NONSMOOTH OPTIMAL REGULATION AND DISCONTINUOUS STABILIZATION

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Abstract

For affine control systems, we study the relationship between an optimal regulation problem on the infinite horizon and stabilizability. We are interested in the case the value function of the optimal regulation problem is not smooth and feedback laws involved in stabilizability may be discontinuous.

Keywords: optimization, infinite horizon, nonlinear systems, discontinuous stabilization.

1 Introduction

We are interested in the relationship between an optimal regulation problem on the infinite horizon and the stabilization problem for systems affine in the control. This relationship is very well understood in the case of the quadratic regulator for linear systems, where the value function turns out to be quadratic (see, for example, [2, 15, 25], and [10] for infinite dimensional systems). The generalization of the linear framework to nonlinear affine systems has been studied in the case the value function of the optimal regulation problem is at least C^1 (see [26, 22, 8, 29, 23]). The main purpose of this paper is to relax this regularity assumption: more precisely, we assume that the value function is locally Lipschitz continuous. In particular, we study to what extent and in what sense solvability of the optimal regulation problem still implies stabilizability. Essential tools for our extension are nonsmooth analysis (especially, the notion of viscosity solution and Clarke gradient) and the theory of differential equations with discontinuous righthand side. Viscosity solutions are used in [20, 21], in order to obtain stabilizability via optimal regulation. However, in [20, 21] the author limits himself to homogeneous systems. When the value function is of class C^1, stabilization via optimal regulation guarantees robustness and stability margin for the control law (to this respect, see [19, 33] and especially the book [29]). The robustness issue is not addressed in the present paper; however, our results indicate that such a development may be possible even in the nonsmooth case.

Let us mention that a very preliminary study of this subject was already performed in [6]. We now describe more precisely the two problems we deal with.

1.1 Feedback stabilization

We consider a system of the form

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, the vector fields $f : \mathbf{R}^n \to \mathbf{R}^n$, $g_i : \mathbf{R}^n \to \mathbf{R}^n$, $i = 1, ..., m$, are of class C^1 and G is the matrix whose columns are $g_1, ..., g_m$. For most of the paper, as admissible inputs we consider piecewise continuous and right continuous functions $u : \mathbf{R} \to \mathbf{R}^m$. We denote by $\mathcal{U}$ the set of admissible inputs and by $\varphi(t; x, u(\cdot))$ the solution of equation (1) corresponding to a fixed control law $u(\cdot) \in \mathcal{U}$ and such that $\varphi(0; x, u(\cdot)) = x$. Let us remark that for every admissible input and every initial condition there exists a Carathéodory solution which is unique. We require that all such solutions are right continuable on $[0, +\infty)$.

We say that system (1) is (globally) *stabilizable* if there exists a map $u = k(x) : \mathbf{R}^n \to \mathbf{R}^m$, called a *feedback law*, such that for the closed loop system

$$\dot{x} = f(x) + G(x)k(x) \tag{2}$$

the following properties hold:

(i) (Lyapunov stability) for all $\epsilon > 0$ there exists $\delta > 0$ such that for each solution $\varphi(\cdot)$ of (2), $|\varphi(0)| < \delta$ implies $|\varphi(t)| < \epsilon$ for all $t \geq 0$,

(ii) (attractivity) for each solution $\varphi(t)$ of (2) one has $\lim_{t \to +\infty} \varphi(t) = 0$.

It is well known that the class of continuous feedbacks is not sufficiently large in order to solve general stabilization problems (see [3, 9, 32]). For this reason in the following we also consider discontinuous feedbacks. Of course, the introduction of discontinuous feedback laws leads to the theoretical problem of defining solutions of the differential equation (2) whose righthand side is discontinuous. In the following we consider Carathéodory and Filippov solutions (see [17] for the definitions of both kinds of solutions). Thus we say that system (1) is either *Carathéodory* or *Filippov stabilizable* according to the fact that we consider either Carathéodory or Filippov solutions of the closed-loop system (2).

1.2 The optimal regulation problem

We associate to system (1) the *cost functional*

$$J(x, u(\cdot)) = \frac{1}{2} \int_0^{+\infty} \left(h(\varphi(t; x, u(\cdot))) + \frac{|u(t)|^2}{\gamma} \right) dt \tag{3}$$

where $h : \mathbf{R}^n \to \mathbf{R}$ is a continuous, radially unbounded function with $h(x) \geq 0$ for all x and $\gamma \in \mathbf{R}^+$. Radially unboundedness means that $\lim_{|x| \to \infty} h(x) = +\infty$; such a property is needed in order to achieve global results, and can be neglected if one is only interested in a local treatment. Occasionally, we will also require that h is *positive definite* i.e., $h(0) = 0$ and $h(x) > 0$ if $x \neq 0$.

We are interested in the problem of minimizing the functional J for every initial condition x. The *value function* $V : \mathbf{R}^n \to \mathbf{R}$ associated to the minimization problem is

$$V(x) = \inf_{u \in \mathcal{U}} J(x, u(\cdot)).$$

We say that *the optimal regulation problem is solvable* if for every x the infimum in the definition of V is actually a minimum. If this is the case, we denote by $u_x^*(\cdot)$ an optimal open-loop control corresponding to the initial condition x; we also write $\varphi_x^*(\cdot)$ instead of $\varphi(t; x, u_x^*(\cdot))$.

In the classical approach, it is usual to assume that the value function is of class C^1. Under this assumption, the following statement is well known: a system for which the optimal regulation problem is solvable can be stabilized by means of a feedback in the so-called *damping form*

$$u = k_\alpha(x) = -\alpha (\nabla V(x) G(x))^{\mathbf{t}} \tag{4}$$

provided that α is a sufficiently large positive real constant. As already mentioned, in this paper we are interested in the case the value function is merely locally Lipschitz continuous. This case is particularly interesting because it is known that if h is locally Lipschitz continuous and if certain restrictive assumptions about the right hand side of (1) are fulfilled, then the value function is locally Lipschitz continuous (see [16]).

1.3 Plan of the paper and description of the results

In Section 2 we generalize the classical necessary conditions which must be fulfilled by optimal controls and by the value function of an optimal regulation problem. We also provide an expression for an optimal control which is reminiscent of the feedback form (4).

The results concerning stabilization are presented in Sections 3 and 4. By combining some well known results about stabilization of asymptotically controllable systems, with the characterizations of optimal controls given in Section 2, in Section 3 we first prove that solvability of the optimal regulation problem implies Carathéodory stabilizability. Then, by assuming that the value function is C-regular, we prove that the solvability of the optimal regulation problem also implies Filippov stabilizability. Unfortunately, by this way we are not able to recover any explicit form of the feedback law. We are so led to directly investigate the stabilizing properties of the feedback (4). To this respect, we prove two theorems in Section 4. Both of them apply when the value function is nonpathological

(in the sense introduced by Valadier in [34]). The first one makes use of a strong condition, actually implying that (4) is continuous. The second theorem is more general, but requires an additional assumption.

In Section 5 we finally prove a nonsmooth version of the optimality principle (see [22, 8, 29]). It turns out to be useful in the analysis of the illustrative examples presented in Section 6. Particularly interesting are Examples 4 and 5, which enlightens some intriguing features of the problem.

Two appendices conclude the paper. In Appendix A we collect some tools of nonsmooth analysis used throughout the paper. These include a new characterization of Clarke regular functions and the proof that semiconcave functions are nonpathological. The proofs of all the results of the present paper are based on several Lemmas, which are stated and proved in Appendix B.

2 Necessary conditions for optimality

It is well known that when the value function is of class C^1, a necessary (as well as sufficient) condition for optimality can be given in terms of a partial differential equation of the Hamilton-Jacobi type. Moreover, optimal controls admit a representation in the feedback form (4), with $\alpha = \gamma$ (see for instance [31]). The aim of this section is to prove analogous results for the case the value function is locally Lipschitz continuous. The optimal regulation problem (3) is naturally associated with the pre-Hamiltonian function

$$\mathcal{H}(x,p,u) = -p \cdot \big(f(x) + G(x)u\big) - \frac{h(x)}{2} - \frac{|u|^2}{2\gamma} \ . \tag{5}$$

For each x and p, $u \mapsto \mathcal{H}(x,p,u)$ is strictly concave. By completing the square, we easily obtain the following expression for the Hamiltonian function

$$H(x,p) \underset{\text{def}}{=} \max_u \mathcal{H}(x,p,u) = \mathcal{H}(x,p,-\gamma(pG(x))^{\mathbf{t}}) = -pf(x) + \frac{\gamma}{2}|pG(x)|^2 - \frac{h(x)}{2} \ . \tag{6}$$

The achievements of this sections are presented in the following Propositions. Comments and remarks are inserted in order to relate our conclusions to the existing literature. The proofs are essentially based on the Dynamic Programming Principle (see [7, 31]), and some Lemmas established in Appendix B; we also exploit certain tools of nonsmooth analysis (see Appendix A for notations and definitions).

Proposition 1 *Assume that the optimal regulation problem is solvable and that the value function $V(x)$ is locally Lipschitz continuous. Let $x \in \mathbf{R}^n$ be fixed. Let $u_x^*(\cdot)$ be an optimal control for x and let $\varphi_x^*(\cdot)$ be the corresponding optimal solution. Then for all $t \geq 0$ there exists $p_0(t) \in \partial_C V(\varphi_x^*(t))$ such that*

(i) $H(\varphi_x^*(t), p_0(t)) = 0$

(ii) $u_x^*(t) = -\gamma(p_0(t)G(\varphi_x^*(t)))^{\mathbf{t}}$.

Proof Lemma 1 and Lemma 2 imply that

$$\forall x \in \mathbf{R}^n, \forall t \geq 0, \exists u_0(t) \in \mathbf{R}^m, \exists p_0(t) \in \partial_C V(\varphi_x^*(t)) \ \text{ such that } \ \mathcal{H}(\varphi_x^*(t), p_0(t), u_0(t)) = 0 \ .$$

On the other hand by Lemma 3, $\mathcal{H}(\varphi_x^*(t), p_0(t), u) \leq 0$ for each $u \in \mathbf{R}^m$. Recalling the definition of H, (i) and (ii) are immediatly obtained. □

Remark 1 Under the assumptions of Proposition 1, we also have

$$\forall x \in \mathbf{R}^n \ \ \exists p_0 \in \partial_C V(x) \quad \text{ such that } \quad H(x,p_0) = 0 \ . \tag{7}$$

This follows from statement (i), setting $t = 0$. □

Proposition 1 is a necessary condition for an open loop control being optimal. In particular, (ii) provides the analogue of the usual feedback form representation of optimal controls. Next Proposition gives necessary conditions for $V(x)$ being the value function of the optimal regulation problem.

Proposition 2 *Given the optimal regulation problem (3), assume that the value function $V(x)$ is locally Lipschitz continuous. Then,*

(i) for each $x \in \mathbf{R}^n$ and for each $p \in \partial_C V(x)$, $H(x,p) \leq 0$.

In addition, assume that the optimal regulation problem is solvable. Then,

(ii) for each $x \in \mathbf{R}^n$ and for each $p \in \underline{\partial} V(x)$, $H(x,p) = 0$.

Proof Statement (i) is an immediate consequence of Lemma 3 and the definition of H; statement (ii) follows by Lemma 4, taking into account statement (i). □

Propositions 1 and 2 can be interpreted in terms of generalized solutions of the Hamilton-Jacobi equation

$$H(x, \nabla V(x)) = 0 \ . \tag{8}$$

Indeed, Proposition 2 implies in particular that V is a viscosity solution of (8) (a similar conclusion is obtained in [16], for a more general cost functional but under restrictive assumptions on the vector fields). Note that (ii) of Proposition 2 cannot be deduced from Theorem 5.6 of [7], since in our case the Hamiltonian function is not uniformely continuous on $\mathbf{R}^n$. Together with (i) of Proposition 2, (7) can be interpreted by saying that $V(x)$ is a solution in extended sense of (8) (since $p \mapsto H(x,p)$ is convex, the same conclusion also follows from Proposition 5.13 of [7]: in fact, we provide a simpler and more direct proof).

Finally, Proposition 2 (i) implies that $V(x)$ is a viscosity supersolution of the equation

$$-H(x, \nabla V(x)) = 0 \ . \tag{9}$$

Remark 2 In general it is not true that $V(x)$ is a viscosity subsolution of (9), unless certain additional conditions such as C-regularity are imposed (see next Corollary 1). This is the reason why the complete equivalence between solvability of the optimal regulation problem, solvability of the Hamilton-Jacobi equation and stabilizability by damping feedback breaks down in the general nonsmooth case. Basically, the main difference between the smooth and the nonsmooth case reduces to it. □

If the value function $V(x)$ satisfies additional assumptions, further facts can be proven. For instance, from Proposition 2 (ii) and Proposition 4 (see Appendix A) we immediately obtain:

Corollary 1 *Assume that the optimal regulation problem is solvable, and let $V(x)$ be the value function. Assume further that $V(x)$ is locally Lipschitz continuous and C-regular. Then*

$$\forall x \in \mathbf{R}^n \quad \forall p \in \partial_C V(x) \ , \qquad H(x,p) = 0 \ . \tag{10}$$

Remark 3 Corollary 1 implies that V is a subsolution of the equation (9), as well. Moreover, when $V(x)$ is C-regular, in Proposition 1(ii) we can choose any $p_0(t) \in \partial_C V(\varphi_x^*(t))$. □

3 Control Lyapunov functions and stabilizability

In this section we show that the value function of the optimal regulation problem can be interpreted as a control Lyapunov function for system (1). Then by using well known results in the literature, we will be able to recognize that a system for which the optimal regulation problem is solvable, can be stabilized both in Carathéodory and Filippov sense. However, by this approach, it is not possible to give an explicit construction of the feedback law.

Since we consider nonsmooth value functions, our definition of control Lyapunov function must make use of some sort of generalized gradient. Actually we need two different kinds of control Lyapunov function, introduced respectively by Sontag and Rifford (see [32, 28]). Let us denote by ∂V a (for the moment unspecified) generalized gradient of a function $V : \mathbf{R}^n \to \mathbf{R}$.

Definition 1 *We say that $V : \mathbf{R}^n \to \mathbf{R}^+$ is a control Lyapunov function for system (1) in the sense of the generalized gradient ∂ if it is continuous, positive definite and radially unbounded and there exist $W : \mathbf{R}^n \to \mathbf{R}$ continuous, positive definite and radially unbounded and $\sigma : \mathbf{R}^+ \to \mathbf{R}^+$ nondecreasing such that*

$$\sup_{x \in \mathbf{R}^n} \max_{p \in \partial V(x)} \min_{|u| \leq \sigma(|x|)} \{p \cdot \big(f(x) + G(x)u\big) + W(x)\} \leq 0$$

i.e.,

$$\forall x \in \mathbf{R}^n,\ \forall p \in \partial V(x),\ \exists \overline{u} :\ |\overline{u}| \leq \sigma(|x|) \ \text{ and } \ p \cdot \big(f(x) + G(x)\overline{u}\big) + W(x) \leq 0.$$

In particular, we say that $V(x)$ is a control Lyapunov function in the sense of the proximal sub-differential *if $\partial = \partial_P$ and we say that $V(x)$ is a* control Lyapunov function in the sense of Clarke generalized gradient *if $\partial = \partial_C$.*

3.1 Carathéodory stabilizability

We now prove the Carathéodory stabilizability result. We get it as a consequence of Ancona and Bressan's result (see [1]) which states that an asymptotically controllable system is Carathéodory stabilizable. The expression obtained for the optimal control in Proposition 1 also plays an important role. Let us first recall the definition of asymptotic controllability.

We say that system (1) is *asymptotically controllable* if
(i) for each x there exists an input $u_x(\cdot) \in \mathcal{U}$ such that $\lim_{t \to +\infty} \varphi(t; x, u_x(\cdot)) = 0$,
(ii) for each $\epsilon > 0$ there exists $\delta > 0$ such that if $|x| < \delta$, there exists a control $u_x(\cdot)$ as in (i) such that $|\varphi(t; x, u_x(\cdot))| < \epsilon$ for each $t \geq 0$.
Moreover, we require that there exist $\delta_0 > 0$ and $\eta_0 > 0$ such that if $|x| < \delta_0$, then $u_x(\cdot)$ can be chosen in such a way that $|u_x(t)| < \eta_0$ for $t \geq 0$.

Theorem 1 *Let system (1) be given, and let $h(x)$ be continuous, radially unbounded and positive definite. If the optimal regulation problem (3) is solvable and if its value function $V(x)$ is locally Lipschitz continuous and radially unbounded, then $V(x)$ is a control Lyapunov function in the sense of the proximal subdifferential, and the system is asymptotically controllable. Moreover, the system is Carathéodory stabilizable.*

Proof Thanks to Theorem D in [32], page 569, system (1) is asymptotically controllable if and only if there exists a control Lyapunov function in the sense of the proximal subdifferential. Thus, the conclusion follows from Lemma 4 (Appendix B) and the fact that $\partial_P V(x) \subseteq \underline{\partial} V(x)$.

Note that the existence of σ such that $|u_x^*(0)| \leq \sigma(|x|)$ is a consequence of the feedback form obtained for the optimal control in Proposition 1 and the fact that the set-valued map $\partial_C V$ is upper semi-continuous with compact values. The second statement is therefore a consequence of Theorem 1 in [1]. □

We remark that since asymptotic controllability has been proven, stabilizability in the sense of the so called sampling solutions may also be deduced (see [13]). A different proof of asymptotic controllability which does not make use of Theorem D in [32] was already given in [6]. There the fact that an optimal control gives asymptotic controllability was proved by means of Lemma 5. From that proof it turns out evidently that the optimal control itself gives asymptotic controllability.

3.2 Filippov stabilizability

We now discuss Filippov stabilizability. In this section we consider the case where the value function $V(x)$ is C-regular. The result is based on the interpretation of the value function as a control Lyapunov function in the sense of Clarke generalized gradient. In the following section the result will be improved: indeed, in Section 4 we will show that under the same assumptions the system can be stabilized just by the damping feedback (4), with α large enough.

Theorem 2 *Let system (1) be given, and let h be continuous, radially unbounded and positive definite. If the optimal regulation problem (3) is solvable and if its value function $V(x)$ is locally Lipschitz continuous, C-regular and radially unbounded, then $V(x)$ is a control Lyapunov function in the sense of Clarke gradient. Moreover, the system is Filippov stabilizable.*

Proof The first statement is a trivial consequence of Lemma 4, the fact that for C-regular functions $\underline{\partial} V(x) = \partial_C V(x)$ for all x (see Proposition 4 in Appendix A), and the feedback form obtained for the optimal control in Proposition 1. Then, the second statement follows from Theorem 2.7 in [28], according to which the existence of a control Lyapunov function in the sense of Clarke gradient guarantees Filippov stabilizability (the differences between our definition of control Lyapunov function in the sense of Clarke generalized gradient and the definition given in [28] are not essential). □

Remark 4 Due to Theorem 2.7 in [28], the existence of a control Lyapunov function in the sense of Clarke generalized gradient for (1) also implies the existence of a C^∞ Lyapunov function. In turn, thanks to Sontag universal formula, this implies the existence of a stabilizing feedback in $C^1(\mathbf{R}^n \backslash \{0\})$ (see also Theorem 2.8 in [28]). □

4 Stabilization by damping feedback

As already mentioned, in this section we improve the result of Theorem 2. More precisely, we discuss the possibility of stabilizing the system by means of an explicit feedback in damping form. For a moment, let us forget the optimal regulation problem, and let $V(x)$ be any locally Lipschitz continuous function. Consider the corresponding feedback law defined by (4). When it is implemented, it gives rise to the closed loop system

$$\dot{x} = f(x) + G(x)k_\alpha(x) = f(x) - \alpha G(x)(\nabla V(x) G(x))^{\mathbf{t}} \ . \tag{11}$$

In general, the righthand side of (11) is not continuous. Indeed, by virtue of Rademacher's Theorem the righthand side of (11) is almost everywhere defined; moreover, it is locally bounded and measurable (see [5]). Nevertheless, under the assumptions of next theorem, the feedback law (4) turns out to be continuous, so that (11) possesses solutions in classical sense.

Let $h : \mathbf{R}^n \mapsto \mathbf{R}$ be any continuous, positive definite, radially unbounded function, and define H according to (6).

Theorem 3 *Let $V(x)$, $h(x)$ and $H(x,p)$ be as above. Assume that*

$$\forall x \in \mathbf{R}^n \quad \forall p \in \partial_C V(x) \ , \qquad H(x,p) = 0 \ . \tag{12}$$

Then, the map $x \mapsto \nabla V(x) G(x)$ admits a continuous extension. If in addition $V(x)$ is positive definite, radially unbounded and nonpathological, the damping feedback (4) with $\alpha \geq \frac{\gamma}{2}$ is a stabilizer (in classical sense) for system (1).

Proof By contradiction, assume that there exists a point $\bar{x}$ where $\nabla V(x) G(x)$ cannot be completed in a continuous way. There must exist sequences $x'_n \to \bar{x}$ and $x''_n \to \bar{x}$ such that

$$\lim_n \nabla V(x'_n) G(x'_n) = c' \neq c'' = \lim_n \nabla V(x''_n) G(x''_n) \ .$$

Since $V(x)$ is locally Lipschitz continuous, its gradient, where it exists, is locally bounded. Possibly taking subsequences, we may assume that the limits

$$p' = \lim_n \nabla V(x'_n) \quad \text{and} \quad p'' = \lim_n \nabla V(x''_n)$$

exist. Of course, $p' \neq p''$. Clearly, $p', p'' \in \partial_C V(\bar{x})$, and hence by assumption (12)

$$-p' f(\bar{x}) + \frac{\gamma}{2}|c'|^2 - \frac{h(\bar{x})}{2} = 0 \quad \text{and} \quad -p'' f(\bar{x}) + \frac{\gamma}{2}|c''|^2 - \frac{h(\bar{x})}{2} = 0 \ . \tag{13}$$

Let $0 < \mu, \nu < 1$, with $\mu + \nu = 1$. From (13) it follows

$$-p f(\bar{x}) + \frac{\gamma}{2}\left[\mu|c'|^2 + \nu|c''|^2\right] - \frac{h(\bar{x})}{2} = 0 \tag{14}$$

where $p = \mu p' + \nu p''$. On the other hand, since $\partial_C V(\bar{x})$ is convex, invoking again assumption (12) we have

$$0 = -p f(\bar{x}) + \frac{\gamma}{2}|pG(\bar{x})|^2 - \frac{h(\bar{x})}{2} < -p f(\bar{x}) + \frac{\gamma}{2}\left[\mu|c'|^2 + \nu|c''|^2\right] - \frac{h(\bar{x})}{2} = 0 \tag{15}$$

where we also used the fact that the map $c \mapsto |c|^2$ is strictly convex. Comparing (14) and (15) we obtain a contradiction, and the first conclusion is achieved.

The second conclusion is based on the natural interpretation of V as a Lyapunov function for the closed loop system. Although we now know that the right hand side of such system is continuous, we cannot apply the usual Lyapunov argument since V is not differentiable. Instead, we invoke Proposition 6 in Appendix A, which is stated in terms of the set-valued derivative of a nonpathological function with respect to a differential inclusion.

Let x be arbitrarily fixed ($x \neq 0$) and let $a \in \dot{\overline{V}}^{(17)}(x)$. Then a is such that there exists $q \in \partial_C V(x)$ such that $a = p \cdot (f(x) - \frac{\gamma}{2} G(x)(qG(x))^{\mathbf{t}})$ for all $p \in \partial_C V(x)$. We have to prove that $a < 0$. If we take $p = q$ we obtain the following expression for a:

$$a = q \cdot f(x) - \frac{\gamma}{2} |qG(x)|^2. \tag{16}$$

By virtue of assumption (12), we get that $a = -\frac{h(x)}{2}$. Finally, we recall that h is positive definite. The statement is so proved for $\alpha = \frac{\gamma}{2}$. The case $\alpha > \frac{\gamma}{2}$ easily follows. □

Coming back to the optimal regulation problem and recalling Corollary 1, we immediatly have the following.

Corollary 2 *The same conclusion of Theorem 3 hold in particular when the optimal regulation problem is solvable and the value function $V(x)$ is locally Lipschitz continuous, C-regular and radially unbounded.*

Remark 5 Theorem 2 and Corollary 2 emphasize the role of C-regular functions. To this respect, it would be interesting to know conditions about the function $h(x)$ which enable us to prove that $V(x)$ is C-regular. The problem seems to be open in general. In Section 6 we show some examples where the function $V(x)$ is C-regular. Moreover, we point out some particular (but not completely trivial) situations where convexity (and hence, C-regularity and Lipschitz continuity) of $V(x)$ is guaranteed.

Assume for instance that system (1) is linear i.e., $f(x) = Ax$ and $G(x) = B$, and that h is convex. Let $x_1, x_2 \in \mathbf{R}^n$, let $0 \leq \nu, \mu \leq 1$ be such that $\nu + \mu = 1$, and let $\varepsilon > 0$. We have

$$\nu V(x_1) + \mu V(x_2) + \varepsilon \geq \frac{1}{2} \left[\int_0^\infty \left(\nu h(\varphi^\varepsilon_{x_1}(t)) + \mu h(\varphi^\varepsilon_{x_2}(t)) \right) dt + \frac{1}{\gamma} \int_0^\infty \left(\nu |u^\varepsilon_{x_1}(t)|^2 + \mu |u^\varepsilon_{x_2}(t)|^2 \right) dt \right]$$

where, according to the definition of V, $u^\varepsilon_{x_i}$ is such that $V(x_i) + \varepsilon \geq J(x_i, u^\varepsilon_{x_i})$, $(i = 1, 2)$. Using the convexity of both h and the quadratic map $u \mapsto |u|^2$, yields

$$\nu V(x_1) + \mu V(x_2) + \varepsilon \geq \frac{1}{2} \left[\int_0^\infty h(\nu \varphi^\varepsilon_{x_1}(t) + \mu \varphi^\varepsilon_{x_2}(t))\, dt + \frac{1}{\gamma} \int_0^\infty |\nu u^\varepsilon_{x_1}(t) + \mu u^\varepsilon_{x_2}(t)|^2\, dt \right] .$$

Finally, by virtue of linearity

$$\nu V(x_1) + \mu V(x_2) + \varepsilon \geq \frac{1}{2} \left[\int_0^\infty h(\varphi_{\nu x_1 + \mu x_2}(t))\, dt + \frac{1}{\gamma} \int_0^\infty |u(t)|^2\, dt \right]$$

where $u(t) = \nu u^\varepsilon_{x_1}(t) + \mu u^\varepsilon_{x_2}(t)$ and $\varphi_x(t) = \varphi(t; x, u(\cdot))$. Since V is an infimum and the choice of ε is arbitrary, we conclude

$$\nu V(x_1) + \mu V(x_2) \geq V(\nu x_1 + \mu x_2) \ .$$

Note that here existence of solutions of the optimal regulation problem, so as a priori information about the value function, are not required. □

Next theorem provides an alternative stabilizability result. Condition (12) of Theorem 3 is weakened, so that the damping feedback (4) is no more expected to be continuous in general. As a consequence, the stability analysis will be carried out in terms of Filippov solutions. Recall that Filippov solutions of (11) coincide with the solutions of the differential inclusion

$$\dot{x} \in f(x) - \alpha G(x)(\partial_C V(x) G(x))^{\mathbf{t}} \tag{17}$$

(see [27], [5]), where the set valued character of the righthand side depends on the presence of Clarke gradient.

Weakening condition (12) is balanced by the introduction of a new assumption. Roughly speaking, this new assumption amounts to say that V is not "too irregular" with respect to the vector fields $g_1, \ldots, g_m$ (in a sense to be precised).

In particular, Theorem 4 focuses on the class of nonpathological functions. The definition is given in Appendix A. We recall that the class of nonpathological functions includes both C-regular and semiconcave functions.

Theorem 4 *Let $V(x)$ be any locally Lipschitz continuous, positive definite, radially unbounded and nonpathological function. Let $h(x)$ be any continuous, positive definite, radially unbounded function. Moreover, let H be defined as in (6), and assume that*

$$\forall x \in \mathbf{R}^n \quad \exists p_0 \in \partial_C V(x) \quad \text{such that} \quad H(x, p_0) = 0 \ . \tag{18}$$

Let α and γ be given positive numbers, and assume that the following condition holds:

[H] *There exists a real constant $R < 1$ such that the following inequality holds:*

$$\gamma(A_1^2 + \ldots + A_m^2) - 2\alpha(A_1 B_1 + \ldots + A_m B_m) - Rh(x) \leq 0 \tag{19}$$

for each $x \in \mathbf{R}^n$ $(x \neq 0)$ and each choice of the real indeterminates $A_1, \ldots, A_m$, $B_1, \ldots, B_m$ subject to the following constraints:

$$A_i, B_i \in [\underline{D}_C V(x, g_i(x)), \overline{D}_C V(x, g_i(x))] \qquad \text{for} \quad i = 1, \ldots, m \ . \tag{20}$$

Then, the feedback law (4) Filippov stabilizes system (1).

Proof As in the proof of Theorem 3 we shall apply Proposition 6 in Appendix A. Let $a \in \overline{\overline{V}}^{(17)}(x)$. By construction, there exists $\bar{q} \in \partial_C V(x)$ such that for each $p \in \partial_C V(x)$ we have

$$a = p \cdot f(x) - \alpha(\bar{q}G(x))(pG(x))^{\mathbf{t}} \ .$$

In order to prove the theorem, it is therefore sufficient to show the following

Claim. *For each $x \neq 0$ there exists $p_0 \in \partial_C V(x)$ such that for each $q \in \partial_C V(x)$*

$$p_0 \cdot f(x) - \alpha(qG(x))(p_0 G(x))^{\mathbf{t}} < 0.$$

Let p_0 be as in (18) and let q be any element in $\partial_C V(x)$. We have

$$\begin{aligned} &p_0 \cdot f(x) - \alpha(qG(x))(p_0 G(x))^{\mathbf{t}} = \\ &= \frac{1}{2}\left[-h(x) + \left(\gamma(p_0 G(x))(p_0 G(x))^{\mathbf{t}} - 2\alpha(qG(x))(p_0 G(x))^{\mathbf{t}}\right)\right] \ . \end{aligned}$$

For each $x \neq 0$, let us interprete $A_1, \ldots, A_n$ as the components of the vector $p_0 G(x)$ and, respectively, $B_1, \ldots, B_n$ as the components of the vector $qG(x)$. Now, (20) is fulfilled and (19) is applicable, so that we finally have

$$p_0 \cdot f(x) - \alpha(qG(x))(p_0 G(x))^{\mathbf{t}} \leq \frac{h(x)}{2}(R-1) < 0 \ . \ \square$$

Taking into account Proposition 1, we immediatly have the following.

Corollary 3 *Let h be positive definite, continuous and radially unbounded. Assume that the optimal regulation problem is solvable and that the value function V is locally Lipschitz continuous, nonpathological and radially unbounded. Assume finally condition* **[H]**. *Then, the feedback law (4) Filippov stabilizes system (1).*

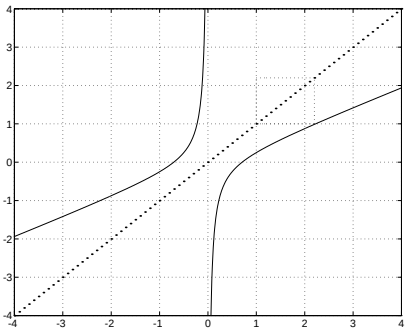

Figure 1: $0 < R < 1$, $\gamma \leq 2\alpha$

In order to grasp the meaning of condition **[H]**, let us focus on the single-input case ($m = 1$). Writing A, B instead of A_1, B_1, conditions (19), (20) reduce to

$$\gamma A^2 - 2\alpha AB - Rh(x) \leq 0 \tag{21}$$

for each $x \in \mathbf{R}^n$ ($x \neq 0$) and each choice of the pair A, B satisfying

$$A, B \in [\underline{D}_C V(x, g(x)), \overline{D}_C V(x, g(x))] \ . \tag{22}$$

In the plane of coordinates A, B, (21) defines a region bounded by the branches of an hyperbola. Our assumptions amounts to say that the square

$$Q = [\underline{D}_C V(x, g(x)), \overline{D}_C V(x, g(x))] \times [\underline{D}_C V(x, g(x)), \overline{D}_C V(x, g(x))]$$

is contained in this region (intuitively, it means that the distance between $\underline{D}_C V(x, g(x))$ and $\overline{D}_C V(x, g(x))$ should be not too large). Note that the "north-est" and the "south-west" corners of Q lie on the line $B = A$.

In order to rewrite the condition in a more explicit way, we distinguish several cases. In the following, we set for simplicity $\underline{D} = \underline{D}_C V(x, g(x))$ and $\overline{D} = \overline{D}_C V(x, g(x))$.

First case. Assume that the condition (21), (22) is verified with $0 < R < 1$, and let $\gamma \leq 2\alpha$. The line $B = A$ is contained in the "good" region (see Figure 1). Let

$$A_0 = \sqrt{\frac{Rh(x)}{\gamma + 2\alpha}}$$

be the abscissa of the intersection between the line $B = -A$ and the right branche of the hyperbola. Then, condition (21), (22) is equivalent to

$$\begin{cases} \underline{D} \geq \dfrac{\gamma \overline{D}^2 - Rh(x)}{2\alpha \overline{D}} & \text{if } \overline{D} \geq A_0 \\ \underline{D} \geq \dfrac{\alpha \overline{D} - \sqrt{\alpha^2 \overline{D}^2 + \gamma Rh(x)}}{\gamma} & \text{if } \overline{D} \leq A_0 \end{cases} \tag{23}$$

(for $\overline{D} = A_0$ the two formulas coincides).

When $\gamma > 2\alpha$, the line $B = A$ cross the hyperbola in two points whose abscissas are $A_1 = \sqrt{\frac{Rh(x)}{\gamma - 2\alpha}}$ and $-A_1$ (see Figure 2). The condition (21), (22) is still reducible to (23), but it can be satisfied only if

$$\overline{D} \leq A_0 \quad \text{or} \quad \underline{D} \geq -A_0 \ .$$

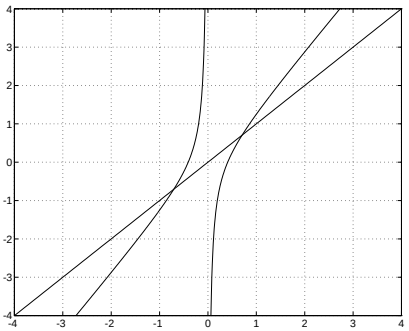

Figure 2: $0 < R < 1$, $\gamma > 2\alpha$

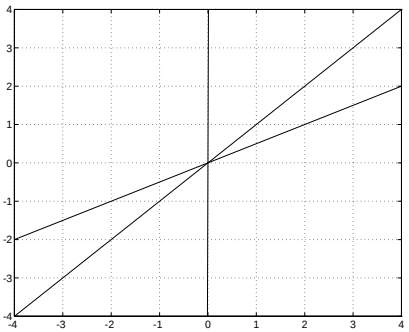

Figure 3: $R = 0$, $\gamma < 2\alpha$

Second case. Assume now that the condition (21), (22) is verified with $R = 0$. In this case the hyperbola degenerates, and the "good" region becomes a cone. It contains the line $B = A$ if and only if $\gamma \leq 2\alpha$. Hence, the condition is never satisfied if $\gamma > 2\alpha$.

If $\gamma = 2\alpha$, the condition is satisfied provided that $\underline{D} = \overline{D}$, and hence in particular when V is smooth.

Finally, if $\gamma < 2\alpha$, conditions (23) simplifies in the following manner (see Figure 3)

$$\begin{cases} \underline{D} \geq \dfrac{\gamma \overline{D}}{2\alpha} & \text{if } \overline{D} \geq 0 \\ \underline{D} \geq \dfrac{2\alpha \overline{D}}{\gamma} & \text{if } \overline{D} < 0 \quad . \end{cases} \tag{24}$$

Third case. Assume finally that the condition (21), (22) is verified with $R < 0$. The "good" regions are now the convex regions bounded by the branches of the hyperbola (see Figure 4).

The condition is never satisfied if $\gamma \geq 2\alpha$. For $\gamma < 2\alpha$, the condition are given by (23). However, the condition cannot be satisfied if

$$0 \leq \overline{D} < A_1 \quad \text{or} \quad -A_1 < \underline{D} \leq 0 \ .$$

Remark 6 Note that in certain cases stabilization is possible even if $2\alpha < \gamma$ (typically, this happens for stabilizable driftless systems). □

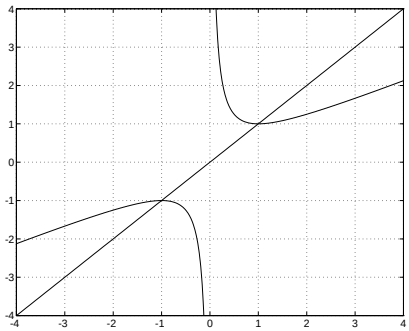

Figure 4: $R < 0$, $\gamma < 2\alpha$

5 Sufficient conditions for optimality

In this section we enlarge the class of admissible inputs to all measurable, locally bounded maps $u(t) : [0, +\infty) \to \mathbf{R}^m$. The aim is to extend the following result, whose proof can be found in [22, 8, 29] in slightly different forms.

Optimality Principle. *If the Hamilton-Jacobi equation (8) admits a positive definite, C^1 solution $V(x)$ such that $V(0) = 0$, and if the feedback (4) with $\alpha = \gamma$ is a global stabilizer for (1), then for each initial state x, trajectories corresponding to the same feedback law minimize the cost functional (3) over all the admissible inputs $u(t)$ for which $\lim_{t\to+\infty} \varphi(t; x, u(\cdot)) = 0$. Moreover, $V(x)$ coincides with the value function.*

As remarked in [29], restricting the minimization to those inputs whose corresponding solutions converge to zero, can be interpreted as incorporating a detectability condition. In this section we explicit the detectability condition by assuming that h is positive definite.

The following result can be seen as a partial converse of Proposition 1. Roughly speaking, it says that if the closed loop system admits a Carathéodory solution satisfying the necessary conditions and driving the system asymptotically to zero, then this solution is optimal.

Theorem 5 *Consider the optimal regulation problem (3) with $h(x)$ continuous, positive definite and radially unbounded, and let $V(x)$ be any locally Lipschitz continuous, radially unbounded and positive definite function. Assume in addition that $V(x)$ is nonpathological. Let H be defined according to (6), and assume that*

(A) $\forall x \in \mathbf{R}^n, \forall p \in \partial_C V(x)$ one has $H(x, p) \leq 0$.

Let $x^o \in \mathbf{R}^n$, and let $u^o(t)$ be any admissible input. Let us write for simplicity $\varphi^o(t) = \varphi(t; x^o, u^o(\cdot))$ and assume that

(B) for a.e. $t \geq 0$ there exists $p^o(t) \in \partial_C V(\varphi^o(t))$ such that

(i) $H(\varphi^o(t), p^o(t)) = 0$

(ii) $u^o(t) = -\gamma(p^o(t)G(\varphi^o(t)))^{\mathbf{t}}$;

(C) $\lim_{t\to+\infty} \varphi^o(t) = 0$.

Then, $u^o(t)$ is optimal for x^o. Moreover, the value function of the optimal regulation problem and $V(x)$ coincides at x^o.

Proof Since $\varphi^o(t)$ is absolutely continuous, by (B)(ii) we have for a.e. $t \geq 0$

$$\dot{\varphi}^o(t) = f(\varphi^o(t)) - G(\varphi^o(t))u^o(t) = f(\varphi^o(t)) - \gamma G(\varphi^o(t))(p^o(t)G(\varphi^o(t)))^{\mathbf{t}} \ .$$

Using (B)(i), we now can compute the cost

$$
\begin{aligned}
J(x^o, u^o(\cdot)) &= \frac{1}{2}\int_0^{+\infty} \left(h(\varphi^o(t)) + \frac{|u^o(t)|^2}{\gamma} \right) dt \\
&= \int_0^{+\infty} -p^o(t)\left[f(\varphi^o(t)) - \gamma G(\varphi^o(t))u^o(t) \right] dt \\
&= \int_0^{+\infty} -p^o(t)\dot{\varphi}^o(t)\, dt = V(x^o)
\end{aligned}
$$

where the last equality follows by virtue of Lemma 6 and (C). In order to complete the proof, we now show that for any other admissible input $u(t)$, we have

$$
V(x^o) = J(x^o, u^o(\cdot)) \leq J(x^o, u(\cdot)) \ .
$$

For simplicity, we use again a shortned notation $\varphi(t) = \varphi(t; x^o, u(\cdot))$. Let us distinguish two cases.

1) The integral in (3) diverges. In this case, it is obvious that $J(x^o, u^o(\cdot)) = V(x^o) < J(x^o, u(\cdot))$.

2) The integral in (3) converges. According to Lemma 5, we conclude that $\lim_{t\to+\infty} \varphi^o(t) = 0$, and since $V(x)$ is radially unbounded, continuous and positive definite, this in turn implies $\lim_{t\to+\infty} V(\varphi(t)) = 0$. Let $p(t)$ be any measurable selection of the set valued map $\partial_C V(\varphi(t))$ (such a selection exists, since $\partial_C V(\varphi(t))$, the composition of an upper semicontinuous set valued map and a continuous single valued map, is upper semicontinuous, hence measurable; see [4]). By (A), and the usual completing the square method, we have

$$
\begin{aligned}
J(x^o, u(\cdot)) &= \frac{1}{2}\int_0^{+\infty} \left(h(\varphi(t)) + \frac{|u(t)|^2}{\gamma} \right) dt \\
&\geq \int_0^{+\infty} \Big[-p(t)f(\varphi(t)) + \frac{\gamma}{2}|p(t)G(\varphi(t))|^2 + \frac{|u(t)|^2}{2\gamma} \Big] dt \\
&= \int_0^{+\infty} \Big[-p(t)f(\varphi(t)) - p(t)G(\varphi(t))u(t) + \frac{1}{2\gamma}|\gamma p(t)G(\varphi(t)) + u(t)|^2 \Big] dt \\
&\geq \int_0^{+\infty} -p(t)\dot{\varphi}(t)\, dt = V(x^o)
\end{aligned}
$$

where we used again Lemma 6. This achieves the proof. In particular, we see that $u^o(t)$ is optimal, and we see that the value function of the minimization problem (3) coincides with $V(x)$ at x^o. □

Note that (C) is actually necessary, since $h(x)$ is positive definite (see Lemma 5). It could be replaced by the assumption that $J(x^o, u^o(\cdot))$ is finite.

Corollary 4 *Let $h(x)$ be continuous, radially unbounded and positive definite. Let $V(x)$ be any locally Lipschitz continuous, radially unbounded and positive definite function. Assume in addition that $V(x)$ is nonpathological. Let finally H be defined according to (6), and assume that (12) holds. Then, for each $x \in \mathbf{R}^n$ there exists a measurable, locally bounded control which is optimal for the minimization problem (3). Moreover, the value function and $V(x)$ coincide at every $x \in \mathbf{R}^n$.*

Proof Let $x^0 \in \mathbf{R}^n$ and let $\varphi^o(t)$ be any solution of the initial value problem

$$
\begin{cases} \dot{x} \in f(x) + G(x)k_\gamma(x) \\ x(0) = x^o \end{cases} \tag{25}
$$

where for a.e. $x \in \mathbf{R}^n$, $k_\gamma(x) = -\gamma(\nabla V(x)G(x))^{\mathbf{t}}$ (that is, at those points where the gradient exists, k_γ is given by (4) with $\alpha = \gamma$). By virtue of Theorem 3, we can assume that $k_\gamma(x)$ is continuous, so that such a $\varphi^o(t)$ exists, and it is a solution in classical sense. From the proof of Theorem 3, it is also clear that $k_\gamma(x) = -\gamma(pG(x))^{\mathbf{t}}$ for each $p \in \partial_C V(x)$ and each $x \in \mathbf{R}^n$. Since $x \mapsto \partial_C V(x)$ is compact convex valued and upper semicontinuous,

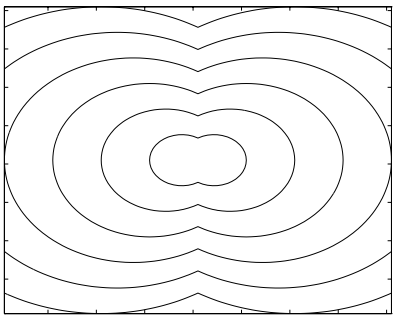

Figure 5: Level curves of $V(x,y) = (4x^2 + 3y^2)^{1/2} - |x|$

by Filippov's Lemma ([4]) there exists a measurable map $p^o(t) \in \partial_C V(\varphi^o(t))$ such that for a.e. $t \geq 0$, one has $k_\gamma(\varphi^o(t)) = -\gamma(p^o(t)G(\varphi^o(t)))^{\mathbf{t}}$.

Let us set $u^o(t) = -\gamma(p^o(t)G(\varphi^o(t)))^{\mathbf{t}}$. Note that $\varphi^o(t)$ is the unique solution of (1) issuing from x^o and corresponding to the admissible input $u^o(t)$.

Theorem 3 also states that $k_\gamma(t)$ is a stabilizing feedback. In conclusion, all the assumptions (A), (B), (C) of Theorem 5 are fulfilled. The statement is proven. □

6 Examples

The results of the previous sections are illustrated by the following examples.

Example 1 Consider the two-dimensional, single input driftless system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = u g(x,y) \quad \text{where} \quad g(x,y) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

(the so-called Artstein's circles example). The function $V(x,y) = \sqrt{4x^2 + 3y^2} - |x|$ is a control Lyapunov function (in the sense of proximal gradient) for this system. As a sum of a function of class C^1 and a concave function, V is semiconcave in $\mathbf{R}^2 \setminus \{(0,0)\}$, but not differentiable when $x = 0$ (the level curves are piecewise arcs of circumferences: see Figure 5).

We want to construct an optimization problem with $\gamma = 1$, whose value function is V. To this purpose, we follow an "inverse optimality" approach (see [29]). Define

$$h(x,y) = \begin{cases} \left[\frac{x}{\sqrt{4x^2+3y^2}} \left(4x^2 + 2y^2 - |x| \sqrt{4x^2 + 3y^2} \right) + (\operatorname{sgn} x) y^2 \right]^2 & \text{if } x \neq 0 \\ y^4 & \text{if } x = 0 \end{cases} . \tag{26}$$

Note that $h(x,y)$ is continuous and positive definite. Equation (8) takes the form

$$(\nabla V(x,y) g(x,y))^2 = h(x,y) \ .$$

A simple computation shows that it is fulfilled in the usual sense if $x \neq 0$. In points where $x = 0$, we have

$$\partial_C V(0,y) = (\, p_1 \,, \sqrt{3} \operatorname{sgn} y \,) \quad \text{where} \quad p_1 \in [-1,1]$$

and the Hamilton-Jacobi equation reduces to $\left(\frac{\partial V}{\partial x} \right)^2 = 1$. Consistently with Propositions 1 and 2, we therefore see that V is a viscosity subsolution, and actually a viscosity solution (note that the subdifferential is empty for $x = 0$, $y \neq 0$), as well as a solution in extended sense. We also see that V is a viscosity supersolution of (9), but not a viscosity subsolution of such equation.

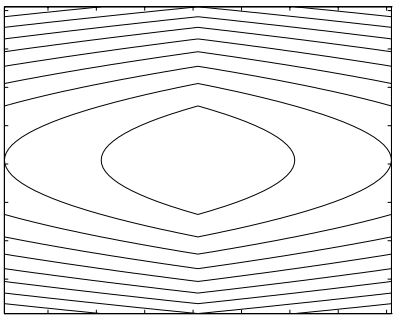

Figure 6: Level curves of $V(x,y) = |x| + y^2$

The damping feedback $k_1(x)$ corresponding to V (i.e., (4) with $\alpha = 1$) is easily computed for $x \neq 0$ (it coincides with minus the expression inside the square brackets in (26)). It turns out to be positive if $x < 0$, and negative if $x > 0$. It is discontinuous along the y-axis. Its construction can be completed in such a way condition (i) of Proposition 1 is preserved. In fact, at the points of the form $(0, y)$ there are two possible choices of the vector p_0. Both of them give rise to a stabilizing feedback, provided that solutions are intended in Carathéodory sense.

Now let $\varphi^o(t)$ be any Carathéodory solution of the closed loop system, and let $u^o(t) = k_1(\varphi^o(t))$. The assumptions (A), (B), (C) of Theorem 5 are fulfilled. Thus, all the solutions of the closed loop system are optimal and V is actually the value function.

Note that in this example optimal controls are not unique. Note also that the damping feedback does not stabilize the system in Filippov sense. On the other hand, it is well known that Artstein's circles system cannot be stabilized in Filippov sense. □

Example 2 Given the two-dimensional, single input linear system

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y + 2u \end{cases}$$

we want to impose the value function $V(x,y) = |x| + y^2$. Note that V is C-regular, but not differentiable for $x = 0$. Its level curves are plotted in Figure 6. Let us set

$$h(x,y) = 2|x| + 12y^2$$

and $\gamma = 1$. The function $h(x)$ is continuous, positive definite and radially unbounded. In points where $x \neq 0$, V is smooth: the Hamilton-Jacobi equation is fulfilled in the usual sense. In points where $x = 0$, we have

$$\partial_C V(0,y) = (\, p_1, 2y\,) \quad \text{where} \quad p_1 \in [-1,1]\ .$$

The Hamilton-Jacobi equation reduces to an identity in these points, so that (12) is satisfied. According to Theorems 3 and 5, the damping feedback is continuous. It takes the form

$$k_\alpha(x,y) = -\alpha \nabla V(x,y) g(x,y) = -4\alpha y\ . \tag{27}$$

Hence, it is a stabilizer for $\alpha \geq \frac{1}{2}$ (actually, we have a larger stability margin: $\alpha > \frac{1}{8}$). Moreover, for $\alpha = 1$, it generates the optimal solutions. Finally, thanks to Corollary 4 $V(x)$ coincides with the value function.

Note that in this example matrix G is constant. Nevertheless, in points of the form $(0,y)$ the Hamiltonian function H is not strictly convex with respect to p. □

Example 3 First we consider the single-input, driftless system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = u g(x,y) \quad \text{where} \quad g(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{28}$$

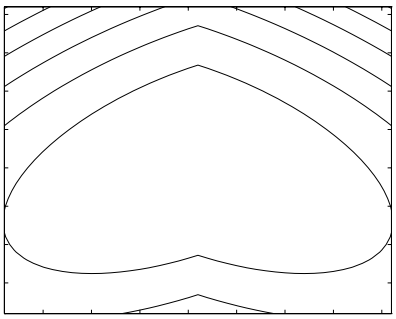

Figure 7: Level curves of $V(x,y) = \frac{7}{4}x^2 + \frac{13}{4}y^2 - \frac{3^{3/2}}{2}|x|y$

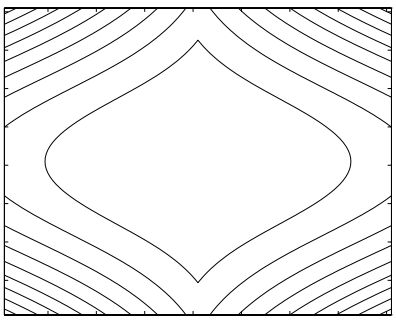

Figure 8: Level curves of $V(x,y) = x^2 + y^2 + |x|y^2$

and choose the semiconcave function V of the Example 1. To interpret V as a value function, we define $\gamma = 1$ and $h(x,y) = V^2(x,y)$. Theorem 3 is applicable, and the damping feedback law is continuous. Optimality of solutions and the fact that V is the value function are guaranteed by Corollary 4. Similar conclusions are obtained if we take the semiconvex function $V(x,y) = \sqrt{4x^2+3y^2} + |x|$.

Finally, we consider system (28) and the associated optimal regulation problem with $\gamma = 1$ and $h(x,y) = \left(\frac{7}{2}x^2 + \frac{13}{2}y^2 - 3\sqrt{3}|x|y\right)^2$. The value function in this case is given by $V(x,y) = \frac{7}{4}x^2 + \frac{13}{4}y^2 - \frac{3\sqrt{3}}{2}|x|y$. Such a function V is neither C-regular nor semiconcave, but it is nonpathological (the levels curves are plotted in Figure 7). Even in this case, Theorem 3 and Corollary 4 are applicable. □

Example 4 In this example we consider the system

$$\begin{cases} \dot{x} = u \\ \dot{y} = -y^3 \end{cases}$$

and the function $V(x,y) = x^2 + y^2 + |x|y^2$ (see Figure 8). By direct computation, it is possible to see that $\partial_C V(x,y) = \underline{\partial} V(x,y)$ at each point, so that V is C-regular, and hence nonpathological. In particular, along the y-axis we have

$$\partial_C V(0,y) = (p_1, 2y) \quad \text{with} \quad p_1 \in [-y^2, y^2] \ .$$

Define $\gamma = 1$ and $h(x,y) = 4x^2 + 5y^2 + 4|x|(y^4+y^2)$. Then the Hamilton-Jacobi equation (8) is satisfied by $V(x)$ in the usual sense when $x \neq 0$. Along the y-axis the Hamiltonian reduces to $\frac{1}{2}(p_1^2 - y^4)$. Thus, we have $H(p,0,y) \leq 0$ for each p in the Clarke gradient, but the equality holds only at the extremal points. Hence, $V(x,y)$ is seen to be a viscosity subsolution of (8) (since V is C-regular but not differentiable along the y-axis, the superdifferential is empty in these points), but not a supersolution.

In particular, condition (12) is not met and Theorem 3 is not applicable. Nevertheless, the system is Filippov stabilized by the (discontinuous) damping feedback $k_1(x,y) = -2x - (\operatorname{sgn} x)y^2$. Indeed, a simple computation shows that condition **[H]** is fulfilled with $3/5 \le R < 1$, so that we can use Theorem 4.

As far as the existence of optimal controls is concerned, we make the following important remark. Given an initial point $(\bar{x}, \bar{y})$, we do not have for sure that there is a Carathéodory solution of the closed loop system issuing from $(\bar{x}, \bar{y})$ and asymptotically going to the origin. In fact, by numerical simulation one realizes that this is actually false, with the exception of the points along the x-axis. As a matter of fact, if $\bar{x} \neq 0$, $\bar{y} \neq 0$, the solution starting from $(\bar{x}, \bar{y})$ hits the y-axis at some point $(0, \hat{y})$ with $\hat{y} \neq 0$. The only way to construct a Carathéodory solution issuing from a point $(0, \bar{y})$, moving along the y-axis and asymptotically going to the origin, is taking $u = 0$. But in this way the necessary conditions for optimality fail. In fact, by direct computation is possible to see that the cost of such a solution is stricty greater than $V(0, \bar{y})$.

In conclusion, according to the theory developed in this paper, we have the following alternative: either V is not the value function of the optimal regulation problem, or there exist no optimal controls (with the exceptions of points along the x-axis).

Example 5 Consider the two-dimensional driftless system with two inputs

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = u_1 g_1(x,y) + u_2 g_2(x,y) \quad \text{where} \quad g_1(x,y) = \begin{pmatrix} x+y \\ x+y \end{pmatrix} \ , \quad g_2(x,y) = \begin{pmatrix} x-y \\ -x+y \end{pmatrix} \ .$$

In order to impose the value function $V(x,y) = |x| + |y|$, we try $h(x,y) = 4(|x| + |y|)^2$. Note that V is locally Lipschitz continuous and C-regular, while h is continuous and positive definite. As before, we set $\gamma = 1$. For $xy \neq 0$, V is differentiable, and the Hamilton-Jacobi equation is satisfied in the usual sense. This allows us to construct a (discontinuous) feedback in damping form

$$\begin{cases} u_1 = k_1(x,y) = -2(|x| + |y|) \\ u_2 = k_2(x,y) = 0 \end{cases} \quad \text{if } xy > 0 \ , \qquad \text{and} \qquad \begin{cases} u_1 = k_1(x,y) = 0 \\ u_2 = k_2(x,y) = -2(|x| + |y|) \end{cases} \quad \text{if } xy < 0 \tag{29}$$

In points of the form $(x, 0)$ we have

$$\partial_C V(x,0) = (\ 1, p_2\ ) \quad \text{where} \quad p_2 \in [-1, 1]$$

so that the Hamilton-Jacobi equation reduces to

$$2x^2(1 + p_2^2) = 4x^2 \ .$$

This equality is satisfied only for $p_2 = \pm 1$. Even in this case we see that V is a viscosity subsolution of (8), but not a supersolution. Unfortunately, if we complete the construction of the feedback (29) according to one of these choices, the closed loop system has no (Carathéodory) solution issuing from $(x, 0)$. In fact, the only way to construct a solution going to the origin for $t \to +\infty$, is to take a strict convex combination of the two vector fields g_1, g_2, but this cannot be done in "optimal" way.

In conclusion, the necessary conditions are not satisfied for points of the form $(x, 0)$, so that we have again the alternative: either V is not the value function or the optimal regulation problem is not solvable for these points. Actually, we conjecture that in this example the optimal regulation problem is solvable only for points lying on the lines $y = \pm x$. □

7 Appendix A (Tools from nonsmooth analysis)

For reader's convenience, we shortly review the definitions of the various extensions of derivatives and gradients used in this paper (see [14, 12] as general references). Moreover we prove two apparently new results on Clarke regular and semiconvex functions.

Given a function $V : \mathbf{R}^n \to \mathbf{R}$, for any $x, v \in \mathbf{R}^n$, for any $h \in \mathbf{R} \backslash \{0\}$, we consider the difference quotient

$$\mathcal{R}(h, x, v) = \frac{V(x + hv) - V(x)}{h}.$$

If there exists $\lim_{h\to 0^+} \mathcal{R}(h,x,v)$, then it is called the *directional derivative* of V at x in the direction v and is denoted by $D^+V(x,v)$.

When V does not admit directional derivative in the direction v, we may substitute this notion with a number of different generalizations. We limit ourselves to the definitions we use in this paper.

The so-called *Dini directional derivatives* associate to each x four numbers, $\underline{D}^+V(x,v)$, $\overline{D}^+V(x,v)$, $\underline{D}^-V(x,v)$, $\overline{D}^-V(x,v)$, where the former is defined as:

$$\underline{D}^+V(x,v) = \liminf_{h\to 0^+} \mathcal{R}(h,x,v),$$

and the others are analogously defined. When $\underline{D}^+V(x,v) = \underline{D}^-V(x,v)$ (respectively, $\overline{D}^+V(x,v) = \overline{D}^-V(x,v)$) we simply write it as $\underline{D}V(x,v)$ (respectively, $\overline{D}V(x,v)$).

If we let vary v as well, we get the so-called *contingent directional derivatives* $\underline{D}_K^+V(x,v)$, $\overline{D}_K^+V(x,v)$, $\underline{D}_K^-V(x,v)$, $\overline{D}_K^-V(x,v)$. More precisely, the *lower right contingent directional derivative* is defined as:

$$\underline{D}_K^+V(x,v) = \liminf_{h\to 0^+,\, w\to v} \mathcal{R}(h,x,w)$$

and the other are defined in similar way. When V is locally Lipschitz continuous, Dini derivatives and contingent derivatives coincide.

Clarke introduced another kind of generalized directional derivative, by letting x vary:

$$\overline{D}_C V(x,v) = \limsup_{h\to 0,\, y\to x} \mathcal{R}(h,y,v) \quad \text{and} \quad \underline{D}_C V(x,v) = \liminf_{h\to 0,\, y\to x} \mathcal{R}(h,y,v).$$

Besides directional derivatives, different generalizations of the differential have been defined in the literature.

The *sub-differential* can be seen as a generalization of Fréchet differential:

$$\underline{\partial}V(x) = \left\{ p\in\mathbf{R}^n \ : \ \liminf_{h\to 0} \frac{V(x+h)-V(x)-p\cdot h}{|h|} \geq 0 \right\}.$$

Analogously the *super-differential* is defined as:

$$\overline{\partial}V(x) = \left\{ p\in\mathbf{R}^n \ : \ \limsup_{h\to 0} \frac{V(x+h)-V(x)-p\cdot h}{|h|} \leq 0 \right\}.$$

These objects can be used in order to define the notions of viscosity super and sub-solutions of partial differential equations of the Hamilton-Jacobi type (see [7, 14]). The sub and super-differentials can be characterized by means of contingent derivatives (see [18]): indeed we have

$$\underline{\partial}V(x) = \{p\in\mathbf{R}^n : p\cdot v \leq \underline{D}_K^+V(x,v) \text{ for all } v\in\mathbf{R}^n\}$$
$$\overline{\partial}V(x) = \{p\in\mathbf{R}^n : p\cdot v \geq \overline{D}_K^+V(x,v) \text{ for all } v\in\mathbf{R}^n\}$$

For each x, $\underline{\partial}V(x)$ (and analogously $\overline{\partial}V(x)$) is a convex and closed set, and it may be empty. If V is differentiable at x, then it coincides with the singleton $\{\nabla V(x)\}$.

The *proximal sub-differential* is defined as:

$$\partial_P V(x) = \{p\in\mathbf{R}^n \ : \ \exists\sigma\geq 0, \exists\delta\geq 0 \text{ s.t. } (z\in\mathbf{R}^n, |z-x|<\delta)$$
$$\Rightarrow \ (V(z)-V(x)\geq p\cdot(z-x)-\sigma|z-x|^2)\}.$$

For each x, $\partial_P V(x)$ is convex, but not necessarily closed. Moreover, $\partial_P V(x)\subseteq\underline{\partial}V(x)$.

The *Clarke generalized gradient* is defined as:

$$\partial_C V(x) = \left\{ p\in\mathbf{R}^n \ : \ p\cdot v \leq \overline{D}_C V(x,v), \ \forall v\in\mathbf{R}^n \right\}$$

or, equivalently,

$$\partial_C V(x) = \{p \in \mathbf{R}^n \ : \ p \cdot v \geq \underline{D}_C V(x,v), \ \forall v \in \mathbf{R}^n\}.$$

It is possible to see that $\overline{D}_C V(x,v) = \sup\{p \cdot v \ : \ p \in \partial_C V(x)\}$ and $\underline{D}_C V(x,v) = \inf\{p \cdot v \ : \ p \in \partial_C V(x)\}$.

If V is Lipschitz continuous, by Rademacher's Theorem its gradient $\nabla V(x)$ exists almost everywhere. Let N be the subset of $\mathbf{R}^n$ where the gradient does not exist. It is possible to characterize Clarke generalized gradient as:

$$\partial_C V(x) = \text{co}\left\{\lim_{x_i \to x} \nabla V(x_i), \ x_i \to x, \ x_i \notin N \cup \Omega\right\}$$

where Ω is any null measure set. By using this characterization, it is obvious that $\partial_C V(x)$ is convex; it is possible to see that it is also compact. Moreover, we have

$$\partial_P V(x) \subseteq \underline{\partial} V(x) \subseteq \partial_C V(x)$$

and also

$$\overline{\partial} V(x) \subseteq \partial_C V(x).$$

Let us now give the definition of semiconcave and Clarke regular (briefly C-regular) functions.

V is said to be a *semiconcave* function (with linear modulus) if there exists $C > 0$ such that

$$\lambda V(x) + (1-\lambda)V(y) - V(\lambda x + (1-\lambda)y) \leq \lambda(1-\lambda)C|x-y|^2$$

for any pair $x, y \in \mathbf{R}^n$ and for any $\lambda \in [0,1]$. Analogously V is said to be *semiconvex* if $-V$ is semiconcave.

In the next proposition (see [11]) a few interesting properties of semiconcave functions are collected.

Proposition 3 *If V is semiconcave then:*

(i) V is Lipschitz continuous,

(ii) for any x, v there exists $D^+V(x,v)$ and $D^+V(x,v) = \underline{D}_C V(x,v)$,

(iii) $\overline{\partial} V(x) = \partial_C V(x)$ for any x.

We say that V is *C-regular* if for all $x, v \in \mathbf{R}^n$ there exists $D^+V(x,v)$ and $D^+V(x,v) = \overline{D}_C V(x,v)$.

C-regular functions form a rather wide class: for instance, semiconvex functions are C-regular. C-regular functions can be characterized in terms of generalized gradients in the following way:

Proposition 4 *Let $V : \mathbf{R}^n \to \mathbf{R}$ be locally Lipschitz continuous. V is C-regular if and only if $\partial_C V(x) = \underline{\partial} V(x)$ for all x.*

Proof Let us first assume V is C-regular. Being V also Lipschitz continuous, we have that

$$\underline{\partial} V(x) = \{p \in \mathbf{R}^n : p \cdot v \leq \underline{D}^+_K V(x,v) \text{ for all } v \in \mathbf{R}^n\} = \{p \in \mathbf{R}^n : p \cdot v \leq \underline{D}^+ V(x,v) \text{ for all } v \in \mathbf{R}^n\} =$$
$$= \{p \in \mathbf{R}^n : p \cdot v \leq \overline{D}^+ V(x,v) \text{ for all } v \in \mathbf{R}^n\} = \{p \in \mathbf{R}^n : p \cdot v \leq \overline{D}_C V(x,v) \text{ for all } v \in \mathbf{R}^n\} = \partial_C V(x)$$

Let us now assume $\partial_C V(x) = \underline{\partial} V(x)$ for all x. Due to the convexity of $\partial_C V(x)$, we get that $\underline{D}^+_K V(x,v) = \overline{D}_C V(x,v)$ for all x and for all v. Moreover we have

$$\underline{D}^+_K V(x,v) \leq \underline{D}^+ V(x,v) \leq \overline{D}^+ V(x,v) \leq \overline{D}_C V(x,v)$$

that implies that there exists

$$D^+ V(x,v) = \overline{D}_C V(x,v). \ \square$$

The proofs of Theorems 3 and 4 rely on the notion of *set-valued derivative of a map V with respect to a differential inclusion*, introduced in [30] and already exploited in [5]. Given a differential inclusion

$$\dot{x} \in F(x) \tag{30}$$

(with $0 \in F(0)$) the set-valued derivative of a map V with respect to (30) is defined as the closed, bounded (possibly empty) interval

$$\overset{\cdot}{V}^{(30)}(x) = \{a \in \mathbf{R} : \ \exists v \in F(x) \text{ s.t.} \quad p \cdot v = a \ , \ \forall p \in \partial_C V(x)\} \ .$$

Such a derivative of the map V can be successfully used in case V is nonpathological, in the sense of the following definition given by Valadier in [34].

A function V is said to be *nonpathological* if for every absolute continuous function $\varphi : \mathbf{R} \to \mathbf{R}^n$ and for a.e. t, the set $\partial_C V(\varphi(t))$ is a subset of an affine subspace orthogonal to $\dot{\varphi}(t)$.

Note that nonpathological functions form a quite wide class which includes C-regular functions. The following proposition can be also easily proven.

Proposition 5 *If V is semiconcave then it is nonpathological.*

Proof Let $\varphi : \mathbf{R} \to \mathbf{R}^n$ be an absolutely continuous function. Since V is semiconcave, then it is also locally Lipschitz continuous. This implies that $V \circ \varphi$ is absolutely continuous and then for almost all t there exists $\frac{d}{dt}V(\varphi(t))$. Let $t \in \mathbf{R}$ be such that there exists both $\dot{\varphi}(t)$ and $\frac{d}{dt}V(\varphi(t))$. Since V is locally Lipschitz we have that

$$\frac{d}{dt}V(\varphi(t)) = \lim_{h \to 0} \frac{V(\varphi(t) + h\dot{\varphi}(t)) - V(\varphi(t))}{h}.$$

On one hand due to Lipschitz continuity of V we have that

$$\begin{aligned}
&\frac{d}{dt}V(\varphi(t)) = \lim_{h \to 0^+} \frac{V(\varphi(t) + h\dot{\varphi}(t)) - V(\varphi(t))}{h} = \\
&= D^+V(\varphi(t), \dot{\varphi}(t)) = \underline{D}_C V(\varphi(t), \dot{\varphi}(t)) = \min\{p \cdot \dot{\varphi}(t), p \in \partial_C V(\varphi(t))\}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\frac{d}{dt}V(\varphi(t)) = \lim_{h \to 0^-} \frac{V(\varphi(t) + h\dot{\varphi}(t)) - V(\varphi(t))}{h} = -\lim_{h \to 0^+} \frac{V(\varphi(t) + h(-\dot{\varphi}(t))) - V(\varphi(t))}{h} = \\
&= -D^+V(\varphi(t), -\dot{\varphi}(t)) = -\underline{D}_C V(\varphi(t), -\dot{\varphi}(t)) = \overline{D}_C V(\varphi(t), \dot{\varphi}(t)) = \\
&= \max\{p \cdot \dot{\varphi}(t), p \in \partial_C V(\varphi(t))\}.
\end{aligned}$$

This means that for almost all t the set $\{p \cdot \dot{\varphi}(t), p \in \partial_C V(\varphi(t))\}$ reduces to the singleton $\left\{\frac{d}{dt}V(\varphi(t))\right\}$, and then $\partial_C V(\varphi(t))$ is a subset of an affine subspace orthogonal to $\dot{\varphi}(t)$. □

Note that, in order to prove Proposition 5, we don't really need semiconcavity, but Lipschitz continuity and property (ii) of Proposition 3 would be sufficient.

The following extension of second Lyapunov theorem to differential inclusions and nonpathological functions holds (see [5] for the case of C-regular functions: the case of nonpathological functions requires minor modifications).

Proposition 6 *Assume that $V : \mathbf{R}^n \to \mathbf{R}$ is positive definite, locally Lipschitz continuous, nonpathological and radially unbounded. Assume further that*

$$\forall x \in \mathbf{R}^n \backslash \{0\} \quad \overset{\cdot}{V}^{(30)}(x) \subseteq \{a \in \mathbf{R} : \ a < 0\} \ .$$

Then

(i) (Lyapunov stability) for all $\epsilon > 0$ there exists $\delta > 0$ such that for each solution $\varphi(\cdot)$ of (30), $|\varphi(0)| < \delta$ implies $|\varphi(t)| < \epsilon$ for all $t \geq 0$,

(ii) (attractivity) for each solution $\varphi(\cdot)$ of (30) one has $\lim_{t \to +\infty} \varphi(t) = 0$.

8 Appendix B (Lemmas)

This appendix contains a number of Lemmas used in the proofs of the results of this paper. For a (real or vector valued) function $\psi(t)$, the right derivative is denoted by $\frac{d^+}{dt}\psi(t)$ or, when convenient, simply by $\dot\psi^+(t)$.

Lemma 1 *Let $I \subseteq \mathbf{R}$, $\varphi : I \to \mathbf{R}^n$ be an absolutely continuous function and $V : \mathbf{R}^n \to \mathbf{R}$ be a locally Lipschitz continuous function. If $\overline{t} \in I$ is such that there exists both $\frac{d^+}{dt}V(\varphi(\overline{t}))$ and $\dot\varphi^+(\overline{t})$, then*

$$\frac{d^+}{dt}V(\varphi(\overline{t})) = \lim_{h\to 0^+} \frac{V(\varphi(\overline{t}) + h\dot\varphi^+(\overline{t})) - V(\varphi(\overline{t}))}{h}.$$

Moreover, there exists $p_0 \in \partial_C V(\varphi(\overline{t}))$ such that $\frac{d^+}{dt}V(\varphi(\overline{t})) = p_0 \cdot \dot\varphi^+(\overline{t})$.

Proof The first statement is a consequence of the Lipschitz continuity of V (see also [5]). As far as the second statement is concerned, let us first remark that

$$\underline{D}_C V(\varphi(\overline{t}), \dot\varphi^+(\overline{t}))) \le \underline{D}^+ V(\varphi(\overline{t}), \dot\varphi^+(\overline{t}))) = \frac{d^+}{dt}V(\varphi(\overline{t})) = \overline{D}^+ V(\varphi(\overline{t}), \dot\varphi^+(\overline{t}))) \le \overline{D}_C V(\varphi(t), \dot\varphi^+(t))) \ .$$

Then we use the fact that $\partial_C V(x)$ is compact and convex at each point, so that the set $\{p\cdot\dot\varphi^+(\overline{t})),\ p \in \partial_C V(\varphi(\overline{t}))\}$ is a bounded and closed interval. □

The following lemmas are essentially based on the Dynamic Programming Principle. The outline of the proof is standard, but some modifications are needed in order to face the lack of differentiability of the value function.

Lemma 2 *Let the optimal regulation problem be solvable and let V be its value function. Then for each $x \in \mathbf{R}^n$, for each optimal solution $\varphi_x^*(\cdot)$ and each $\overline{t} \ge 0$ the derivative $\frac{d^+}{dt}V(\varphi_x^*(\overline{t}))$ exists. In addition,*

$$\frac{d^+}{dt}V(\varphi_x^*(\overline{t})) = -\frac{1}{2}h(\varphi_x^*(\overline{t})) - \frac{|u_x^*(\overline{t})|^2}{2\gamma}. \tag{31}$$

Proof Let $\overline{t} \ge 0$ and let $\eta = \varphi_x^*(\overline{t})$. The solution $\varphi_x^*(t)$ fot $t \ge \overline{t}$ provides an optimal trajectory issuing from η. Hence, it is sufficient to prove (31) for $\overline{t} = 0$. Let $T > 0$. Due to the definition of the value function and the Dynamic Programming Principle, we have that

$$\frac{V(\varphi_x^*(T)) - V(x)}{T} = -\frac{1}{2}\frac{1}{T}\int_0^T \left(h(\varphi_x^*(t)) + \frac{|u_x^*(t)|^2}{\gamma}\right) dt.$$

By the continuity of h and φ_x^*, and right-continuity of $u_x^*(\cdot)$, there exists

$$\lim_{T\to 0^+} -\frac{1}{2}\frac{1}{T}\int_0^T \left(h(\varphi_x^*(t)) - \frac{|u_x^*(t)|^2}{\gamma}\right) dt = -\frac{1}{2}h(x) - \frac{|u_x^*(0)|^2}{2\gamma}.$$

Then there exists also the limit on the left hand side, that is $\lim_{T\to 0^+} \frac{V(\varphi_x^*(T)) - V(x)}{T} = -\frac{1}{2}h(x) - \frac{|u_x^*(0)|^2}{2\gamma}$. □

Lemma 3 *Let the value function V of the optimal regulation problem be locally Lipschitz continuous. Then*

$$\forall x \in \mathbf{R}^n,\ \forall u \in \mathbf{R}^m,\ \forall p \in \partial_C V(x) \qquad p\cdot(f(x) + G(x)u) \ge -\frac{1}{2}h(x) - \frac{|u|^2}{2\gamma}. \tag{32}$$

Proof Let us fix a control value u_0, and an instant $T > 0$. Let y be an arbitrary point in a neighbourhood of x, and let $\eta = \varphi(T; y, u_0)$. By the Dynamic Programming Principle we have that

$$V(y) \le \frac{1}{2}\int_0^T \left(h(\varphi(t; y, u_0)) + \frac{|u_0|^2}{\gamma}\right) dt + V(\eta)$$

and then

$$\liminf_{T\to 0^+,\, y\to x}\left(-\frac{1}{2}\frac{1}{T}\int_0^T\left(h(\varphi(t;y,u_0))+\frac{|u_0|^2}{\gamma}\right)dt\right)\le \liminf_{T\to 0^+,\, y\to x}\frac{V(\eta)-V(y)}{T}. \tag{33}$$

Let us consider the two sides of the inequality separately.

Lefthand side. There exists $\theta\in[0,T]$ such that

$$-\frac{1}{2}\frac{1}{T}\int_0^T\left(h(\varphi(t;y,u_0))+\frac{|u_0|^2}{\gamma}\right)dt=-\frac{1}{2}h(\varphi(\theta;y,u_0))-\frac{|u_0|^2}{2\gamma}.$$

By continuous dependence of solutions of Cauchy problem on initial data we have that $\liminf_{T\to 0^+,\, y\to x}\varphi(\theta;y,u_0)=x$ and then

$$\liminf_{T\to 0^+,\, y\to x}\left(-\frac{1}{2}\frac{1}{T}\int_0^T\left(h(\varphi(t;y,u_0))+\frac{|u_0|^2}{\gamma}\right)dt\right)=-\frac{1}{2}h(x)-\frac{|u_0|^2}{2\gamma}.$$

Righthand side. Let us consider the quotient

$$\frac{V(\varphi(T;y,u_0))-V(y)}{T}=\frac{V(\varphi(T;y,u_0))-V(y+(f(x)+G(x)u_0)T)}{T}+\frac{V(y+(f(x)+G(x)u_0)T)-V(y)}{T}. \tag{34}$$

By Lipschitz continuity of V there exists $L>0$ such that

$$\left|\frac{V(\varphi(T;y,u_0))-V(y+(f(x)+G(x)u_0)T)}{T}\right|\le\frac{L}{T}|\varphi(T;y,u_0)-y-(f(x)+G(x)u_0)T)|=$$

$$=\frac{L}{T}|x+\frac{\partial\varphi}{\partial t}(0;x,u_0)T+\frac{\partial\varphi}{\partial y}(0;x,u_0)\cdot(y-x)+o(T,|y-x|)-y-(f(x)+G(x)u_0)T)|\le$$

$$\le\frac{L}{T}o(T,|y-x|)\ .$$

Then there exists

$$\lim_{T\to 0^+,y\to x}\frac{V(\varphi(T;y,u_0))-V(y+(f(x)+G(x)u_0)T)}{T}=0\ .$$

Coming back to (34), we therefore have

$$\liminf_{T\to 0^+,\, y\to x}\frac{V(\varphi(T;y,u_0))-V(y)}{T}=\liminf_{T\to 0^+,\, y\to x}\frac{V(y+(f(x)+G(x)u_0)T)-V(y)}{T}=$$

$$=\underline{D_C}V(x,f(x)+G(x)u_0)=\min\{p\cdot(f(x)+G(x)u_0),p\in\partial_C V(x)\}.$$

By comparing the two sides of the inequality (33), the conclusion of the lemma follows. □

Lemma 4 *If the optimal regulation problem is solvable and its value function is locally Lipschitz continuous then*

$$\forall x,\ \forall p\in\underline{\partial}V(x)\qquad p\cdot(f(x)+G(x)u_x^*(0))=-\frac{1}{2}h(x)-\frac{|u_x^*(0)|^2}{2\gamma}. \tag{35}$$

Proof Let us consider $\frac{d^+}{dt}V(\varphi_x^*(t))$. On one hand, by Lemma 2, we have that

$$\frac{d^+}{dt}V(\varphi_x^*(0))=-\frac{1}{2}h(x)-\frac{|u_x^*(0)|^2}{2\gamma}.$$

On the other hand, from the characterization of the subdifferential by means of the contingent derivatives (see Appendix A) and Lemma 1, it follows that for all $p \in \underline{\partial} V(x)$

$$p \cdot (f(x) + G(x)u_x^*(0)) \leq \underline{D}_K^+ V(x, f(x) + G(x)u_x^*(0)) = \underline{D}^+ V(x, f(x) + G(x)u_x^*(0)) =$$

$$= \liminf_{t \to 0^+} \frac{V(x + t(f(x) + G(x)u_x^*(0))) - V(x)}{t} = \frac{d^+}{dt} V(\varphi_x^*(0)).$$

Finally we get that for all x and for all $p \in \underline{\partial} V(x)$

$$p \cdot (f(x) + G(x)u_x^*(0)) \leq -\frac{1}{2}h(x) - \frac{|u_x^*(0)|^2}{2\gamma} .$$

Since $\underline{\partial} V(x) \subseteq \partial_C V(x)$, the opposite inequality is provided by Lemma 3. □

In the next lemma, we denote by $\mathcal{K}_0$ the class of functions $a : [0, +\infty) \to [0, +\infty)$ such that $a(\cdot)$ is continuous, strictly increasing and $a(0) = 0$.

Lemma 5 *Let x_0 be fixed. Let $a \in \mathcal{K}_0$ be such that $h(x) \geq a(|x|)$ for each $x \in \mathbf{R}^n$ (such a function exists, if h is continuous and positive definite). Assume also that $J(x_0, u(\cdot)) < \infty$ for some admissible input $u(t)$. Then,*

$$\lim_{t \to +\infty} \varphi(t; x_0, u(\cdot)) = 0 .$$

Proof Since $u(t)$ and x_0 are fixed, we shall write simply $\varphi(t)$ instead of $\varphi(t; x_0, u(\cdot))$. From the assumption, it follows that both the integrals

$$\int_0^{+\infty} h(\varphi(t))\, dt \quad \text{and} \quad \int_0^{+\infty} |u(t))|^2\, dt \tag{36}$$

converge. It follows in particular that $u(t)$ is square integrable on $[0, +\infty)$ and on every subinterval of $[0, +\infty)$. It also follows that $\liminf_{t \to +\infty} h(\varphi(t)) = 0$, and since $h(x)$ is continuous, positive definite and bounded from below by a class $\mathcal{K}_0$ function, this in turn implies

$$\liminf_{t \to +\infty} |\varphi(t)| = 0 .$$

Assume, by contradiction, that $\limsup_{t \to +\infty} |\varphi(t)| > 0$, and let

$$l = \min\{1, \limsup_{t \to +\infty} |\varphi(t)|\} .$$

Let L be a Lipschitz constant for $f(x)$, valid on the sphere $|x| \leq l$. Moreover, let $b > 0$ be a bound for the norm of the matrix $G(x)$ for $|x| \leq l$. By the definition of l, there exists a strictly increasing, divergent sequence $\{t_j\}$ such that for each j,

$$|\varphi(t_j)| > \frac{3l}{4} .$$

Without loss of generality, we can assume that $t_{j+1} - t_j > 1/(4L)$ for each $j \in \mathbf{N}$. The existence of some $\tau > 0$ and $k \in \mathbf{N}$ such that for each $j > k$ and each $t \in [t_j - \tau, t_j]$

$$||\varphi(t)|| \geq \frac{l}{4}$$

is excluded, since in this case the first integral in (36) would be divergent (here again, we use the facts that h is bounded from below by $a(\cdot)$). Hence, for $\tau = 1/(4L)$ and each $k \in \mathbf{N}$, we can find an index $j_k > k$ and an instant s_k such that

$$t_{j_k} - \tau \leq s_k \leq t_{j_k} \quad \text{and} \quad |\varphi(s_k)| < \frac{l}{4} .$$

Since the solution $\varphi(t)$ is continuous, for each k there exist two instants σ_k, θ_k such that

$$s_k < \sigma_k < \theta_k < t_{j_k} \ , \qquad |\varphi(\sigma_k)| = \frac{l}{4} \ , \qquad |\varphi(\theta_k)| = \frac{3l}{4} \ ,$$

and

$$\frac{l}{4} < |\varphi(t)| < \frac{3l}{4} \ , \qquad \forall t \in (\sigma_k, \theta_k) \ .$$

We have, for each k,

$$|\varphi(\theta_k) - \varphi(\sigma_k)| \geq |\varphi(\theta_k)| - |\varphi(\sigma_k)| = \frac{l}{2} \ . \tag{37}$$

On the other hand,

$$|\varphi(\theta_k) - \varphi(\sigma_k)| \leq \int_{\sigma_k}^{\theta_k} |f(\varphi(t))| \, dt + \int_{\sigma_k}^{\theta_k} |G(\varphi(t))| \cdot |u(t)| \, dt \ .$$

By construction, for $t \in [\sigma_k, \theta_k]$ we have $|\varphi(t)| \leq l$. Hence, on the interval $[\sigma_k, \theta_k]$ the following inequalities hold:

$$|f(\varphi(t))| \leq L|\varphi(t)| \quad \text{and} \quad |G(x)| \leq b \ .$$

This yields

$$|\varphi(\theta_k) - \varphi(\sigma_k)| \leq lL\tau + b \int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)| \, dt \ . \tag{38}$$

Now, taking into account (37), (38) and recalling that $\tau = 1/(4L)$, we infer

$$\frac{l}{2} \leq \frac{l}{4} + b \int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)| \, dt$$

that is,

$$\int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)| \, dt \geq \frac{l}{4b} \ .$$

Using Hölder inequality and the fact that $u(t)$ is square integrable on $[t_{j_k} - \tau, t_{j_k}]$, we also have

$$C \left(\int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)|^2 \, dt \right)^{\frac{1}{2}} \geq \int_{t_{j_k}-\tau}^{t_{j_k}} \|u(t)\| \, dt$$

where C is a positive constant independent of $u(\cdot)$. This yields

$$\int_{t_{j_k}-\tau}^{t_{j_k}} |u(t)|^2 \, dt \geq \frac{l^2}{16b^2C^2} > 0 \ .$$

But this is impossible, since $u(t)$ is square integrable on $[0, +\infty)$ by virtue of (36). Thus, we conclude that

$$\liminf_{t \to +\infty} |\varphi(t)| = \limsup_{t \to +\infty} |\varphi(t)| = 0$$

which implies

$$\lim_{t \to +\infty} |\varphi(t)| = 0$$

as required. □

Lemma 5 is reminiscent of the so-called Barbalat's Lemma ([24], p. 491). However, it does not reduce to Barbalat's Lemma since we have to take into account the input variable and we cannot use uniform continuity of solutions.

Lemma 6 *Let $V(x) : \mathbf{R}^n \to \mathbf{R}$ be locally Lipschitz continuous and nonpathological. Let $x \in \mathbf{R}^n$, and let $u(\cdot)$ be any admissible input. Let us write, for simplicity, $\varphi(t) = \varphi(t; x, u(\cdot))$. Let finally $p(t)$ be any measurable function such that $p(t) \in \partial_C V(\varphi(t))$ a.e.. Then,*

$$\int_{t_1}^{t_2} p(t) \cdot \dot{\varphi}(t)\, dt = V(\varphi(t_2)) - V(\varphi(t_1)) \ .$$

Proof Under our assumptions, for a.e. $t \in \mathbf{R}$ there exists $p_0 \in \partial_C V(\varphi(t))$ such that the right derivative $\frac{d^+}{dt} V(\varphi(t))$ exists and it is equal to $p_0 \cdot \frac{d^+}{dt}\varphi(t)$ (see Lemma 1). The conclusion follows by virtue of the definition of nonpathological function. □

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