ON SMALL CARNOT-CARATHÉODORY SPHERES

Yu. Baryshnikov

EURANDOM, Eindhoven University of Technology, Eindhoven, The Netherlands

Abstract. The sphericity of small cc-spheres is proved for homogeneous and 2-generating distributions.

0. Introduction

0.1. The Carnot-Carathéodory or sub-Riemannian or non-holonomic geometry deals with the (length) metrics associated to the pairs (totally nonintegrable distribution, norm on the distribution plane) and is prominent in applications such as control theory or mechanics. An overview of the theory can be found e.g. in [**G, GV**].

The zero-order properties of this metric (like estimates of ball shapes and volumes) are quite intuitive and easy to recover. The related differential-geometric properties are, on the contrary, rather intricate. Thus it seems to be unknown (though widely believed) that small spheres in Carnot-Carathéodory geometry are homeomorphic to the sphere. The standard in Riemannian geometry proof via the exponential map does not work as it is not a diffeomorphism in any vicinity of the origin for cc-geometry.

0.2. In this note I show that the cc-spheres are spheres indeed, at least when either the *distribution* admits a 1-parametric group of contractions which leaves it invariant, or when the distribution is *of length 2* (this is the generic case when the inequality $N \leq n(n+1)/2$ relating the dimensions of the space and of the distribution holds).

The method is to replace the contraction along the rays in the tangent space by the contracting along the trajectories of a self-similarity flow in the space itself. The homogeneity thus imposed is by no way canonic but is sufficient to shorten curves.

0.3. Acknowledgements. I am thankful to R. Montgomery (who told me about this problem) for informative discussion, to M. Gromov for support and to IHES (where this work was begun) for hospitality.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Setup and main results

1.1. Let $U \ni o$ be a vicinity of the origin in the Euclidean space $V \cong \mathbb{R}^N$ (we identify V with the tangent space to U at o). Consider a totally nonintegrable distribution H, that is a (locally) trivial subbundle of the tangent bundle such that the sections of this subbundle generate the whole Lie algebra of vector fields on U. The sheafs generate by the k-fold commutators of tangent vector fields (denoted as $H_k, k \geq 1$) form a filtration of the tangent sheaf. The growth vector (at x) is the vector of dimensions of the associated graded vector space $GV_x = \oplus_j H_{j+1,x}/H_{j,x}$. A projection p to V_1 gives a local trivialization $p_x^* : H_x \to T_{px}V_1 \cong V_1$. The plane $T_oH = V_1 \subset V$ will be called *horizontal*.

1.2. The (piece-wise) differentiable curves tangent to H at their smooth points will be called *horizontal*. If the fibers of p are transversal to H (which will be assumed to be the case in U), then one can lift any piece-wise smooth curve γ in the horizontal plane V_1 to a horizontal curve $L\gamma$ (the *lift of* γ) with the left endpoint at the origin.

1.3. Assume that a norm on H is given, that is a (smooth) family of fiber-wise norms $g(x, \cdot) : H_x \to \mathbb{R}$. The situation when g is fiber-wise Euclidean is often referred to as sub-Riemannian.

The length of a horizontal curve $\gamma : [0, 1] \to U$ given by $l(\gamma) = \int_0^1 g(\gamma(s), \gamma'(s))\, dt$ is reparametrization invariant; henceforth it will be assumed that all horizontal curves are parameterized by the unit interval. The corresponding length distance is called *Carnot-Carathéodory* (or *cc-*) distance. The *cc-ball* of radius r with the center at the origin (that is the set of endpoints of horizontal curves of length at most r starting at the origin) contains an open vicinity of the origin (Rashevsky-Chow lemma). The topological boundary of the cc-ball is called the *cc-sphere*.

1.4. For a *regular* distribution, that is distribution with a constant growth vector, there exists a limiting nilpotent graded Lie algebra approximating it (cf [**GV**]). The standard construction is the following: take any splitting $V = \oplus_{i\geq 1} V_i$ such that the flag $\{F_j = \oplus_{i\geq j} V_i\}$ coincides with the flag $\{H_j\}_o$ and consider the family of dilatations $\delta_\lambda, 0 < \lambda \leq 1$ equal to λ^{-k} on V_k. The family of distributions $H(\lambda) = \delta_\lambda^* H$ extends smoothly to $\lambda = 0$.

Lifts (using p) of the constant vector fields on V_1 to $H(0)$ are vector fields which generate a graded nilpotent Lie algebra $L(H)$ (isomorphic to GV as a graded vector space) generated by V_1.

Properly rescaled, the norm g converges to the $L(H)$-left invariant norm on $H(0)$ (again, just the lift to $H(0)$ of the g-norm on V_1). The dilatation flow δ_λ leaves $H(0)$ invariant and multiplies the length element by λ. It follows that the cc-distance is degree 1 homogeneous with respect to δ and therefore the cc-sphere is homeomorphic to S^{N-1}.

1.5. Below we consider two situations where this sphericity persists. One is the case when there exists a (globally contracting) flow which preserves the distribution but does not necessarily act conformally on the norm, merely contracts it. This case covers, e.g. all contact distributions (regardless of the norm).

Such distributions are rarely generic: the germs of distributions diffeomorphic to a homogeneous one (with respect to the Euler vector field) have infinite codimension in the relevant functional space for all dimension except for $n = N - 1$ and $n = 2, N = 4$ (again, see **[GV]**). As a partial remedy to this we prove that small perturbations of distributions with 2-step nilpotent limit (that is the distributions with the growth vector (n, N)) preserve the sphericity. This is the main result of this note:

Theorem. *Small cc-spheres for distributions of length 2 are homeomorphic to spheres.*

2. Self-similar distributions

2.1 Notations and assumptions. Denote the Euclidean norm on V by $|\cdot|$ and assume the constant dimension of the H (but not necessarily the regularity).

We use the trivialization $p : U \times H_o \to H$ to fix the coordinates $(x, h), x \in U, h \in H_o$ on H.

We assume the (Finsler) metric defined by $g : H \to \mathbb{R}$ to be C^1 (actually, Lipschitz is enough). More precisely, the following estimate will be used:

A. $$|(\partial_x g \cdot v)\,(x, h)| \le A|v|\,|h|, v \in T_xU, h \in H_o$$

for some positive A (here ∂_x is the partial derivative with respect to x).

The length of a curve in the standard (Euclidean) metric will be denoted as l_e. Obviously, for a horizontal curve, $C^{-1}l_e \le l \le Cl_e$ for some positive C.

Let v be a vector field defined on U with the unique equilibrium point at the origin. We assume that the shift along v takes U into itself and that the 1-parameter (multiplicative) semigroup $\delta_\lambda = \lambda^v, 1 \le \lambda$ generated by v satisfies the following properties:

B. The diffeomorphisms δ_λ preserve the distribution H.

We denote by Δ_λ the induced (fiberwise linear) transfromation of H, $\Delta_\lambda : (x, h) \mapsto (\delta_\lambda x, (\delta_\lambda)_* h)$. Its derivative with respect to λ at $\lambda = 1$ defines an endomorphism of H denoted by $D = \frac{\partial}{\partial\lambda}\big|_{\lambda=1}\Delta_\lambda$.

The linear vectorfield $\nu = D(o, \cdot)$ on H_o is the restriction of the linearization of v at the origin to H_o.

C. The norm $g(o, \cdot)$ is a strict Lyapunov function for ν, i.e.

$$L_\nu g(o, \cdot) < -a|\cdot|$$

for some positive a. By continuity this inequality is valid also in some vicinity of o.

2.2. The crucial fact is that given these three properties, A, B and C, the flow δ_t *shortens short horizontal curves.*

Lemma. *Assuming* **A**, **B**, **C** *above,*

$$\frac{\partial}{\partial\lambda}\Big|_{\lambda=1} l(\delta_\lambda\gamma) \leq -\alpha l(\gamma)$$

for some positive α and short enough horizontal curve γ.

Proof. Here is the calculation. The derivative above is given by

$$\int_0^1 \left(\partial_x g[\gamma(s),\gamma'(s)]\cdot v[\gamma(s)] + \partial_h g[\gamma(s),\gamma'(s)]\cdot D[\gamma(s),\gamma'(s)]\right) \mathrm{d}s.$$

(Here ∂_x, ∂_h are the partial derivatives with respect to x, h correspondingly).

Estimating the first summand in absolute value using A and the second summand using **C** (and using $|v(x)| \leq B|x|$ in U), we arrive at

$$\text{(S)} \qquad \frac{\partial}{\partial\lambda}\Big|_{\lambda=1} l\delta_\lambda\gamma \leq \int_0^1 \left(ABl_e(\gamma)|\gamma'(s)| - a|\gamma'(s)|\right) \mathrm{d}s$$

which is less than $-(a/2)C^{-1}l(\gamma)$ if $l(\gamma)$ is small enough. □

2.3. Now the last assumption on v:

D. The origin is Lyapunov stable equilibrium point for v.

Proposition. *Let H is totally nonintegrable at the origin and g, v satisfy the conditions* **A**–**D** *above. Then for r small enough, the cc-spheres of radius r are homeomorphic to the sphere S^{n-1}.*

Proof. As the origin is Lyapunov stable, one can choose some small ellipsoid S such that each (nonconstant) trajectory of v intersects S just once and transversally. We want to show that each trajectory of v intersects the cc-sphere of small enough radius $r > 0$ just once and that the intersection point depends on the trajectory continuously. This, clearly, will prove the claim.

Let γ be a horizontal curve with endpoints o, x. Consider he family of curves $\delta_\lambda\gamma$. The right endpoint of $\delta_\lambda\gamma$ runs along a trajectory of the vector field v.

The estimate of Lemma 2.2 shows that the cc-lengths of $\delta_\lambda\gamma$ decreases as λ grows.

As the estimate (S) is independent of the curve γ, it follows that the cc-metric is a strict Lyapunov function for v, which implies the result we need. □

2.4. An immediate corollary of this result is the sphericity of the cc-spheres for the contact distributions (and arbitrary sub-Finsler metric). Indeed, by Darboux' theorem, all such distributions are diffeomorphic, locally, to the distribution $ann\{dz + xdy\}$ for which the Euler vector field $v = -x\partial/\partial x - y\partial/\partial y - 2z\partial/\partial z$ satisfies the assumptions of the Proposition 2.3.

More generally, for a graded-nilpotent finite-dimensional Lie algebra $L = \oplus_i L_i$ (meaning that $[L_i, L_j] \subset L_{i+j}$ and L is generated by L_1), the totally non-holonomic distribution of right translates of L_1 on the corresponding Lie group and the Euler vector field $v = -\sum w(q)x_q\partial/\partial x_q$ (where the weight $w(q) = i$ if $e_q \in L_i$) satisfy the conditions of Proposition 2.3.

3. Distributions of length 2

3.1. As mentioned above, situations when there exists a contracting flow leaving the distribution invariant are rare. One might hope that in the regular case, the condition of transversality of the Euler vector field to the cc-spheres satisfied in the limiting graded nilpotent Lie case survives small perturbations. While in general certainly false, this holds for the distributions of length 2. Recall that H is *of length 2* (at o) if the growth vector is (n, N) in a vicinity of o.

3.2. We consider a smooth deformation $H(z)$ of the distributions $H(0)$, where $H(0)$ is left invariant on a 2-step graded nilpotent algebra $V = V_1 \oplus V_2$, $H_o(0) = v_1$. We consider only small deformation parameters z so that all $H(z)$'s are of length 2. Without restriction of generality assume that $H_o(0) = V_1$ for all z.

Also we take $g(z)$ to be the deformation of the left-invariant norm on $H(0)$. In this section we slightly strengthen our assumption on g requiring its level surface in fibers to be strictly convex (or, equivalently, the level surface of the associated Hamiltonians to be smooth). For simplicity sake assume also that all norms $g(z)$ agree on H_o.

3.3. The Hamiltonian $h : T^*U \to \mathbb{R}$ for a non-holonomic normed distribution (H, g) is defined as

$$h(z)[x,p] = \sup_{h \in H_x} ph - g(x,h).$$

Denote the 1-time shift along the Hamiltonian flow corresponding to $(H(z), g(z))$ as $\Gamma(z)$ and the level set of the Hamiltonian $h(z)$ in T_o^*U (topologically a cylinder) as C. The (time 1) wave front $W(z) \in U$ is just the projection to U of $\Gamma(z)C$, a (singular) hypersurface.

3.4. Two facts about distributions of length 2 will be crucial for us:

E. The cc-sphere of radius 1 is a subset of the wave front. Moreover, it belongs to the image of a compact subset of the cylinder C. This compact subset C_c can be chosen large enough to suffice for all z close enough to 0. These results are proved in [**AS**].

F. For some $c > 0$, the (Euclidean) $c\epsilon$-neighborhood of the cc-sphere of radius 1 belongs to the cc-ball of radius $1 + \epsilon$, for ϵ small enough. This is, probably, also deducible from the results of [**AS**], but is easy to prove anyway, by considering appropriate variations.

3.5. Let $\delta_\lambda, 1 \leq \lambda$ be the 1-parameter semigroup of homogeneous contractions of V given by $\delta_\lambda : (v_1, v_2) \mapsto (\lambda^{-1}v_1, \lambda^{-2}v_2)$, $v_i \in V_i$. Let γ be a horizontal curve. Project it to V_1, contract there (by δ_λ) and lift back to a horizontal curve $L\delta_\lambda p\gamma$. This defines a deformation of γ parameterized by λ. The right ends of the curves in the deformation run along a curve $c_\gamma(\lambda) \in V$ (projecting to the straight segment in V_1). Call $\frac{\partial}{\partial\lambda}|_{\lambda=1} c_\gamma(\lambda)$ the *standard variation* vector. Clearly, for a homogeneous distribution H, the standard variation is just the Euler vector field at the endpoint, independent of the curve.

3.6. For $p \in C$ consider the 1-time trajectory of the Hamilton flow with Hamiltonian $h(z)$ starting at (o,p) and denote its projection to U as $\gamma_p(z)$, its right endpoint as $x_p(z)$ and the standard variation vector as $v_p(z) \in T_{x_p(z)}U$.

Let $\psi(z,p,\lambda)$ be the cc-length (with respect to $g(z)$) of $L\delta_\lambda p\gamma_p(z)$ (lift with respect to $H(z)$). One has $\psi(z,p,1) \equiv 1$ and $\psi(0,p,\lambda) \equiv \lambda$. As ψ is smooth, it follows that there exists $a > 0$ such that for z close enough to 0,

$$(S') \qquad \psi(z,p,\lambda) \leq 1 + a(\lambda - 1) \text{ for all } p \in C_c.$$

Restrict z to the vicinity of 0 where (S') holds.

3.7. Let $S(z)$ be the cc-sphere of radius 1 for the cc-pair $(H(z), g(z))$. By property **E**, there exists $p \in C_c$ such that $\gamma_p(z)$ has the cc-length 1. By (S'), the cc-distance to zero decreases at least linearly along $c_{\gamma_p(z)}$. By property **F**, the same is valid for all curves starting at $x_p(z)$ with tangent vectors within $c \times |v_p(z)|$-distance to the standard variation vector $v_p(z)$. The vectors $v_p(z)$ depend smoothly on p, z and $v_p(0)$ is just the Euler vector at $x_p(0)$. Hence, for all z close enough to 0, the Euler vector at $x_p(z)$ is within $c \times |v_p(z)|$-distance to $v_p(z)$, for all $p \in C_c$. Therefore the cc-distance to zero, with respect to $(H(z), g(z))$ is the Lyapunov function for the Euler vector field which implies the main result:

Proposition 3.8. *Small cc-spheres for distributions of length 2 are homeomorphic to spheres.*

□

References

[AS] A. Agrachev. A. Sarychev *Sub-Riemannian Metrics: Minimality of Abnormal Geodesics vs. Subanalyticity.* Preprint, 1998, available electronically at `http://www.botik.ru/~duzhin/as-papers/`.

[G] M. Gromov. Carnot-Carathodory spaces seen from within. *In: Sub-Riemannian geometry*, 79–323, Progr. Math., **144**, Birkhäuser, Basel, 1996.

[GV] V. Gershkovich, A. Vershik. Nonholonomic manifolds and nilpotent analysis. *In: Geometry and Physics, S. Gindikin and I.M. Singer, eds.*, 407-452, North Holland, Amsterdam, 1991.

[M] R. Montgomery. Generic distributions and Lie algebras of vector fields. *J. Differential Equations*, **103** (1993), no. 2, 387–393.