OPTIMAL AND ROBUST CONTROL FOR LINEAR SYSTEMS WITH STATE DELAY

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Abstract This paper presents the optimal regulator for a linear system with state delay and a quadratic criterion. The optimal regulator equations are obtained using the maximum principle. Performance of the obtained optimal regulator is verified in the illustrative example against the best linear regulator available for linear systems without delays. Simulation graphs demonstrating better performance of the obtained optimal regulator are included. The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial time moment is designed. The obtained robust control algorithm is verified by simulations in the illustrative example.

Keywords: Optimal control, robust control, integral sliding mode, linear time-delay system

1 Introduction

Although the optimal control (regulator) problem for linear system states was solved in 1960s (see [1, 2]), the optimal control problem for linear systems with delays is still open, depending on the delay type, specific system equations, criterion, etc. A detailed comment on the up-to-date state of the control theory for time-delay systems is given in [3, 4]. Comprehensive reviews of theory and algorithms for time delay systems can be found in [5, 6, 7, 8, 9].

The first part of this paper concentrates on the solution of the optimal control problem for a linear system with state delay and a quadratic criterion. Using the maximum principle [10, 11], the solution to the stated optimal control problem is obtained in a closed form, i.e., it is represented as a linear in state control law, whose gain matrix satisfies an ordinary differential (quasi-Riccati) equation, which does not contain time-advanced arguments and does not depend on the state variables. The obtained optimal regulator makes an advance with respect to general optimality results for time delay systems (such as given in [12, 13, 14, 15, 16]), since (a) the optimal control law is given explicitly and not as a solution of a system of integro-differential or PDE equations, and (b) the quasi-Riccati equation for the gain matrix does not contain any time advanced arguments and does not depend on the state variables and, therefore, leads to a conventional two points boundary-valued problem generated in the optimal control problems with quadratic criterion and finite horizon (see, for example, [1]). Thus, the obtained optimal regulator is realizable using two delay-differential equations. Taking into account that the state space of a delayed system is infinite-dimensional [5], this seems to be a significant advantage.

Performance of the obtained optimal control for a linear system with state delay and a quadratic criterion is verified in the illustrative example against the best linear regulator available for linear systems without delays. The simulation results show a definitive (three and half times) advantage of the obtained optimal regulator in the criterion value.

The second part of the paper presents an integral sliding mode regulator robustifying the optimal regulator for linear systems with state delay and a quadratic criterion. The idea is to add a compensator to the known optimal control to suppress external disturbances deteriorating the optimal system behavior [17, 18]. The integral sliding mode compensator is realized as a relay control in a such way that the sliding mode motion starts from the initial moment, thus eliminating the matched uncertainties from the beginning of system functioning. This constitutes the crucial advantage of the integral sliding modes in comparison to the conventional ones. Note that in the framework of this modified (with respect to [17, 3]) integral sliding mode approach, the optimal control is not required to be differentiable and the sliding mode manifold matrix is always invertible. Other original modifications of the sliding mode control technique applicable to disturbance suppression were suggested in [19, 20].

The paper is organized as follows. Section 2 states the optimal control problem for a linear system with state delay. The solution to the optimal control problem is given in Section 3. The proof of the obtained results, based on the maximum principle [10, 11], is given in Appendix. The paper then presents a robustification algorithm for the obtained optimal regulator based on integral sliding mode compensation of disturbances [17]. Section 4 outlines the new general principles of the integral sliding mode compensator design, which yield the basic control algorithm oriented to time-delay systems. This basic algorithm is then applied to robustify the optimal regulator. As a result, the sliding mode compensating control leading to suppression of the disturbances from the initial time moment is designed. Section 5 presents an illustrative example.

2 Optimal control problem for linear state delay system

Consider a linear system with time delay in the state

$$\dot{x}(t) = a_0(t) + a(t)x(t-h) + B(t)u(t), \qquad (1)$$

with the initial condition $x(s) = \varphi(s)$, $s \in [t_0 - h, t_0]$, where $x(t) \in R^n$ is the system state, $u(t) \in R^m$ is the control variable, and $\phi(s)$ is a piecewise continuous function given in the interval $[t_0 - h, t_0]$. Existence of the unique solution of the equation (1) is thus assured by the Carathéodory theorem (see, for example, [21]). The quadratic cost function to be minimized is defined as follows:

$$J = \frac{1}{2}[x(T)]^T \psi[x(T)] +$$

$$\frac{1}{2}\int_{t_0}^{T} u^T(s)R(s)u(s)ds + \frac{1}{2}\int_{t_0}^{T} x^T(s)L(s)x(s)ds, \qquad (2)$$

where R is positive and ψ, L are nonnegative definite symmetric matrices, and $T > t_0$ is a certain time moment.

The optimal control problem is to find the control $u(t)$, $t \in [t_0, T]$, that minimizes the criterion J along with the trajectory $x^*(t)$, $t \in [t_0, T]$, generated upon substituting $u^*(t)$ into the state equation (1). The solution to the stated optimal control problem is given in the next section and then proved using the maximum principle [10, 11] in Appendix.

3 Optimal control problem solution

The solution to the optimal control problem for the linear system with state delay (1) and the quadratic criterion (2) is given as follows. The optimal control law is given by

$$u^*(t) = (R(t))^{-1}B^T(t)Q(t)x(t), \qquad (3)$$

where the matrix function $Q(t)$ satisfies the matrix equation

$$\dot{Q}(t) = L(t) - Q(t)M_1(t)a(t) - a^T(t)M_1^T(t)Q(t) - \qquad (4)$$

$$Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = \psi$. The auxiliary matrix $M_1(t)$ is defined as $M_1(t) = (\partial x(t-h)/\partial x(t))$, whose value is equal to zero, $M_1(t) = 0$, if $t \in [t_0, t_0 + h)$, and is determined as $M_1(t) = \Phi^{-1}(t, t-h) = \Phi(t-h, t) = \exp\left(-\int_{t-h}^{t} B(s)R^{-1}(s)B^T(s)Q(s)ds\right)$, if $t \geq t_0 + h$, where $\Phi(t, \tau)$ satisfies the matrix equation

$$\frac{d\Phi(t,\tau)}{dt} = B(t)R^{-1}(t)B^T(t)Q(t)\Phi(t,\tau),$$

with the initial condition $\Phi(t,t) = I$, and I is the identity matrix.

Upon substituting the optimal control (3) into the state equation (1), the optimally controlled state equation is obtained

$$\dot{x}(t) = a_0(t) + a(t)x(t-h) + B(t)R^{-1}(t)B^T(t)Q(t)x(t), \qquad (5)$$

with the initial condition $x(s) = \varphi(s)$, $s \in [t_0 - h, t_0]$.

The results obtained in this section by virtue of the duality principle are proved in Appendix using the general equations of the Pontryagin maximum principle [10, 11].

4 Robust control problem

Consider a nominal control system with state delay

$$\dot{x}(t) = f(x(t-h)) + B(t)u(t), \qquad (6)$$

where $u(t) \in R^m$ is the control input, the rank of matrix $B(t)$ is complete and equal to m for any $t > 0$, and the pseudoinverse matrix of B is uniformly bounded:

$$\|B^+(t)\| \leq b^+,\ b^+ = const > 0,\ B^+(t) := [B^T(t)B(t)]^{-1}B^T(t),$$

and $B^+(t)B(t) = I$, where I is the m-dimensional identity matrix.

Suppose that there exists a state feedback control law $u_0(x(t),t)$, such that the dynamics of the nominal closed loop system takes the form

$$\dot{x}_0(t) = f(x_0(t-h)) + B(t)u_0(x_0(t),t), \qquad (7)$$

and has certain desired properties. However, in practical applications, system (6) operates under uncertainty conditions that may be generated by parameter variations and external disturbances. Let us consider the real trajectory of the disturbed closed loop control system

$$\dot{x}(t) = f(x(t-h)) + B(t)u(t) + g_1(x(t),t) + g_2(x(t-h),t), \quad (8)$$

where g_1, g_2 are smooth uncertainties presenting perturbations and nonlinearities in the system (6). For g_1, g_2, the standard matching and conditions are assumed to be held: $g_1, g_2 \in \mathrm{span}B$, or, in other words, there exist smooth functions γ_1, γ_2 such that

$$g_1(x(t),t) = B(t)\gamma_1(x(t),t),$$

$$g_2(x(t-h),t) = B(t)\gamma_2(x(t-h),t),$$

$$||\gamma_1(x(t),t)|| \leq q_1||x(t)|| + p_1,\ q_1, p_1 > 0,$$

$$||\gamma_2(x(t-h),t)|| \leq q_2||x(t-h)|| + p_2,\ q_2, p_2 > 0.$$

The last two conditions provide reasonable restrictions on the growth of the uncertainties.

The following initial conditions are assumed for system (6)

$$x(\theta) = \varphi(\theta), \qquad (9)$$

where $\varphi(\theta)$ is a piecewise continuous function given in the interval $[t_0 - h, t_0]$.

Thus, the control problem now consists in robustification of control design in system (7) with respect to uncertainties g_1, g_2: to find such a control law that the trajectories of system (8) with initial conditions (9) coincide with the trajectories $x_0(t)$ with the same initial conditions (9).

4.1 Design principles

Let us redesign the control law for system (6) in the form

$$u(t) = u_0(x(t),t) + u_1(t), \qquad (10)$$

where $u_0(x(t),t)$ is the ideal feedback control designed for (6), and $u_1(t) \in R^m$ is the relay control generating the integral sliding mode in some auxiliary space to reject uncertainties g_1, g_2. Substitution of the control law (10) into the system (6) yields

$$\dot{x}(t) = f(x(t-h)) + B(t)u_0(x(t),t) + B(t)u_1(t) + \qquad (11)$$

$$g_1(x(t),t)+g_2(x(t-h),t).$$

Define the auxiliary function

$$s(t)=z(t)+s_0(x(t),t), \tag{12}$$

where $s_0(x(t),t)=B^+(t)x(t)$, and $z(t)$ is an auxiliary variable defined below. Then,

$$\dot{s}(t)=\dot{z}(t)+G(t)[f(x(t))+B(t)u_0(x(t),t)+ \tag{13}$$

$$B(\gamma_1(x(t),t))+\gamma_2(x(t-h),t))+B(t)u_1(t)]+(\partial s_0(x(t),t)/\partial t),$$

$G(t)=\partial s_0(x(t),t)/\partial x=B^+(t)$ and $\partial s_0(x(t),t)/\partial t=d(B^+(t))/dt)x(t)$. Note that in the framework of this modified (with respect to [17, 3]) integral sliding mode approach, the optimal control $u_0(x(t))$ is not required to be differentiable and the sliding mode manifold matrix $GB=B^+B=I$ is always invertible.

The philosophy of integral sliding mode control is the following: in order to achieve $x(t)=x_0(t)$ at all $t\in[t_0,\infty)$, the sliding mode should be organized on the surface $s(t)=0$, since the following disturbance compensation should have been obtained in the sliding mode motion

$$B^+(t)B(t)u_{1eq}(t)=-B^+(t)B(t)\gamma_1(x(t),t)-$$

$$B^+(t)B(t)\gamma_2(x(t-h),t),$$

that is

$$u_{1eq}(t)=-\gamma_1(x(t),t)-\gamma_2(x(t-h),t).$$

Note that the equivalent control $u_{1eq}(t)$ can be unambiguously determined from the last equality and the initial condition for $x(t)$.

Define the auxiliary variable $z(t)$ as the solution to the differential equation

$$\dot{z}(t)=-B^+(t)[f(x(t-h))+B(t)u_0(x(t),t)]+d(B^+(t))/dt)x(t),$$

with the initial conditions $z(t_0)=-s_0(t_0)=-B^+(t_0)\varphi(t_0)$. Then, the sliding manifold equation takes the form

$$\dot{s}(t)=B^+(t)[B(t)(\gamma_1(x(t),t))+\gamma_2(x(t-h),t))+B(t)u_1(t)]=$$

$$=\gamma_1(x(t),t)+\gamma_2(x(t-h),t)+u_1(t)=0.$$

Finally, to realize sliding mode, the relay control is designed

$$u_1(t)=-M(x(t),x(t-h),t)sign[s(t)], \tag{14}$$

$$M=q(||x(t)||+||x(t-h)||)+p,$$

$q>q_1,q_2$, $p>p_1+p_2$.

The convergence to and along the sliding mode manifold $s(t)=0$ is assured by the Lyapunov function $V(t)=s^T(t)s(t)/2$ for the system (11) with the control input $u_1(t)$ of (14):

$$\dot{V}(t)=s^T(t)[\gamma_1(x(t),t)+\gamma_2(x(t-h),t)+u_1(t)]\le$$

$$-|s(t)|([q(||x(t)||+||x(t-h)||)+p]+$$

$$[\gamma_1(x(t),t)+\gamma_2(x(t-h),t)])<0,$$

where $|s(t)|=\sum_{i=1}^{m}|s_i(t)|$.

The next subsection presents the robustification of the designed optimal control (3). This robust regulator is designed assigning the sliding mode manifold according to (12)–(13) and subsequently moving to and along this manifold using relay control (14).

4.2 Robust sliding mode control design for linear state delay system

Consider the disturbed linear state delay system (1), whose behavior is affected by uncertainties g_1,g_2 presenting perturbations and nonlinearities in the system

$$\dot{x}(t)=a_0(t)+a(t)x(t-h)+B(t)u(t)+g_1(x(t),t)+g_2(x(t-h),t). \tag{15}$$

It is also assumed that the uncertainties satisfy the standard matching and growth conditions given in the beginning of Section 4, and the quadratic cost function (2) is the same as in Section 2.

The problem is to robustify the obtained optimal control (3), using the method specified by (12)–(13). Define this new control in the form (10): $u(t)=u_0(x(t),t)+u_1(t)$, where the optimal control $u_0(x(t),t)$ coincides with (3) and the robustifying component $u_1(t)$ is obtained according to (14)

$$u_1(t)=-M(x(t),x(t-h),t)sign[s(t)],$$

$$M=q(||x(t)||+||x(t-h)||)+p,$$

$q>q_1,q_2$, $p>p_1+p_2$. Consequently, the sliding mode manifold function $s(t)$ is defined as

$$s(t)=z(t)+s_0(x(t),t), \tag{16}$$

where

$$s_0(x(t),t)=B^+(t)x(t), \tag{17}$$

and the auxiliary variable $z(t)$ satisfies the delay differential equation

$$\dot{z}(t)=-B^+(t)[a_0(t)+a(t)x(t-h)+B(t)u_0(x(t),t)], \tag{18}$$

with the initial conditions $z(t_0)=-B^+(t_0)\varphi(t_0)$.

5 Example

This section presents an example of designing the optimal regulator for a system (1) with a criterion (2), using the scheme (3)–(5), and comparing it to the regulator where the matrix Q is selected as in the optimal linear regulator for a system without delays, disturbing the obtained regulator by a noise, and designing a robust sliding mode compensator for that disturbance, using the scheme (16)–(18).

Consider a scalar linear system

$$\dot{x}(t)=10x(t-0.25)+u(t), \tag{19}$$

with the initial conditions $x(s)=1$ for $s\in[-0.1,0]$. The control problem is to find the control $u(t)$, $t\in[0,T]$, $T=0.5$, that minimizes the criterion

$$J=\frac{1}{2}[\int_0^T u^2(t)dt+\int_0^T x^2(t)dt]. \tag{20}$$

In other words, the control problem is to minimize the overall energy of the state x using the minimal overall energy of control u.

Let us first construct the regulator where the control law and the matrix $Q(t)$ are calculated in the same manner as for the optimal linear regulator for a linear system without delays, that is $u(t) = R^{-1}(t)B^T(t)Q(t)x(t)$ (see [1] for reference). Since $B(t) = 1$ in (19) and $R(t) = 1$ in (20), the optimal control is actually equal to

$$u(t) = Q(t)x(t), \tag{21}$$

where $Q(t)$ satisfies the Riccati equation

$$\dot{Q}(t) = -a^T(t)Q(t) - Q(t)a(t) + L(t) - Q(t)B(t)R^{-1}(t)B^T(t)Q(t),$$

with the terminal condition $Q(T) = \psi$. Since $a(t) = 10$, $B(t) = 1$ in (19), and $L(t) = 1$ and $\psi = 0$ in (20), the last equation turns to

$$\dot{Q}(t) = 1 - 20Q(t) - Q^2(t), \quad Q(0.5) = 0. \tag{22}$$

Upon substituting the control (21) into (19), the controlled system takes the form

$$\dot{x}(t) = 10x(t-0.25) + Q(t)x(t). \tag{23}$$

The results of applying the regulator (21)–(23) to the system (19) are shown in Fig. 1, which presents the graphs of the criterion (20) $J(t)$ and the control (21) $u(t)$ in the interval $[0,T]$. The value of criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 15.94$.

Let us now apply the optimal regulator (3)–(5) for linear states with time delay to the system (19). The control law (3) takes the same form as (21)

$$u^*(t) = Q^*(t)x(t), \tag{24}$$

where $Q^*(t)$ satisfies the equation

$$\dot{Q}^*(t) = 1 - 20Q^*(t)M_1(t) - Q^{*2}(t), \quad Q^*(0.5) = 0, \tag{25}$$

where $M_1(t) = 0$ for $t \in [0, 0.25)$ and $M_1(t) = \exp\left(-\int_{t-0.25}^{t} Q^*(s)ds\right)$ for $t \in [0.25, 0.5]$. Since the solution $Q^*(t)$ of the equation (25) is not smooth, it has been numerically solved with the approximating terminal condition $Q^*(0.5) = 0.04$, in order to avoid chattering.

Upon substituting the control (24) into (19), the optimally controlled system takes the same form as (23)

$$\dot{x}(t) = 10x(t-0.25) + Q^*(t)x(t). \tag{26}$$

The results of applying the regulator (24)–(26) to the system (19) are shown in Fig. 2, which presents the graphs of the criterion (20) $J(t)$ and the control (24) $u^*(t)$ in the interval $[0,T]$. The value of the criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 4.63$. There is a definitive improvement (three and half times) in the values of the criterion to be minimized in comparison to the preceding case, due to the optimality of the regulator (3)–(5) for linear states with time delay.

The next task is to introduce a disturbance into the controlled system (26). This deterministic disturbance is realized as a constant: $g(t) = 100$. The matching conditions are valid, because state $x(t)$ and control $u(t)$ have the same dimension: $dim(x) = dim(u) = 1$. The restrictions on the disturbance growth hold with $q_1 = q_2 = p_2 = 0$ and $p_1 = 100$, since $||g(t)|| = 100$. The disturbed system equation (26) takes the form

$$\dot{x}(t) = 100 + 10x(t-0.25) + Q^*(t)x(t). \tag{27}$$

The system state behavior significantly deteriorates upon introducing the disturbance. Figure 3 presents the graphs of the criterion (20) $J(t)$ and the control (24) $u(t)$ in the interval $[0,T]$. The value of the the criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 398.68$. The deterioration of the criterion value in comparison to that obtained using the optimal regulator (24) is more than 80 times.

Let us finally design the robust integral sliding mode control compensating for the introduced disturbance. The new controlled state equation should be

$$\dot{x}(t) = 100 + 10x(t-0.25) + Q^*(t)x(t) + u_1(t), \tag{28}$$

where the compensator $u_1(t)$ is obtained according to (14)

$$u_1(t) = -M(x(t), x(t-h), t)sign[s(t)], \tag{29}$$

and $M = 100.4 > p_1 = 100$. The sliding mode manifold $s(t)$ is defined by (21)

$$s(t) = z(t) + s_0(x(t), t),$$

where

$$s_0(x(t), t) = B^+(t)x(t) = x(t),$$

and the auxiliary variable $z(t)$ satisfies the delay differential equation

$$\dot{z}(t) = -B^+(t)[10x(t-0.25) + u_0(t)] =$$

$$-[10x(t-0.25) + Q^*(t)x(t)],$$

with the initial conditions $z(0) = -x(0) = -1$.

Upon introducing the compensator (29) into the state equation (28), the system state behavior is much improved. Figure 4 presents the graphs of the criterion (20) $J(t)$ and the control (24) $u(t)$, after applying the compensator (29), in the interval $[0,T]$. The value of the criterion (20) at the final moment $T = 0.5$ is $J(0.5) = 4.64$. Thus, the criterion value after applying the compensator (29) is only slightly different from the criterion value given by the optimal regulator (24)–(25) for linear state delay systems.

6 Appendix

Proof of the optimal control problem solution. Define the Hamiltonian function [10, 11] for the optimal control problem (1),(2) as

$$H(x,u,q,t) = \frac{1}{2}(u^T R(t)u + x^T L(t)x) + q^T[a_0(t) + a(t)x_1 + B(t)u], \tag{30}$$

where $x_1(x) = x(t-h)$. Applying the maximum principle condition $\partial H/\partial u = 0$ to this specific Hamiltonian function (30) yields

$$\partial H/\partial u = 0 \Rightarrow R(t)u(t) + B^T(t)q(t) = 0,$$

and the optimal control law is obtained as

$$u^*(t) = -R^{-1}(t)B^T(t)q(t).$$

Taking linearity and causality of the problem into account, let us seek $q(t)$ as a linear function in $x(t)$

$$q(t) = -Q(t)x(t), \tag{31}$$

where $Q(t)$ is a square symmetric matrix of dimension n. This yields the complete form of the optimal control

$$u^*(t) = R^{-1}(t)B^T(t)Q(t)x(t). \tag{32}$$

Note that the transversality condition [10, 11] for $q(T)$ implies that $q(T) = -\partial J/\partial x(T) = -\psi x(T)$ and, therefore, $Q(T) = \psi$.

Using the co-state equation $dq(t)/dt = -\partial H/\partial x$ and denoting $(\partial x_1(t)/\partial x) = M_1(t)$ yields

$$-dq(t)/dt = L(t)x(t) + a^T(t)M_1^T(t)q(t), \qquad (33)$$

and substituting (31) into (33), we obtain

$$\dot{Q}(t)x(t) + Q(t)d(x(t))/dt = L(t)x(t) - a^T(t)M_1^T(t)Q(t)x(t). \qquad (34)$$

Substituting the expression for $\dot{x}(t)$ from the state equation (1) into (34) yields

$$\dot{Q}(t)x(t) + Q(t)a(t)x(t-h) + Q(t)B(t)u(t) = \qquad (35)$$

$$L(t)x(t) - a^T(t)M_1^T(t)Q(t)x(t). \qquad (35)$$

In view of linearity of the problem, differentiating the last expression in x does not imply loss of generality. Upon substituting the optimal control law (32) into (35), taking into account that $(\partial x(t-h)/\partial x(t)) = M_1(t)$, and differentiating the equation (35) in x, it is transformed into the quasi-Riccati equation

$$\dot{Q}(t) = L(t) - Q(t)M_1(t)a(t) - a^T(t)M_1^T(t)Q(t) - \qquad (36)$$

$$Q(t)B(t)R^{-1}(t)B^T(t)Q(t).$$

with the terminal condition $Q(T) = \psi$.

Let us now obtain the value of $M_1(t)$. By definition, $M_1(t) = (\partial x(t-h)/\partial x(t))$. Substituting the optimal control law (32) into the equation (1) gives

$$\dot{x}(t) = a_0(t) + a(t)x(t-h) + B(t)R^{-1}(t)B^T(t)Q(t)x(t), \quad (37)$$

with the initial condition $x(s) = \phi(s)$, $s \in [t_0 - h, t_0]$. Integrating (37) yields

$$x(t_0 + h) = x(t_0) + \int_{t_0}^{t_0+h} (a_0(s) + a(s)x(s-h))ds + \qquad (38)$$

$$\int_{t_0}^{t_0+h} B(s)R^{-1}(s)B^T(s)Q(s)x(s)ds.$$

Analysis of the formula (38) shows that $x(t)$ does not depend on $x(t-h)$, if $t \in [t_0, t_0 + h)$. Therefore, $M_1(t) = 0$ for $t \in [t_0, t_0 + h)$. On the other hand, if $t \geq t_0 + h$, the following Cauchy formula is valid for the solution $x(t)$ of the equation (37)

$$x(t) = \Phi(t, t-h)x(t-h) + \int_{t-h}^{t} \Phi(t,s)(a_0(s) + a(s)x(s-h))ds, \qquad (39)$$

where $\Phi(t, \tau)$ satisfies the matrix equation

$$\frac{d\Phi(t,\tau)}{dt} = B(t)R^{-1}(t)B^T(t)Q(t)\Phi(t,\tau),$$

with the initial condition $\Phi(t,t) = I$, and I is the identity matrix. The expression (39) immediately implies that $M_1(t) = \Phi^{-1}(t, t-h) = \Phi(t-h, t) = \exp(-\int_{t-h}^{t} B(s)R^{-1}(s)B^T(s)Q(s)ds)$ for $t \geq t_0 + h$. The optimal control problem solution is proved.

References

[1] H. Kwakernaak, R. Sivan, Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.

[2] W. H. Fleming, R. W. Rishel, Deterministic and Stochastic Optimal Control, Springer, New York, 1975.

[3] M. V. Basin, L. M. Fridman, P. Acosta, J. Rodriguez-Gonzalez, Robust integral sliding mode regulator for linear systems with time delay in control input, in Proc. American Control Conference 2003 (Denver, CO, June 4–6, 2003), 2003, pp. 2138-2143.

[4] M. V. Basin, L. M. Fridman, J. Rodriguez-Gonzalez, P. Acosta, Optimal and robust sliding mode control for linear systems with multiple time delays in control input, Asian Journal of Control, 5 (2003) 557-567.

[5] M. Malek-Zavarei, M. Jamshidi, Time-Delay Systems: Analysis, Optimization and Applications, North-Holland, Amsterdam, 1987.

[6] V. B. Kolmanovskii, L. E. Shaikhet, Control of Systems with Aftereffect, American Mathematical Society, Providence, 1996.

[7] V.B. Kolmanovskii, A.D. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer, New York, 1999.

[8] M. S. Mahmoud, Robust Control and Filtering for Time-Delay Systems, Marcel Dekker, New York, 2000.

[9] E.-K. Boukas, Z.-K. Liu, Deterministic and Stochastic Time-Delay Systems, Birkhauser, Boston, 2002.

[10] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1962.

[11] G. L. Kharatashvili, A maximum principle in external problems with delays, In Mathematical Theory of Control (Eds. A. V. Balakrishnan and L. W. Neustadt), Academic Press, New York, 1967.

[12] M. N. Oguztoreli, Time-Lag Control Systems, Academic Press, New York, 1966.

[13] D. H. Eller, J. K. Aggarwal, H. T. Banks, Optimal control of linear time-delay systems, IEEE Trans. Aut. Contr., AC-14 (1969) 678-687.

[14] Y. Alekal, P. Brunovsky, D. H. Chyung, E. B. Lee, The quadratic problem for systems with time delays. IEEE Trans. Automat. Contr., AC-16 (1971) 673-687.

[15] M. C. Delfour, The linear quadratic control problem with delays in space and control variables: a state space approach, SIAM. J. Contr. Optim., 24 (1986) 835–883.

[16] K. Uchida, E. Shimemura, T. Kubo, N. Abe, The linear-quadratic optimal control approach to feedback control design for systems with delay, Automatica, 24 (1988) 773-780.

[17] V.I. Utkin, J. Guldner, J. Shi, Sliding Mode Control in Electromechanical Systems, Taylor and Francis, London, 1999.

[18] W.-J. Cao, J.-X. Xu, Nonlinear integral type sliding surfaces for both matched and unmatched uncertain systems, in Proc. American Control Conference 2001 (Arlington, VA, June 2001), 2001, pp. 4369-4374.

[19] Y. B. Shtessel, A. S. I. Zinober, I. Shkolnikov, Sliding mode control for nonlinear systems with output delay via method of stable system center, ASME Trans. J. Dyn. Syst. Measur. Contr., 25 (2003), 253-257.

[20] A. S. Poznyak, Y. B. Shtessel, C. J. Gallegos, Min-max sliding-mode control for multimodel linear time varying systems, IEEE Trans. Automat. Contr., 48 (2003), 2141-2150.

[21] A. F. Filippov, Differential Equations with Discontinuous Right-Hand Sides, Kluwer, New York, 1989.

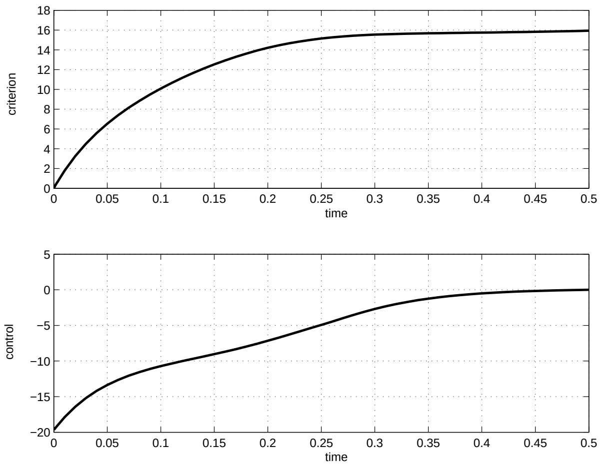

Figure 1: Best linear regulator available for linear systems without state delay. Graphs of the criterion (20) $J(t)$ and the control (21) $u(t)$ in the interval $[0, 0.5]$.

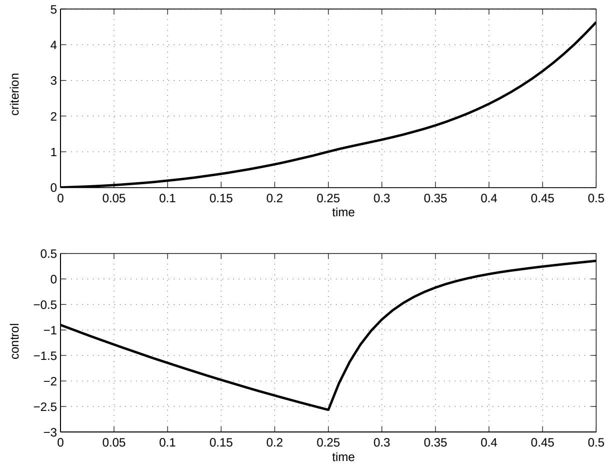

Figure 2: Optimal regulator obtained for linear systems with state delay. Graphs of the criterion (20) $J(t)$, and the optimal control (24) $u^*(t)$ in the interval $[0, 0.5]$.

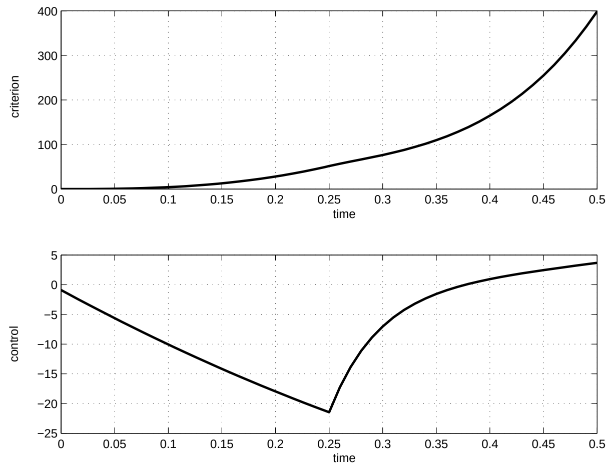

Figure 3: Controlled system in the presence of disturbance. Graphs of the criterion (20) $J(t)$ and the control (24) $u(t)$ in the interval $[0, 0.5]$.

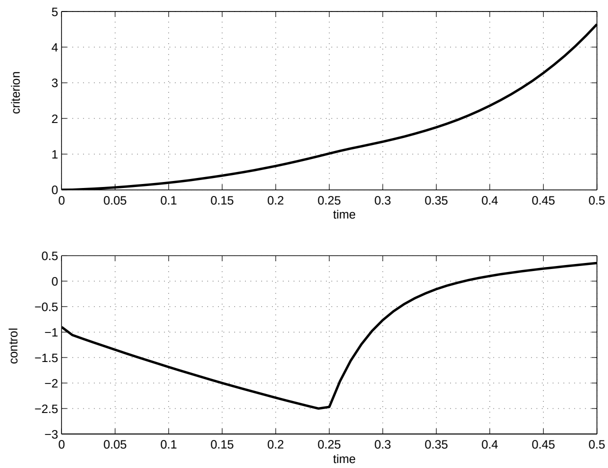

Figure 4: Controlled system after applying robust integral sliding mode compensator. Graphs of the criterion (20) $J(t)$ and the control (24) $u(t)$ in the interval $[0, 0.5]$.