A Note on the Notion of Geometric Rough Paths

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Abstract

We use simple sub-Riemannian techniques to prove that an arbitrary geometric p-rough path in the sense of [15] is the limit in sup-norm of a sequence of canonically lifted smooth paths, which are uniformly bounded in p-variation, clarifying the two different defintions of a geometric p-rough [15,16]. Our proofs are based on fine estimates in terms of control functions and are sufficiently general to include the case of Hölder- and modulus-type regularity [6,7]. This allows us to extend a few classical results on Hölder-spaces [3,19] and p-variation spaces [4,22] to the non-commutative setting necessary for the theory of rough paths.

Key words: Rough Paths, Sub-Riemannian geometry

Introduction

Over the last years T. Lyons developed a general theory of integration and differential equations of the form

$$dy_t = f(y_t)dx_t \tag{1}$$

where the driving signal $x_t \in V$, $t \in [0,1]$, is a Banach space valued path of finite p-variation. For $p < 2$ this leads to a (pathwise) differential equation theory based on Young integrals, but it was not observed before [14] that $x_\cdot \mapsto y_\cdot$ is actually continuous in p-variation topology (and also in some more refined topologies). Most recently [12], Fréchet smoothness was established under natural conditions on f. The situation is much more complicated for

$p \geq 2$ and was successfully worked out in [15]. Clearly, this case is the one needed for applications to stochastic differential equations. The path x driving the differential equation (1) needs to be lifted to a path X of finite p-variation with values in $G^{[p]}(V)$, the free nilpotent group of step $[p]$ over V. The theory of rough paths then gives a solution y to the differential equation, and actually also automatically lifts y to a path Y of finite p-variation with values in a free nilpotent group of step $[p]$. Moreover, the map $X \to Y$ is continuous using an appropriate $p$variation distance (and actually also in more refined topologies).

A smooth V-valued path x can be canonically lifted to a $G^{[p]}(V)$-valued path X. The solution of equation (1) driven by such a canonical X is simply the canonical lift of the classical solution y of the corresponding ODE. If x is a Brownian motion and X its Stratonovich lift to a geometric p-rough path, then y is the solution of the corresponding Stratonovich SDE.

The theory of rough paths tells us that the signal in control differential equations of type (1) are paths with values in a free nilpotent group, satisfying some p-variation constraints. The set of such signals is called the set of geometric p-rough paths.

There has been some confusions on the precise definition of a geometric p-rough path: in [15], a geometric p-rough path is defined as the set of paths with values in $G^{[p]}(V)$ which has finite p-variation, computed with a natural metric associated to the group. It is then falsely claimed that an equivalent definition is the p-variation closure of the canonical lift of smooth paths to paths with values in the group. The latter definition was the one chosen in the more recent monograph of Lyons and Qian [16] and we shall follow its notation.

This paper studies precisely the difference between these two definitions. In the first section, we reintroduce the algebra and analysis needed to explain the theory of rough paths: free nilpotent groups and their homogeneous norms. The second section deals with some basic results on path space. There are at least two possible notions of generalization of Hölder distance between two group valued paths. We show that these two notions lead to the same topology. Then, we obtain some classical interpolation results. The third and fourth section study precisely the set of paths with values in $G^{[p]}(V)$ which have finite p-variation, and the set of geometric p-rough paths. We will see that if Y is a $G^{[p]}(V)$-valued path with finite p-variation, it is the limit in sup-norm of a sequence uniformly bounded in p-variation norm of signature of smooth paths. These smooth paths are constructed using (almost) sub-riemannian geodesics [18]. If Y is indeed a geometric p-rough path, this sequence is shown to converge in p-variation distance. The same results holds replacing p-variation by $1/p$-Hölder, or some more general modulus norms.

We also give a characterization of the set of geometric p-rough paths in the spirit of the Wiener class [4,22]. When translated into $1/p$-Hölder topology, the Wiener class relates to a characterization due to Ciesielski in the vector space case [3,19]. Finally, we precise which of the spaces under consideration are Polish.

C in this paper denotes a constant, which may vary from line to line.

1 Algebraic Preliminaries

We refer to [20] for more details on free nilpotent groups, and [5,18] on homogeneous norms and Carnot Caratheodory distance.

1.1 Free Nilpotent Groups

We fix a real Banach space $(V, \|.\|)$, that we assume finite dimensional. Let $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the tensor algebra over V. $T(V)$ equipped with standard addition $+$, tensor multiplication $\otimes$ and scalar product is an associative algebra. $T^{(m)}(V)$, the quotient algebra of $T(V)$ by the ideal $\bigoplus_{n=m+1}^{\infty} V^{\otimes n}$, inherits this algebra structure. One can define on $T^{(m)}(V)$ a Lie bracket by the

formula

$$[a, b] = a \otimes b - b \otimes a,$$

which makes $T^{(m)}(V)$ into a Lie algebra. Let $\mathcal{G}^m(V)$ be the Lie subalgebra of $T^{(m)}(V)$ generated by elements in V. Note that

$$\mathcal{G}^m(V) \simeq \bigoplus_{i=1}^{m} V_i,$$

where

$$V_1 = V \text{ and } V_{i+1} = [V, V_i]. \tag{2}$$

$\mathcal{G}^m(V)$ is the free nilpotent Lie algebra of step m [15,16,20]. The exponential, logarithm and inverse function are defined on $T^{(m)}(V)$ by mean of their power series. We denote by $G^m(V) = \exp(\mathcal{G}^m(V))$. By the Baker-Campbell-Hausdorff formula, $(G^m(V), \otimes)$ is a connected nilpotent Lie group, called the free nilpotent Lie group of step m over V, with Lie algebra $\mathcal{G}^m(V)$. We also define $\widetilde{T}^{(m)}(V)$ to be the set of elements in $\widetilde{T}^{(m)}$ such that the term in $V^{\otimes 0} = \mathbb{R}$ is equal to 1. $\widetilde{T}^{(m)}(V)$ with the product $\otimes$ of $T^{(m)}(V)$ is a Lie group. Note that $G^m(V)$ is a subgroup of $\widetilde{T}^{(m)}(V)$. For an element $g = 1 + v_1 + \ldots + v_m \in \widetilde{T}^{(m)}(V)$, with $v_i \in V^{\otimes i}$, we define, for $t \in \mathbb{R}$,

$$\delta_t g = 1 + t v_1 + \ldots + t^m v_m.$$

δ is called the dilation operator.

1.2 Homogeneous Norms

We are now going to equip $G^m(V)$ with a (symmetric sub-additive) homogeneous norm [5], i.e. a function $\|.\|_{G^m(V)} : G^m(V) \to \mathbb{R}^+$ such that

- (i) $\|.\|_{G^m(V)}$ if and only if $g = 1$,
- (ii) $\|\delta_t g\|_{G^m(V)} = |t| \, \|g\|_{G^m(V)}$,
- (iii) for all $g, h \in_{G^m(V)}$, $\|g \otimes h\|_{G^m(V)} \leq \|g\|_{G^m(V)} + \|h\|_{G^m(V)}$,
- (iv) for all g, $\|g\|_{G^m(V)} = \|g^{-1}\|_{G^m(V)}$.

We define on the group the Carnot-Caratheodory homogeneous norm $||.||_{G^m(V)}$ with the help of the formula

$$||g||_{G^m(V)} = \inf \left(\int_0^1 |\dot{y}_r| \, dr \right),$$

where the infimum is taken over all smooth paths $y : [0,1] \to V$ such that

$$S_m(y)_{0,1} = g. \tag{3}$$

Here S_m denotes the m-signature of y in $G^m(V)$ between the time r and s, that is

$$S_m(y)_{r,s} = \left(1, y_{r,s} = \int_r^s dy_u, \int_r^s y_{r,u} \otimes dy_u, ..., \int_{r<u_1<\cdots<u_m<s} dy_{u_1} \otimes ... \otimes dy_{u_m}\right).$$

The fact that there exists a smooth path y which satisfies (3) is precisely Chow's theorem [18]. Chen's theorem [2] asserts that $S_m(y)_{0,r} \otimes S_m(y)_{r,s} = S_m(y)_{0,s}$. Note that $s \to S_m(y)_{0,s}$ is equivalently defined as the solution of the ordinary differential equation in $T^{(m)}(V)$

$$dS_m(y)_{0,s} = S_m(y)_{0,s} \otimes dy_s.$$

Proposition 1 *Let z be a path $[0,1] \to V$ in $W^{1,1}$ (i.e. with derivatives in L^1). Then*

$$\|S_m(z)_{s,t}\|_{G^m(V)} \leq \int_s^t |\dot{z}| \, dr.$$

Proof. Obvious by the definition of the Carnot-Caratheodory norm. ■

We now fix $|.|_i$ be some norms on $V^{\otimes i}$ such that for all $(a^i, a^j) \in V^{\otimes i} \times V^{\otimes j}$, $|a^i \otimes a^j|_{i+j} \leq |a^i|_i + |a^j|_j$. To simplify notations, we will write $|.|$ for all these norms. For $x \in \widetilde{T}^{(m)}(V)$,

$$\|x\|_{\widetilde{T}^{(m)}(V)} = \max_{i=1,\ldots,m} \left(i! \left|x^i\right|\right)^{1/i},$$

where $x = 1 + x^1 + \ldots + x^m$, $x^i \in V^{\otimes i}$. Then $x \in \widetilde{T}^{(m)}(V) \to \|x\|_{\widetilde{T}^{(m)}(V)}$ defines a subadditive homogeneous norm on $\widetilde{T}^{(m)}(V)$ [1]. When restricted to $G^m(V)$, it is also symmetric [15]. For $g = \exp(\ell^1 + \ldots + \ell^m)$, with $\ell_i \in \mathcal{L}_i$, we also define

$$\|g\|_{\mathcal{L}^{(m)}(V)} = \max_{i=1,\ldots,m} \left|\ell^i\right|^{1/i}.$$

$\|.\|_{\mathcal{L}^{(m)}(V)}$ is a symmetric homogeneous norm which is equivalent, even when V is of infinite dimension, to the homogeneous norm $\|.\|_{\widetilde{T}^{(m)}(V)}$ restricted to the group $G^m(V)$ [17]. When V is finite dimensional, as all homogeneous norms are equivalent [8], $\|.\|_{G^m(V)}$, $\|.\|_{\widetilde{T}^{(m)}(V)}$ restricted to the group $G^m(V)$, and $\|.\|_{\mathcal{L}^{(m)}(V)}$ are equivalent. Therefore, when no confusion arises, we will not distinguish between these homogeneous norms and we will denote them $\|.\|_m$. We define a left invariant distance: $d_m(g,h) = \|g^{-1} \otimes h\|_m$.

2 Group Valued Paths

By a G-valued path, where $(G, \cdot)$ is a Lie group, we will always mean a continuous function from $[0,1]$ into G, starting at the neutral element of the group. We denote this set by $C_0([0,1], G)$. Moreover, if Y is such a path, we will use the *notation* throughout the paper $Y_{s,t} = Y_s^{-1} \cdot Y_t$.

[1] Note that $g \in \widetilde{T}^{(m)}(V) \to \|g\|_{\widetilde{T}^{(m)}(V)} + \left\|g^{-1}\right\|_{\widetilde{T}^{(m)}(V)}$ defines a subadditive symmetric homogeneous norm, which is equivalent to $\|.\|_{\widetilde{T}^{(m)}(V)}$. Indeed, if $g \in \widetilde{T}^{(m)}(V)$, $g = g^0 + g^1 + ... + g^m$, $g^i \in V^{\otimes i}$, (with $g^0 = 1$), then, for $k \geq 1$

$$\left(g^{-1}\right)^k = \sum_{j=1}^{k} (-1)^j \sum_{\substack{i_1,\cdots,i_j \in \{1,\cdots,m\} \\ i_1+\cdots+i_j=k}} g_{i_1} \otimes \cdots \otimes g_{i_j}. \tag{4}$$

This easily implies that there exists a constant C_m,which depends only on m, such that

$$\left\|g^{-1}\right\|_{\widetilde{T}^{(m)}(V)} \leq C_m \|g\|_{\widetilde{T}^{(m)}(V)}.$$

Lemma 2 *Let g, h be two elements of $\widetilde{T}^{(m)}(V)$, with $g = 1 + g^1 + ... + g^m$, $g^i \in V^{\otimes i}$; we use similar notations for h. The following equation holds in $V^{\otimes k}, k = 1, ..., m$*

$$\left(g^{-1} \otimes h\right)^k = h^k - g^k + \sum_{i=1}^{k-1} \left(g^{-1}\right)^{k-i} \otimes (h^i - g^i) \tag{5}$$

Proof. Set $g^0 = h^0 = 1$. By definition of the tensor product in $\widetilde{T}^{(m)}(V)$, $\left(g^{-1} \otimes h\right)^k = \sum_{i=0}^k \left(g^{-1}\right)^{k-i} \otimes h^i$.The result follows from subtracting to the previous expression $0 = \left(g^{-1} \otimes g\right)^k = \sum_{i=0}^k \left(g^{-1}\right)^{k-i} \otimes g^i$. ■

Proposition 3 *Let $\varepsilon \in (0, 1)$. Given $g \in \widetilde{T}^{(m)}(V)$ there exists a constant $C_m > 0$ such that:*
(i) If $\max_{i=1,\ldots,m} |h^i - g^i| \leq 1$,

$$d_m(g, h) \leq C_m \max\{1, \|g\|_m\} \left(\max_{i=1,\ldots,m} \left|h^i - g^i\right|\right)^{1/m}. \tag{6}$$

(ii)If $\|g^{-1} \otimes h\|_m \leq 1$,

$$\max_{i=1,\ldots,m} \left|h^i - g^i\right| \leq C_m \max\{1, \|g\|_m^m\} d_m(g, h). \tag{7}$$

Proof. From formula (5), one easily sees that

$$\left|\left(g^{-1} \otimes h\right)^k\right| \leq m \max\left\{1, \left\|g^{-1}\right\|_m^k\right\} \max_{i=1,\ldots,m} \left|h^i - g^i\right|.$$

Hence,

$$\begin{aligned}\left|\left(g^{-1} \otimes h\right)^k\right|^{1/k} &\leq C_m \max\{1, \|g\|_m\} \left(\max_{i=1,\ldots,m} \left|h^i - g^i\right|\right)^{1/k} \\ &\leq C_m \max\{1, \|g\|_m\} \left(\max_{i=1,\ldots,m} \left|h^i - g^i\right|\right)^{1/m}\end{aligned}$$

which gives inequality (6).
Reciprocally, assume that $\|g^{-1} \otimes h\|_m \leq 1$. We are going to show by induction that there exists a constant C_m such that for all $i \in \{1, \cdots, m\}$,

$$\left|h^i - g^i\right| \leq C_m \max\left\{1, \|g\|_m^i\right\} \left\|g^{-1} \otimes h\right\|_m. \tag{8}$$

$h^1 - g^1 = \left(g^{-1} \otimes h\right)^1$ so the initial step is easy. Assume now that (8) is true up to a fixed index i. Inequality (5) then gives

$$\begin{aligned}\left|h^{i+1}-g^{i+1}\right| &\leq \left\|g^{-1}\otimes h\right\|_m^k + C_m \sum_{j=1}^{i} \|g\|_m^{i+1-j}\left|h^j-g^j\right| \\ &\leq \left\|g^{-1}\otimes h\right\|_m + C_m \sum_{j=1}^{i} \|g\|_m^{i+1-j} \max\left\{1, \|g\|_m^j\right\}\left\|g^{-1}\otimes h\right\|_m \\ &\leq C_m \left\|g^{-1}\otimes h\right\|_m \max\left\{1, \|g\|_m^{i+1}\right\}.\end{aligned}$$

A straight-forward modification of the above proof yields the same result in terms of the right invariant distance. ■

We obtain the following:

Proposition 4 *Define a right invariant distance* $d_{r,m}(g,h) = \|g\otimes h^{-1}\|_m$ *based on a homogeneous norm* $\|.\|_m$. *If* g_n, $n\in\mathbb{N}$, *and* g *are elements in* $\widetilde{T}^{(m)}(V)$ *then the following is equivalent*
(i): $\lim_{n\to\infty} d_{r,m}(g_n,g)=0$.
(ii): $\lim_{n\to\infty}\max_{i=1,\ldots,m}|g_n^i-g^i|=0$.
(iii): $\lim_{n\to\infty} d_m(g_n,g)=0$.

Corollary 5 *Let* X,Y *be* $G^m(V)$*-valued paths with p-variation controlled by* ω. *Let* d *denote* d_m *or* $d_{r,m}$. *There exists a constant* $c=c\left(\|X\|_\infty, m\right)$ *such that for* ε *small enough, namely* $0<\varepsilon<1/\left(\omega(0,1)\vee 1\right)$, *we have*

$$\max_{k=1,\cdots,m}\frac{\left|X_{s,t}^k-Y_{s,t}^k\right|}{\omega(s,t)^{k/p}}\leq\varepsilon \Longrightarrow d\left(X_{s,t},Y_{s,t}\right)\leq c\varepsilon^{1/m}\omega^{1/p}(s,t)$$

$$d\left(X_{s,t},Y_{s,t}\right)\leq\varepsilon\omega^{1/p}(s,t)\Longrightarrow \max_{k=1,\cdots,m}\frac{\left|X_{s,t}^k-Y_{s,t}^k\right|}{\omega(s,t)^{k/p}}\leq c\varepsilon.$$

Proof. Define using the dilation operator δ on $T^{(m)}$,

$$\tilde{X}_{s,t}=\delta_\gamma(X_{s,t}) \text{ with } \gamma=1/\omega(s,t).$$

As δ commutes with $\otimes$ (and $^{-1}$) so that

$$\begin{aligned}d_m\left(\tilde{X}_{s,t},\tilde{Y}_{s,t}\right) &= \left\|\delta_\gamma(X_{s,t}^{-1}\otimes Y_{s,t})\right\|_m \\ &= \gamma d_m\left(X_{s,t},Y_{s,t}\right)\end{aligned}$$

This reduction allows us to consider without loss of generality $\omega(s,t)=1$ and the proposition follows from the results above. ■

The last corollary implies that the topologies induced by the distances on $G^m(V)$-valued path space

$$\sup_{0\le s<t\le 1} \max_{k=1,\cdots,m} \frac{\left|X_{s,t}^k - Y_{s,t}^k\right|}{\omega(s,t)^{k/p}}$$

and

$$\sup_{0\le s<t\le 1} \frac{d_m\left(X_{s,t}, Y_{s,t}\right)}{\omega(s,t)^{1/p}}$$

are the same (here $\omega(s,t)$ is a control [2] equal to 0 only on the diagonal). The first distance is the one used by Lyons and Lyons/Qian for the continuity results of integration and Itô map, the second one is the authors' favorite one. Any continuity result can therefore be stated in either distances.

2.2 p-Variation and Modulus Distances

A path x in a $G^m(V)$ is said to have finite p-variation if for all subdivision $D = (0 = t_0 < \cdots < t_n = 1)$ of $[0,1]$,

$$\sum_{i=0}^{n-1} \left\| x_{t_i,t_{i+1}} \right\|_m^p < \infty.$$

It can easily be seen to be equivalent the existence of a control function ω such that for all $s \le t$, $\|x_{s,t}\|_m^p \le \omega(s,t)$. We define the following metric on the space of $G^m(V)$-valued paths:

$$d_{\omega,p}(x,y) = \sup_{0\le s<t\le 1} \frac{d_m(x_{s,t}, y_{s,t})}{\omega(s,t)^{1/p}}.$$

Note than when $\omega(s,t) = t-s$, $d_{\omega,p}$ is just the $1/p$-Hölder distance. We introduce a class of "nice" controls:

Condition 6 *A control is said to satisfy the condition (H_p) if it is not identical equal to 0 and if there exists C such that for all $r < s < t < u$, $\frac{\omega(r,u)}{(u-r)^p} \le C\frac{\omega(s,t)}{(t-s)^p}$. Note that it implies in particular that $(t-s)^p \le C\omega(s,t)$.*

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[2] ω is a control if

$$\begin{array}{ll} \text{(i)} & \omega : \{(s,t), 0\le s\le t\le 1\} \to \mathbb{R}^+ \text{ is continuous.} \\ \text{(ii)} & \omega \text{ is super-additive, i.e. } \forall\ s<t<u,\ \omega(s,t)+\omega(t,u) \le \omega(t,u). \\ \text{(iii)} & \omega(t,t) = 0 \text{ for all } t\in[0,1] \end{array} \qquad (9)$$

The control $(s,t) \to t-s$, as well as the controls introduced in [7], satisfy condition (H_p).

We will also look at the p-variation distance:

$$d_{p-var}(x,y) = \sup_{D=(0=t_0<\cdots<t_n=1)} \left(\sum_{i=0}^{n-1} d_m(x_{t_i,t_{i+1}}, y_{t_i,t_{i+1}})^p \right)^{1/p}.$$

We define $\|x\|_{\omega,p} = d_{\omega,p}(x,0)$ and $\|x\|_{p-var} = d_{p-var}(x,0)$.

Definition 7 *We define the following path-spaces*

$$C^{p-var}(G^m(V)) = \left\{ x \in C_0([0,1], G^m(V)) \text{ such that } \|x\|_{p-var} < \infty \right\},$$
$$C^{\omega,p}(G^m(V)) = \left\{ x \in C_0([0,1], G^m(V)) \text{ such that } \|x\|_{\omega,p} < \infty \right\}.$$

$C^{0,p-var}(G^m(V))$ (resp. $C^{0,\omega,p}(G^m(V))$) is defined as the d_{p-var}-closure (resp. $d_{\omega,p}$-closure) of the set $\{S_m(x),\ x$ smooth V-valued path$\}$.

$C^{0,p-var}(G^{[p]}(V))$ is precisely the set of geometric p-rough paths, according to the definition of [16], while $C^{p-var}(G^{[p]}(V))$ is the set of geometric p-rough paths, according to [15]. There has indeed been some confusions in the seminal paper [15] between the two sets $C^{0,p-var}(G^{[p]}(V))$ and $C^{p-var}(G^{[p]}(V))$. Here, we propose to study and characterize these sets precisely. Studying their subset $C^{0,\omega,p}(G^m(V))$ and $C^{\omega,p}(G^m(V))$ is also of interest, as the continuity results of the theory of rough paths can involve the distance $d_{\omega,p}$.

We will need some interpolation results.

2.3 Interpolations

Proposition 8 *Let $Y(n)$ be a sequence of equi-continuous $G^m(V)$-valued paths converging pointwise to a continuous path Y. Then $Y(n)$ converges uniformly on $[0,1]$ to Y, i.e.*

$$\sup_t d_m(Y(n)_t, Y_t) \to 0.$$

Proof. Standard Arzela-Ascoli argument. ■

Proposition 9 $\widetilde{d}_\infty(Y(n),Y) = \sup_t d_m(Y_t(n), Y_t) \to_{n\to\infty} 0$ *if and only if*

$$d_\infty(Y(n),Y) = \sup_{s,t} d_m(Y(n)_{s,t}, Y_{s,t}) \to_{n\to\infty} 0.$$

Proof. Clearly, if $d_\infty(Y(n),Y)$ goes to 0 as $n\to\infty$, then so does $\tilde{d}_\infty(Y(n),Y)$. Reciprocally,

$$d_\infty(Y(n)_{s,t},Y_{s,t}) \le \sup_{s,t} d_m(Y(n)_{s,t},Y(n)_s^{-1}\otimes Y_t) + \sup_{s,t} d_m(Y(n)_s^{-1}\otimes Y_t, Y_{s,t}).$$

But $\sup_{s,t} d_m(Y(n)_{s,t},Y(n)_s^{-1}\otimes Y_t) = \sup_t d_m(Y(n)_t,Y_t)$ goes, by assumption, to 0 when $n\to\infty$. Moreover, by corollary 4,

$$\lim_{n\to\infty}\sup_{s,t} d_m(Y(n)_s^{-1}\otimes Y_t, Y_{s,t}) = 0$$

if and only if

$$\lim_{n\to\infty}\sup_{s,t} d_{r,m}(Y(n)_s^{-1}\otimes Y_t, Y_{s,t}) = 0.$$

But the latter is true as

$$\begin{aligned} d_{r,m}(Y(n)_s^{-1}\otimes Y_t, Y_{s,t}) &= d_{r,m}(Y(n)_s^{-1}, Y_s^{-1}) \\ &= d_m(Y(n)_s, Y_s). \end{aligned}$$

■

Remark 10 *$\tilde{d}_\infty$ and d_∞ are not equivalent distances, but induce the same topology. The following inequalities are classical, at least for the Hölder norms [10,21] and p-variation norms [11].*

Proposition 11 *Let $1\le p<p'<\infty$. Then for all $G^m(V)$-valued paths Y,Z*

$$d_{\omega,p'}(Y,Z) \le d_\infty(Y,Z)^{1-p/p'} d_{\omega,p}(Y,Z)^{\frac{p}{p'}}. \tag{10}$$

In particular, if $Y(n)$ converges pointwise to Y and $\sup_n \|Y(n)\|_{\omega,p}<\infty$ then

$$d_{\omega,p'}(Y(n),Y)\to 0.$$

Proof. For all $s<t$,

$$\begin{aligned} \frac{d_m(Y_{s,t},Z_{s,t})^{p'}}{\omega(s,t)} &= d_m(Y_{s,t},Z_{s,t})^{p'-p}\frac{d_m(Y_{s,t},Z_{s,t})^{p}}{\omega(s,t)} \\ &\le d_\infty(Y,Z)^{p'-p} d_{\omega,p}(Y,Z)^p, \end{aligned}$$

which gives inequality (10). $\sup_n \|Y(n)\|_{\omega,p}<\infty$ and the pointwise convergence of $Y(n)$ to Y implies that $\|Y\|_{\omega,p}<\infty$. Then, by proposition 8, we obtain that $Y(n)$ converges uniformly to Y, and we obtain our result by applying inequality (10). ■

A similar proof gives the following proposition:

Proposition 12 *Let $1 \leq p < p' < \infty$. Then for all $G^m(V)$-valued paths Y, Z,*

$$d_{p'-var}(Y,Z) \leq d_\infty(Y,Z)^{1-p/p'} d_{p-var}(Y,Z)^{\frac{p}{p'}}. \tag{11}$$

In particular, if $Y(n)$ converges uniformly to Y and $\sup_n \|Y(n)\|_{p-var} < \infty$ then

$$d_{p-var}(Y(n),Y) \to 0.$$

3 The Spaces $C^{p-var}(G^m(V))$ and $C^{\omega,p}(G^m(V))$

We are going to prove that elements in $C^{p-var}(G^m(V))$ and $C^{\omega,p}(G^m(V))$ are still limit in some sense of signature of smooth paths. We will use the following proposition.

Proposition 13 *For every $g \in G^m(V)$, there exists a smooth path $h_g(.) = y(.)$ of constant speed $|\dot{y}|$ such that $S_m(y)_{0,1} = g$ and such that*

$$\|\dot{y}\|_{L^\infty} \leq 2\|g\|_m.$$

As a consequence,

$$\|S_m(y)_{s,t}\|_m \leq 2\|g\|_m (t-s).$$

Proof. Without lost of generalities, we can assume that $\|g\| > 0$. (Indeed, if $\|g\| = 0$, the path $y : [0,1] \to V$, $u \to 0$ will do.)

By definition of the Carnot-Caratheodory norm, for every $\varepsilon > 0$, there exists a smooth path y, which we may take of constant speed (by time reparametrization), such that $S_m(y)_{0,1} = g$ and such that the sup norm of $\dot{y}$ is bounded by $\|g\| + \varepsilon$. Taking $\varepsilon = \|g\|$ finishes the first part. The second part follows from Proposition 1. ■

We would have liked to define h_g as a geodesic associated to g, e.g. the shortest connection of the neutral element in $G^m(V)$ with g w.r.t. the Carnot-Caratheodory distance $d_{G^m(V)}$. Unfortunately, smoothness of such geodesics is still an open problem for $m \geq 3$ (personal communication, R.Montgomery). An affirmative answer for the case $m = 2$ is found in [13].

Theorem 14 *Let ω be a control satisfying condition (H_p). A path Y belongs to $C^{\omega,p}(G^m(V))$ if and only if there exists a sequence of smooth V-valued paths $y(n)$ such that*
(i): $\sup_n \|S_m(y(n))\|_{\omega,p} < \infty$.
(ii): $S_m(y(n))$ converges pointwise to Y.
In particular, $S_m(y(n))$ converges to Y in the topology induced by $d_{\omega,q}$, whenever $q > p$.

Proof. The fact that the existence of a sequence of smooth paths y_n satisfying conditions (i) and (ii) implies that $\|Y\|_{\omega,p} < \infty$ is obvious. We prove the reverse implication.
We let ϕ be a non-decreasing function in $C^\infty([0,1],\mathbb{R})$ such that

$$\begin{aligned} &\phi(0)=0,\\ &\phi(1)=1,\\ &\quad\forall k\geq 1, \phi^{(k)}(0) = \phi^{(k)}(1) = 0. \end{aligned}$$

Fix a subdivision of $[0,1]$,$D = \{0 = t_0 < t_1 < ... < t_n = 1\}$. ¿From this subdivision, we construct a smooth path $y(D)$, one time-interval after the other: first $y(D)_0 = 0$. Then, for $t \in [t_{i-1}, t_i]$, we let $y(D)_{t_i,t} = h_{Y_{t_i,t_{i+1}}}\left(\phi\left(\frac{t-t_i}{t_{i+1}-t_i}\right)\right)$ (h_g has been defined in the previous proposition). Thanks to our choice of the function ϕ, $y(D)$ is a smooth path.

Using Proposition 13, for $t_{i-1} \leq s \leq t \leq t_i$,

$$\begin{aligned} \left\|S_m\left(y(D)\right)_{s,t}\right\|_m &= \left\|S_m\left(h_{Y_{t_i,t_{i+1}}}\right)_{\phi\left(\frac{s-t_i}{t_{i+1}-t_i}\right),\phi\left(\frac{t-t_i}{t_{i+1}-t_i}\right)}\right\|_m \\ &\leq 2\left\|Y_{t_i,t_{i+1}}\right\|_m\left(\phi\left(\frac{t-t_i}{t_{i+1}-t_i}\right) - \phi\left(\frac{s-t_i}{t_{i+1}-t_i}\right)\right) \\ &\leq 2\left|\phi'\right|_\infty \frac{t-s}{t_{i+1}-t_i}\left\|Y_{t_i,t_{i+1}}\right\|_m \\ &\leq C\frac{t-s}{t_{i+1}-t_i}\left\|Y\right\|_{\omega,p}\left[\omega(t_i,t_{i+1})\right]^{1/p} \\ &\leq C\left\|Y\right\|_{\omega,p}\omega(s,t)^{1/p} \qquad \text{using condition (H).} \end{aligned}$$

Note that

$$S_m\left(y(D)\right)_{t_i} = Y_{t_i} \text{ for all } i = 0, ..., n. \tag{12}$$

Then for all $t_{i-1} \leq s \leq t_i \leq t_j < t \leq t_{j+1}$,

$$\begin{aligned} \|S_m(y(D))_{s,t}\|_m^p &= \left\|S_m(y(D))_{s,t_i} \otimes Y_{t_i,t_j} \otimes S_m(y(D))_{t_j,t}\right\|_m^p \\ &\leq 3^{p-1}\left(\|S_m(y(D))_{s,t_i}\|_m^p + \left\|Y_{t_i,t_j}\right\|_m^p + \left\|S_m(y(D))_{t_j,t}\right\|_m^p\right) \\ &\leq C\left\|Y\right\|_{\omega,p}^p\left(\omega(s,t_i) + \omega(t_i,t_j) + \omega(t_j,t)\right) \\ &\leq C\left\|Y\right\|_{\omega,p}^p\omega(s,t). \end{aligned}$$

Moreover, for t as above, from equality (12) and the left invariance of the distance d, we get that if $t_j \leq t \leq t_{j+1}$,

$$
\begin{aligned}
d_m(S_m(y(D))_t, Y_t) &= d_m(S_m(y(D))_{t_j,t}, Y_{t_j,t}) \\
&\leq C \left\| S_m(y(D))_{t_j,t} \right\|_m + \left\| Y_{t_j,t} \right\|_m \\
&\leq C \left(\left\| Y_{t_j,t_{j+1}} \right\|_m + \left\| Y_{t_j,t} \right\|_m \right) \\
&\leq C \sup_{\substack{s,t\in[0,1] \\ |t-s|\leq\|D\|}} \left\| Y_{s,t} \right\|_m .
\end{aligned} \tag{13}
$$

Y is continuous and defined on a compact ($[0,1]$), hence by Heine-Cantor's theorem, it is uniformly continuous. Therefore, for all $\varepsilon > 0$, there exists η such that $|D| < \eta \Rightarrow d_m(S_m(y(D))_t, Y_t) < \varepsilon$. We have just shown that if $(D_n)_n$ is a family of subdivision of $[0,1]$, whose mesh's size goes to 0 when $n \to \infty$, then $y(D_n)$ is a sequence of smooth path satisfying conditions (i) and (ii). The last statement is just a corollary of inequality (10). ■

Corollary 15 *Let Y be a $G^m(V)$-valued path. Then Y is $1/p$-Hölder if and only if there exists a sequence of infinitely differentiable V-valued paths $y(n)$ such that*
(i): The $1/p$-Hölder norm of $S_m(y(n))$ is uniformly bounded.
(ii): $S_m(y(n))$ converges pointwise to Y.
In particular, given an $\alpha = 1/p$ Hölder regular $G^m(V)$-valued path Y there is a sequence of signature of smooth paths that converge in α'-Hölder topology to Y, for any $\alpha' < \alpha$.

Proof. Apply the previous theorem with the control $(s,t) \mapsto t - s$. ■

We now consider the p-variation distance. First, if Y are paths of finite p-variation, we define

$$
\delta_Y^p(s,t) = \sup_{D=(s\leq t_0<\cdots<t_n\leq t)} \sum_{i=0}^{n-1} \left\| Y_{t_i,t_{i+1}} \right\|_m^p . \tag{14}
$$

In other words, δ_Y^p is the smallest control of the p-variation of Y.

Theorem 16 *Y belongs to $C^{p-var}(G^m(V))$ if and only if there exists a sequence of infinitely differentiable V-valued paths $y(n)$ such that*
(i): $\|S_m(y(n))\|_{p-var}$ is uniformly bounded.
(ii): $S_m(y(n))$ converges pointwise to Y.
In particular, $S_m(y(n))$ converges in q-variation to Y, whenever $q > p$.

Proof. We construct $y(D)$ from Y as in the proof of theorem 14 with the help of a subdivision $D = \{0 = t_0 < t_1 < ... < t_n = 1\}$. Define the control

$$
\omega_D(s,t) = \left(\frac{t-s}{t_{i+1}-t_i} \right)^p \delta_Y^p(t_i, t_{i+1}) \text{ for } t_i \leq s \leq t \leq t_{i+1},\ 0 \leq i \leq n-1;
$$

and for $0 \leq i < j \leq n-1$, and $t_{i-1} \leq s \leq t_i \leq t_j \leq t \leq t_{j+1}$,

$$\omega_D(s,t) = \omega_D(s,t_i) + \delta_Y^p(t_i,t_j) + \omega_D(t_j,t). \tag{15}$$

It is easy to check that ω_D is a control (but does not necessarily satisfies condition (H_p)). Then, from the proof of theorem 14, we see that for $t_i \leq s \leq t \leq t_{i+1}$,

$$\begin{aligned} \|S_m(y(D))_{s,t}\|^p &\leq C\left(\frac{t-s}{t_{i+1}-t_i}\right)^p \delta_Y^p(t_i,t_{i+1}) \\ &\leq C\omega_D(s,t). \end{aligned}$$

Then, if $t_{i-1} \leq s \leq t_i \leq t_j < t \leq t_{j+1}$,

$$\begin{aligned} \|S_m(y(D))_{s,t}\|_m^p &\leq C\left(\|S_m(y(D))_{s,t_i}\|_m^p + \left\|Y_{t_i,t_j}\right\|_m^p + \left\|S_m(y(D))_{t_j,t}\right\|_m^p\right) \\ &\leq C\left(\omega_D(s,t_i) + \delta_Y^p(t_i,t_j) + \omega_D(t_j,t)\right) \\ &= C\omega_D(s,t). \end{aligned}$$

Therefore,

$$\|S_m(y(D))\|_{p-var} \leq C\omega_D(0,1)^{1/p} = C\delta_Y^p(0,1)^{1/p} = C\|Y\|_{p-var}.$$

Hence, if $\left(D_n = \left\{0 \leq t_1^n < \cdots \leq t_{\#D_n}^n\right\}\right)_n$ is a sequence of subdivision of $[0,1]$ whose mesh's size tends to 0, we have just proved that $(y(D_n))_n$ satisfies condition (i); condition (ii) is treated just like before, using inequality (13).
The last statement is a corollary of inequality 11, once we prove that $S_m(y(D_n))$ converges uniformly to Y. To do so, define $h_{D_n}(\delta) = \sup_{|t-s|\leq\delta} \omega_{D_n}(s,t)$, and $h_\infty(\delta) = \sup_{|t-s|\leq\delta} \omega(s,t)$. By Heine-Cantor's theorem, ω is uniformly continuous. As it is zero on the diagonal, we obtain that $h_{D_n}(\delta) \to_{\delta\to 0} 0$. If $|D_n| \leq \delta$, for all $s < t$ such that $|t-s| \leq \delta$, there exists $1 \leq i \leq \#D_n - 2$ such that $t_i \leq s < t \leq t_{i+2}$. Hence, by definition of

$$\omega_{D_n}(s,t) \leq \omega(t_i^n, t_{i+2}^n) \leq h_\infty(2|D_n|) \leq h_\infty(2\delta).$$

Hence, given an $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\delta \leq \delta_0 \Rightarrow h_\infty(2\delta) \leq \varepsilon$. Let N be such that $\sup_{n\geq N} |D_n| \leq \delta_0$. Then, there exists $\delta_1 > 0$ such that $\delta \leq \delta_1 \Rightarrow \max_{n<N} h_{D_n}(\delta) \leq \varepsilon$. In particular, $\delta \leq \min\{\delta_0,\delta_1\} \Rightarrow \sup_{n\in\mathbb{N}\cup\{\infty\}} h_{D_n}(\delta) \leq \varepsilon$, i.e. $\sup_{n\in\mathbb{N}\cup\{\infty\}} h_{D_n}$ goes to 0 at 0. We have therefore proved that $S_m(y(D_n))$ is equicontinuous, which implies the wanted uniform convergence by lemma 8. ■

Remark 17 *We could have obtained the above theorem replacing smooth paths by paths of bounded variation, by a simple change of time. Indeed, a path of finite p-variation can be reparametrized into a 1/p-Hölder path.*

Remark 18 *These convergences results, more precisely the (almost) geodesic approximation versus piecewise linear, can be compared with probabilistic constructions in the context of Brownian rough paths ([16,6,7]). There it is essential to base approximations on nested (or dyadic) subdivisions of* $[0,1]$ *which is not required here.*

4 The Spaces $C^{0,p-var}(G^m(V))$ and $C^{0,\omega,p}(G^m(V))$

Recall the definitions of these space made earlier. We are first going to give an equivalent definition of these sets. We will then prove that these spaces are separable.

4.1 A Ciesielski/Museliak-Semadini Type Result

Ciesielski and Museliak-Semadini proved at similar times with different techniques the following theorem, in the case of Hölder real valued paths. Taking $\omega(s,t) = t - s$, and $m = 1$, $V = \mathbb{R}$ in the theorem below gives (some) of their results. [3,19].

Theorem 19 *Let* Y *be an element of* $C^{\omega,p}(G^m(V))$*. We assume that* ω *satisfies condition* (H_p) *and that*

$$\lim_{\delta\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\ge\delta}} \frac{t-s}{\omega(s,t)^{1/p}} = 0. \tag{16}$$

Then, Y *belongs to* $C^{0,\omega,p}(G^m(V))$ *if and only if*

$$\lim_{\delta\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s\le\delta}} \frac{\|Y_{s,t}\|}{\omega(s,t)^{1/p}} = 0.$$

Note that the condition (16) implies that $p > 1$. Under condition (H_p), we have

$$\overline{\lim_{\delta\to 0}} \sup_{\substack{0\le s<t\le 1\\ t-s\ge\delta}} \frac{t-s}{\omega(s,t)^{1/p}} =: C < \infty.$$

Then condition 16 simply reads $C = 0$.

Proof. Assume that Y belongs to $C^{0,\omega,p}(G^m(V))$. Then, by definition, there exists a sequence of signature of smooth paths $(S_m(y_n))_n$ such that

$$\lim_{n\to\infty} d_{\omega,p}(S_m(y_n), Y) = 0.$$

For all $s < t$,

$$\frac{\|Y_{s,t}\|}{\omega(s,t)^{1/p}} \leq d_{p,\omega}(Y, S_m(y_n)) + \frac{\|S_m(y_n)_{s,t}\|}{\omega(s,t)^{1/p}},$$

hence for all n,

$$\lim_{\delta\to 0} \sup_{\substack{0\leq s<t\leq 1\\ t-s\geq\delta}} \frac{\|Y_{s,t}\|}{\omega(s,t)^{1/p}} \leq d_{p,\omega}(Y, S_m(y_n)) + \lim_{\delta\to 0} \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} \frac{\|S_m(y_n)_{s,t}\|}{\omega(s,t)^{1/p}}.$$

But as y_n is smooth, $S_m(y_n)$ is Lipschitz, hence, by the assumption on the control ω, we obtain that $\lim_{\delta\to 0} \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} \frac{\|S_m(y_n)_{s,t}\|}{\omega(s,t)^{1/p}} = 0$. Therefore, for all n,

$$\lim_{\delta\to 0} \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} \frac{\|Y_{s,t}\|}{\omega(s,t)^{1/p}} \leq d_{p,\omega}(Y, S_m(y_n)),$$

i.e. this limit is equal to 0.

Reciprocally, assume that

$$\lim_{\delta\to 0} \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} \frac{\|Y_{s,t}\|}{\omega(s,t)^{1/p}} = 0.$$

We define

$$\vartheta_Y(\delta) = \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} \frac{\|Y_{s,t}\|}{\omega(s,t)^{1/p}} \in [0, \|Y\|_{\omega,p}] \text{ for } \delta \in [0,1].$$

We now define $y(D)$ from Y as in the proof of theorem 14, where D is a given subdivision of $[0,1]$. With techniques similar to the one used in the proof of theorem 14, we see that $\vartheta_{S_m(y(D))}(\delta) \leq C \|Y\|_{\omega,p} \vartheta_Y(\delta)$ for a universal constant C. Then, for $s < t$ such that $|t-s| \geq \delta$,

$$\begin{aligned}\frac{d(Y_{s,t}, S_m(y(D))_{s,t})}{\omega(s,t)^{1/p}} &\leq \frac{d_\infty(Y, S_m(y(D)))}{\inf_{|t-s|\geq\delta} \omega(s,t)^{1/p}} \\ &\leq C \frac{d_\infty(Y, S_m(y(D)))}{\delta}, \qquad \text{using condition } (H_p)\end{aligned}$$

For $s < t$ such that $|t-s| < \delta$,

$$\begin{aligned}\frac{d(Y_{s,t}, S_m(y(D))_{s,t})}{\omega(s,t)^{1/p}} &\leq \frac{\|Y_{s,t}\|_m + \|S_m(y(D))_{s,t}\|_m}{\omega(s,t)^{1/p}} \\ &\leq C \|Y\|_{\omega,p} \vartheta(\delta).\end{aligned}$$

Hence, for all $\delta > 0$,

$$d_{\omega,p}(Y, S_m(y(D))) \leq C \max\left\{\frac{d_\infty(Y, S_m(y(D)))}{\delta}, \|Y\|_{\omega,p}\,\vartheta(\delta)\right\}.$$

Now, once again, consider a sequence of subdivisions $(D_n)_n$ of $[0,1]$ whose mesh's size tends to 0. For a given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|Y\|_{\omega,p}\,\vartheta(\delta) \leq \varepsilon/C$ (as by assumption, $\lim_{\delta\to 0}\vartheta(\delta) = 0$). For such a δ and ε, there exists N such that if $n \geq N$, $\frac{d_\infty(Y,S_m(y(D_n)))}{\delta} \leq \varepsilon/C$. We have therefore proved that for all $\varepsilon > 0$, there exists N such that for all $n \geq N$, $d_{\omega,p}(Y, S_m(y(D_n))) \leq \varepsilon$. ■

4.2 A Wiener Type Result

We now prove a similar theorem, but in p-variation topology rather than modulus topology. p-variation closure of step functions has been characterized by Wiener [22,19], for real valued functions, but not necessarily continuous. We obtain this result here in the simpler case of continuous paths, but harder case of group valued paths.

Theorem 20 *Let Y be an element of $C^{p-var}(G^m(V))$, $p > 1$. Then, Y belongs to $C^{0,p-var}(G^m(V))$ if and only if*

$$\lim_{\delta\to 0} \sup_{\substack{D=(0=t_0<\cdots<t_n=1)\\ |D|\leq\delta}} \sum_{i=0}^{n-1} \delta_Y^p(t_i, t_{i+1}) = 0,$$

with δ_Y^p defined as earlier (equation (14)).

Proof. If Y belongs to $C^{0,p-var}(G^m(V))$, there exists a sequence of smooth paths y_n such that $S_m(y_n)$ converges in the topology induced by d_{p-var} to Y. Then, if $D = (0 = t_0 < \cdots < t_l = 1)$ is a partition of $[0,1]$,

$$\left(\sum_{i=0}^{l-1} \delta_Y^p(t_i, t_{i+1})\right)^{1/p} \leq \left(\sum_{i=0}^{l-1} \delta_{S_m(y_n)}^p(t_i, t_{i+1})\right)^{1/p} + d_{p-var}(Y, S_m(y_n)).$$

Because y_n is smooth and $p > 1$,

$$\lim_{\delta\to 0} \sup_{\substack{D=(0=t_0<\cdots<t_l=1)\\ |D|\leq\delta}} \sum_{i=0}^{l-1} \omega_{p,S_m(y_n)}(t_i, t_{i+1}) = 0.$$

We therefore obtain that

$$\lim_{\delta\to 0}\sup_{\substack{D=(0=t_0<\cdots<t_l=1)\\|D|\le\delta}}\sum_{i=0}^{l-1}\delta_Y^p(t_i,t_{i+1})=0.$$

Reciprocally, we let $\pi_n=\{k2^{-n},k\in\{0,\cdots,2^n\}\}$ be a (very specific for simplicity) sequence of subdivisions of $[0,1]$, whose mesh's size goes to 0. We let $y_n=y(\pi_n)$, obtained from π_n and Y, as in the proof of theorems 16. We also define $\omega_n(s,t)=\omega_{\pi_n}(s,t)$ (ω_{π_n} is defined as in equation (15)), $\omega_\infty=\delta_Y^p$ and for $n\in\mathbb{N}\cup\{\infty\}$,

$$g_n(\delta)=\sup_{\substack{D=(0=t_0<\cdots<t_l=1)\\|D|\le\delta}}\sum_{i=0}^{l-1}\omega_n(t_i,t_{i+1}).$$

By assumption, g_∞ tends to 0 at 0. We are now going to prove that g_n, $n\in\mathbb{N}$, share the same property. Consider a subdivision $D=(0=t_0<\cdots<t_l=1)$ of $[0,1]$ with mesh's size less than $\delta<2^{-n}$. From the definition of ω_n, we see that when we compute $\sum_{i=0}^{l-1}\omega_n(t_i,t_{i+1})$, we obtain the same result if we add to our subdivision D all the point $\{k2^{-n},0\le k\le 2^n\}$. Having done so, we can write

$$\begin{aligned}\sum_{i=0}^{l-1}\omega_n(t_i,t_{i+1}) &= \sum_{k=0}^{2^n-1}\omega_\infty\left(\frac{k}{2^n},\frac{k+1}{2^n}\right)\sum_{i,\ \frac{k}{2^n}\le t_i\le t_{i+1}\le\frac{k+1}{2^n}}\left(\frac{t_{i+1}-t_i}{2^{-n}}\right)^p\\ &\le\sum_{k=0}^{2^n-1}\omega_\infty\left(\frac{k}{2^n},\frac{k+1}{2^n}\right)\sum_{i,\ \frac{k}{2^n}\le t_i\le t_{i+1}\le\frac{k+1}{2^n}}\left(\frac{t_{i+1}-t_i}{2^{-n}}\right)(\delta 2^n)^{p-1}\\ &=(\delta 2^n)^{p-1}\sum_{k=0}^{2^n-1}\omega_\infty\left(\frac{k}{2^n},\frac{k+1}{2^n}\right)\\ &\le(\delta 2^n)^{p-1}g_\infty(2^{-n}).\end{aligned}\tag{17}$$

This proves that $\lim_{\delta\to 0}g_n(\delta)=0$. We claim that $\lim_{\delta\to 0}\sup_{n\in\mathbb{N}\cup\{\infty\}}g_n(\delta)=0$. Again, fix a subdivision $D=(0=t_0<\cdots<t_l=1)$ of $[0,1]$. Assume first that the mesh's size of D is greater than or equal to 2^{-n}. Let $I=\{0\le i\le 2^n,D\cap[i2^{-n},(i+1)2^{-n})\ne\emptyset\}$, and s_i the smallest element of $D\cap[i2^{-n},(i+1)2^{-n})$ for $i\in I$. Letting $I=\left\{i_1,\cdots,i_{|I|}\right\}$, $\left(0=s_{i_1}<\cdots<s_{i_{|I|}}=1\right)$ is then a subdivision of $[0,1]$ included in D, such that for all $1\le j\le|I|$, $\left|s_{i_{j+1}}-s_{i_j}\right|\le|D|+2^{-n}\le 2|D|$. In particular, this implies, as $\frac{i_j}{2^n}\le s_{i_j}\le\frac{i_j+1}{2^n}$, that

$$i_{j+1}2^{-n}-(i_j+1)\,2^{-n}\le s_{i_{j+1}}-s_{i_j}\le 2\,|D|\,.$$

By the super-additivity property of a control,

$$\sum_{i=0}^{l-1} \omega_n(t_i, t_{i+1}) \leq \sum_{j=1}^{|I|-1} \omega_n(s_{i_j}, s_{i_{j+1}})$$

. Moreover,

$$\begin{aligned}\sum_{j=1}^{|I|-1} \omega_n(s_{i_j}, s_{i_{j+1}}) &= \sum_{j=1}^{|I|-1} \left(\omega_n\left(s_{i_j}, \frac{i_j+1}{2^n}\right) + \omega_n\left(\frac{i_{j+1}}{2^n}, s_{i_{j+1}}\right)\right) \\ &\quad + \sum_{j=1}^{|I|-1} \omega_\infty\left(\frac{i_j+1}{2^n}, \frac{i_{j+1}}{2^n}\right) \qquad \text{by definition of } \omega_n \\ &= \sum_{j=1}^{|I|-1} \omega_n\left(s_{i_j}, \frac{i_j+1}{2^n}\right) + \sum_{j=2}^{|I|} \omega_n\left(\frac{i_j}{2^n}, s_{i_j}\right) \\ &\quad + \sum_{j=1}^{|I|-1} \omega_\infty\left(\frac{i_j+1}{2^n}, \frac{i_{j+1}}{2^n}\right) \\ &\leq \sum_{j=1}^{|I|-1} \left(\omega_\infty\left(\frac{i_j}{2^n}, \frac{i_j+1}{2^n}\right) + \omega_\infty\left(\frac{i_j+1}{2^n}, \frac{i_{j+1}}{2^n}\right)\right) \\ &\leq g_\infty\left(2\,|D|\right).\end{aligned}$$

When the mesh's size of D is less than or equal to 2^{-n}, we have already seen (equation (17)) that

$$\sum_{i=0}^{l-1} \omega_n(t_i, t_{i+1}) \leq \left(\frac{|D|}{2^{-n}}\right)^{p-1} g_\infty\left(2^{-n}\right) \leq g_\infty\left(2^{-n}\right).$$

Therefore, $g_n(\delta) \leq \max\{g_\infty(2\delta), g_\infty(2^{-n})\}$. Let $\varepsilon > 0$. There exists n_0 such that $n \geq n_0 \implies g_\infty(2^{-n}) \leq \varepsilon$. Then, using the observation above that

$$\lim_{\delta\to 0} g_n(\delta) = 0$$

for fixed n, there exists δ_0 such that for all $\delta \leq \delta_0$,

$$\max\{g_\infty(2\delta), g_n(\delta), n = 0, \cdots, n_0 - 1\} \leq \varepsilon.$$

In particular, for all $\delta \leq \delta_0$, $\sup_{n\in\mathbb{N}\cup\{\infty\}} g_n(\delta) \leq \varepsilon$, as

$$\sup_n g_n(\delta) = \max\left\{\max_{n<n_0} g_n(\delta), \sup_{n\geq n_0} g_n(\delta)\right\}.$$

In other words,

$$\lim_{\delta\to 0} \sup_{n\in\mathbb{N}\cup\{\infty\}} g_n(\delta) = 0.$$

We consider again a subdivision $D = (0 = t_0 < \cdots < t_l = 1)$ of $[0,1]$. As

$$d(Y_{t_i,t_{i+1}}, S_m(y_n)_{t_i,t_{i+1}}) \leq \min\left\{d_\infty(S_m(y_n), Y), \omega_n(t_i,t_{i+1})^{\frac{1}{p}} + \omega_\infty(t_i,t_{i+1})^{\frac{1}{p}}\right\},$$

we obtain that

$$\begin{aligned}\sum_{i=0}^{l-1} d(Y_{t_i,t_{i+1}}, S_m(y_n)_{t_i,t_{i+1}})^p &\leq \sum_{\substack{i\in\{0,\cdots,l-1\}\\ |t_{i+1}-t_i|>\delta}} d_\infty(S_m(y_n),Y)^p \\ &\quad +2^{p-1} \sum_{\substack{i\in\{0,\cdots,n-1\}\\ |t_{i+1}-t_i|\leq\delta}} (\omega_n(t_i,t_{i+1}) + \omega_\infty(t_i,t_{i+1})) \\ &\leq \frac{d_\infty(S_m(y_n),Y)^p}{\delta} + 2^p \sup_{n\in\mathbb{N}\cup\{\infty\}} g_n(\delta).\end{aligned}$$

Therefore, $d_{p-var}\left(Y, S_m(y_n)\right)^p \leq \frac{d_\infty(S_m(y_n),Y)^p}{\delta} + 2^p \sup_{n\in\mathbb{N}\cup\{\infty\}} g_n(\delta)$. That gives us our result. ■

One can then see the following equality of spaces:

$$\begin{aligned} C^{0,\omega,p}(G^m(V)) &= \overline{\cup_{q>p} C^{0,\omega,q}\left(G^m(V)\right)}^{d_{p,\omega}} \\ C^{0,p-var}(G^m(V)) &= \overline{\cup_{q>p} C^{0,q-var}\left(G^m(V)\right)}^{d_{p-var}} \end{aligned}$$

4.3 Polishness

4.3.1 Separability

Theorem 21 *Assume that ω is such that for all $s < t$, $\omega(s,t) \geq K\,(t-s)^p$.*[3] *Then, $C^{0,\omega,p}(G^m(V))$ is separable.*

Proof. We know that the space $C_0^1([0,1],V)$ of continuously differentiable paths is separable. Let D be a countable set of $C_0^1([0,1],V)$ such that its Lipschitz closure is $C_0^1([0,1],V)$. We claim that the $d_{\omega,p}$-closure of $S_m(D) = \{S_m(y), y \in D\}$ is dense in $C^{0,\omega,p}(G^m(V))$. Indeed, if $Y \in C^{0,\omega,p}(G^m(V))$, there exists a sequence $(y_n)_n$ of elements in $C_0^1([0,1],V)$, such that

$$\lim_{n\to\infty} d_{\omega,p}\left(S_m(y_n), Y\right) = 0.$$

For all n, there exists $\widetilde{y_n} \in D$ such that the Lipschitz distance between y_n and $\widetilde{y_n}$ goes to 0 with $n \to \infty$. By theorem 1 with its continuity statement in

[3] which is true if ω satisfies condition (H_p).

[15], we deduce that the Lipschitz distance between $S_m(y_n)$ and $S_m(\widetilde{y_n})$ goes to 0 when n tends to infinity. In particular, $d_{\omega,p}(S_m(y_n), S_m(\widetilde{y_n}))$ tends to 0 when $n \to \infty$. The triangle inequalitiy show that $S_m(\widetilde{y_n})$ converges to Y in the topology induced by $d_{\omega,p}$. ■

The same proof gives the following theorem:

Theorem 22 *$C^{0,p-var}(G^m(V))$ is separable.*

We now look at the space $C^{p-var}(G^m(V))$ and $C^{\omega,p}(G^m(V)$.

Theorem 23 *$C^{p-var}(G^m(V))$ and $C^{\omega,p}(G^m(V))$ are not separable.*

Proof. The proof is pretty simple. If they were separable, it would mean, projecting $G^m(V)$ onto V, that $C^{p-var}(V)$ and $C^{\omega,p}(V)$ are separable. They are not, see [3,19]. ■

4.3.2 Completeness

Theorem 24 *$C^{p-var}(G^m(V))$ is complete.*

Proof. We suppose that X^n is a Cauchy sequence. Then, it is also a Cauchy sequence for the sup norm, therefore, it converges when n tends to infinity, in sup norm to, say, X. For a given $\varepsilon > 0$, there exists N, such that $n, m \geq N$ implies that for all subdivision $D = (0 \leq t_0 < t_1 < \cdots < t_l \leq 1)$ of $[0,1]$,

$$\sum_{i=0}^{l-1} d\left(X^n_{t_i,t_{i+1}}, X^m_{t_i,t_{i+1}}\right)^p < \varepsilon.$$

In particular, letting m tends to infinity, we obtain that

$$\sum_{i=0}^{l-1} d\left(X^n_{t_i,t_{i+1}}, X_{t_i,t_{i+1}}\right)^p < \varepsilon,$$

this being true for all subdivisions. That proves our assertion. ■

A similar and somewhat simpler proof gives the following:

Theorem 25 *Let ω be a control which is zero only on the diagonal. Then $C^{\omega,p}(G^m(V))$ is complete.*

Of course, $C^{0,\omega,p}(G^m(V))$ and $C^{0,p-var}(G^m(V))$ are complete, being closed subsets of complete sets.

5 Conclusion

We have precisely characterized the spaces $C^{0,p-var}(G^m(V))$, $C^{p-var}(G^m(V))$, $C^{0,\omega,p}(G^m(V))$ and $C^{0,\omega,p}(G^m(V))$. The separability statement, incidentally, proves that the inclusions

$$C^{0,p-var}(G^m(V)) \subset C^{p-var}(G^m(V))$$

$$C^{0,p-var}(G^m(V)) \subset C^{p-var}(G^m(V))$$

are strict. We let $\omega(s,t) = t - s$ for the rest of this discussion. The function

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ t &\to \sum_{i=1}^{\infty} 2^{-i/p} \sin(2^i t) \end{aligned}$$

provides a concrete proof of the strict inclusions just mentionned [4]. Also, if $V = \mathbb{R}^2$ is generated by a basis $\{e_1, e_2\}$, $X_t = \exp(t[e_1, e_2])$ is an element of $C^{\omega,2}(G^m(V)) \backslash C^{0,\omega,2}(G^m(V))$ and $C^{2-var}(G^m(V)) \backslash C^{0,2-var}(G^m(V))$. The function

$$\begin{aligned} h : \mathbb{R}^+ &\to \mathbb{R} \\ t &\to \begin{cases} \frac{t^{1/p}}{\log t} \cos^2\left(\frac{\pi}{t}\right) \text{ if } t > 0, \\ 0 \text{ if } t = 0. \end{cases} \end{aligned}$$

proves that the inclusions

$$\cup_{q<p} C^{q-var}(G^m(V)) \subset C^{0,p-var}(G^m(V))$$

$$\cup_{q<p} C^{\omega,q}(G^m(V)) \subset C^{0,\omega,p}(G^m(V))$$

are strict [4].

$C^{0,\omega,p}(G^m(V))$ and $C^{0,p-var}(G^m(V))$ are therefore Polish space. In particular, the (Stratonovich enhanced) Brownian motion takes values in a Polish space. Many important probabilistic theorems (e.g. Prohorov's theorem) rely on Polishness.

Finally, we want to point out the fact that the approximations of a rough path X that we have introduced (the (almost) geodesic one) may be very useful in various area. For example, in the field of stochastic numerical analysis, consider $\mathbf{B}$ the Stratonovich enhanced Brownian Motion lying above a standard d-dimensional Brownian motion B. Then let B^n be the geodesic approximation based on the subdivision $\pi(n) = \left\{\frac{k}{n},\ k = 0, \cdots, n\right\}$ of $[0,1]$, and $B^{(n)}$ the linear path which coincides with B at the points $\frac{k}{n}$, and linear in the intervals $[\frac{k-1}{n}, \frac{k}{n}]$, $k = 0, \cdots, n$. Consider the Stratonovich differential equation

$$dX_t = V_0(t, X_t)dt + V(t, X_t) \circ dB_t$$

and its approximations

$$\begin{aligned} dX_t^n &= V_0(t, X_t^n)dt + V(t, X_t^n)dB_t^n, \\ dX_t^{(n)} &= V_0(t, X_t^{(n)})dt + V(t, X_t^{(n)})dB_t^{(n)}. \end{aligned}$$

Then, in the L^2 sense, X^n converges to X with a speed of convergence proportional to $\frac{1}{\sqrt{n}}$, while $X^{(n)}$ converges to X with a speed of convergence proportional to $\frac{1}{n}$. Lifting $\mathbf{B}$ to a $G^{(m)}(R^d)$-valued process and considering the almost geodesic approximation in this group would lead to a speed of convergence proportional to $n^{-m/2}$. See [1,9] for a discussion involving speed of convergence for algorithm involving the use of iterated integrals.

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