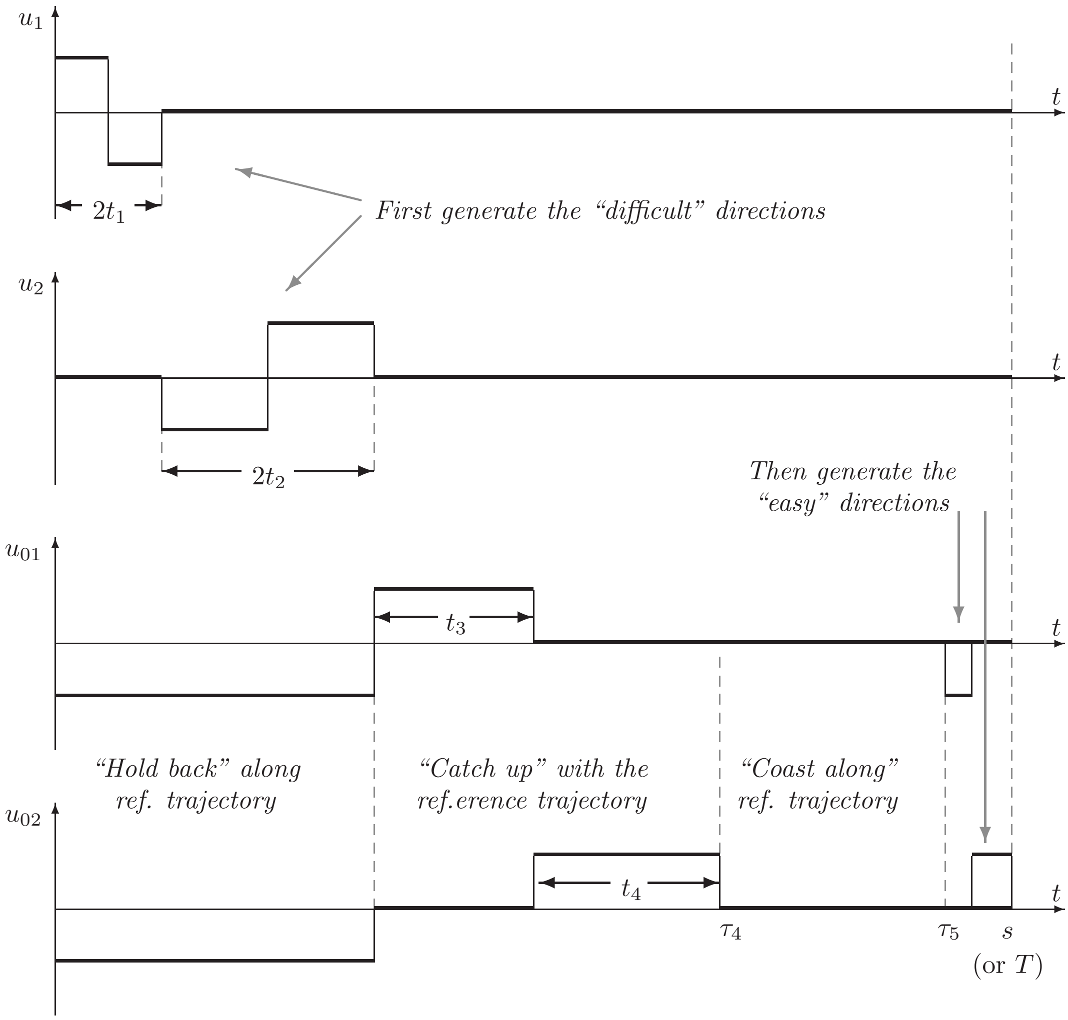

FIG. 4.1. *Schematic design of the needle variations for the proof of theorem 4.1*

LEMMA 4.2. *If $0 < a < M^5$ and $0 < b < M^8$ then the unique positive root t_0 of the scalar function $f(t) = b + at^3 - \frac{1}{8}t^8$ satisfies $t_0 < 2M$.*

Proof. The existence of a unique positive root is elementary. Set $t_{00} = (8a)^{1/5}$, which is the first positive root of $t \mapsto at^3 - \frac{1}{8}t^8$. Since $f''(t) < 0$ for all $t \geq t_{00}$, the linear estimate $0 \overset{!}{=} f(t_0) \leq f(t_{00}) + f'(t_{00})(t_0 - t_{00}) = b - 10 \cdot 2^{1/5}a^{7/5}(t_0 - (8a)^{1/5})$ yields the bound

$$t_0 \leq t_{00} + \frac{2^{1/5}b}{10a^{7/5}} = (8a)^{1/5} + \frac{2^{1/5}b}{a^{7/5}} \tag{4.5}$$

In the case that $b < a^{8/5}$, this immediately yields the desired estimate $t_0 < 2a^{1/5} \leq 2M$. Alternatively, if a is too small, i.e. if $b \geq a^{8/5}$, we find directly that $f(2b^{1/8}) \leq b + b^{5/8}\left(2b^{1/8}\right)^3 - \frac{1}{8}\left(2b^{1/8}\right)^8 = -25b < 0$, and hence the first positive zero t_0 of f

again satisfies $x_0 \leq 2b^{1/8} < 2M$. ■

Theorem 4.3. *For every $T > 0$ sufficiently small, the reachable set $\mathcal{R}(T)$ of system* (4.1) *contains the open ball $B^\infty_{x^*(T)}((2^{-5}T)^{40/3})$ centered at $x^*(T)$.*

Corollary 4.4. *The system* (4.1) *is STLC about the reference trajectory $x^*(t) = (0,0,t,t,0,0)$.*

Corollary 4.5. *The approximating cone $\overline{\mathcal{K}}_0^{40/3}$ of system* (4.1) *is the whole tangent space $\overline{\mathcal{K}}_0^{40/3} = T_0\mathbb{R}^6$.*

Proof. Similar to the proof of theorem 4.1 we explicitly construct for any given time $T > 0$ and any given point q sufficiently close to $x^*(T)$ a control $u = u_q\colon [0,T] \mapsto [-1,1]^4$ which steers the system to q. The main differences to the prior work is that now we need to also consider the case when $q_1 \neq 0$ and $q_2 \neq 0$. Note that since suitable open mapping principles for approximating cones for this setting are not available, we cannot rely on just generating tangent vectors.
Suppose $0 < T < 1$ and $q \in B_0^\infty(2^{-40}T^{40/3})$ are given. We consider only the case that both $q_5 < 0$ and $q_6 < 0$. All other cases can be handled analogously, albeit with considerably simpler constructions. The basic construction for moving into the *difficult* directions $(0,0,0,0,*,*)$ is the same as in earlier proofs. These are followed by simple control actions that move into the *easy* directions. However, these later moves also have some effect on the difficult directions. Thus we first calculate these side-effects, and modify the initial controls to *overshoot* the targets in the difficult directions. The required corrections are calculated to cancel the integrals that result from (4.11), also compare (4.12).

$$\Delta q_{31} = q_1^2|q_2| = \int_0^{|q_2|} q_1^2\,dt, \qquad \Delta q_3 = \tfrac{1}{3}|q_1|^3 + |q_2|q_1^2 = \Delta q_{31} + \int_0^{|q_1|} (\sigma_1 t)^2 dt \tag{4.6}$$

$$\Delta q_4 = \tfrac{1}{3}|q_2|^3 = \int_0^{|q_2|} t^2 dt \tag{4.7}$$

$$\begin{aligned}\Delta q_{51} &= q_1^2|q_2|(T+q_4) - \tfrac{1}{2}q_1^2 q_2^2 - \tfrac{1}{4}q_1^2 q_2^4 - q_1^7|q_2| \\ &= \int_0^{|q_2|} \big((\tau_8 + q_4 - \Delta q_4 + (t + \tfrac{1}{3}t^3))q_1^2 - q_1^7\big)\, dt\end{aligned} \tag{4.8}$$

$$\begin{aligned}\Delta q_5 &= \Delta q_{51} + \tfrac{1}{3}|q_1|^3(T+q_4) - \tfrac{1}{3}|q_1|^3|q_2| - \tfrac{1}{9}|q_1|^3|q_2|^3 - \tfrac{1}{12}q_1^4 - \tfrac{1}{8}|q_1|q_1^7 \\ &= \Delta q_{51} + \int_0^{|q_1|} \big((\tau_7 + t + q_4 - \Delta q_4)(\sigma_1 t)^2 + (\sigma_1 t)^7\big)\, dt\end{aligned} \tag{4.9}$$

$$\begin{aligned}\Delta q_6 &= \tfrac{1}{3}|q_2|^3 T + \tfrac{1}{3}|q_2|^3 q_3 - \tfrac{1}{12}q_1^2 q_2^4 - \tfrac{1}{12}q_2^4 - \tfrac{1}{8}|q_2|q_2^7 \\ &= \int_0^{|q_2|} \big((\tau_8 + q_3 - \Delta q_{31} + (q_1^2+1)t)(\sigma_2 t)^2 - (\sigma_2 t)^7\big)\, dt\end{aligned} \tag{4.10}$$

What really matters in the subsequent construction is the size of these correction terms – and clearly they are all of order at least 3 in $\|q\|_\infty$. Next let $t_1 = (-4(q_5 - \Delta q_5))^{\frac{1}{8}}$, $t_2 \geq 0$ be the smallest nonnegative number such that $(q_6 - \Delta q_6) = \frac{2}{9}t_1^3 t_2^3 - \frac{1}{8}t_2^8$ (which is unique in the challenging case of $(q_6 - \Delta q_6) < 0$). Let $t_3 = 2t_1 + 2t_2 - \frac{2}{3}t_1^3$, and $t_4 = 2t_1 + 2t_2 - \frac{2}{3}t_2^3$. Write $\tau_3 = 2t_1 + 2t_2 + t_3$, $\tau_4 = 2t_1 + 2t_2 + t_3 + t_4$ and working backwards $\tau_8 = T - |q_2|$, $\tau_7 = \tau_8 - |q_1|$, $\tau_6 = \tau_7 - |q_4 - \Delta q_4|$, $\tau_5 = \tau_6 - |q_3 - \Delta q_3|$. Also use the abbreviations $\sigma_i = \operatorname{sign}(q_i)$ for $i = 1, 2$ and $\sigma_i = \operatorname{sign}(q_i - \Delta q_i)$ for $i = 3, 4$. Define the control $u = u_{q,T}\colon [0, T] \mapsto [-1, 1]^4$ by

$$
u(t) = \begin{cases}
(\;1,\; 0,\; -1,\; -1) & \text{if } \quad 0 \leq t < t_1 \\
(-1,\; 0,\; -1,\; -1) & \text{if } \quad t_1 \leq t < 2t_1 \\
(\;0,\; 1,\; -1,\; -1) & \text{if } \quad 2t_1 \leq t < 2t_1 + t_2 \\
(\;0,\; -1,\; -1,\; -1) & \text{if } \quad 2t_1 + t_2 \leq t < 2t_1 + 2t_2 \\
(\;0,\; 0,\; 1,\; 0) & \text{if } \quad 2t_1 + 2t_2 \leq t < \tau_3 \\
(\;0,\; 0,\; 0,\; 1) & \text{if } \quad \tau_3 \leq t < \tau_4 \\
(\;0,\; 0,\; 0,\; 0) & \text{if } \quad \tau_4 \leq t < \tau_5 \\
(\;0,\; 0,\; \sigma_3,\; 0) & \text{if } \quad \tau_5 \leq t < \tau_6 \\
(\;0,\; 0,\; 0,\; \sigma_4) & \text{if } \quad \tau_6 \leq t < \tau_7 \\
(\sigma_1,\; 0,\; 0,\; 0) & \text{if } \quad \tau_7 \leq t < \tau_8 \\
(\;0,\; \sigma_2,\; 0,\; 0) & \text{if } \quad \tau_8 \leq t < T.
\end{cases}
\tag{4.11}
$$

It is essential to verify that $\tau_4 \leq \tau_5$. Start with the crude estimate $|\Delta q_i| \leq 2\|q\|_\infty^3 \leq 2\|q\|_\infty$ for $i = 1, 2, 5, 6$. Next $t_1 \leq 2\|q\|_\infty^{\frac{1}{8}}$. The critical estimate is for t_2 – note the sign reversal in the defining equation in this construction compared to the one in the proof of theorem 4.1: The earlier construction allowed one to estimate $s^8 q_6 = \frac{2}{9}t_1^3 t_2^3 + \frac{1}{8}t_2^8 \overset{!}{\geq} \frac{1}{8}t_2^8$ yielding $t_2 \leq (8s^8 q\_6)^{1/8}$. Now in the interesting case $0 \geq (q_6 - \Delta q_6) = \frac{2}{9}t_1^3 t_2^3 - \frac{1}{8}t_2^8$ lemma 4.2 applies with $a = \frac{2}{9}t_1^3 \leq \frac{4}{9}\|q\|_\infty^{3/8}$ $b = (\Delta q_6 - q_6) \leq 3\|q\|_\infty$, and $M = \|q\|_\infty^{3/40}$, yielding $t_2 \leq 2\|q\|_\infty^{3/40}$. Easily $t_i \leq 8\|q\|_\infty^{3/40}$ for $i = 3, 4$, and thus $\tau_4 \leq 24\|q\|_\infty^{3/40}$.

On the other hand, $T - \tau_5 \leq 4\|q\|_\infty + |\Delta q_5| + |\Delta q_6| \leq 8\|q\|_\infty \leq 8\|q\|_\infty^{3/40}$. Together with $0 < T < 1$ and $\|q\|_\infty < (2^{-5}T)^{\frac{40}{3}}$ this yields $\tau_5 - \tau_4 \geq T - 32\|q\|_\infty^{3/40} \geq 0$.

Note that $x_1(t) = x_2(t) = 0$ for all $t \in [2t_1 + 2t_2, \tau_7]$. Thus $x_5(\cdot)$ and $x_6(\cdot)$ are constant on this interval. It is straightforward [15] to verify that at selected switching times the trajectory passes through

$$
\begin{array}{llllllll}
x(t_1) & =(t_1, & 0, & \frac{1}{3}t_1^3, & 0, & -\frac{1}{8}t_1^8, & 0) \\
x(2t_1) & =(0, & 0, & \frac{2}{3}t_1^3, & 0, & q_5 - \Delta q_5, & 0) \\
x(2t_1 + t_2) & =(0, & t_2, & \frac{2}{3}t_1^3, & \frac{1}{3}t_2^3, & q_5 - \Delta q_5, & -\frac{1}{8}t_2^8 + \frac{2}{9}t_1^3 t_2^3) \\
x(2t_1 + 2t_2) & =(0, & 0, & \frac{2}{3}t_1^3, & \frac{2}{3}t_2^3, & q_5 - \Delta q_5, & q_6 - \Delta q_6) \\
x(\tau_3) & =(0, & 0, & \tau_3, & t_3 + \frac{2}{3}t_2^3, & q_5 - \Delta q_5, & q_6 - \Delta q_6) \\
x(\tau_4) & =(0, & 0, & \tau_4, & \tau_4, & q_5 - \Delta q_5, & q_6 - \Delta q_6) \\
x(\tau_5) & =(0, & 0, & \tau_5, & \tau_5, & q_5 - \Delta q_5, & q_6 - \Delta q_6) \\
x(\tau_6) & =(0, & 0, & \tau_6 + q_3 - \Delta q_3, & \tau_6, & q_5 - \Delta q_5, & q_6 - \Delta q_6) \\
x(\tau_7) & =(0, & 0, & \tau_7 + q_3 - \Delta q_3, & \tau_7 + q_4 - \Delta q_4, & q_5 - \Delta q_5, & q_6 - \Delta q_6) \\
x(\tau_8) & =(q_1, & 0, & \tau_8 + q_3 - \Delta q_{31}, & \tau_8 + q_4 - \Delta q_4, & q_5 - \Delta q_{51}, & q_6 - \Delta q_6) \\
x(T) & =(q_1, & q_2, & T + q_3, & T + q_4, & q_5, & q_6). \quad \blacksquare
\end{array}
$$

(4.12)

Note that the construction above was given for a fixed terminal time T. But it is easily modified to yield families of control variations $u_s\colon [0, s] \mapsto U$ at zero by replacing T by

s and q by $s^{40/3}q$. This yields the curve of endpoints $x(s,u_s)=(0,0,s,s,0,0)+s^{40/3}q$ yielding $q\in\overline{\mathcal{K}}_0^{40/3}$.

5. Delicate structural instability. This section analyzes small modifications of the systems considered in the preceding sections. The main result is that what at first appear to be *"at a higher order"* perturbations, they nonetheless can destroy the delicate controllability exhibited in the earlier sections. More specifically, we exhibit a polynomial cascade system which has nonconvex approximating cones generated by needle variations, and which is not controllable.

Returning to the discussion of the system (4.1) recall that the cones $\overline{\mathcal{K}}_0^{8}$ are not convex. While the vectors $(0,0,0,0,-1,0)\in\overline{\mathcal{K}}_0^{8}$ and $(0,0,0,0,0,-1)\in\overline{\mathcal{K}}_0^{8}$ could be generated by needle variations at zero as 8-th order tangent vectors, it was shown that it is impossible to obtain convex combinations of these as tangent vectors of the same order. However, such convex combinations can be generated as even higher order tangent vectors, e.g. $(0,0,0,0,-\frac{1}{2},-\frac{1}{2})\in\overline{\mathcal{K}}_0^{40/3}$ by fairly delicate constructions. As we illustrate next, such constructions can be rendered impossible through quite innocent looking perturbations. Similar to section 3, we start again with an affine system with output and a stationary reference trajectory.

$$(5.1)\qquad \left\{\begin{array}{lcl} \dot z_1 &=& u_1\\ \dot z_2 &=& u_2\\ \dot z_3 &=& z_1^2\\ \dot z_4 &=& z_2^2\\ \dot z_5 &=& z_4z_1^2-z_1^7+z_1^{10}+z_2^{10}\\ \dot z_6 &=& z_3z_2^2-z_2^7+z_1^{10}+z_2^{10}\end{array}\right. \qquad \begin{array}{rcl} |u_1(\cdot)| &\le& 1\\ |u_2(\cdot)| &\le& 1\\ z(0) &=& 0\\ \varphi(z) &=& (z_1,z_2,z_5,z_6)\end{array}$$

It is clear that the added 10-th order terms do not at all affect the construction given in the proof of theorem 3.2 and corollary 3.4, and thus it is still true that the 8-th order cone is nonconvex.

Proposition 5.1. *For the system* (5.1), $(0,0,a,b)\in\overline{\mathcal{K}}_\varphi^{8}$ *if and only if* $a\ge 0$ *or* $b\ge 0$. However, we will show that the perturbations destroy the controllability, and the tangent cones of all orders $m\ge 8$ are not convex.

Theorem 5.2. *For* $0\le T<1$ *the image* $\varphi(\mathcal{R}(T))$ *of the reachable set of system* (5.1) *does not contain any points of the form* $(0,0,\eta_3,\eta_4)$ *with* $\eta_3<0$ *and* $\eta_4<0$.

Corollary 5.3. *The system* (5.1) *is not small-time locally output controllable* (STLOC) *about* $y=0$.

Corollary 5.4. *For the system* (5.1) *and* $m\ge 8$, *both* $(0,0,-1,0)\in\overline{\mathcal{K}}_\varphi^{m}$ *and* $(0,0,0,-1)\in\overline{\mathcal{K}}_\varphi^{m}$ *are generated by needle variations, but if* $\eta_3,\eta_4<0$ *then* $(0,0,\eta_3,\eta_4)\notin\overline{\mathcal{K}}_\varphi^{m}$ *for all* $m>0$. *In particular, for all* $m>0$, *the convex combination* $(0,0,-\frac{1}{2},-\frac{1}{2})\notin\overline{\mathcal{K}}_\varphi^{m}$.

Proof. We write z and x for the corresponding solutions of the perturbed and unperturbed systems (3.1) and (5.1). I.e. $z_i(t,u)=x_i(t,u)$ for $i=1,2,3,4$, while for $i=5,6$

$$(5.2)\qquad z_i(t,u)=x_i(t,u)+\int_0^t\big(x_1^{10}(\tau,u)+x_2^{10}(\tau,u)\big)\,d\tau \quad \text{for all } \ 0\le t\le T.$$

W.l.o.g. suppose $x_6(T) < 0$. Then using lemma 3.7 and the key estimate (3.17)

$$(5.3)\ \int_0^T x_2^{10}(t)\,dt = \int_0^T |x_2(t)|^{10} dt \geq \tfrac{2}{11}\,\xi_2^{11} \geq \tfrac{2}{11}\left(\left(\frac{-x_5(T)}{T}\right)^{\frac{3}{35}}\right)^{11} \geq |x_5(T)| > 0.$$

If $T < 1$, clearly $|x_5(T)| < 1$. Together with $\frac{33}{35} < 1$ this shows that $z_5(T) = x_5(T) + \int_0^T |x_2(t)|^{10} dt \geq 0$ if $z_6(T) < 0$. ■

For corresponding results for loss of controllability about a nonstationary reference trajectory consider the system

$$(5.4)\quad \begin{cases} \dot z_1 = u_1 & |u_1(\cdot)| \leq 1 \\ \dot z_2 = u_2 & |u_2(\cdot)| \leq 1 \\ \dot z_3 = z_1^2 + (1 + u_{01}) & |u_{01}(\cdot)| \leq 1 \\ \dot z_4 = z_2^2 + (1 + u_{02}) & |u_{02}(\cdot)| \leq 1 \\ \dot z_5 = z_4 z_1^2 - z_1^7 + z_1^{10} + z_2^{10} & z(0) = 0 \\ \dot z_6 = z_3 z_2^2 - z_2^7 + z_1^{10} + z_2^{10} & z^*(t) = (0,0,t,t,0,0) \end{cases}$$

By combining the arguments from the previous sections, one obtains readily the following statements. In particular, the 10^{th}-order terms in (5.4) do not affect the constructions of the 8^{th}-order tangent vectors, compare the proof of theorem 3.2.

Proposition 5.5. *For the system* (5.4)*,* $(0,0,0,0,a,b) \in \overline{\mathcal{K}}_0^8$ *if and only if* $a \geq 0$ *or* $b \geq 0$. On the other hand the estimate (5.3) in the proof of theorem 5.2 applies also to the system with nonstationary reference trajectory.

Theorem 5.6. *For* $0 \leq T < 1$ *the reachable set* $\mathcal{R}(T)$ *of system* (5.4) *does not contain any points of the form* $(0,0,0,0,q_5,q_6)$ *with* $q_5 < 0$ *and* $q_6 < 0$.

Corollary 5.7. *The system* (5.4) *is not small-time locally controllable* (STLC) *about* $z^*(t) = (0,0,t,t,0,0)$. The control variations constructed earlier that generate the tangent vectors $(0,0,0,0,-1,0)$ and $(0,0,0,0,0,-1)$ satisfy the requirements of definition 2.5. Together with theorem 5.6 this allows to state the last corollary for both notions of tangent vectors, $\mathcal{K}_0^m$ and $\mathcal{K}_T^m$.

Corollary 5.8. *For the system* (5.4)*,* $m \geq 8$*, and* $0 \leq T < 1$*, none of the cones* $\overline{\mathcal{K}}_T^m$ *are convex. In particular,* $(0,0,0,0,-1,0) \in \overline{\mathcal{K}}_T^m$ *and* $(0,0,0,0,0,-1) \in \overline{\mathcal{K}}_T^m$ *are generated by needle variations at zero, but* $(0,0,0,0,-\frac{1}{2},-\frac{1}{2}) \notin \overline{\mathcal{K}}_T^m$ *for all* $m > 0$ *and* $T \geq 0$.

6. Conclusion and further outlook. Summarizing, the analysis of the custom-designed systems in this article has yielded two major insights: The immediate result is a counterexample for a long-standing natural conjecture about needle variations.

- The most general notion of tangent vector (which works well for stationary reference trajectories) does not necessarily yield convex cones for neither nonstationary reference trajectories nor for systems with output.
- Narrower notions of tangent vectors, that yield convex cones, e.g. by requiring explicitly that needle variations must be *movable* by fixed times, may miss the controllability of some systems. (Recall in section 4 the systems were controllable, but this cannot be detected with needle variations that require standard *movability*.)
- The common *technical* conditions on *needle variations* are **not** stated only because of *not yet discovered* stronger theorems, but they are essential even

for some of the most benign systems. In a nutshell, the *time*(s) at which needle variations are generated must be movable – though not necessarily by a fixed distance, but by a distance of the same order of magnitude as the duration as the needle variation. Moveability by a smaller amount, e.g. some power larger than one is not sufficient, compare remark 3.11. Otherwise, one should expect a lack of convexity of the *cones* of tangent vectors, and a consequent failure to satisfy the hypotheses for open mapping theorems that are essential for making the cones into meaningful *"approximating cones"*.

On a deeper level these examples cast further doubts on the structural stability and robustness of the small-time local controllability property. It is well-known that simple Taylor approximations do not need preserve controllability (see e.g. [3]). For example, the system $\dot{x}_1 = u$, $\dot{x}_2 = x_1$, $\dot{x}_3 = x_2^2 + x_1^3$ [21] is STLC about $x(0) = 0$ and $|u(\cdot)| \leq 1$, but its quadratic Taylor approximation $\dot{y}_1 = u$, $\dot{y}_2 = y_1$, $\dot{y}_3 = y_2^2$ is not STLC. Instead, many controllability results of the past two decades rely on nilpotent approximating systems [4, 11] which are based on *graded structures*: For a fixed set of local coordinates $(x_1, \ldots x_n)$ on $\mathbf{R}^n$ and weights $r_1, \ldots r_n \geq 1$ consider the family of dilations $\Delta\colon \mathbb{R}^+ \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by $\Delta_s(x) = (s^{r_1} x_1, \ldots, s^{r_n} x_n)$. A function $\varphi\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is called homogeneous of order m w.r.t. Δ if $\varphi \circ \Delta_s = s^m \varphi$. A vector field f is called homogeneous of order k if (considered as a first order partial differential operator) it maps (smooth) homogeneous functions of any order m to homogeneous functions of order $m+k$. To *approximate* the system (2.1) by a *nilpotent* system of the same form, replace each vector field f_i by the *principal part* g_i of the expansion of f_i into a (generally infinite) series of homogeneous vector fields. If all vector fields g_i have negative degrees of homogeneity, then all are polynomial vector fields of *cascade form*, and they generate a *nilpotent* Lie algebra, compare the survey [11] for details. Roughly speaking, the fundamental result is that if such *homogeneous* nilpotent approximating system with vector fields g_i is STLC, then the original system is *STLC* e.g. [4, 22]. By varying the construction of the local coordinates and dilation exponents r_i several major results for STLC follow, e.g. [4, 21, 22]. On the other hand, it is known that there exist polynomial systems that are STLC, but for whom any such homogeneous nilpotent approximating system is not STLC [14].

Returning to the systems (3.1) and (4.1) discussed in this article which are STLOC and STLC, respectively, these are similar to the system in [14] as any of the standard nilpotent approximation schemes will simply delete the terms x_1^7 and x_2^7 on the right hand side, yielding homogeneous nilpotent approximating systems that are not STLOC or STLC. Since the example constructed in [14], one has been searching for more general approximating schemes that would *recognize* the need to preserve terms such as x_1^7 and x_2^7 in the above systems which are essential for their controllability, and which would only truncate *truly higher-order* terms that are inessential for controllability. However, the analysis in the preceding section of the systems (5.1) and (5.4) raises new questions as the inclusion of the even higher order terms x_1^{10} and x_2^{10} again causes loss of controllability. One naturally wonders whether it is possible to construct an infinite such chain, which could possibly lead to a negative answer to the long-standing question of whether STLC (of analytic systems) is *finitely determined* [2].

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