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Adaptive Stabilization of Second-Order Systems with Minimum or Non-minimum Phase Zeros[1]

by

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Abstract

We consider output feedback adaptive stabilization for second-order systems. The assumptions we make are standard, namely, the system order, relative degree and sign of the high frequency gain are known. However, we extend the existing literature by presenting a controller which requires no knowledge of the zero, which may be either minimum phase or non-minimum phase. The controller has the form of an 8th-order dynamic compensator for the relative-degree = 1 case, and a 6th-order for the relative-degree = 2 case. The proof of convergence is based on a variation of Lyapunov's method in which the Lyapunov derivative is shown to be asymptotically nonpositive. Application of the controller to the Van der Pol and Duffing oscillators shows that the controller is effective for nonlinear systems as well.

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1 Introduction

There are many applications of control in which a reliable model of the dynamical system is not available. This can occur if the system is not amenable to analytical modeling due to unknown physics, or if identification is not feasible due to disturbance and sensor noise, lack of accessibility, poor repeatability, unpredictable changes, or high cost. Under such high levels of uncertainty, robust control may be ineffective and adaptive control is warranted.

For implementation, adaptive controllers generally require some knowledge about the plant in the form of parameter or transfer function estimates, and this knowledge may be available prior to operation due to analytical modeling or off-line identification, or it may be determined during operation through concurrent identification. The former case is usually termed direct adaptive control, while the latter is termed indirect adaptive control. In addition, the analysis of adaptive control methods is generally based on structural assumptions about the plant, for example, passivity and relative degree.

In this paper we consider the problem of adaptive stabilization for second-order systems under output feedback. As in [4, 5] we assume that the order of the system, relative degree and sign of the high frequency gain are known. However, our results extend the literature and the results of [4, 5] in two distinct ways. Firstly, we do not require the plant to be minimum phase nor do we need to know the sign of the zero. Secondly, the theory does not rely on any positivity assumptions (or the KY Lemma). For relative-degree 1 case we derive an 8th-order dynamic compensator which guarantees convergence of plant states (Theorem 1). For relative-degree 2 case we derive an 6th-order dynamic compensator which is presented in Theorem 2. In either case, the controller is essentially made of three constituents, namely, a filter, a state estimator and a parameter estimator. Moreover, for relative-degree 2 case, the control u can be explicitly written as an expression involving state estimates, parameter estimates and filter states. In this case, the overall controller, in this case, has the form of a 6th-order dynamic compensator with quadratic, cubic and quartic nonlinearities. This form of the controller is distinct from the expressions given in [4, 5]. In addition, our proof of convergence is Lyapunov based. In particular, we develop a variation of Lyapunov's method in which the Lyapunov derivative is shown to be asymptotically nonpositive.

The contents of the paper are as follows. In Section 2 we introduce the stabilization problem for second order systems with relative degree 1. We state and prove Theorem 1 which provides the 8th-order dynamic compensator which guarantees convergence of states and boundedness of parameter estimates for second order systems with relative degree 1. Theorem 2 in Section 3 provides the corresponding result for plants with relative degree 2.

In Section 4 we present several numerical examples involving linear and nonlinear plants. In particular, we apply the controller to the Van der Pol and Duffing systems to show that the controller is effective for nonlinear systems as well.

2 Adaptive Stabilization Problem: Relative Degree 1

Consider the second-order system

$$\ddot{q} + a_1\dot{q} + a_2 q = \frac{1}{m}\dot{u} + \frac{z_0}{m}u, \tag{1}$$

where a_1, a_2, z_0 and m are constant. We make the following assumptions about (1). We assume that a_1, a_2 and z_0 are completely unknown. Furthermore, we assume that m is nonzero and $\mathrm{sign}(m)$ is known but m is otherwise unknown. Finally, we assume that only q is available for feedback. The stabilization problem constitutes finding a control input u such that q, $\dot{q}$ converge to 0 as $t \to \infty$.

Theorem 1 Let λ, f_1, f_2, r_1, r_2, g_1, g_2 and ε be positive constants and let $p_1 = r_1/(2f_2)$ and $p_2 = (r_1 + f_2 r_2)/(2f_1 f_2)$. Consider the dynamic compensator

$$\begin{aligned}
\dot{q}_\mathrm{f} &= -\lambda q_\mathrm{f} + q, &(2)\\
\ddot{\hat{q}}_\mathrm{f} + \hat{a}_1\dot{\hat{q}}_\mathrm{f} + \hat{a}_2\hat{q}_\mathrm{f} &= (f_1 - \hat{a}_1)\dot{e} + (f_2 - \hat{a}_2)e + \bar{u}_\mathrm{f}, &(3)\\
\dot{u}_\mathrm{f} &= -(\sqrt{z} + \varepsilon)u_\mathrm{f} + \hat{m}\bar{u}_\mathrm{f}, &(4)\\
\dot{\hat{a}}_1 &= -\tilde{e}\dot{q}_\mathrm{f}, &(5)\\
\dot{\hat{a}}_2 &= -\tilde{e}q_\mathrm{f}, &(6)\\
\dot{\hat{m}} &= -\mathrm{sign}(m)\tilde{e}\bar{u}_\mathrm{f}, &(7)\\
\dot{z} &= -\mathrm{sign}(m)\tilde{e}u_\mathrm{f}\sqrt{z}, \ \ z(0) > 0, &(8)
\end{aligned}$$

where $e \triangleq q_\mathrm{f} - \hat{q}_\mathrm{f}$, $\tilde{e} \triangleq p_1 e + p_2\dot{e}$ and

$$\bar{u}_\mathrm{f} \triangleq (\hat{a}_1 - g_1)\dot{\hat{q}}_\mathrm{f} + (\hat{a}_2 - g_2)\hat{q}_\mathrm{f}. \tag{9}$$

Let the control input u be given by

$$u = (\lambda - \sqrt{z} - \varepsilon)u_\mathrm{f} + \hat{m}\bar{u}_\mathrm{f}. \tag{10}$$

Then q, $\dot{q}$, q_f, $\dot{q}_\mathrm{f}$, $\hat{q}_\mathrm{f}$ and $\dot{\hat{q}}_\mathrm{f} \to 0$ as $t \to \infty$. Furthermore, $\hat{a}_1$, $\hat{a}_2$, $\hat{m}$ and z are bounded. □

Proof. It follows from (4) and (10) that u_f satisfies

$$\dot{u}_\mathrm{f} = -\lambda u_\mathrm{f} + u. \tag{11}$$

Next, defining

$$H \triangleq \ddot{q}_\mathrm{f} + a_1\dot{q}_\mathrm{f} + a_2 q_\mathrm{f} - \frac{1}{m}\dot{u}_\mathrm{f} - \frac{z_0}{m}u_\mathrm{f}, \tag{12}$$

and using (1), (2) and (11), it follows that H satisfies $\dot{H} = -\lambda H$. Hence $H(t) = H(0)e^{-\lambda t}$. Equation (12) can be rewritten as

$$\ddot{q}_\mathrm{f} + a_1\dot{q}_\mathrm{f} + a_2 q_\mathrm{f} = \frac{1}{m}\dot{u}_\mathrm{f} + \frac{z_0}{m}u_\mathrm{f} + H. \tag{13}$$

Subtracting (3) from (13) yields

$$\ddot{e} + f_1\dot{e} + f_2 e = \tilde{a}_1\dot{q}_{\rm f} + \tilde{a}_2 q_{\rm f} + \frac{\tilde{m}}{m}\bar{u}_{\rm f} + \frac{z_0 - \sqrt{z} - \varepsilon}{m}u_{\rm f} + H, \tag{14}$$

where $\tilde{a}_1 \triangleq \hat{a}_1 - a_1$, $\tilde{a}_2 \triangleq \hat{a}_2 - a_2$ and $\tilde{m} \triangleq \hat{m} - m$. Using (8), note that $\dot{z} = 0$ at $z = 0$. Hence if $z(0) > 0$, $z(t) \geq 0$ for all t. Therefore $\sqrt{z}$ is defined for all t.

Next, (14) can be rewritten as

$$\dot{E} = FE + \begin{bmatrix} 0 \\ \tilde{a}_2 q_{\rm f} + \tilde{a}_1\dot{q}_{\rm f} + \frac{\tilde{m}}{m}u_{\rm f} + \frac{z_0-\sqrt{z}-\varepsilon}{m}u_{\rm f} + H \end{bmatrix}, \tag{15}$$

where $E \triangleq [e \quad \dot{e}]^{\rm T}$ and $F \triangleq \left[\begin{smallmatrix} 0 & 1 \\ -f_2 & -f_1 \end{smallmatrix}\right]$. Let P denote the solution of the Lyapunov equation $F^{\rm T}P + PF = -R$ where $R = \left[\begin{smallmatrix} r_1 & 0 \\ 0 & r_2 \end{smallmatrix}\right]$. Then P is given by $P = \left[\begin{smallmatrix} p_0 & p_1 \\ p_1 & p_2 \end{smallmatrix}\right]$, where $p_0 = (f_1 r_1 + f_1 f_2 r_2 + f_1^2 r_2)/(2 f_1 f_2)$.

Next, we define $\tilde{x} \triangleq [e, \dot{e}, \tilde{a}_1, \tilde{a}_2, \tilde{m}, z]^{\rm T}$ and the positive definite function

$$V(\tilde{x}) \triangleq \frac{1}{2}E^{\rm T}PE + \frac{1}{2}\tilde{a}_1{}^2 + \frac{1}{2}\tilde{a}_2{}^2 + \frac{1}{2|m|}\tilde{m}^2 + \frac{1}{|m|}(\sqrt{z} + \varepsilon - z_0)^2. \tag{16}$$

Then using parameter update laws (5)-(8) and using $eHp_1 \leq e^2 r_1/4 + H^2 p_1^2/r_1$ and $\dot{e}Hp_2 \leq \dot{e}^2 r_2/4 + H^2 p_2^2/r_2$, we obtain

$$\begin{aligned}
\dot{V}(\tilde{x}) &= -\frac{1}{2}E^{\rm T}RE + \tilde{a}_1(\dot{\hat{a}}_1 + p_1 e\dot{q}_{\rm f} + p_2\dot{e}\dot{q}_{\rm f}) + \tilde{a}_2(\dot{\hat{a}}_2 + p_1 e q_{\rm f} + p_2\dot{e}q_{\rm f}) + \frac{\tilde{m}}{m}(\dot{\hat{m}} + {\rm sign}(m)(p_1 e + p_2\dot{e})u_{\rm f}) \\
&\quad + \frac{\sqrt{z}+\varepsilon-z_0}{|m|}\left(\frac{\dot{z}}{\sqrt{z}} - {\rm sign}(m)(p_1 e + p_2\dot{e})u_{\rm f}\right) + H(p_1 e + p_2\dot{e}) \\
&= -\frac{1}{2}r_1 e^2 - \frac{1}{2}r_2\dot{e}^2 + H(p_1 e + p_2\dot{e}) \\
&\leq -\frac{1}{4}r_1 e^2 - \frac{1}{4}r_2\dot{e}^2 + \left(\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2}\right)H^2 \\
&\leq \left(\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2}\right)H^2(0)e^{-2\lambda t}.
\end{aligned}$$

Let $V_0 \triangleq V + \frac{\alpha}{2\lambda}e^{-2\lambda t}$, where $\alpha = (\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2})H^2(0)$. Then

$$\dot{V}_0 \;=\; \dot{V} - \alpha e^{-2\lambda t} \leq \alpha e^{-2\lambda t} - \alpha e^{-2\lambda t} = 0$$

Since $\lambda > 0$ and $\alpha \geq 0$, it follows that V_0 is positive definite and monotonically decreasing. Hence V_0 has a limit as $t \to \infty$. Since $\lim_{t\to\infty}\frac{\alpha}{2\lambda}e^{-2\lambda t} = 0$, V has a limit as $t \to \infty$. Hence all components of $\tilde{x}$, namely e, $\dot{e}$, $\tilde{a}_1$, $\tilde{a}_2$, z and $\tilde{m}$ are bounded.

Next, note that

$$\int_0^T \frac{1}{4}r_1 e^2 dt + \int_0^T \frac{1}{4}r_2\dot{e}^2 dt \leq V(\tilde{x}(0)) - V(\tilde{x}(T)) + \left(\frac{p_1^2}{r_1} + \frac{p_2^2}{r_2}\right)\int_0^T H(t)^2 dt, \tag{17}$$

along the trajectories of the closed-loop system. Noting that the right-hand-side of inequality (17) is bounded as $T \to \infty$, it follows that $e(t)$ and $\dot{e}(t)$ are square integrable on $[0, \infty)$. Since $\dot{e}(t)$ is bounded for all time, $e(t) \to 0$ as $t \to \infty$ (see Lemma A.1).

To prove convergence of q, $\dot{q}$, q_{f}, $\dot{q}_{\mathrm{f}}$, $\hat{q}_{\mathrm{f}}$ and $\dot{\hat{q}}_{\mathrm{f}}$, we first prove boundedness. Using (9), (3) yields

$$\ddot{\hat{q}}_{\mathrm{f}} + g_1\dot{\hat{q}}_{\mathrm{f}} + g_2\hat{q}_{\mathrm{f}} = (f_2 - \hat{a}_2)e + (f_1 - \hat{a}_1)\dot{e}. \tag{18}$$

Note that since $\tilde{x}$ is bounded, the right-hand side of (18) is bounded. Since the polynomial $p^2 + g_1p + g_2$ is Hurwitz, $\hat{q}_{\mathrm{f}}$ and $\dot{\hat{q}}_{\mathrm{f}}$ are bounded. Furthermore, since e and $\dot{e}$ are bounded, the states q_{f} and $\dot{q}_{\mathrm{f}}$ are bounded. Next, noting that $\sqrt{z} + \varepsilon \geq \varepsilon > 0$, it follows that the $u_{\mathrm{f}} = 0$ is an asymptotically stable equilibrium of the internal controller dynamics

$$\dot{u}_{\mathrm{f}} = -(\sqrt{z} + \varepsilon)u_{\mathrm{f}}.$$

Since $\bar{u}_{\mathrm{f}}$ is bounded, using (4) it follows that u_{f} is bounded as well. Therefore all terms on the right-hand-side of (14) are bounded, which implies that $\ddot{e}$ is bounded. Since $\dot{e}$ is square integrable, it follows that $\dot{e} \to 0$ as $t \to \infty$ (see Lemma A.1). Consequently, the right-hand side of (18) converges to 0 as $t \to \infty$. We, thus, conclude that estimator states $\hat{q}_{\mathrm{f}}$ and $\dot{\hat{q}}_{\mathrm{f}}$ converge to 0 as $t \to \infty$. Hence the filter states q_{f} and $\dot{q}_{\mathrm{f}}$ converges to 0 as $t \to \infty$ and hence using (2) it follows that $q \to 0$. Using (9) it follows that $\bar{u}_{\mathrm{f}} \to 0$ as $t \to \infty$. Therefore, using (4), we conclude that $u_{\mathrm{f}} \to 0$ as $t \to \infty$. Furthermore, from (13) we note that $\ddot{q}_{\mathrm{f}} \to 0$ as $t \to \infty$. Differentiating (2) yields

$$\ddot{q}_{\mathrm{f}} = -\lambda\dot{q}_{\mathrm{f}} + \dot{q}, \tag{19}$$

which implies $\dot{q} \to 0$ as $t \to \infty$. □

The controller (2)-(10) can be identified as the combination of three essential modules. First, a stable filter represented by (2), filters the available feedback variable q. The filter state q_{f} mimics the plant dynamics as indicated by (13). Second, the estimator (3),(4) with states $\hat{q}_{\mathrm{f}}$, $\dot{\hat{q}}_{\mathrm{f}}$ facilitates the use of a certainty-equivalence control input given by (9). Last, equations (5)-(8) constitute the parameter update laws.

Since q is available for feedback and q_{f} is obtained from the filter (2), $\dot{q}_{\mathrm{f}}$ can be computed using (2). Since $\hat{q}_{\mathrm{f}}$ and $\dot{\hat{q}}_{\mathrm{f}}$ are available from (3), the quantities e and $\dot{e}$ can be computed. Therefore, the parameter update laws (5)-(8) and equation (9) are implementable. Hence, no differentiation or improper realization is required to implement the controller.

Finally, the controller implementation only requires q for feedback and does not need to know a_1, a_2, z_0 or the value of m. However, knowledge of sign(m) is required to implement (7) and 8.

3 Adaptive Stabilization Problem: Relative Degree 2

Consider the second-order system

$$\ddot{q} + a_1\dot{q} + a_2q = \frac{1}{m}u, \tag{20}$$

where a_1, a_2 and m are constant. We make the following assumptions about (20). We assume that a_1 and a_2 are completely unknown. Furthermore, we assume that m is nonzero and $\operatorname{sign}(m)$ is known but m is otherwise unknown. Finally, we assume that only q is available for feedback. The stabilization problem constitutes finding a control input u such that q, $\dot{q}$ converge to 0 as $t \to \infty$.

Theorem 2 Let λ, f_1, f_2, r_1, r_2, g_1 and g_2 be positive constants and let $p_1 = r_1/(2f_2)$ and $p_2 = (r_1 + f_2 r_2)/(2 f_1 f_2)$. Consider the dynamic compensator

$$\begin{aligned}
\dot{q}_{\mathrm{f}} &= -\lambda q_{\mathrm{f}} + q, && (21)\\
\ddot{\hat{q}}_{\mathrm{f}} + f_1 \dot{\hat{q}}_{\mathrm{f}} + f_2 \hat{q}_{\mathrm{f}} &= (f_1 - \hat{a}_1)\dot{q}_{\mathrm{f}} + (f_2 - \hat{a}_2) q_{\mathrm{f}} + \bar{u}_{\mathrm{f}}, && (22)\\
\dot{\hat{a}}_1 &= -(p_1 e + p_2 \dot{e})\dot{q}_{\mathrm{f}}, && (23)\\
\dot{\hat{a}}_2 &= -(p_1 e + p_2 \dot{e}) q_{\mathrm{f}}, && (24)\\
\dot{\hat{m}} &= -\operatorname{sign}(m)(p_1 e + p_2 \dot{e})\bar{u}_{\mathrm{f}}, && (25)
\end{aligned}$$

where $e \triangleq q_{\mathrm{f}} - \hat{q}_{\mathrm{f}}$ and $\bar{u}_{\mathrm{f}} \triangleq (\hat{a}_1 - g_1)\dot{\hat{q}}_{\mathrm{f}} + (\hat{a}_2 - g_2)\hat{q}_{\mathrm{f}}$. Let the control input u be given by

$$\begin{aligned}
u \;=\; & \hat{m}(\hat{a}_1 - g_1)\ddot{\hat{q}}_{\mathrm{f}} + \left(\hat{m}[\dot{\hat{a}}_1 + \hat{a}_2 - g_2] + (\dot{\hat{m}} + \lambda \hat{m})(\hat{a}_1 - g_1)\right)\dot{\hat{q}}_{\mathrm{f}} \\
& + \left(\hat{m}\dot{\hat{a}}_2 + (\dot{\hat{m}} + \lambda \hat{m})(\hat{a}_2 - g_2)\right)\hat{q}_{\mathrm{f}}. \qquad (26)
\end{aligned}$$

Then q, $\dot{q}$, q_{f}, $\dot{q}_{\mathrm{f}}$, $\hat{q}_{\mathrm{f}}$ and $\dot{\hat{q}}_{\mathrm{f}} \to 0$ as $t \to \infty$. Furthermore, $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ are bounded. □

Proof. It follows from (26) that $u_{\mathrm{f}} \triangleq \hat{m}\bar{u}_{\mathrm{f}}$ satisfies

$$\dot{u}_{\mathrm{f}} = -\lambda u_{\mathrm{f}} + u. \qquad (27)$$

The proof then proceeds as the proof for Theorem 1. The error equation corresponding to (14) is given by

$$\ddot{e} + f_1 \dot{e} + f_2 e = \tilde{a}_1 \dot{q}_{\mathrm{f}} + \tilde{a}_2 q_{\mathrm{f}} + \frac{\tilde{m}}{m}\bar{u}_{\mathrm{f}} + H, \qquad (28)$$

where $\tilde{a}_1 \triangleq \hat{a}_1 - a_1$, $\tilde{a}_2 \triangleq \hat{a}_2 - a_2$ and $\tilde{m} \triangleq \hat{m} - m$. In this case, the positive definite function $V(\tilde{x})$ corresponding to the step (16) is chosen as

$$V(\tilde{x}) \triangleq \frac{1}{2}E^{\mathrm{T}} P E + \frac{1}{2}\tilde{a}_1{}^2 + \frac{1}{2}\tilde{a}_2{}^2 + \frac{1}{2|m|}\tilde{m}^2. \qquad (29)$$

The remainder of the proof is identical to the proof of Theorem 1 □

In this case, the control input u can be evaluated explicitly in terms of the available states and is given by

$$\begin{aligned}
u \;=\; & \hat{m}\Big[\hat{a}_2 g_1 q_{\mathrm{f}} - f_2 g_1 q_{\mathrm{f}} - \hat{a}_2 g_1 \hat{q}_{\mathrm{f}} + f_2 g_1 \hat{q}_{\mathrm{f}} + g_1 g_2 \hat{q}_{\mathrm{f}} + \hat{a}_2 \lambda \hat{q}_{\mathrm{f}} - g_2 \lambda \hat{q}_{\mathrm{f}} - p_2 q \hat{q}_{\mathrm{f}}^2 - p_2 q_{\mathrm{f}} \hat{q}_{\mathrm{f}}^2 + p_2 \hat{q}_{\mathrm{f}}^3 + \hat{a}_2 \dot{\hat{q}}_{\mathrm{f}} \\
& - \hat{a}_2 g_1 \dot{\hat{q}}_{\mathrm{f}} + g_1^2 \dot{\hat{q}}_{\mathrm{f}} - g_2 \dot{\hat{q}}_{\mathrm{f}} - g_1 \lambda \dot{\hat{q}}_{\mathrm{f}} + p_2 \hat{q}_{\mathrm{f}}^2 \dot{\hat{q}}_{\mathrm{f}} - p_2 q \dot{\hat{q}}_{\mathrm{f}}^2 - p_2 q_{\mathrm{f}} \dot{\hat{q}}_{\mathrm{f}}^2 + p_2 \hat{q}_{\mathrm{f}} \dot{\hat{q}}_{\mathrm{f}}^2 + p_2 \dot{\hat{q}}_{\mathrm{f}}^3 + p_2 q_{\mathrm{f}} \hat{q}_{\mathrm{f}}^2 r + p_2 q_{\mathrm{f}} \dot{\hat{q}}_{\mathrm{f}}^2 r \\
& + \hat{a}_1^2(-q + q_{\mathrm{f}} r) + f_1 g_1(-q + \dot{\hat{q}}_{\mathrm{f}} + q_{\mathrm{f}} r) + \hat{a}_1(g_1 q - \hat{a}_2 q_{\mathrm{f}} + f_2 q_{\mathrm{f}} + \hat{a}_2 \hat{q}_{\mathrm{f}} - f_2 \hat{q}_{\mathrm{f}} - g_2 \hat{q}_{\mathrm{f}} + \hat{a}_2 \dot{\hat{q}}_{\mathrm{f}} \\
& - g_1 \dot{\hat{q}}_{\mathrm{f}} + \lambda \dot{\hat{q}}_{\mathrm{f}} - g_1 q_{\mathrm{f}} r + f_1[q - \dot{\hat{q}}_{\mathrm{f}} - q_{\mathrm{f}} r])\Big] + (\hat{a}_2 \hat{q}_{\mathrm{f}} - g_2 \hat{q}_{\mathrm{f}} + (\hat{a}_1 - g_1)\dot{\hat{q}}_{\mathrm{f}})(-g_1 q + \hat{a}_2 q_{\mathrm{f}} - g_2 q_{\mathrm{f}} \\
& + g_1 q_{\mathrm{f}} r + \hat{a}_1[q - q_{\mathrm{f}} r])\Big[p_2(-q_{\mathrm{f}} + \hat{q}_{\mathrm{f}}) + p_2(-q + \dot{\hat{q}}_{\mathrm{f}} + q_{\mathrm{f}} r)\Big]\operatorname{sign}(m). \qquad (30)
\end{aligned}$$

Note that the control contains square, cubic and quartic nonlinear terms.

4 Numerical Examples

We consider the minimum-phase and non-minimum phase unstable systems

$$\ddot{q} - 4\dot{q} + 10.5q = 4.5(\dot{u} + 1.0u), \tag{31}$$

$$\ddot{q} - 4\dot{q} + 10.5q = 4.5(\dot{u} - 1.0u). \tag{32}$$

For simulations, we choose $\lambda = 10$, $f_1 = g_1 = 11$, $f_2 = g_2 = 36$ and $\varepsilon = 1.0$. The closed-loop response shown in Figure 1 indicates that the algorithm successfully stabilizes the system in (31). The controller is turned on at $t = 0.1$ sec. The time history of the parameter estimates $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ (Figure 2) shows that the estimates converge to constant values which are not the true values of the parameters a_1, a_2 and m. This is consistent with Theorem 1. The corresponding results for the non-minimum phase plant (32) are shown in Figures 3 and 4, respectively. The controller is not based on knowledge of the zero and therefore is able to stabilize both plants.

Next, we consider the unstable system with relative degree 2 given by

$$\ddot{q} - 4\dot{q} + 10.5q = -0.5u. \tag{33}$$

For controller implementation, we choose $\lambda = 10$, $f_1 = g_1 = 11$ and $f_2 = g_2 = 36$. The closed-loop response shown in Figure 5 indicates that the algorithm successfully stabilizes the unstable system. The controller is turned on at $t = 2.0$ sec. The time history of the parameter estimates $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ is shown in Figure 6.

Next, we investigate the effects of a change in the value of m by changing m from $m = -2$ to $m = -25$ at time $t = 3.0$ sec. The controller is turned on at $t = 1.6$ sec (Figure 7). After the change in value of m, we observe a small transient due to a reduction of control authority, following which the output of the system converges to 0. The parameter estimates $\hat{a}_1$, $\hat{a}_2$ and $\hat{m}$ (Figure 8) converge to constant values.

Next, we consider Van der Pol's oscillator given by

$$\ddot{q} + \varepsilon(q^2 - 1)\dot{q} + \omega^2 q = bu. \tag{34}$$

The controller is turned on when the system approaches the limit cycle as indicated by the phase portrait of the system (Figure 10). Although Theorem 2 applies to second-order systems with constant coefficients, Figure 9 shows that the controller is able to suppress the limit cycle oscillations.

Lastly, we consider a second order mass-spring-damper system with a nonlinear (Duffing) spring described by the dynamical equation

$$\ddot{q} + \varepsilon\dot{q} + (q^2 - 1)q = bu. \tag{35}$$

The uncontrolled system has three equilibria, namely $(q, \dot{q}) = (0, 0), (-1, 0), (1, 0)$. The origin is an unstable (saddle) equilibrium, whereas the equilibria $(\pm 1, 0)$ are stable (foci). In the simulation, the system is allowed to approach to one of the stable equilibria before the controller is turned on (Figure 11, Figure 12). The phase portrait of the system indicates that the controller is able to bring the system to the origin.

A Proofs and Lemmas

Lemma A.1 *Let $x : [0, \infty) \to \mathbb{R}$ be C^1 and square integrable on $[0, \infty)$, and assume that $\dot{x}$ is bounded. Then $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $M > 0$ satisfy $|\dot{x}(t)| \leq M$ for all $t \geq 0$. Next, note that

$$|x^3(t+h) - x^3(t)| \quad \leq \quad \int_t^{t+h} 3|x^2(t)\dot{x}(t)|dt \leq 3M \int_t^{t+h} |x^2(t)|dt \tag{36}$$

Since x is square integrable, it follows that, for all $\varepsilon > 0$, we can choose sufficiently large $T > 0$ such that

$$\int_T^{T+h} |x^2(t)| \leq \frac{\varepsilon}{3M}. \tag{37}$$

Hence $|x^3(t+h) - x^3(t)| \leq \varepsilon$ for all $t > T$. Since h is arbitrary, it follows that $x^3(t)$ has a limit L as $t \to \infty$. Hence $x(t) \to L^{\frac{1}{3}}$ as $t \to \infty$. Since x is square integrable, L must be 0. □

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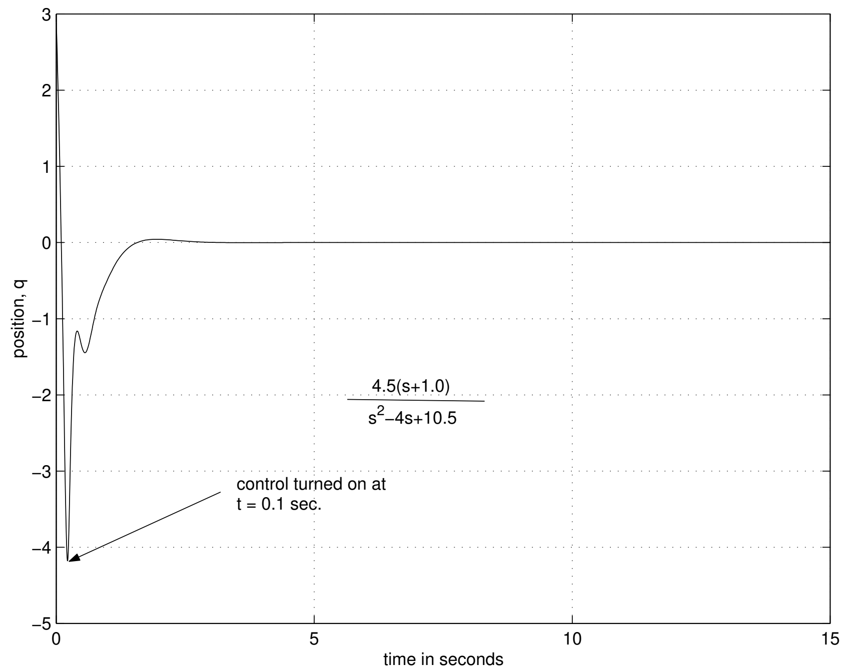

Figure 1: Closed-loop response of the unstable second-order minimum-phase plant (31).

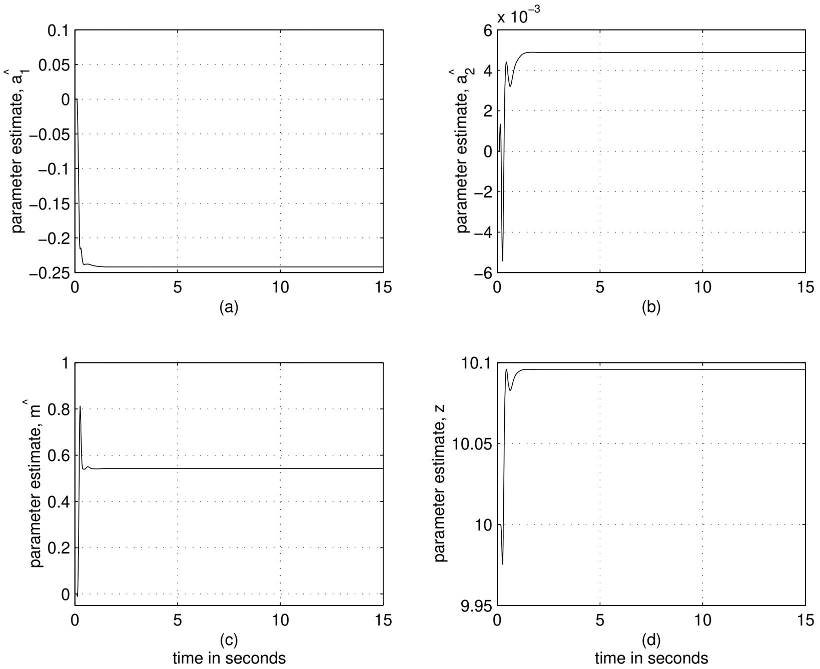

Figure 2: Time history of parameter estimates (a) $\hat{a}_1$, (b) $\hat{a}_2$, (c) $\hat{m}$ and (d) z corresponding to Figure 1.

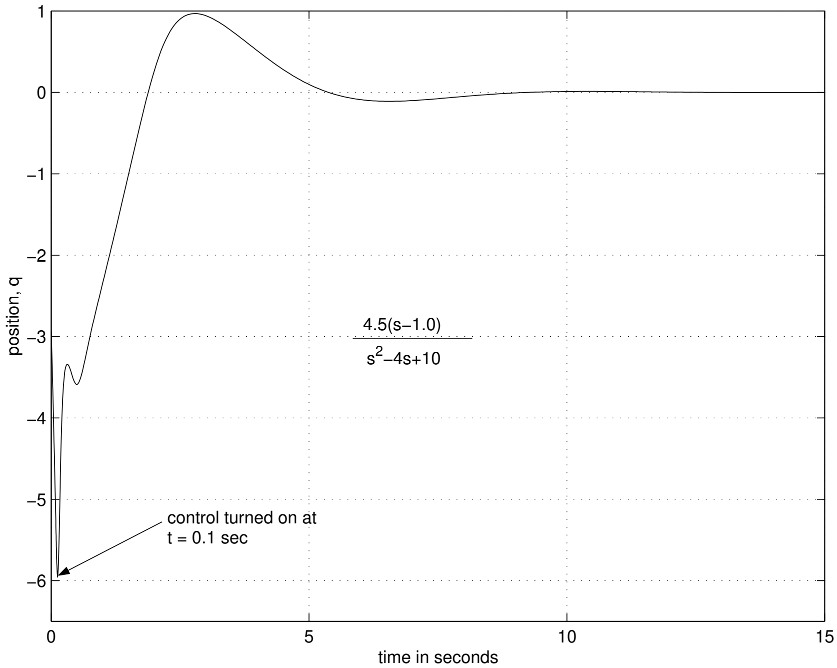

Figure 3: Closed-loop response of the unstable second-order non-minimum phase plant (32).

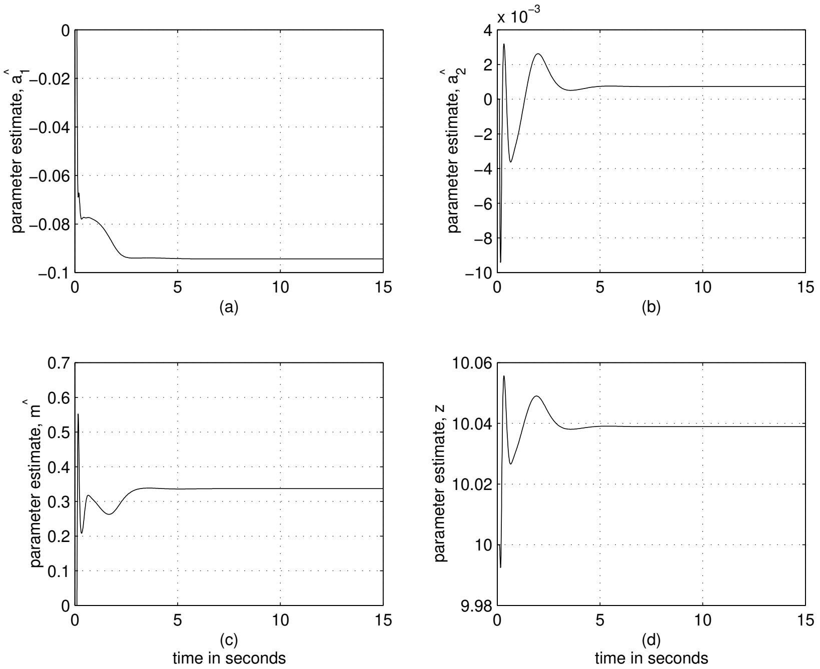

Figure 4: Time history of parameter estimates (a) $\hat{a}_1$, (b) $\hat{a}_2$, (c) $\hat{m}$ and (d) z corresponding to Figure 3.

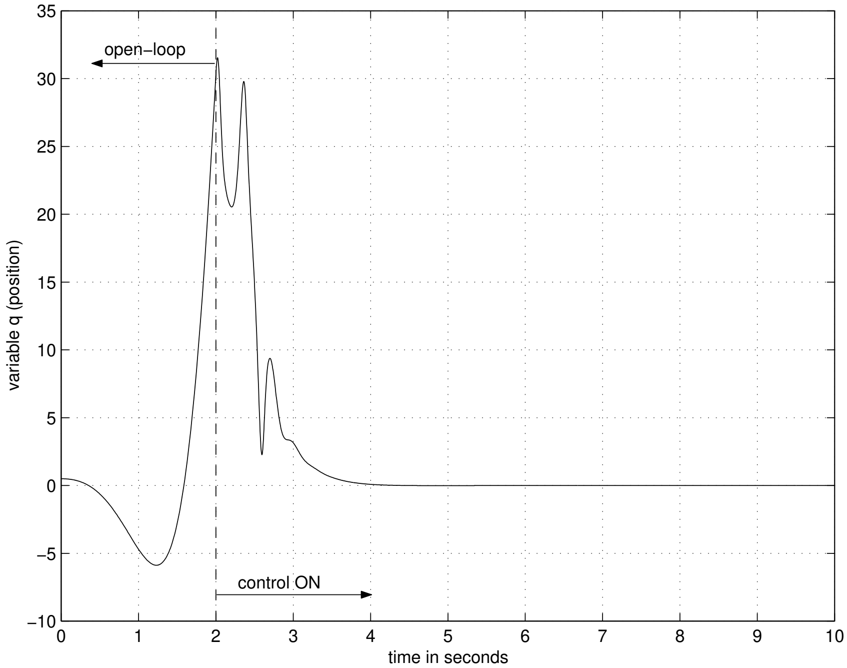

Figure 5: Closed-loop response of the unstable second-order plant (33).

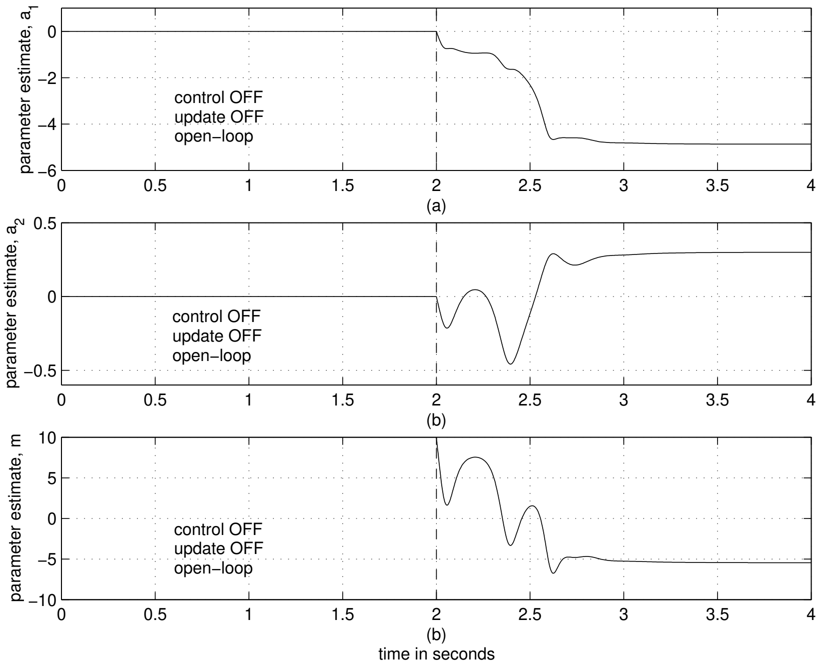

Figure 6: Time history of parameter estimates (a) $\hat{a}_1$, (b) $\hat{a}_2$ and (c) $\hat{m}$ corresponding to Figure 5.

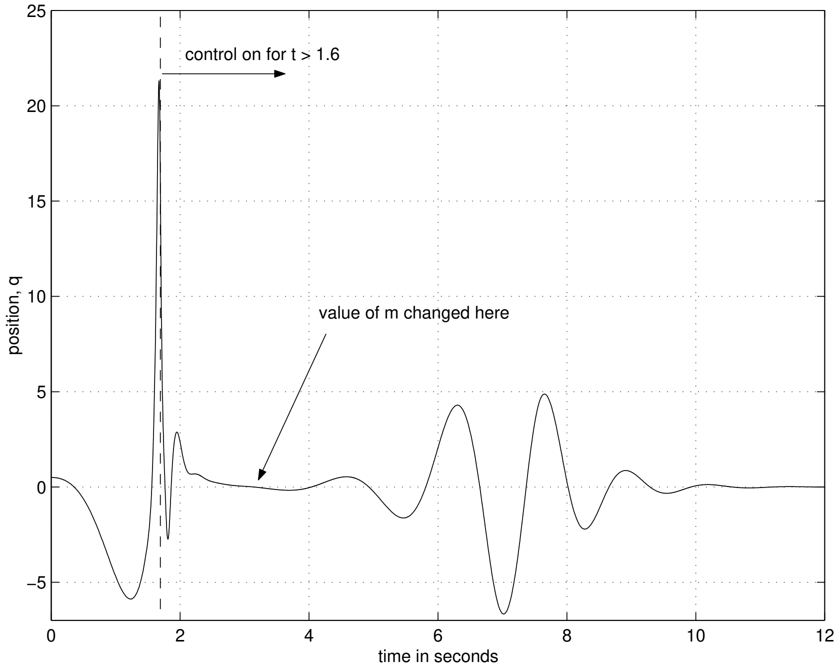

Figure 7: Closed-loop response of the unstable second-order plant (33) with change in m from $m = -2$ to $m = -25$ at arbitrary time $t = 3.0$ sec.

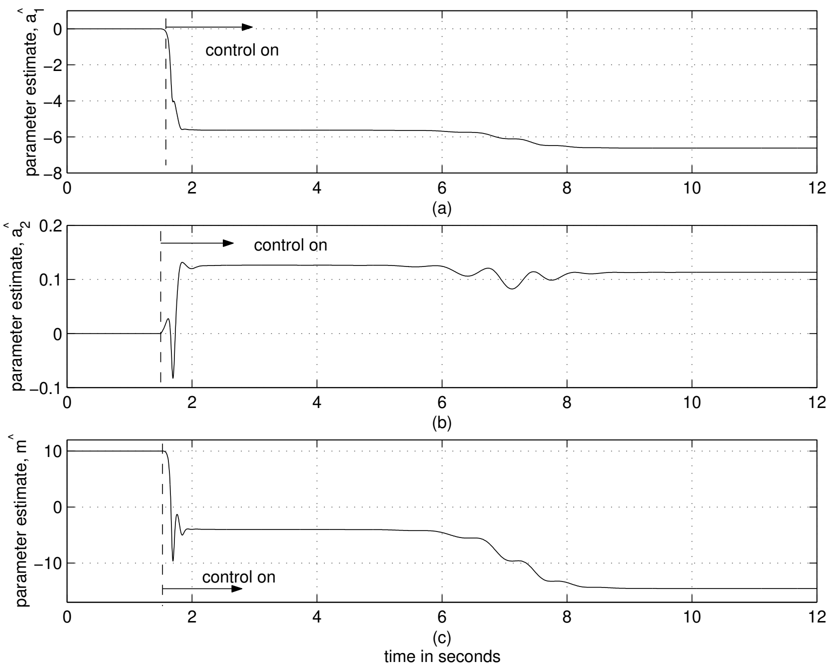

Figure 8: Time history of parameter estimates (a) $\hat{a}_1$, (b) $\hat{a}_2$ and (c) $\hat{m}$ corresponding to Figure 7.

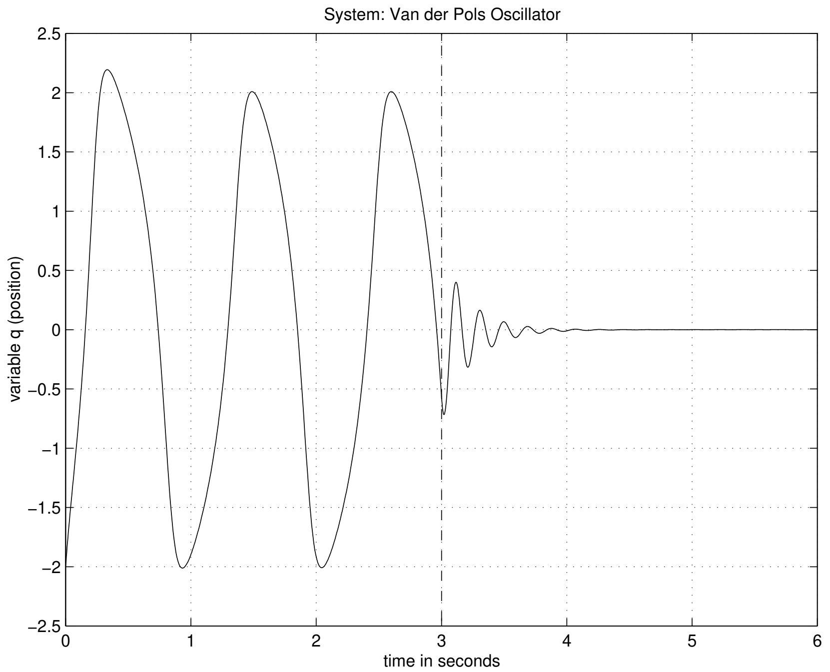

Figure 9: Response of Van der Pol's oscillator (34). The controller is turned on at $t = 3.0$ sec.

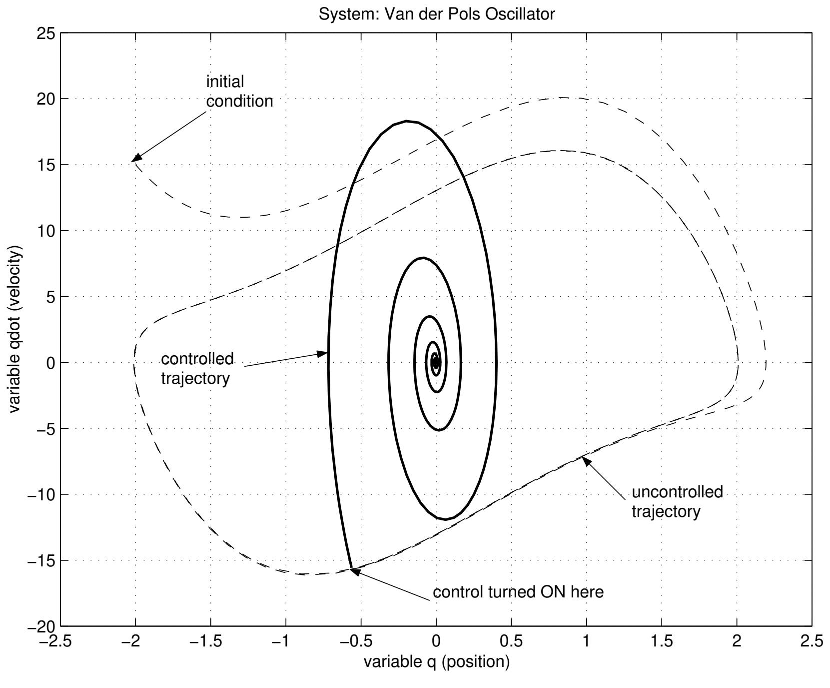

Figure 10: Phase portrait of Van der Pol's oscillator (34). The system is allowed to approach the limit cycle before the controller is turned on at $t = 3.0$ sec.

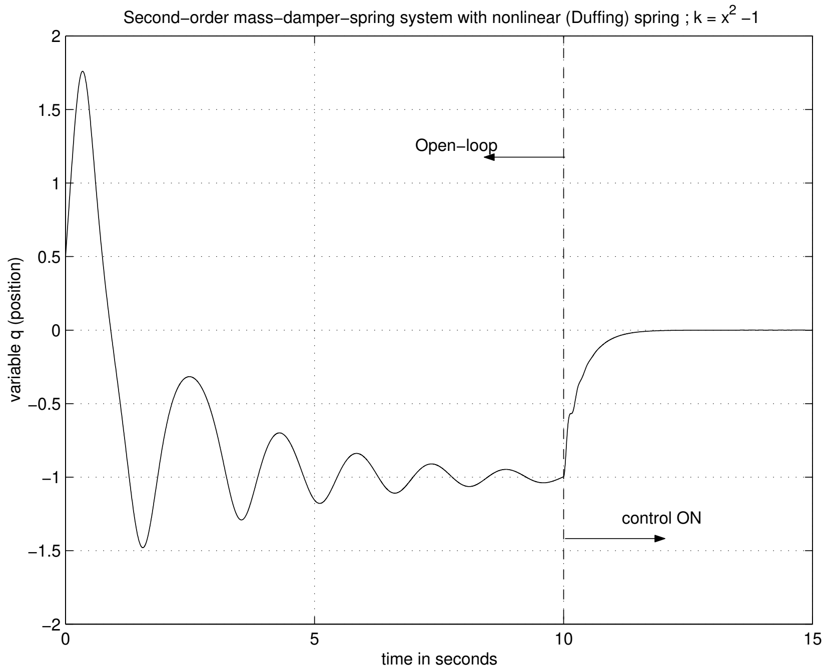

Figure 11: Response of Duffing's mass-spring-damper system with nonlinear spring (35). The controller is turned on at $t = 10.0$ sec.

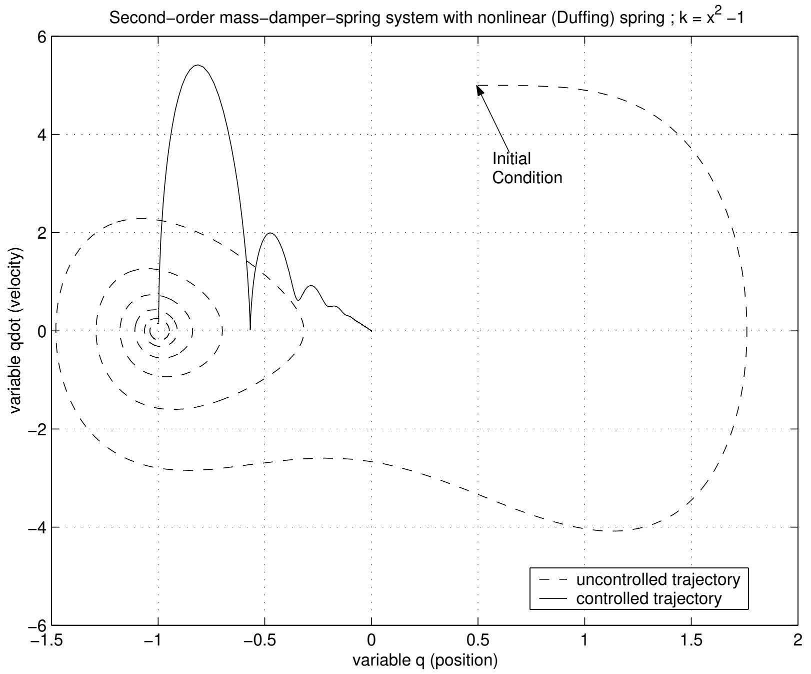

Figure 12: Phase portrait of mass-spring-damper system with nonlinear spring (35). The system is allowed to approach the equilibrium point $(-1, 0)$ before the controller is turned on at $t = 10.0$ sec.