

An efficient algorithm to solve optimal control problem for nonlinear switched hybrid systems

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Abstract

This paper focuses on the gain in efficiency that may be obtained when using strong variations-like algorithms to solve optimal control problems for switched hybrid nonlinear systems. After a review of existing algorithms, a simple version of a strong variations algorithm is proposed together with some new convergence results. A key features in the proposed algorithm are : 1) the absence of a priori assumptions on the number of switches. 2) The complexity is independent of the state's dimension. 3) The algorithm uses a unified approach for both continuous input and discrete switching strategy. Several examples including a system with 8 states and 7 configurations are used to illustrate the efficiency of the proposed algorithm.

1 Introduction

Hybrid systems paradigm is a key issue in process engineering. Since the early paper of Witsenhausen [29] that pointed out some features of hybrid systems. Many works have been done to properly define such systems [8, 27, 20]. Then attention has been focused on possible characterizations of optimality enabling an exhaustive search over the whole set of possibilities to be avoided. Such characterizations may be directly obtained by using the Bellman principle via a dynamic programming approach [4, 15]. Although such characterization is universal and hence directly applies to hybrid systems, its use for systems with high state's dimension is cumbersome since the complexity of the unknown function V increases exponentially with the state dimension in the general nonlinear case. That is the reason why generalizations of the Pontryagin Maximum Principle (**MP**) have been attempted. [27, 23, 28, 20].

As for concrete optimal control computational algorithms, an excellent almost exhaustive recent survey of existing attempts can be found in [31]. Here, the basic methodologies are briefly described in order to underline the novelty of the proposed approach.

In [14], the dynamic programming principle is used to compute the optimal cost function. The complexity is maintained at a low level by projecting the computed cost function at the end of each iteration on a low dimensional functional basis. This projection is done under precision-related constraint yielding a controlled approximation. For truly complicated optimal cost function $V(x)$ however, and despite this clever trick, the methodology remains limited to system with low number of states. The dynamic programming paradigm is also invoked in [9, 10] using level sets and behavioural programming with roughly the same unfortunate explosion of complexity with the state dimension.

Another approach that gains increasing number of followers everyday is the one based on Mixed-Integer Logical Dynamics (**MILD**) formulation where basically linear dynamics are used with extended state containing the continuous and binary variables [7, 5]. The approach is very attractive since systematic tools begin to appear enabling an easy implementation and because of the wide class of hybrid features and constraint that it may conceptually tackle. however, when it is used to solve truly nonlinear problems, the dimension of

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the decision variable increases exponentially with the state dimension and the solution needs branch & bound-like approaches to explore the huge combinatorial set of possibilities.

In ([13, 12, 30, 6], a two stage approach has been proposed. In the first stage, the total number of switches is a priori fixed as well as the sequence of active subsystems. By doing this, the cost function is only function of the switching instants. Therefore, finding the optimal switching instants for the given number of switches and the sequence of active subsystems is a classical nonlinear programming problem. In the second stage, the a priori given data, namely, the number of switches and the sequence of active subsystems are updated to improve the optimal solution that would be obtained by the first stage.

The work by [25, 26] uses the same two-stage structure but uses explicitly the maximum-principle in deriving the updated value of the switching instants. In [18] optimizing the switching instants when the number of switches and the sequence of active configurations is given is achieved using either constrained nonlinear programming of the maximum principle.

In this paper, it is shown that as long as switched nonlinear hybrid systems with only external switching controls are concerned, strong variations algorithms enables a unified approach that iterates on both continuous and logical variables and avoid the use of a priori assumptions on the number of switches or the sequence of active configurations.

Strong variations algorithms are based on a leading paper of R. V. Gamkrelidze [11] where it has been shown that variations in the control profiles that are small in the L_2 norm but not in the extremum norm may be used in the iterations when looking for an optimal control profile. This makes possible to imagine algorithms for optimal control problems in which the admissible control sets are not necessarily convex [19, 21, 16, 3]. In particular, discrete or boolean sets can be naturally handled by such algorithms. This is of a crucial interest for our concern. The use of strong variations algorithms to solve hybrid optimal control problems has already been suggested in [1]. The present papers confirms the conjecture stated there about the efficiency that such algorithms may have in tackling this problem.

The paper is organized as follows: First the problem in question is clearly stated in section 2 and re-formulated in a rather standard form in section 3. In section 4 a simple algorithm is proposed with some convergence results. The algorithm may be viewed as a free-disturbance version of an existing algorithm that has been proposed in [3] to solve nonlinear differential games over non convex sets in the context of robust model predictive control of batch processes. However, the fact that only the free-uncertainty case is considered here, new stronger convergence results are derived here that might not hold for the min-max version of the algorithm given in [3]. Finally, in section 5, the algorithm is applied on several examples of switched hybrid systems to assess its efficiency in solving the optimal control problem under concern. It is particularly shown that for relatively large state's dimension, no combinatoric effects appear nor tree search-like task is needed. The paper ends with some concluding remarks suggesting some ideas for further investigations.

2 Problem statement

Let us consider a nonlinear system that may be in Q different configurations. Let $q \in \underline{Q} := \{1, \dots, Q\}$ be a discrete state variable used to designate the active configuration of the system. It is assumed that both the continuous state of and the control input, denoted respectively by $x \in \mathbb{R}^n$ and $v \in \mathcal{V} \subset \mathbb{R}^m$ are uniquely defined whatever is the active configuration q . When the system is at configuration $q \in \underline{Q}$, its dynamics is described by

$$\dot{x} = f_q(x, v) \quad ; \quad x \in \mathbb{R}^n \quad ; \quad v \in \mathcal{V} \subset \mathbb{R}^m \quad ; \quad q \in \underline{Q} \quad (1)$$

where $\mathcal{V} \subset \mathbb{R}^m$ is some set of admissible controls.

To complete the system's definition, one has to describe how the system switches from one configuration to another. In the general hybrid systems framework, this may be induced by external inputs (controlled

switches) and/or when the state crosses some switching boundary (autonomous switches). In the present paper, interest is focused on hybrid systems with only controlled switches. Furthermore, the jump is assumed to be completely free in the sense that whatever is the present configuration, the system may jump to any other configuration. The aim of this paper is to propose a heuristic algorithm with appropriate convergence properties that provides a *good sub-optimal* switching strategy $q^*(\cdot)$ between configurations and a corresponding *good sub-optimal* control $v^*(\cdot)$ such that the following functional is minimized

$$J(x_0, q(\cdot), v(\cdot)) := \int_0^T L_{q(\tau)}(x(\tau), v(\tau)) d\tau \quad (2)$$

where for all $q \in \underline{Q}$, $L_q : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_+$ is some penalty function depending on the system's configuration.

3 Problem's re-formulation

The system's dynamics can be clearly re-written as follows

$$\dot{x} = \sum_{q \in \underline{Q}} \alpha_q f_q(x, v) \quad (3)$$

provided that for all $q^0 \in \underline{Q}$, α_{q^0} is equal to 1 if q^0 is the "active" configuration and $\alpha_{q^0} = 0$ otherwise. More precisely, the α_q 's have to meet the following requirements

$$\forall q \in \underline{Q} \quad , \quad \alpha_q \in \{0, 1\} \quad \text{and} \quad \sum_{q \in \underline{Q}} \alpha_q = 1 \quad (4)$$

namely, one and only one of the α_q 's is equal to 1 while the others are all 0's. In what follows, the constraint (4) is formally denoted by $\alpha = (\alpha_1 \dots \alpha_Q)^T \in \mathcal{A}$ where $\mathcal{A} \subset \{0, 1\}^Q$ is a discrete subset with $\text{card}(\mathcal{A}) = Q$. with this notation, equation (3) becomes

$$\dot{x} = F(x, v, \alpha) := \sum_{q \in \underline{Q}} \alpha_q f_q(x, v) \quad ; \quad v \in \mathcal{V} \quad \text{and} \quad \alpha \in \mathcal{A} \quad (5)$$

and by concatenating (v, α) in a single control vector $u := (v, \alpha) \in \mathcal{V} \times \mathcal{A}$ it comes that the class of hybrid systems under study may be written in the following general form

$$\dot{x} = f(x, u) \quad ; \quad u \in U := \mathcal{V} \times \mathcal{A} \subset \mathbb{R}^m \times \{0, 1\}^Q \quad (6)$$

with a non convex set U of admissible controls.

Similarly, the cost function may be rewritten as follows

$$J(x_0, u) := \int_0^T \sum_{q=1}^Q \alpha_q(\tau) L_q(x(\tau), v(\tau)) d\tau =: \int_0^T L(x(\tau), u(\tau)) d\tau \quad ; \quad u := (v, \alpha) \quad (7)$$

In the following section, a simple algorithm is proposed to find "good" sub-optimal solutions $u(\cdot)$ to the problem of minimizing (7) under (6) with controls that belong to the non convex set U .

4 The proposed algorithm

In this section, a general purpose algorithm is given to solve optimal control problems in which the control input may be restricted to possibly non convex admissible sets. Consider the following constrained optimization problem

$$\text{Minimize} \quad J(u) = \int_{t_0}^{t_f} L(x(t), u(t), t) dt \quad (8)$$

subject to the system equations

$$\dot{x}(t) = f(x(t), u(t), t) \quad ; \quad x(t_0) = x_0 \quad (9)$$

and the control constraint

$$u(t) \in U \quad \forall t \in [t_0, t_f] \quad (10)$$

where U is not necessarily convex. Assume that the following assumption holds

ASSUMPTION 1

- The functions f and L are twice continuously differentiable. (Note that for the original problem this can be guaranteed by assuming twice continuous differentiability of f_q and L_q for all $q \in \underline{Q}$).
- There is a positive real $M > 0$ such that for all admissible control u , the corresponding state trajectory $x(t; t_0; x_0, u)$ satisfies the following inequality

$$\forall t \in [t_0, t_f] \quad , \quad \|x(t; t_0; x_0, u)\| \leq M \quad (11)$$

To solve the above problem for non convex admissible sets U , several existing algorithms may be used such as the ones given in [16, 19, 21]. However, these strong variations algorithms are rather tedious to properly encode. The proposed algorithm, based on a variable penalty technique is very simple while enabling convergence results to be derived.

4.1 The algorithm

It is the Maximum-Principle characterization of the optimal solution that will be used in our algorithm. It is clear that the maximum principle is an infinite dimensional sufficient condition in the sense that it must hold for all $t \in [t_0, t_f]$. An implementable algorithm, however, must be finite dimensional and all convergence investigations are to be done on some finite dimensional approximation of the optimal control problem.

One natural way to define a finite dimensional version of the above problem is to define a sufficiently dense grid

$$t_0 = t_1 < t_2 < \dots < t_N = t_f \quad ; \quad t_{k+1} = t_k + h \quad ; \quad h = \frac{t_f - t_0}{N - 1} \quad (12)$$

over the time interval $[t_0, t_f]$ in order to approximate the solutions of the state and the co-state differential equations with a sufficiently high precision using a second order integration method. More specifically, given an $(N - 1)m$ -dimensional vector \bar{u}

$$\bar{u} = [\bar{u}^T(1), \dots, \bar{u}^T(N - 1)] \in \bar{U} := U \times \dots \times U \subset \mathbb{R}^{(N-1)m}$$

we shall identify the vector $\bar{u} \in \mathbb{R}^{(N-1)m}$ to the corresponding piece-wise constant control defined over $[t_0, t_f]$ by

$$u(t_k + \tau) = \bar{u}(k) \quad ; \quad k = 1 \dots N \quad ; \quad \tau \in [0, h[\quad (13)$$

Furthermore, the following finite dimensional approximations

$$\bar{x} := [\bar{x}^T(1), \dots, \bar{x}^T(N)]^T \in \mathbb{R}^{Nn} \quad ; \quad \bar{\lambda} := [\bar{\lambda}^T(1), \dots, \bar{\lambda}^T(N)]^T \in \mathbb{R}^{Nn}$$

are defined by

$$\bar{x}(k + 1) := \bar{x}(k) + \frac{h}{2} \left[f(\bar{x}(k), \bar{u}(k), t_k) + f(\bar{x}(k) + h f(\bar{x}(k), \bar{u}(k), t_{k+1})) \right] \quad ; \quad \bar{x}(1) = x_0 \quad (14)$$

$$\begin{aligned} \bar{\lambda}(k - 1) := & \bar{\lambda}(k) + \frac{h}{2} \left[H_x(\bar{x}(k - 1), \bar{u}(k - 1), \bar{\lambda}(k) + h H_x(\bar{x}(k), \bar{u}(k - 1), \bar{\lambda}(k), t_k), t_{k-1}) \right. \\ & \left. H_x(\bar{x}(k), \bar{u}(k - 1), \bar{\lambda}(k), t_k) \right] \quad ; \quad \bar{\lambda}(N) = 0 \end{aligned} \quad (15)$$

With the discretization scheme defined above, an approximate value of the cost function can be obtained for all $\bar{u} \in \bar{U}$ using $\bar{J}(\bar{u})$ given by

$$\bar{J}(\bar{u}) = h \sum_{k=1}^{N-1} L(\bar{x}(k), \bar{u}(k), t_k) \quad (16)$$

Given the above definitions, consider the following algorithm

ALGORITHM

- **Step 0:** Fix some small $\epsilon_u > 0, \epsilon_J > 0$, some integer i_{max} and two reals $d\mu > 0$ and $\gamma > 1$. Choose $\mu^0 \geq 0$ and some initial admissible guess $\bar{u}^0 \in \bar{U}$.
- **Step 1:** Compute \bar{x}^0 solution of (14) with $\bar{u} = \bar{u}^0$, let $i = 1$,
- **Step 2:** Compute $\bar{\lambda}^{i-1}$ solution of (15) with \bar{u}^{i-1} and \bar{x}^{i-1} already computed,
- **Step 3:** Compute \bar{u}^i and \bar{x}^i such that
 - \bar{x}^i is solution of (14) with $\bar{u} = \bar{u}^i$ such that
 - $\bar{u}^i(k) := \text{Arg} \min_{u \in U} \left[H(\bar{x}^i(k), u, \bar{\lambda}^{i-1}(k), t_k) + \mu^{i-1} \|u - \bar{u}^{i-1}(k)\|^2 \right]$,
- **Step 4:** If $\left(\bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$ and $\left(\|\bar{u}^i - \bar{u}^{i-1}\| > \epsilon_u \right)$ **then** let $\mu^{i-1} = \max(\mu^{i-1} + d\mu, \gamma\mu^{i-1})$ and return to **Step 3**,
- **Step 5:** If $\left(\bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$ then $\mu^i = \max(0, \min(\mu^{i-1} - d\mu, \mu^{i-1}/\gamma))$,
- **Step 6:** If $\left(\|\bar{u}^i - \bar{u}^{i-1}\| \leq \epsilon_u \right)$ and $\left(i \geq i_{max} \right)$ **Then** stop **else** let $i = i + 1$ and return to **Step 2**.

This algorithm may be understood in the light of the following remarks. Rigorous convergence results are given in the following subsection and the corresponding proofs are given in the appendix.

- ✓ The algorithm begins with some initial guess \bar{u}^0 . The key problem is that such an arbitrary choice is generically incompatible with the maximum principle's necessary condition (??).
- ✓ **Step 3** produces then a *chattering* behaviour between the proceeding step's guess \bar{u}^{i-1} and the one that minimizes the corresponding *modified* Hamiltonian in the construction of which, \bar{x}^{i-1} and $\bar{\lambda}^{i-1}$ are used.
- ✓ The modification of the Hamiltonian by the penalty term $\mu^{i-1} \|u - \bar{u}^{i-1}(k)\|^2$ enables the successive iterations to be stabilized. Indeed, when μ^{i-1} is high, u will be different from \bar{u}^{i-1} only if it dramatically decreases H .
- ✓ Note that **Step 4** leads to high values of μ^i if the algorithm remains between **Step 3** and **Step 4**. These high values stabilize the iterations as explained above and force the algorithm to leave this loop after a finite number of iterations (see corollary 2 hereafter). The aim of **Step 5** is then to reduce μ^i that may have been temporary increased (in **Step 4**) in order to avoid some local difficulties. This is because, from the maximum principle viewpoint, the minimization in **Step 3** has sense only for $\mu^{i-1} = 0$.
- ✓ The condition $\left(\bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$ and $\left(\|\bar{u}^i - \bar{u}^{i-1}\| > \epsilon_u \right)$ in **Step 4** is to be interpreted as follows:
If \bar{u}^i fails to decrease significantly the cost function, that is $\left(\bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J \right)$ **and if** there is still evolution margin (that is $\left(\|\bar{u}^i - \bar{u}^{i-1}\| > \epsilon_u \right)$) **then** return to **Step 3** with the new value of μ^i .
- ✓ Note that both additive and multiplicative factors are used to decrease μ . Indeed, the multiplicative factor guarantees a fast decrease that may be necessary if very high values are needed to stabilise the iterations. The additive term enables to avoid asymptotic decrease of μ that would never exactly reach the key value 0 (see point 2. of corollary 2 hereafter).

4.2 Some convergence results

In this section, the convergence results are stated. No proofs are reported because of the lack of space, interested readers may consult the technical report [2] for complete proofs.

The following proposition is the key convergence result. It concerns the behavior of the approximate cost function values at successive iterations

PROPOSITION 1 [Behavior of successive cost function values]

there are positive reals $r > 0$ and $\sigma > 0$ such that, for sufficiently small step size h , the solutions of successive iterations satisfy the following inequality

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h(r - \mu^{i-1}) \sum_{k=1}^{N-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 + \sigma \quad (17)$$

Note that inequality (17) clearly shows that if

$$\sum_{k=1}^{N-1} \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|^2 \neq 0$$

then one can always decrease the cost function by letting μ^{i-1} taking sufficiently high values. But this is exactly what is **Step 4** dedicated to. Now since J^i cannot indefinitely decreases, the quantity $\|\bar{u}^i(k) - \bar{u}^{i-1}(k)\|$ must vanish.

From proposition 1, we can easily derive the following corollaries

COROLLARY 1 [Convergence of the cost function]

Let (\bar{x}^i, \bar{u}^i) be a sequence generated by the algorithm. The corresponding sequence of cost values $\bar{J}(\bar{u}^i)$ are monotonically decreasing. Moreover, if $i_{max} = \infty$, the infinite sequence $\bar{J}(\bar{u}^i)$ is convergent. ♠

Finally, the following is the basic result of the present paper

COROLLARY 2 [Convergence of the control sequence]

Let (\bar{x}^i, \bar{u}^i) be a sequence generated by the algorithm.

1. Suppose that $i_{max} = \infty$ in order to generate an infinite sequence $\bar{u}^i \in \bar{U}$. There is an integer \bar{i} such that

$$\forall i \geq \bar{i} \quad : \quad \|\bar{u}^i(k) - \bar{u}^{i-1}(k)\| = 0 \quad ; \quad k = 1 \dots N-1 \quad (18)$$

Moreover, if 0 is an accumulation point for the sequence μ^{i-1} then there is an accumulation point \bar{u}^* of the sequence \bar{u}^i that satisfies the maximum principle on the grid points, that is

$$H(\bar{x}^*(k), \bar{u}^*(k), \bar{\lambda}^*(k), t_k) \leq H(\bar{x}^*(k), \bar{u}, \bar{\lambda}^*(k), t_k) \quad ; \quad \text{for all } \bar{u} \in \bar{U} \quad (19)$$

2. In particular, if for some \bar{i} , one has $\bar{u}^{\bar{i}} = \bar{u}^{\bar{i}-1} =: \bar{u}^*$ with $\mu^{\bar{i}-1} = 0$ then the control \bar{u}^* satisfies the maximum principle (19) at the grid points.
3. In the case where i_{max} is finite, the algorithm stops after a finite number of iterations. Namely, it cannot be trapped by the loop in **(Step 3)-(Step 4)**

5 Illustrative examples

In this section, some examples are used to illustrate the efficiency of the proposed algorithm. Experiments have been conducted using a FORTRAN 90-compiler on a PENTIUM III-600Mhz Personal Computer.

Iterations	Cost function	μ	$\ \bar{u}^i - \bar{u}^{i-1}\ _\infty$
1	146.83	10	20.0
2	47.43	8.33	11.0
3	43.47	6.94	3.0
4	43.24	5.79	1.0
5	43.16	4.79	1.0
6	43.10	3.79	1.0
\vdots	\vdots	\vdots	\vdots
9	42.99	0.79	1.0
10	42.99	0.00	0.00

Table 1: Convergence results for example 1. (Execution time ≈ 0.2 s)

5.1 Example 1 [17, 22, 24, 16]

Consider the switched nonlinear hybrid system given by

$$\dot{x} = f_q(x) := \begin{pmatrix} x_2 \\ -x_1 + (1.4 - 0.14x_2^2)x_2 + 4(\frac{2(q-1)}{Q-1} - 1) \end{pmatrix} \quad ; \quad q \in \{1, \dots, Q\} \quad (20)$$

having $Q \in \mathbb{N}$ distinct configurations in each of which the system is autonomous, namely, no continuous control input v appears in the general form (1). The optimal control problem is defined by taking

$$L_q(x) = x_1^2 + \left[\frac{2(q-1)}{Q-1} - 1 \right]^2 \quad ; \quad T = 2.5 \text{ s} \quad (21)$$

It can be easily seen that when Q tends to infinity, the corresponding hybrid optimal control problems tends to the following classical one

$$\min_u \int_0^{2.5} (x_1^2(t) + u^2(t)) dt \quad \text{under} \quad \dot{x} = \begin{pmatrix} x_2 \\ -x_1 + (1.4 - 0.14x_2^2)x_2 + 4u \end{pmatrix} \quad \text{and} \quad |u(t)| \leq 1 \quad (22)$$

This optimal control problem has been studied in [17, 22, 24, 16] and the optimal value is known to be equal to 42.8 that can be viewed as a lower bound of the optimal solution of the hybrid optimal control problem. The hybrid optimal control problem corresponding to $Q = 21$ has been solved using $N = 201$ in (12), that is $h = 2.5/200$ s. Consequently, the corresponding decision variable is basically of dimension $\boxed{21^{200}}$. The parameters in step 0 of the algorithm are as follows $\epsilon_u = \epsilon_J = 0.001$, $d\mu = 1$, $\gamma = 1.2$ and $\mu_0 = 10$ while the initial guess $\bar{u}^0(\cdot) \equiv -1$ has been used. The convergence results of the algorithm are shown on table 1 with the obvious notation $u = q$ in accordance with the notations used in the problem's reformulation since there is no continuous input (see section 3). Note the monotonic decrease of the cost function to a value that is quite close to the lower bound 42.8 mentioned above. Note also that according to point 2. of corollary 2, the iterations stop at an extremal point satisfying the maximum principle at the grid points since for $i = 10$, one obtains $\mu^i = 0$ and $\bar{u}^i - \bar{u}^{i-1} = 0$.

The switching strategies obtained at different iterations as well as the initial and the optimal state trajectories are given on figure 1. Note that during the optimal scenario, about 13 controlled switches take place. This suggests that strategies based on the use of an a-priori given number of switches have to explore a 21^{13} -dimensional tree corresponding to all the possible sequences of active configurations for each choice of the 13 switching instants.

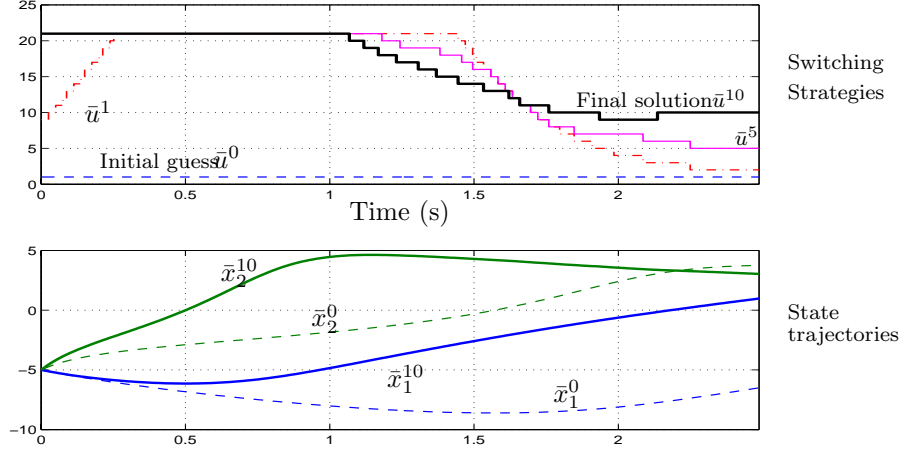


Figure 1: Switching strategies and state trajectories at different iterations (Example 1)

5.2 Example 2 [30]

In [30], an algorithm has been proposed to solve optimal control problems on switched hybrid systems. The following bilinear system has been considered to illustrate the proposed algorithm

$$\begin{aligned}
 \text{subsystem 1: } & \begin{cases} \dot{x}_1 = -x_1 + 2x_1v \\ \dot{x}_2 = x_2 + x_2v \end{cases} \\
 \text{subsystem 2: } & \begin{cases} \dot{x}_1 = x_1 - 3x_1v \\ \dot{x}_2 = 2x_2 - 2x_2v \end{cases} \\
 \text{subsystem 3: } & \begin{cases} \dot{x}_1 = 2x_1 + x_1v \\ \dot{x}_2 = -x_2 + 3x_2v \end{cases}
 \end{aligned}$$

with the following cost function

$$J = \frac{1}{2}(x_1(T) - 2)^2 + \frac{1}{2}(x_2(T) - 2)^2 + \frac{1}{2} \int_0^T \left[(x_1(t) - 2)^2 + (x_2(t) - 2)^2 + v^2(t) \right] dt \quad ; \quad T = 3 \quad (23)$$

with the initial state $x(0) = (1 \ 1)^T$. A sub-optimal solution of the hybrid optimal control problem above has been obtained in [30] based on the a priori assumption of 2 switching instants. The following sub-optimal cost has been obtained The sub-optimal J achieved by [30] is 3.625. The sub-optimal switching strategy is characterized by one switch at instant $t^* = 1.7244$ taking the system from configuration $q = 2$ to configuration $q = 3$. This can be seen on the dotted lines plots of figure 2. Let us see how the problem above can be put in the framework of the present paper. The system admits three different configurations ($Q = 3$) and can be put in the general form (1) by using the following notations

$$f_1(x, v) = \begin{pmatrix} -x_1 + 2x_1v \\ x_2 + x_2v \end{pmatrix} ; f_2(x, v) = \begin{pmatrix} x_1 - 3x_1v \\ 2x_2 - 2x_2v \end{pmatrix} ; f_3(x, v) = \begin{pmatrix} 2x_1 + x_1v \\ -x_2 + 3x_2v \end{pmatrix} \quad (24)$$

By the same, the cost function can be put in the standard form (2) by using the following straightforward definition of L_q for $q \in \{1, 2, 3\}$

$$L_q(x, v) := \frac{1}{2} \left[(x_1 - 2)^2 + (x_2 - 2)^2 + v^2(t) \right] + \frac{\partial \Psi}{\partial x}(x) \cdot f_q(x, v) \quad ; \quad \Psi(x) = \frac{1}{2}(x_1 - 2)^2 + \frac{1}{2}(x_2 - 2)^2$$

where the final penalty in (23) has been transformed into an integral form. The proposed algorithm has been successfully used to solve the corresponding hybrid optimal control problem using the following parameters $\epsilon_u = \epsilon_J = 0.001$, $d\mu = 0.5$, $\gamma = 1.5$ and $\mu_0 = 10$. The optimal cost achieved by the algorithm is given by $\hat{J}_{\text{proposed algorithm}} = 0.3742$ to be compared to 3.625 obtained in [30]. The convergence history for example 2 is shown on figure 3. A careful look on figure 3 enables the following fact to be underlined

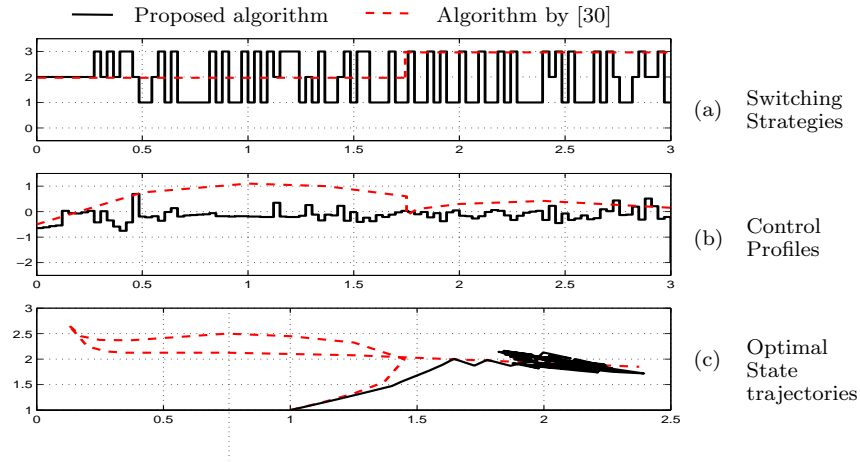


Figure 2: switching strategies, control profiles and state trajectories for example 2. (Dotted lines - -) the sub-optimal solution achieved by [30]. (Solid lines -) The sub-optimal solution achieved by the proposed algorithm.

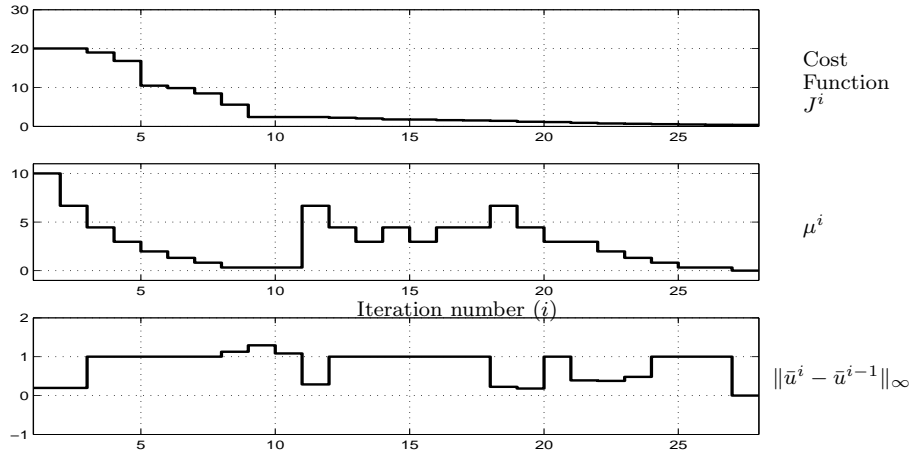


Figure 3: Convergence results for example 2 (Execution time $\approx 1s$)

- ✓ At iteration 28, the condition required by corollary 2 are satisfied, namely $\mu^i = 0$; $\|\bar{u}^i - \bar{u}^{i-1}\|_\infty = 0$. Therefore, according to corollary 2, the sequence so obtained satisfies the maximum principle at the grid points.
- ✓ The evolution of μ^i is not monotonic, the dynamics of this penalty is a key issue in the convergence of the iterations.

As for the optimal trajectories, they can be viewed on figure 2 where the optimal switching strategy, the optimal control input and the optimal states trajectories (in the phase plane) are given in solid lines. These are to be compared with the dotted lines that summarize the results of [30]. Figure 2 suggests the following comments

- ✓ Note the high number of switches "*asked by the proposed algorithm*". This is because, in the absence of switches, keeping the state near the point $(2, 2)$ as implied by the cost function would need a control v that is far from 0 whatever is the configuration being used and this is penalized by the squared penalty term on the control v . That is why a 2-switches strategy necessarily corresponds to a high cost. However, by using cleverly and intensively the switches, the algorithm suggests to use the different dynamics of the system to keep the state "*turning around the target point $(2, 2)$* " while using few continuous control v . This can be clearly seen on figure 2.(b) and 2.(c)

5.3 Example 3

Let us consider the dynamic equations of an induction machine with the norm of flux and the torque as regulated outputs. The voltage input to this machine comes from a two stage power source. The first one is the rectifier that delivers a DC voltage from an AC source and the second is the inverter that transform this DC voltage into a dynamic voltage by changing the positions of the three inverter commutators. This leads to 7 different possible values of the voltage at the input of the induction machine. This is shown on figure 4 in the (α, β) -plane. As a consequence, the system may be viewed as a hybrid system with 7 configurations ($Q = 7$) in each of which, the system is autonomous.

$$\frac{d}{dt} \begin{pmatrix} I_{s\alpha} \\ I_{s\beta} \\ \phi_{s\alpha} \\ \phi_{s\beta} \end{pmatrix} = \begin{pmatrix} -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -p\Omega & \frac{R_r}{z} & \frac{p\Omega L_r}{z} \\ p\Omega & -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -\frac{p\Omega L_r}{z} & \frac{R_r}{z} \\ -R_s & 0 & 0 & 0 \\ 0 & -R_s & 0 & 0 \end{pmatrix} \begin{pmatrix} I_{s\alpha} \\ I_{s\beta} \\ \phi_{s\alpha} \\ \phi_{s\beta} \end{pmatrix} + \begin{pmatrix} \frac{V_0 L_r}{z} & 0 \\ 0 & \frac{V_0 L_r}{z} \\ V_0 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} V_{s\alpha}(q) \\ V_{s\beta}(q) \end{pmatrix} \quad (25)$$

$$y = \begin{pmatrix} \|\Phi\|^2 \\ \Gamma \end{pmatrix} := \begin{pmatrix} \phi_{s\alpha}^2 + \phi_{s\beta}^2 \\ p(I_{s\beta}\phi_{s\alpha} - I_{s\alpha}\phi_{s\beta}) \end{pmatrix} \quad (26)$$

in which,

$$z := L_s L_r - M^2 \quad ; \quad \sigma = 1 - M^2 / (L_s L_r) \quad ; \quad T_s = \frac{L_s}{R_s} \quad ; \quad T_r = \frac{L_r}{R_r}$$

where L_s , L_r , R_s and R_r are the stator and rotor inductances and resistances while Ω is the motor speed and $p = 2$. The optimal control problem in question here is to find a switching strategy $q^*(\cdot)$ such that

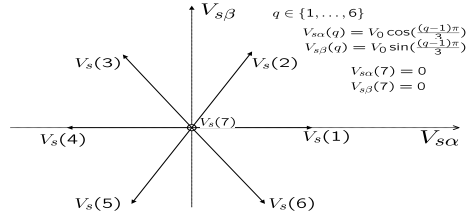


Figure 4: The input voltage of the induction machine as a function of the configuration $q \in \{1, \dots, 7\}$

the nonlinear output (26) is regulated around some reference signal $h_r(t)$ while minimizing the harmonics of the stator current at some precise frequency ω_0 . To tackle such requirements, the system is extended using additional states as follows

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -p\Omega & \frac{R_r}{z} & \frac{p\Omega L_r}{z} \\ p\Omega & -(\frac{1}{\sigma T_r} + \frac{1}{\sigma T_s}) & -\frac{p\Omega L_r}{z} & \frac{R_r}{z} \\ -R_s & 0 & 0 & 0 \\ 0 & -R_s & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} \frac{V_0 L_r}{z} & 0 \\ 0 & \frac{V_0 L_r}{z} \\ V_0 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} V_{s\alpha}(q) \\ V_{s\beta}(q) \end{pmatrix} \quad (27)$$

$$\dot{x}_5 = x_1 \cos \omega_0 t \quad ; \quad x_5(0) = 0 \quad (28)$$

$$\dot{x}_6 = x_1 \sin \omega_0 t \quad ; \quad x_6(0) = 0 \quad (29)$$

$$\dot{x}_7 = x_2 \cos \omega_0 t \quad ; \quad x_7(0) = 0 \quad (30)$$

$$\dot{x}_8 = x_2 \sin \omega_0 t \quad ; \quad x_8(0) = 0 \quad (31)$$

$$y = h(x) = \begin{pmatrix} \|\Phi\|^2 \\ \Gamma \end{pmatrix} := \begin{pmatrix} x_3^2 + x_4^2 \\ p(x_2 x_3 - x_1 x_4) \end{pmatrix} \quad (32)$$

Indeed, in doing so, the quantity

$$\Psi(x(T)) := \frac{1}{T^2} \left[x_5^2(T) + x_6^2(T) + x_7^2(T) + x_8^2(T) \right] \quad (33)$$

is a truncated approximation of the spectral power of the stator current at the prescribed frequency ω_0 and therefore, the following cost function may be used to reflect the control specifications stated above

$$J(x_0, q) = w_f \Psi(x(T)) + \int_0^T \|h(x(\tau)) - h_r(\tau)\|^2 d\tau \quad (34)$$

where w_f is a weighting coefficient penalizing the current power spectrum at the frequency ω_0 . This cost function can be put in the standard form (2) by using the following straightforward definition of L_q for $q \in \{1, \dots, 7\}$

$$L_q(x, t) := \frac{\partial \Psi}{\partial x}(x) f_q(x) + \|h(x) - h_r(t)\|^2 \quad (35)$$

where $f_q(x)$ is clearly defined by the r.h.s of (27)-(31). The numerical values of the parameters used in the simulations are given on table 2. while the values of the prediction horizon $T = 0.2$ s and the sampling period $h = 50 \times 10^{-6}$ s are used leading to a number of discretization instants $N = 4000$. Harmonics attenuation

L_s	L_r	M	R_s	R_r	p	Ω
0.031747H	0.0323H	0.031H	0.07Ω	0.052Ω	2	2π100rad/sec

Table 2: Numerical values of the induction machine's parameters

is tested for the value $\omega_0 = 2\pi \times 170$ rad/s. Finally the reference trajectories on the output are given by the following constant set-points $h_r(t) = \begin{pmatrix} 1 \\ 100 \end{pmatrix}$. The parameters of the algorithm have been taken as follows $\epsilon_u = \epsilon_J = 0.001$, $d\mu = 1$, $\gamma = 1.2$ and $\mu_0 = 0$. Figure 5 shows the optimal trajectories for both the regulated outputs and the state vector when no penalty is used to insure harmonic rejection ($w_f = 0$). Figure 6 shows the same results when the penalty $w_f = 10^4$ is used in order to reject the harmonics corresponding to $w_0 = 2\pi \times 170$ (rad/s). The corresponding power spectrum in the two cases above is given on figure 7 where the effect of the penalty term at the prescribed frequency can be clearly appreciated. Finally, figure 8 shows an example of the optimal switching strategy in the case ($w_f = 10^4$) over the whole time interval (figure 8.(a)) and over a shorter time window enabling the switches to be clearly identified (figure 8.(b)).

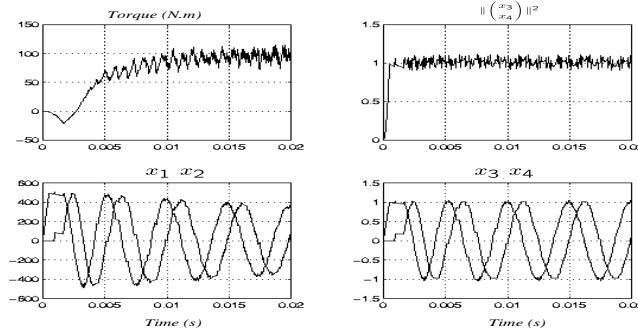


Figure 5: Optimal trajectories without harmonics attenuation ($w_f = 0$).

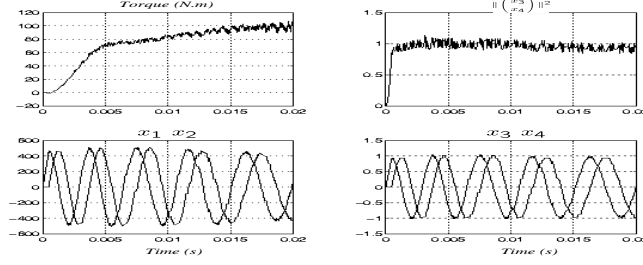


Figure 6: Optimal trajectories without harmonics attenuation ($w_f = 10^4$).

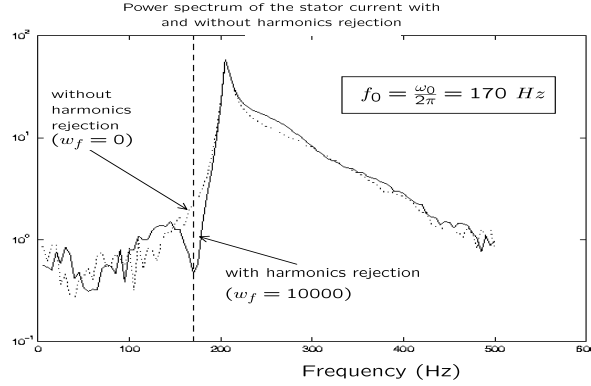


Figure 7: Power spectrum of the stator current for $w_f = 0$ (dotted lines) and $w_f = 10^4$ (solid lines).

6 Conclusion and future work

In this paper, an algorithm is proposed to solve optimal control problems for switched hybrid systems. These are hybrid systems with no autonomous switches in which the controlled switches are free. The basic features of the algorithm are the unified framework used in updating both continuous control and switching strategy and the fact that no combinatoric search is needed whatever is the number of switches. Future works concern the generalization of the proposed scheme to hybrid systems with autonomous switches and/or controlled switched with admissible target configurations that may depend on the active configuration.

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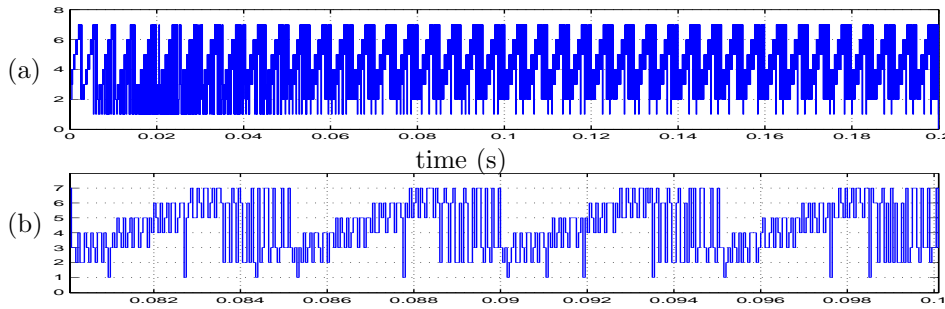


Figure 8: The optimal switching strategy for the case $w_f = 10^4$.

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