Nonlinearity measures: Definition, computation and applications

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Abstract:

This paper forms the first part out of a series of two papers treating the quantification of open loop nonlinearity of dynamic systems. A generic definition of a nonlinearity measure is presented on the basis of the "best" linear approximation of a nonlinear system. Generalizing an approach of Allgöwer (1995,1996*b*), the measure can be applied both to the analysis of steady state operating points of continuously operated processes as well as to a trajectory dependent analysis of batch or other transient processes. An approximative computational strategy transferring the original infinite dimensional nested optimization problem into a convex finite dimensional minimization problem is discussed. The applications in this paper focus on operating point dependent analysis. Three continuously operated stirred tank reactor examples are investigated including a benchmark CSTR. The latter is also used to illustrate a computationally efficient lower bound approximation of the proposed nonlinearity measure. The additional difficulties associated with a trajectory rather than an operating point dependent analysis will be discussed in the forthcoming second part of this communication treating transient reaction processes (Helbig *et al.*, 1999*b*).

Keywords: Nonlinearity measures, system analysis, CSTR.

Introduction

Without doubt process nonlinearity is one of the most relevant factors in characterizing process control problems. In recent years several approaches have been proposed to assess nonlinearity of static systems (Guay *et al.*, 1995; Ogunnaike *et al.*, 1993) as well as dynamic systems (Nikolaou and Hanagandi, 1994; Allgöwer, 1996*a*; Guay *et al.*, 1997; Sun and Kosanovich, 1998).

Research has predominantly treated open-loop I/O-systems in the vicinity of some stationary operating point either through nonlinearity tests or through nonlinearity measures. Nonlinearity tests are well known in the field of system identification (Haber, 1985; Haber, 1989). They evaluate the deviation from a property which is known to hold for linear systems e.g. through coherence analysis (Carter, 1993). Nonlinearity measures are a means of quantifying the "degree" of nonlinearity in the input-output (I/O) behavior of nonlinear systems. Such measures allow a direct comparison of the nonlinearity of different processes or different operating points of a single process. Early work in this area goes back to Desoer and Wang (1981). Definitions relying on the calculation of the "best" linear approximation of a nonlinear I/O-system have been proposed by several authors. Nikolaou (1993, 1994) evaluates the variations of the best linear approximations for different input signals while the best linear approximation for the "worst" input signal is determined by Allgöwer (1995, 1996*a*). A different approach is taken by Guay (1996, 1997) defining a "curvature measure" on the basis of the first and second order sensitivities relative to some norm of the process output signals.

Considering the open-loop system only may be misleading in order to determine the effect of process nonlinearity on controller requirements. In order to answer such important questions as to whether application of nonlinear control methods should be investigated, the "control relevant" nonlinearity has to be determined. No mature solution, however, currently exists to this problem. Available approaches transform the problem to the assessment of open-loop nonlinearity for a different "control relevant" dynamic system. Guay *et al.* (1997) proposed the inclusion of a PI controller in the scaling operator for the considered curvature measure. Stack and Doyle (1997*a*, 1997*b*) suggest the assessment of control relevant nonlinearity through structural analysis of the optimal controller. They apply coherence analysis to assess the I/O-nonlinearity of the so-called "optimal control structure", but other measures of dynamic process nonlinearity could also be applied.

Compared to continuous processes operated at steady state, nonlinearity analysis for transient processes has not found much attention up to now.

This type of processes is characterized by intentionally time-varying process variables (Helbig *et al.*, 1999*a*). Examples include start-up, shut-down, and grade changes of continuously operated plants or batch processes. Hardly any work exists that tries to bridge the gap between trajectory design for such transient processes and the difficulty of the arising closed loop control problem (Abel *et al.*, 1998). In the field of nonlinearity measures, to the authors' knowledge the work of Guay (1996) comprises the first approach to assess nonlinearity about a nominal trajectory rather than a steady state operating point. The approach of Stack and Doyle has already been applied to a small batch reactor example (1997*b*) with a single zero-order reaction approaching a steady state. The considered control problem of optimally driving the process into its steady state is, however, not generally applicable to batch or other transient processes.

This communication proposes a new definition of a measure for open-loop nonlinearity assessment of general dynamic systems and treats applications to different chemical reactors. Being the first out of a series of two, this paper in the next section gives the generic definition of the nonlinearity measure in a state space setting . The approach taken generalizes an earlier concept of Allgöwer (1995; 1996*b*). Computational issues will be discussed the subsequent section. In the remainder, analysis of continuous processes at fixed operating points is treated as a special case of the generic definition. The proposed concepts for operating point dependent analysis are illustrated with three CSTR examples. Trajectory dependent analysis for transient processes will be dealt with in the forthcoming second part of this communication.

Definition of a nonlinearity measure

In the following, a generic definition of a nonlinearity measure for dynamic systems will be presented. The discussion will be restricted to finite dimensional multivariable systems described by ordinary differential equations. The derived concepts, however, carry over to other nonlinear dynamic input/output-systems including finite dimensional systems in discrete time, differential algebraic systems as well as infinite dimensional systems with lumped inputs and outputs.

Preliminaries

Nonlinearity will be assessed for a process with the following state space description:

$$\dot{\boldsymbol{x}}_N(t) \quad = \quad \boldsymbol{f}\left(\boldsymbol{x}_N(t), \boldsymbol{u}_{N,in}(t)\right)\ , \tag{1}$$

$$\boldsymbol{x}_N(0) = \boldsymbol{x}_{N,0}\,, \qquad 0 \le t \le t_f\,,$$

$$\boldsymbol{y}_{N,out}(t) \;=\; \boldsymbol{h}\left(\boldsymbol{x}_N(t), \boldsymbol{u}_{N,in}(t)\right)\,,$$

with $\boldsymbol{f}$ being a vector field and $\boldsymbol{h}$ being a vector function. The time interval of interest may be finite or infinite ($t_f \le \infty$). $\boldsymbol{u}_{N,in}$, $\boldsymbol{x}_N$, and $\boldsymbol{y}_{N,out}$ are the process inputs, states, and outputs, respectively. The subscript N indicates that this is the <u>n</u>onlinear process to be evaluated. The input signals $\boldsymbol{u}_{N,in}$ are assumed to belong to some space $\mathcal{U}_{in,a}$ of admissible input signals. The state signals $\boldsymbol{x}_N$ are elements of a space $\mathcal{X}$ and $\boldsymbol{x}_{N,0}$ belongs to the set of admissible initial conditions $\mathcal{X}_{0,a}$. The output signals $\boldsymbol{y}_{N,out}$ of the process are elements of a normed space of output signals $\mathcal{Y}_{out}$.

System (1) represents the dynamic process in its "natural" I/O-coordinates. The rather unusual indices *in* and *out* are introduced in order to distinguish these variables from the input and output variables used for nonlinearity analysis. For instance, $\boldsymbol{y}_{N,out}$ might comprise a process temperature associated with certain units like $[K]$ or $[^\circ C]$. For a general transient process, a specific solution $\boldsymbol{x}_{N_{nom}}$, $\boldsymbol{y}_{N,out_{nom}}$ defined through a nominal input signal $\boldsymbol{u}_{N,in_{nom}}$ and initial condition $\boldsymbol{x}_{N,0_{nom}}$ will be called the *nominal trajectory* associated with system (1)[2]. A specific form of a nominal trajectory is given

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[2] In the notation chosen a symbol without time argument represents the entire trajectory of a time function on $[0, t_f)$. The time argument will be added if the value of a function

by the *steady state* solution $\boldsymbol{x}_{N_s}$, $\boldsymbol{y}_{N,out_s}$ of the nonlinear system for a specific constant input $\boldsymbol{u}_{N,in_s}$ and with $\boldsymbol{x}_{N,0} = \boldsymbol{x}_{N_s}$.

Depending on the specific control scheme to be used, a definition of new "control relevant" inputs and outputs is necessary. For this purpose, deviation variables with respect to some reference signals $u_{N,ref}$ and $y_{N,ref}$ are introduced for the process inputs and outputs (Figure):

$$\begin{aligned} \boldsymbol{u}_N &= \boldsymbol{u}_{N,in} - \boldsymbol{u}_{N,ref} \;, & (2) \\ \boldsymbol{y}_N &= \boldsymbol{y}_{N,out} - \boldsymbol{y}_{N,ref} \;. & (3) \end{aligned}$$

The reference in- and outputs are e.g. steady state solutions or nominal input and output trajectories as discussed above. See below for a further discussion of this. System (1) extended by equations (2) and (3) defines a nonlinear dynamic operator N that maps input signals $\boldsymbol{u}_N \in \mathcal{U}_a$ and initial conditions $\boldsymbol{x}_{N,0} \in \mathcal{X}_{0,a}$ into output signals $\boldsymbol{y}_N \in \mathcal{Y}$, which we denote by

$$\boldsymbol{y}_N = N\left[\boldsymbol{u}_N, \boldsymbol{x}_{N,0}\right] \;, \qquad (4)$$

which is not necessarily unbiased. The corresponding input and output spaces of N are defined through

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at a certain time instant is considered.

$$\begin{aligned}\mathcal{U}_a &= \{\boldsymbol{u}_N : (\boldsymbol{u}_N + \boldsymbol{u}_{N,ref}) \in \mathcal{U}_{in,a}\} \ , \\ \mathcal{Y} &= \left\{\boldsymbol{y}_N : (\boldsymbol{y}_N + \boldsymbol{y}_{N,ref}) \in \mathcal{Y}_{out}\right\} \ .\end{aligned}$$

Nonlinearity of the dynamic operator N will be considered through comparison with a "best" linear dynamic system of equal input and output dimensionality (see next subsection). The corresponding operator, denoted G, will be an element of the space $\mathcal{G}$ of continuous linear time invariant dynamic systems. As with the nonlinear system, a state space description will be applied for the definition of the linear system. Any $G \in \mathcal{G}$ maps input signals $\boldsymbol{u}_G \in \mathcal{U}_G$ and initial conditions $\boldsymbol{x}_{G,0} \in \mathcal{X}_{0,G}$ into output signals $\boldsymbol{y}_G \in \mathcal{Y}_G$ through

$$\boldsymbol{y}_G = G\left[\boldsymbol{u}_G, \boldsymbol{x}_{G,0}\right] \ , \tag{5}$$

with system dynamics being described by

$$\begin{aligned}\dot{\boldsymbol{x}}_G(t) &= \boldsymbol{A}\,\boldsymbol{x}_G(t) + \boldsymbol{B}\,\boldsymbol{u}_G(t)\,, \quad \boldsymbol{x}_G(0) = \boldsymbol{x}_{G,0}\,, \\ \boldsymbol{y}_G(t) &= \boldsymbol{C}\,\boldsymbol{x}_G(t) + \boldsymbol{D}\,\boldsymbol{u}_G(t)\ , \quad 0 \le t \le t_f\ .\end{aligned} \tag{6}$$

$\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are constant matrices of appropriate dimensionality. $\boldsymbol{u}_G$, $\boldsymbol{x}_G$, and $\boldsymbol{y}_G$ are the system inputs, states, and outputs, respectively, and $\boldsymbol{x}_{G,0}$ is the initial condition of the linear system. Note that the state dimension of the "best" linear approximation needs (and usually will) not be equal to the one of system (1).

Generic definition

In order to provide a broadly applicable means to quantify the nonlinearity of a generic dynamic system, the following measure is proposed:

Definition 1: *The nonlinearity measure* $\phi_N^{\mathcal{Y}_a}$ *of a nonlinear dynamic system* $N : \mathcal{U}_a \times \mathcal{X}_{0,a} \rightarrow \mathcal{Y}$ *with output signals* $\boldsymbol{y}_N \in \mathcal{Y}_a \subseteq \mathcal{Y}$ *is defined by the nonnegative number*

$$\begin{aligned} \phi_N^{\mathcal{Y}_a}(t_f) \; = \; & \inf_{G \in \mathcal{G}} \; \sup_{(\boldsymbol{u},\boldsymbol{x}_{N,0}) \in \mathcal{S}} \; \inf_{\boldsymbol{x}_{G,0} \in \mathcal{X}_{0,G}} \\ & \frac{\|G[\boldsymbol{u}, \boldsymbol{x}_{G,0}] - N[\boldsymbol{u}, \boldsymbol{x}_{N,0}]\|}{\|N[\boldsymbol{u}, \boldsymbol{x}_{N,0}]\|} \end{aligned} \tag{7}$$

with

$$\begin{aligned} \mathcal{S} = \; & \{(\boldsymbol{u}, \boldsymbol{x}_{N,0}) : \\ & \boldsymbol{u} \in \mathcal{U}_a, \; \boldsymbol{x}_{N,0} \in \mathcal{X}_{0,a}, \; N[\boldsymbol{u}, \boldsymbol{x}_{N,0}] \in \mathcal{Y}_a\} \; , \end{aligned}$$

where $G : \mathcal{U}_a \times \mathcal{X}_{0,G} \rightarrow \mathcal{Y}$ *is a linear dynamic operator belonging to the space of linear operators* $\mathcal{G}$*.* $\mathcal{U}_a$*,* $\mathcal{X}_{0,a}$*,* $\mathcal{Y}_a$ *are the spaces of admissible inputs, initial conditions, and outputs, respectively,* $||\cdot||$ *is a suitable norm in* $\mathcal{Y}$*. All signals and norms are defined over a (finite or infinite) time interval* $[0, t_f)$*. The existence of* $\boldsymbol{y}$ *and the boundedness of all norms in the above definition are assumed for all* $(\boldsymbol{u}, \boldsymbol{x}_{N,0}) \in \mathcal{S}$*.*

Through the above definition nonlinearity of a dynamic system is defined as the normalized largest difference between the nonlinear process and a linear time-invariant system. This difference is calculated for the best linear approximation G^* of N with respect to the "worst-case" combination of inputs $\boldsymbol{u}^*$ and initial conditions $\boldsymbol{x}^*_{N,0}$ keeping the outputs of N in $\mathcal{Y}_a$, and with respect to the best choice of the initial condition $\boldsymbol{x}^*_{G,0}$ of the linear system. An application-relevant signal norm like e.g. the L_2-norm should be applied. The chosen setting for system analysis shows some similarities to the set based stability and performance analysis carried out by Nikolaou and Manousiouthakis (1989), although initial conditions are treated differently. If for all relevant initial conditions the nonlinear system is stable over the set of admissible inputs according to their stability definition, then the assumptions on boundedness of all norms in Definition 1 are fulfilled.

The nonlinearity measure always takes a value between zero and one. The latter is clear from the fact that $\phi_N^{y_a} = 1$ can be achieved through a linear system producing a zero output. Obviously, the value of the measure is zero if N is a linear dynamic operator in the operating regime characterized by $\mathcal{S}$. On the other hand, from $\phi_N^{y_a} = 0$ it may be concluded that N behaves like a linear process on $\mathcal{S}$.

The above definition generalizes an approach of Allgöwer (1995; 1996*a*) which was restricted to the analysis of dynamic processes in the vicinity of stable steady state operating points. Here, no specific assumption is made concerning the relevance, location and stability of a steady state for the analysis. This especially affects the initial condition $\boldsymbol{x}_{N,0}$ which is neither restricted to steady state conditions nor to a single state at all. Rather, both input signal and initial condition may be varied in order to define the "worst-case" scenario. Consequently, non-vanishing initial conditions $\boldsymbol{x}_{G,0}$ have to be considered for the linear approximation. The optimal adjustment of $\boldsymbol{x}_{G,0}$ comprises the most significant extension to the nonlinearity measure proposed in (Allgöwer, 1996*b*), allowing the treatment of transient processes. The determination of $\boldsymbol{x}_{G,0}$ is carried out through an additional inner *inf*-operator rather than through the outer *inf*-operator since latter would result in determining a single optimal initial condition for all possible inputs.

Definition 1 relates the nonlinearity measure $\phi_N^{y_a}$ to the best achievable signal fit on some time interval $[0, t_f)$. Taking a control perspective, one might additionally require that the output of nonlinear and linear model match exactly at time zero. This leads to a slightly different nonlinearity measure definition, which might be more useful in certain cases:

Definition 2: *The nonlinearity measure* $\hat{\phi}_N^{\mathcal{Y}_a}$ *of a nonlinear dynamic system* $N : \mathcal{U}_a \times \mathcal{X}_{0,a} \rightarrow \mathcal{Y}$ *with output signals* $\boldsymbol{y}_N \in \mathcal{Y}_a \subseteq \mathcal{Y}$ *is defined by the nonnegative number*

$$\begin{aligned} \hat{\phi}_N^{\mathcal{Y}_a}(t_f) \;=\; & \inf_{G \in \mathcal{G}} \sup_{(\boldsymbol{u}, \boldsymbol{x}_{N,0}) \in \mathcal{S}} \inf_{\boldsymbol{x}_{G,0} \in \mathcal{X}_{0,G}} \\ & \frac{\|G[\boldsymbol{u}, \boldsymbol{x}_{G,0}] - N[\boldsymbol{u}, \boldsymbol{x}_{N,0}]\|}{\|N[\boldsymbol{u}, \boldsymbol{x}_{N,0}]\|} \\ s.t. \quad & \boldsymbol{0} = G[\boldsymbol{u}, \boldsymbol{x}_{G,0}](0) - N[\boldsymbol{u}, \boldsymbol{x}_{N,0}](0) \end{aligned} \tag{8}$$

with

$$\begin{aligned} \mathcal{S} = \; & \{(\boldsymbol{u}, \boldsymbol{x}_{N,0}) : \\ & \boldsymbol{u} \in \mathcal{U}_a, \; \boldsymbol{x}_{N,0} \in \mathcal{X}_{0,a}, \; N[\boldsymbol{u}, \boldsymbol{x}_{N,0}] \in \mathcal{Y}_a\Big\} \;, \end{aligned}$$

where all variables are defined as in Definition 1 and the same assumptions have to hold.

Application specific setup

The given nonlinearity measure definitions provide a very generic framework for the quantification of open-loop nonlinearity of a dynamic process. It is up to the user to specify the several degrees of freedom in such a way that a meaningful application specific problem is stated, e.g. in a control context. In order to use the most appropriate setup, the following aspects have to be considered.

The first aspect concerns the selection of the reference signals, i.e. the choice of the I/O-“coordinate system” for N. For *continuous processes* operated at some steady state operation point, $\boldsymbol{y}_{N,ref} = \boldsymbol{y}_{N,out_s}$ and $\boldsymbol{u}_{N,ref} = \boldsymbol{u}_{N,in_s}$ clearly is the method of choice, since control is normally realized in deviation variables from a steady state. The equivalent for *transient processes* would be the selection of deviation variables from the nominal trajectory as control relevant inputs and outputs. This, however, is only applicable to such control concepts which directly exploit the nominal input and output signals $\boldsymbol{u}_{N,in_{nom}}$, $\boldsymbol{y}_{N,out_{nom}}$ in output tracking. Since $\boldsymbol{u}_{N,in_{nom}}$ is often not known, such concepts are rarely applied in today’s industrial practice. Many tracking control schemes use a constant reference input (bias) instead. However, if the process in its “natural I/O-coordinates” (1) is linear, a coordinate change through $\boldsymbol{y}_{N,ref} = \boldsymbol{y}_{N,out_{nom}}$ and $\boldsymbol{u}_{N,ref} = const \neq \boldsymbol{u}_{N,in_{nom}}$ will result in a nonlinear operator N. The nonlinearity measure will then detect a nonlinear control problem even though the process is linear. This is, however, totally consistent with the standard notion of linearity/nonlinearity.

The second aspect is related to a meaningful specification of the admissible signal spaces in $\mathcal{S}$. This specification may be solely based on the selection of the admissible input signals which should comprise signals being relevant for the closed loop system. The initial condition may be fixed or vary within certain bounds. In addition, limitations on the considered operating range may be put on the system outputs e.g. by requiring their boundedness by a tube or ball around the nominal trajectory or steady state.

Finally, an appropriate time interval has to be selected. For the analysis in the vicinity of some stable operating point, the selection is only dependent on the "time constants" of N. The interval should be sufficiently large to cover all dynamics of the process. For finite time transient processes an additional time constant given by the duration of the transient has to be obeyed. For nonlinearity analysis e.g. of batch processes, t_f may for instance cover the entire batch duration, but a piecewise analysis of parts of the entire trajectory might also be meaningful.

A number of further considerations are related to the computation and approximation scheme applied. These aspects will be dealt with in the discussion of the different specific application areas of the nonlinearity measure.

It should be stressed that in order to get meaningful and application-specific results, proper specification especially of the reference signals and parameters describing the operating regime is crucial. The selection has to match a specific control scheme which has to be chosen prior to the analysis. If e.g. a gain-scheduling or multi-model control scheme is selected for a transient process, the calculation of nonlinearity measures for pieces of the entire nominal trajectory is a natural choice. Consequently, the nonlinearity measures on the basis of time-varying linear models would be an appropriate consideration if time-varying control is applied. In principle, a measure definition analogue to (7) could be used with G being an element of the space of linear time-variant operator $\tilde{\mathcal{G}}$. Such a measure is, however, not further considered here since no appropriate parameterization for elements of $\tilde{\mathcal{G}}$ is currently known which would allow the calculation through the approach described in the next section.

Computation of the nonlinearity measure

The direct calculation of the nonlinearity measure $\phi_N^{\mathcal{Y}_a}$ according to definition (7) requires the solution of an infinite dimensional nonlinear min-max-min problem which is practically infeasible. Therefore, an approximative computational scheme is required to make the measure applicable. Ideally, the problem is transformed into a convex optimization problem in order to make global optimization feasible. Additionally, the computational scheme should allow a decomposition into a preprocessing step for the nonlinear system and a parameter optimization step for the linear system in order to avoid the necessity to use the – probably high dimensional – nonlinear process model in an optimization procedure. In the following such a scheme is derived for SISO systems.

The computation scheme is based on transforming the original problem (7) into a finite dimensional optimization problem. For this purpose a suitable parameterization of the space of linear time-invariant operators has to be chosen. G is then represented by a weighted sum of basis functions G_i through

$$G[u, \boldsymbol{x}_{G,0}] = \sum_{i=1}^{m} d_i \cdot G_i[u, \boldsymbol{x}_{G,0_i}] \ . \tag{9}$$

Ideally, the space spanned by the functions G_i should be dense in $\mathcal{G}$. To the authors' knowledge this problem has only been solved so far for the space of *stable* linear time invariant systems. This space is spanned by the following functions (Scherer, 1995), defined in the frequency domain:

$$G_i(s) = \frac{1}{(1+T\cdot s)^i}\,,\ i=\{0,1,\ldots\}\,,\ T\in\mathcal{R}^+. \tag{10}$$

Experience has shown, however, that for stable systems choosing a set of first order lags with appropriate time constants is often a more efficient choice. Throughout all presented examples, the linear operator will be parameterized by stable and occasionally also unstable linear systems

$$G_i(s) = \frac{1}{1+T_i\cdot s} \tag{11}$$

with time constants T_i. Furthermore, a direct feedthrough term is added. The output of the linear system is then given by

$$\begin{aligned} G[u,\boldsymbol{x}_{G,0}] &= \sum_{i=1}^{m-1} d_i\cdot G_i[u,x_{G,0_i}] + d_m\cdot u && (12)\\ &= \sum_{i=1}^{m-1} d_i\cdot G_i[u,0] + d_m\cdot u && (13)\\ &\quad + \sum_{i=1}^{m-1} d_i\cdot x_{G,0_i}\cdot G_i[0,1] &&\\ &\quad + d_m\cdot u\,. && \end{aligned}$$

The output y_G thus consists of two parts reflecting the effects of input signal and initial condition, respectively. The linear model (13) is bilinear in the free parameters d_i and $x_{G,0_i}$. Under the reasonable assumption that at the optimum none of the weighting factors d_i, $1 \leq i \leq m-1$, is exactly zero, this problem can be remedied by introducing

$$e_i = d_i \cdot x_{G,0_i} \ , 1 \leq i \leq m-1 \tag{14}$$

as free parameters instead of $x_{G,0_i}$. The obtained linear model class parameterized by $\boldsymbol{d}$ and $\boldsymbol{e}$ will be denoted by

$$G[u, \boldsymbol{x}_{G,0}] = G_{d,e}[u] \ . \tag{15}$$

Using this parameterization for G, the two *inf*-operators in (7) can be replaced by two *min*-operators with $\boldsymbol{d}$ and $\boldsymbol{e}$ as free variables, respectively.

The second major step in the derivation of the computational scheme concerns $\mathcal{S}$ which is approximated by a set $\mathcal{S}_c$ with a finite number of elements, n. The *sup*-operator in (7) is than replaced my a *max*-operator with a finite dimensional search space. Applying a standard procedure for transformation of such *min-max*-problems into constrained *min*-problems (Gill *et al.*, 1981), an approximation $\theta_N^{\mathcal{S}_c}$ of $\phi_N^{\mathcal{Y}_a}$ can now be calculated through the solution of the following parameter optimization problem:

$$
\begin{aligned}
\theta_N^{\mathcal{S}_c} \quad &= \min_{z \in \mathcal{R},\, \boldsymbol{d} \in \mathcal{R}^m} z && (16)\\
s.t. \quad & \forall\ (u_j, \boldsymbol{x}_{N,0_j}) \in \mathcal{S}_c \\
& \frac{||G_{d,e_j}[u_j] - N[u_j, \boldsymbol{x}_{N,0_j}]||}{||N[u_j, \boldsymbol{x}_{N,0_j}]||} - z \leq 0 \\
with \quad & \boldsymbol{e}_j = \operatorname*{argmin}_{\boldsymbol{e_j} \in \mathcal{R}^{m-1}} \frac{||G_{d,e_j}[u_j] - N[u_j, \boldsymbol{x}_{N,0_j}]||}{||N[u_j, \boldsymbol{x}_{N,0_j}]||} \,.
\end{aligned}
$$

The nested optimization problem (16) can be rearranged to a single convex minimization problem with n constraints and free variables z, $\boldsymbol{d}$ and $\boldsymbol{e_1} \ldots \boldsymbol{e_n}$:

$$
\begin{aligned}
\theta_N^{\mathcal{S}_c} \quad &= \min_{z \in \mathcal{R},\, \boldsymbol{d} \in \mathcal{R}^m,\, \boldsymbol{e_1} \ldots \boldsymbol{e_n} \in \mathcal{R}^{m-1}} z && (17)\\
s.t. \quad & \frac{||G_{d,e_j}[u_j] - N[u_j, \boldsymbol{x}_{N,0_j}]||}{||N[u_j, \boldsymbol{x}_{N,0_j}]||} - z \leq 0 \\
& \forall\ (u_j, \boldsymbol{x}_{N,0_j}) \in \mathcal{S}_c \ .
\end{aligned}
$$

This optimization problem is structurally identical to the convex optimization scheme derived for the nonlinearity measure in (Allgöwer, 1996*b*) and therefore also convex. It can be efficiently solved by standard optimization software. Since the response of N to any of the n $(u, \boldsymbol{x}_{N,0})$-tuples contained in $\mathcal{S}_c$ has to be evaluated only once prior to optimization, the calculation of $\theta_N^{\mathcal{S}_c}$ is tractable also for large scale systems. If Definition 2 is applied, n additional constraints for the initial point have to be added in order to calculate an approximation $\hat{\theta}_N^{\mathcal{S}_c}$ of $\hat{\phi}_N^{\mathcal{Y}_a}$.

Various alternative parameterizations of G could be applied, e.g. including also higher order elements, as long as the convexity property for the resulting optimization problem is preserved. Furthermore, the presented approach directly carries over to the MIMO case (Odenthal and Allgöwer, 1995). Clearly, the approximation quality of the above approach is dependent on the type and number of basis functions G_i as well as on the input signals and initial conditions considered in $\mathcal{S}_c$. Their selection and other specific approximation schemes relying on a single type of inputs or a fixed linear model will be discussed with the applications below.

Nonlinearity measures for continuous processes at fixed operating points

Analysis of process nonlinearity in the vicinity of some steady state operating point forms a specific application area for the generic nonlinearity measure. The specific setting associated with such an analysis will first be considered for stable operating points.

Stable operating points

Formulation

In case the system under consideration is BIBO-stable the boundedness of all norms in Definition 1 is directly given. With regard to the required application specific setup discussed above, the following setting has proven to be meaningful:

1. Reference signals: The steady state process input and output values are applied, thus $\boldsymbol{y}_{N,ref} = \boldsymbol{y}_{N,out_s}$ and $\boldsymbol{u}_{N,ref} = \boldsymbol{u}_{N,in_s}$.

2. Operating regime: Assuming the process to be initially at steady state is reasonable especially when considering an infinite time horizon. Therefore $\mathcal{X}_{0,a} = \{\boldsymbol{x}_{N,s}\}$.

   The region of operation $\mathcal{Y}_a$ considered for an operating point dependent analysis can be selected implicitly through appropriate bounds on the inputs expressed by $\mathcal{U}_a$. An additional specification of $\mathcal{Y}_a$ is usually not required.

3. Time interval: Due to the stability properties, analysis can be carried out over an infinite time horizon. Practically, t_f should be “sufficiently large” as discussed above.

Since initial conditions are of no importance in this type of analysis the inner *inf*-operator in (7) can be left out, fixing $\boldsymbol{x}_{G,0}$ to zero. The specific operating point dependent version $\phi_{N_{OP}}^{\mathcal{U}_a}$ of $\phi_N^{\mathcal{Y}_a}$ is therefore given by

$$\phi_{N_{OP}}^{\mathcal{U}_a} = \inf_{G \in \mathcal{G}} \sup_{u \in \mathcal{U}_a} \frac{||G[\boldsymbol{u}, \boldsymbol{0}] - N[\boldsymbol{u}, \boldsymbol{x}_{N,s}]||}{||N[\boldsymbol{u}, \boldsymbol{x}_{N,s}]||} \tag{18}$$

and equivalent to the measure given in (Allgöwer, 1996*b*). An approximation $\theta_{N_{OP}}^{\mathcal{U}_c}$ can be calculated through (17) at reduced computational burden by eliminating $\boldsymbol{e}$ from the optimization ($\boldsymbol{e}_1 \ldots \boldsymbol{e}_n = \boldsymbol{0}$). A reasonable choice for the set $\mathcal{U}_c$ of considered admissible inputs comprises L_2-signals with bounded amplitude. Pulse functions, Laguerre polynomials or stochastic signals have proven to be appropriate input signals for analyzing continuous processes in the vicinity of a stable operating point as long as they contain the relevant amplitude and frequency information. Experience has shown that the value of the nonlinearity measure – but not the obtained "best" linear approximation G^* – is rather insensitive with respect to the type of input signals chosen (Allgöwer, 1996*a*; Allgöwer, 1996*b*).

Computation of a lower bound

The computational effort for the analysis of stable operating points can further be reduced significantly if analysis is restricted to the computation of a lower bound $\chi_N^{\mathcal{U}_s}$ for $\phi_{N_{OP}}^{\mathcal{U}_a}$. The following discussion will be restricted to SISO systems.

The simplification of the computation is based on restricting $\mathcal{U}_a$ to a set $\mathcal{U}_s$ of sine-inputs only:

$$\mathcal{U}_s = \{u \mid u(t) = A \cdot \sin(\omega t),\ A \in \mathcal{A},\ \omega \in \Omega\}, \tag{19}$$

where the amplitude A and the frequency ω are taken from the sets

$$A \in \mathcal{A} \;=\; \{A \in \mathcal{R}^+ \mid 0 \leq A \leq A_{max}\}\ , \tag{20}$$

$$\omega \in \Omega \;=\; \{\omega \in \mathcal{R}^+ \mid \omega_{min} \leq \omega \leq \omega_{max}\}\ . \tag{21}$$

This restriction and an interchange of the remaining *sup* and *inf* in (7) leads to the value $\chi_N^{\mathcal{U}_s}$ that for $\mathcal{U}_s \subseteq \mathcal{U}_a$ can be proven to be a lower bound of $\phi_N^{\mathcal{Y}_a}$ (Allgöwer, 1996*b*). Evaluating the periodic steady state response ($t_f \to \infty$) of the nonlinear system, this measure can be defined as:

$$\chi_N^{\mathcal{U}_s} \;=\; \sup_{\boldsymbol{u}\in\mathcal{U}_s} \inf_{G\in\mathcal{G}} \lim_{T\to\infty} \frac{||G[\boldsymbol{u},\boldsymbol{0}] - N[\boldsymbol{u},\boldsymbol{0}]||}{||N[\boldsymbol{u},\boldsymbol{0}]||}\ . \tag{22}$$

Here, the norm $||\cdot||$ is given by

$$||y|| = \lim_{T\to\infty} \sqrt{\frac{1}{T}\int_0^T y^2(t)\, dt}\ . \tag{23}$$

A computationally inexpensive calculation of $\chi_N^{\mathcal{U}_s}$ can be realized on the basis of a Fourier expansion of the periodic steady state response. The following equation holds (Allgöwer, 1996*b*):

$$\chi_N^{\mathcal{U}_s} \;=\; \sup_{\substack{\boldsymbol{A}\in\mathcal{A}\\ \omega\in\Omega}} \sqrt{1-\frac{A_1^2(\omega,A)}{2A_0^2(\omega,A)+\sum\limits_{k=1}^{\infty}\frac{A_k^2(\omega,A)}{2}}}\,, \tag{24}$$

where $A_k(\omega, A)$ are the amplitude coefficients of the k^{th} harmonics of the periodic steady state response of the nonlinear system to inputs $u(t) = A \cdot \sin(\omega t)$.

The computation of $\chi_N^{\mathcal{U}_s}$ reduces to the determination of Fourier coefficients and is thus far less involved than the computation of $\theta_{N_{OP}}^{\mathcal{U}_c}$. It can be carried out for very complex systems as well as for measured data directly obtained from the plant. In addition, Harris *et al.* (1997) have recently presented a computationally attractive approach to calculate the parameter dependence of $\chi_N^{\mathcal{U}_s}$ on the basis of functional expansions. If, however, only sinusoidal inputs are considered in the convex optimization approach ($\mathcal{U}_c = \mathcal{U}_s$) then $\theta_{N_{OP}}^{\mathcal{U}_s}$ is still a tighter bound on $\phi_{N_{OP}}^{\mathcal{U}_a}$ than $\chi_N^{\mathcal{U}_s}$ ($\theta_{N_{OP}}^{\mathcal{U}_s} \geq \chi_N^{\mathcal{U}_s}$).

For the lower bound calculation an extension of satisfying theoretical soundness going beyond SISO systems is only possible for the single-input/multi-output case. For MIMO problems, a heuristic approach has to be chosen.

Independent of the chosen computational scheme scaling of inputs and outputs is necessary for multi-variable problems.

Unstable operating points

The problem setting for stable operating points can in principle also be applied in order to compute nonlinearity measures $\phi_{N_{OP}}^{\mathcal{U}_a}$ or $\theta_{N_{OP}}^{\mathcal{U}_c}$, respectively, for unstable operating points. The main difference must be seen in the time interval under consideration. For unstable systems t_f cannot be chosen to go to infinity in order to keep the signal norms bounded. Consequently, an equivalent to the lower bound $\chi_N^{\mathcal{U}_s}$ does not exist in the case of unstable steady states. Further discussions can be found in example 3 which considers analysis of an unstable example CSTR.

Applications

In this section applications of the given nonlinearity measure to operating point dependent analysis of three different continuously stirred tank reactors (CSTR) are discussed.

Example 1: A mildly nonlinear CSTR

A first example for nonlinearity quantification in the vicinity of a stable operating point treats a CSTR model given by Pottmann and Seborg (1992). Menold *et al.* (1997*a*) have studied this process in the context of model suitability measures which contains the operating point dependent nonlinearity measure $\phi^{\mathcal{U}_a}_{N_{OP}}$ as a special case. The well understood problem is revisited as test case for the more general nonlinearity measure according to Definition 1.

In the considered CSTR, a single irreversible exothermic reaction $A \longrightarrow B$ takes place. The dynamic behavior of the reactor content can be described by the following differential equations for the concentrations c_A of component A and for the reactor temperature T:

$$\begin{aligned} \frac{dc_A}{dt} &= \frac{q}{V}(c_{Af} - c_A) - k_0 c_A e^{\left(\frac{-E}{RT}\right)} , && (25) \\ \frac{dT}{dt} &= \frac{q}{V}(T_f - T) - \frac{(-\Delta H) k_0 c_A}{\rho C_p} e^{\left(\frac{-E}{RT}\right)} && (26) \\ &\quad + \frac{\rho_c C_{pc}}{\rho C_p V} q_c \left[1 - e^{\left(\frac{-hA}{q_c \rho_c C_{pc}}\right)}\right] (T_c - T) . \end{aligned}$$

For the model parameters the reader is referred to (Pottmann and Seborg, 1992). The stable steady state under consideration is given by $\boldsymbol{x}_{N_s} = \left(8.235 \cdot 10^{-2} \, \frac{mol}{l};\ 441.81 \, K\right)^T$. The process input is the coolant flowrate ($u_{N,in} = q$) with $u_{N,in_s} = 100\,\frac{l}{s}$ and the process output $y_{N,out}$ is c_A.

In (Menold *et al.*, 1997*a*) nonlinearity is assessed in a vicinity of the steady state defined through an input region with $||u|| \leq 10\,\frac{l}{s}$. Approximations to $\phi_{N_{OP}}^{\mathcal{U}_a}$ have been calculated by means of the convex optimization approach in a discrete time setting, applying a moving average structure (step response model) for the linear system G. The process is analyzed on a $70\,min$ time interval. The considered input set consists of 8 steps and 132 random sequences ($n = 140$). For the latter, the input value changes randomly every $0.1\,min$ with a uniform amplitude distribution. A measure $\theta_{N_{OP}}^{\mathcal{U}_c} = 0.244$ is obtained with 40 degrees of freedom in the linear system (Menold *et al.*, 1997*a*). The value is confirmed with a second set of input sequences on the basis of Laguerre polynomials.

Here, the linear (continuous-time) operator is parameterized by 20 stable first order lags with logarithmically equal distribution of the time constants T_i between $2\pi \cdot 10^{-3}\,min$ and $2\pi \cdot 10^{3}\,min$ and a direct feedthrough term. Applying the same random input sequences from (Menold *et al.*, 1997*a*) yields $\theta_{N_{OP}}^{\mathcal{U}_c} = 0.248$. This confirms the suitability of the first order lag parameterization of the linear system.

If additionally 20 unstable first order lags with logarithmically equal distribution of the time constants between $2\pi \cdot 10\,min$ and $2\pi \cdot 10^3$ are added, the resulting nonlinearity measure remains practically unchanged (0.247). As expected, the unstable elements are neither needed here nor do they cause computational problems as long as time constants are not too small.

Finally, a nonlinearity measure can also be calculated through (17) with free initial conditions of the linear system. Using again 20 stable lag elements and a feedthrough term, the total number of free variables for the calculation of $\theta_N^{\mathcal{S}_c}$ is then 2822 compared to only 22 for $\theta_{N_{OP}}^{\mathcal{U}_c}$, since $n \cdot (m-1) = 140 \cdot 20 = 2800$ initial conditions have to be determined. This leads to a dramatic increase in computation time. The obtained valued of $\theta_N^{\mathcal{S}_c} = 0.215$ shows that the time horizon was chosen long enough in the above computations in order to practically remove the effect of initial conditions on the nonlinearity measure. This example confirms that for continuous processes operated around a fixed stable operating point, the special nonlinearity measure for stable operating points makes indeed sense and is consistent with the more general Definition 1.

Example 2: A highly nonlinear CSTR

Open loop nonlinearity of a second CSTR will be assessed comparing different operating points. The considered process is a benchmark problem for nonlinear controller design and exhibits interesting dynamic characteristics (Chen *et al.*, 1995). The chemical reactions are described by the van de Vusse reaction scheme:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

$$2A \xrightarrow{k_3} D \ .$$

The reactor is equipped with a cooling system. The temperature of the recirculated cooling medium is manipulated through a heat flow $\dot{Q}$. The inlet flow $\dot{V}$ with temperature T_0 contains reactant A with a concentration c_{A0}. Reactor dynamics can be described by the following differential equations for the concentrations c_A, c_B of A and B, the reactor temperature T, and the cooling medium temperature T_c:

$$\begin{aligned}
\frac{dc_A}{dt} &= \frac{\dot{V}}{V_R}(c_{A0} - c_A) - k_1 c_A - k_3 {c_A}^2 \ , && (27) \\
\frac{dc_B}{dt} &= -\frac{\dot{V}}{V_R} c_B + k_1 c_A - k_2 c_B \ , && (28) \\
\frac{dT}{dt} &= \frac{\dot{V}}{V_R}(T_0 - T) - \frac{1}{\rho C_p}(k_1 c_A \Delta H_{R_{AB}} && (29) \\
& \quad + k_2 c_B \Delta H_{R_{BC}} + k_3 {c_A}^2 \Delta H_{R_{AD}}) \\
& \quad + \frac{k_w A_R}{\rho C_p V_R}(T_c - T) \ , \\
\frac{dT_c}{dt} &= \frac{1}{m_c C_{pc}}(\dot{Q} + k_w A_R (T - T_c)) \ , && (30)
\end{aligned}$$

with the reaction rate constants described by

$$k_i = k_{i0} e^{\left(\frac{E_i}{T}\right)} \;, \quad i = 1, 2, 3 \; . \tag{31}$$

In a SISO setting the concentration $c_B = y_{N,out}$ is controlled by the scaled volumetric inlet flow $\frac{\dot{V}}{V_R} = u_{N,in}$ with

$$3\,h^{-1} \leq \frac{\dot{V}}{V_R} \leq 35\,h^{-1} \; . \tag{32}$$

Two different stable operating points are considered: an optimal operation point (OP) which is characterized by maximum yield with respect to the desired product B and a sub-optimal operating point (SP) with a lower yield. For the CSTR model parameters and steady state data the reader is referred to Chen *et al.* (1995).

In order to get a first qualitative impression of the dynamic behavior at the two operating points, step responses of the product concentration c_B with respect to the manipulated variable are studied. Equally spaced step amplitudes with values in the range $-11.19\,h^{-1} \leq \frac{\Delta\dot{V}}{V_R} \leq 11.19\,h^{-1}$ are applied and the results are shown in Figures and for OP and SP, respectively. Since the step responses obviously violate the superposition principle, system dynamics are clearly nonlinear in both cases, but a quantification of the "degree" of nonlinearity is hardly possible based on these plots. Intuitively, one may consider OP as the more nonlinear operating point.

For the computation of the nonlinearity measure the linear operator is parameterized by 20 stable first order lags with logarithmically equal distribution of the time constants T_i between $2\pi/1000\,h$ and $2\pi\,h$. The set of inputs considered in $\mathcal{S}_c$ consists of 108 pulse functions with amplitudes A between $\pm\,1\,\frac{1}{h}$ and $\pm\,11.19\,\frac{1}{h}$ and pulse lengths T_u between $0.01\,h$ and ∞ (=step). These settings result in

$$\theta_{N_{OP}}^{\mathcal{U}_c}(OP) = 1.00\ , \qquad \theta_{N_{OP}}^{\mathcal{U}_c}(SP) = 0.37$$

revealing a significant difference in nonlinearity between the two operating points. The required CPU time is about 10 minutes on a (rather slow) VAX 4090. For the calculation of the lower bound $\chi_N^{\mathcal{U}_s}$ an input set $\mathcal{U}_s$ with 357 sinusoidal functions is chosen for each operating point with ω varying from $2\pi/500\,\frac{1}{h}$ to $2\pi/0.01\,\frac{1}{h}$. The same amplitudes are chosen as for the calculation of $\theta_N^{\mathcal{U}_c}$. The obtained nonlinearity measures

$$\chi_N^{\mathcal{U}_s}(OP) = 0.92\ , \qquad \chi_N^{\mathcal{U}_s}(SP) = 0.33$$

are in good agreement with the above results and verify the high nonlinearity of the system at the optimal operating point. Computation of $\chi_N^{\mathcal{U}_s}$ is more than ten times faster compared to $\theta_{N_{OP}}^{\mathcal{U}_c}$.

The obtained results can be interpreted by system theoretical arguments: At the optimal operating point the stability characteristics of the zero dynamics change, leading to a sign change in the steady state gain. For all systems with this characteristics $\theta_{N_{OP}}^{\mathcal{U}_c}$ can be proven to be equal to one (Menold and Allgöwer, 1999). The strong nonlinearity of this operating point also manifests itself by the fact that a linear controller with integral action will not be able to robustly stabilize the reactor and accomplish satisfying performance (not integral controllable). A value for the nonlinearity measure of 0.33 also indicates a strong nonlinearity for the sub-optimal operating point that is, however, by far not as pronounced as the nonlinearity at the optimal operating point. Without any further investigations or without a preliminary controller design, this is an indication that more severe problems have to be anticipated for a controller design at the optimal operating point. Thus, the nonlinearity analysis gives valuable insight into the process.

Due to the sign change at the optimal operating point, highly nonlinear behavior of the reactor can be expected already in a small vicinity of the operating point. This characteristic is correctly revealed by a frequency and amplitude dependent analysis of the nonlinearity measure. For the lower bound $\chi_N^{\mathcal{U}_s}$ the results are given in Figures and . As expected, the nonlinearity measure at OP reaches its maximum value already for very low signal amplitudes whilst the measure increases with the amplitude at the sub-optimal operating point. Similar results can be obtained by a frequency dependent analysis of $\theta_{N_{OP}}^{\mathcal{U}_c}$ (Odenthal, 1995).

Example 3: Unstable CSTR

The following model describes a CSTR with an auto-catalytic main reaction $A \rightarrow B$ and two side reactions $A \rightarrow D$ and $B \rightarrow C$:

$$\begin{aligned}
\frac{dc_A}{dt} &= q(c_{A0} - c_A) - r_1(\vartheta, c_B) - r_2(\vartheta, c_A) \ , && (33)\\
\frac{dc_B}{dt} &= -qc_B + r_1(\vartheta, c_B) - r_3(\vartheta, c_B) \ , && (34)\\
\frac{d\vartheta}{dt} &= q(\vartheta_0 - \vartheta) - \frac{1}{\rho_c}(r_1(\vartheta, c_B)\Delta H_1 && (35)\\
&+ r_2(\vartheta, c_A)\Delta H_2 + r_3(\vartheta, c_B)\Delta H_3) + \frac{1}{\rho_c} j_Q \ .
\end{aligned}$$

with the reaction rates described by

$$\begin{aligned}
r_1(\vartheta, c_B) &= (a_1 + b_1\vartheta)\, c_B^2 \ , && (36)\\
r_2(\vartheta, c_A) &= (a_2 + b_2\vartheta)\, c_A^2 \ , && (37)\\
r_3(\vartheta, c_B) &= k_3 c_B \ . && (38)
\end{aligned}$$

For the model parameters the reader is referred to (Allgöwer, 1996*b*) where the process has been studied in the context of approximately I/O-linearizing control. The considered steady state is given by $\boldsymbol{x}_{N_s} = \left(1.0\,\frac{mol}{l};\ 1.5\,\frac{mol}{l};\ 100.0\,^{\circ}C\right)^T$. The volume specific cooling energy j_Q is the process input $u_{N,in}$ with $u_{N,in_s} = -10\,\frac{kJ}{l\,min}$ and the reactor temperature ϑ is the process output $y_{N,out}$. The process is unstable and nonminimum-phase with the poles of the Jacobi-linearized model being located at $s_1 = 1.63\frac{1}{min}$, $s_2 = -0.97\frac{1}{min}$, and $s_3 = -0.15\frac{1}{min}$. The process zeroes are located at $z_1 = 0.27\frac{1}{min}$ and $z_2 = -0.3\frac{1}{min}$.

The same set of 140 random and step input sequences is applied as in Example 1, but signals have been scaled by different values in order to compare different operating regions. In order not to "blow up" the process, only small time intervals can be chosen. The step length is therefore reduced to 3 s such that $t_f = 3.5\,min$. The linear operator is parameterized by 40 stable and 40 unstable first order lags with logarithmically equal distribution of the time constants between $2\pi \cdot 10^{-3}\,min$ and $2\pi \cdot 10e^3\,min$ for the stable and between $2\pi \cdot 0.05\,min$ and $2\pi \cdot 10e^3$ for the unstable elements. A direct feedthrough term is also used.

The results for inputs of different magnitude are summarized in Table 1 also showing the covered output range for the different cases. Clearly, already for small output deviations from the steady state the system is highly nonlinear.

Discussion

The presented computational schemes for process analysis in the vicinity of stable operating points allow effective calculations of measures of open-loop nonlinearity both for an approximative analysis (lower bound) and a detailed analysis up to any desired level of exactness (convex optimization). The proper selection of input signals with respect to the covered dynamic range (frequency and amplitude) is of predominant importance, whilst the type of input signal and computational method are less important. The obtained information is clearly relevant in order to decide whether application of nonlinear control techniques should be investigated. The application of the operating point dependent measure is, however, not restricted to this type of analysis. Further application areas include

- nonlinearity tests (yes/no) in structure identification,
- nonlinear model validation and comparison of different nonlinear models,
- control structure selection (for example, selection of inputs and outputs),

- nonlinearity assessment of the "optimal control structure" (Stack and Doyle III, 1997*b*), and
- analysis and design of I/O-linearizing controllers.

The last aspect is of high importance. The operating point dependent nonlinearity measure has been successfully utilized in design and robustness analysis of approximately I/O-linearizing control schemes (Allgöwer, 1996*b*; Allgöwer and Doyle III, 1998). An extension of the concept of stable operating point dependent nonlinearity measures to a more general system structure assessment with so-called suitability measures can be found in (Menold *et al.*, 1997*a*; Menold *et al.*, 1997*b*; Pearson *et al.*, 1997)

Nonlinearity assessment for unstable operating points necessarily treats a finite time process and can therefore also be interpreted as a special case of finite time transient process analysis. The inclusion of unstable elements in the "best" linear system for transient processes may be necessary. It is well known that e.g. for many batch processes the Jacobi-linearization along the nominal trajectory is typically stable at the beginning and the end of the process, but "goes unstable" in between. The presented operating point dependent analysis must be seen as an initial step to tackle analysis of this observation. Due to the rather heuristic approach chosen for the parameterization of the linear system this measure lacks, however, the sound theoretical foundation available for stable systems.

Conclusions

The paper presented a new general definition of a nonlinearity measure for dynamic systems. The definition provides a unifying framework for nonlinearity assessment of both stationary and transient processes. Application examples have been restricted to operating point dependent analysis of stationary processes. It could be shown that useful process insight and control relevant information can be gained by computationally relatively easily tractable means. The main reason for the computational simplicity of operating point dependent analysis arises from the fact that initial conditions are known for the nonlinear and linear system. This situation does no longer hold in the general case of trajectory dependent analysis of transient processes. This application area of nonlinearity measures will be discussed in the forthcoming second part of this communication (Helbig *et al.*, 1999*b*).

Acknowledgment:

This work has partly been supported by the Human Capital and Mobility program of the European Commission within the Network in Chemical Process Control, contract ERBCHRXCT940672. Contributions from M. Klander, D.

Odenthal, W. Schobbert, and especially P. Menold are greatfully acknowledged.

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$\|u\|_{max}$ $[\frac{kJ}{l\,min}]$	$y_{min}[^{\circ}C]$	y_{max} $[^{\circ}\mathrm{C}]$	$\theta^{\mathcal{U}_c}_{N_{OP}}$
0.05	-4.1	8.1	0.454
0.1	-6.4	25.2	0.470
0.5	-9.0	265.8	0.940

Table 1: Results for unstable CSTR.

Captions to Figures

Figure 1: Process schematic.

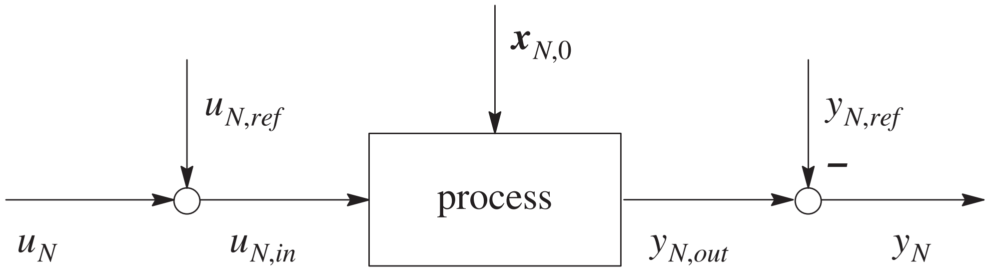
Figure 2: CSTR step responses at the optimal operating point OP.

Figure 3: CSTR step responses at the sub-optimal operating point SP.

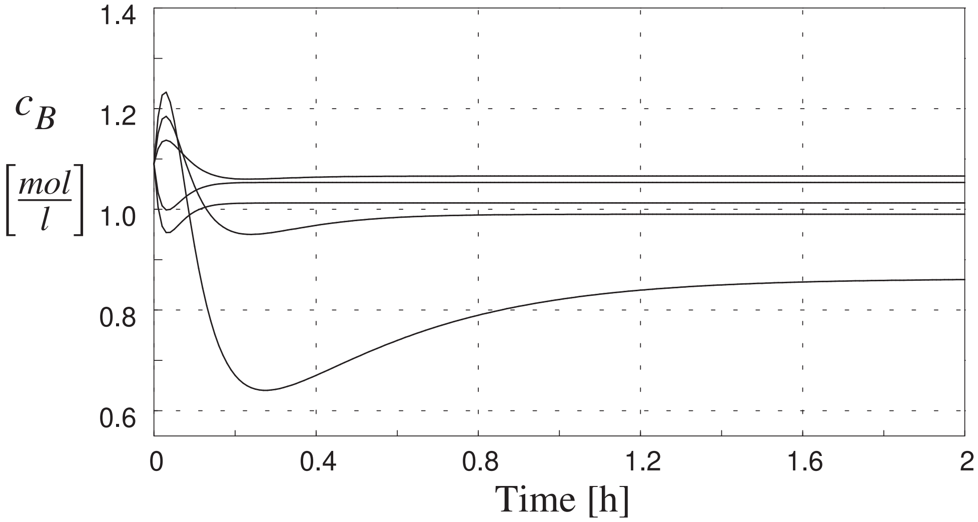
Figure 4: Dependency of $\chi_N^{\mathcal{U}_s}$ on amplitude and frequency of sinusoidal input signal, OP.

Figure 5: Dependency of $\chi_N^{\mathcal{U}_s}$ on amplitude and frequency of sinusoidal input signal, SP.

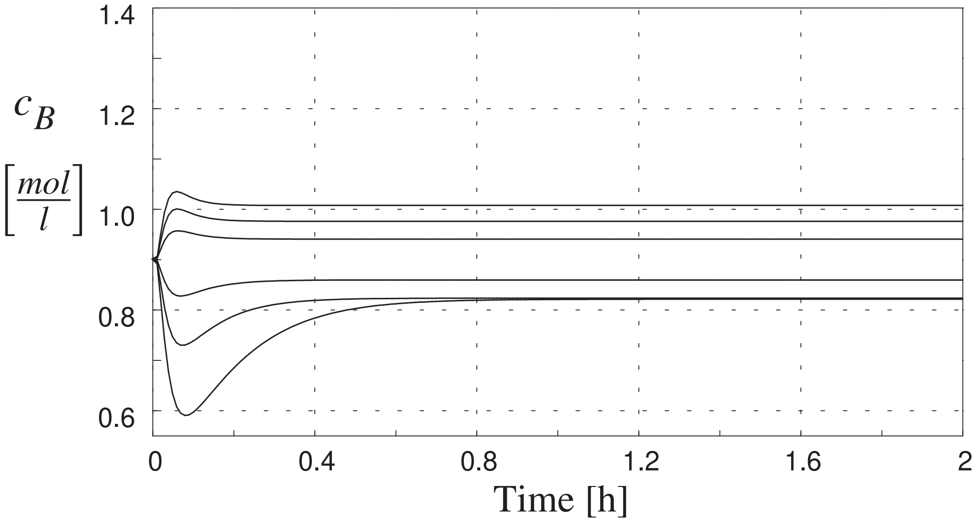
Journal of Process Control, Helbig et al., Figure 1

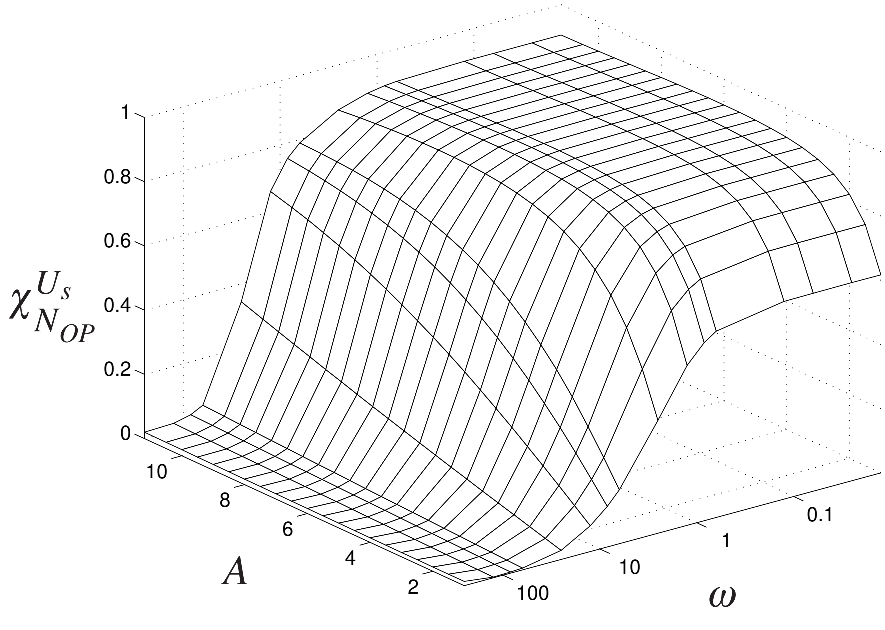


Journal of Process Control, Helbig et al., Figure 2



Journal of Process Control, Helbig et al., Figure 3




$\chi_{N_{OP}}^{U_s}$
0
0.2
0.4
0.6
0.8
1
10
8
6
4
2
A
100
10
1
0.1
ω

Journal of Process Control, Helbig et al., Figure 5