Commutativity and higher order controllability for nonsmooth vector fields

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Abstract

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1 Introduction

Among the classic results concerning families of vector fields one has:

- the characterization of commutative vector fields in terms of vanishing of their Lie bracket;
- the several versions of Chow's theorem on controllability, including the one providing the degree of Holder-continuity of the Optimal Time function near a target point.

Here we shall give akin versions of such results involving an extended notion of Lie bracket.

2 Flows on manifolds: basic facts and notations

2.1 Locally Lipschitz continuous maps

Unless otherwise specified, the term *manifold* will mean "finite-dimensional, second countable, Hausdorff, differentiable manifold of class C^∞."

Let M, N be manifolds of dimension m and n, respectively, and let $x \in M$. A map $F : M \mapsto N$ is *locally Lipschitz continuous near* x if it is continuous at x and, for some pair (φ, ψ) of charts $\varphi : U \mapsto \mathbb{R}^m$, $\psi : V \mapsto \mathbb{R}^n$ defined on open neighborhoods U and V of x and $F(x)$, respectively, such that $F(U) \subseteq V$, the composite map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \mapsto \psi(V)$ is Lipschitz continuous near $\varphi(x)$ in the usual sense. (Notice that in this case $\psi \circ F \circ \varphi^{-1}$ is Lipschitz continuous near $\varphi(x)$ for *every* pair of charts (φ, ψ).) We call F *locally Lipschitz continuous* if it is Lipschitz continuous at x for every $x \in M$.

Remark 2.1 Since all Riemannian metrics are locally equivalent, it is easy to check that a map $F : M \mapsto N$ is locally Lipschitz continuous if and only if it is locally Lipschitz continuous in the usual sense when M and N are endowed with arbitrary Riemannian metrics.

The following version of Rademacher's Theorem on manifolds is an obvious consequence of the standard version on Euclidean spaces. (Let us recall that the concept of a "null subset" —that is, a subset of measure zero— of a manifold M is well defined: if $E \subseteq M$, then E is a null subset if $\varphi(E)$ is a null subset of $\varphi(U)$ for every chart $\varphi : U \mapsto \mathbb{R}^m$ of M.)

Theorem 2.2 *Let M, N be manifolds and let F be a locally Lipschitz map. Then there exists a null subset E of M such that F is differentiable at each point of $M \backslash E$.*

2.2 Notation on manifolds

In [1] the well-known fact that the set $\mathcal{E}(M)$ of C^∞ functions on M completely determines M has been exploited in order to set up a notation and formal rules which highly simplify some computations involving vector fields, brackets, flows and their derivatives, an so on. We shall use this notation, so, for the sake of self-consistency, let us recall some basic facts from [1].

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If M is a manifold, we use $\mathcal{E}(M)$ to denote the commutative algebra —over $\mathbb{R}$— of smooth real-valued functions on M, topologized in the usual way. (A sequence $\{\varphi_j\}$ converges to a limit φ in $\mathcal{E}(M)$ iff for every $k \geq 0$ and every k-tuple $(X_1, \ldots, X_k)$ of smooth vector fields on M the functions $X_1 X_2 \ldots X_k \varphi_j$ converge to $X_1 X_2 \ldots X_k \varphi$ uniformly on compact sets.) We let $\mathcal{E}'(M)$ denote the dual space of $\mathcal{E}(M)$, i.e. the space of compactly supported Schwartz distributions on M.

It is well known that M is completely determined by $\mathcal{E}(M)$, in the following sense. A point p of M gives rise to a linear functional $\delta_p \in \mathcal{E}'(M)$ —the Dirac delta function at p— defined by $\delta_p(\varphi) = \varphi(p)$. This linear functional is *multiplicative*, i.e. is a homomorphism of $\mathbb{R}$-algebras. For any commutative algebra A over a field $\mathbf{k}$, let us use $\sigma(A)$ to denote the *spectrum* of A, i.e. the set of all maps $\mu : A \to \mathbf{k}$ that are algebra homomorphisms and do not vanish identically. Then *the map $p \to \delta_p$ is a bijection from M to $\sigma(\mathcal{E}(M))$*. So a manifold can be canonically identified with the spectrum of its algebra of smooth functions.

Via the map $p \to \delta_p$, we regard M as embedded in $\mathcal{E}'(M)$. Moreover, many other objects related to M can also be naturally regarded as members of $\mathcal{E}'(M)$. For example, TM is embedded in $\mathcal{E}'(M)$ as follows: a tangent vector v at a point is a linear functional on $\mathcal{E}(M)$, which maps $\varphi \in \mathcal{E}(M)$ to $v\varphi$. Since this functional is clearly continuous, v belongs to $\mathcal{E}'(M)$.

If X and Y are smooth vector fields, then the product XY, evaluated at p, is a well defined member of $\mathcal{E}'(M)$, namely, the map $\varphi \to (XY\varphi)(p)$. So the product XY —i.e. the map $p \to XY(p)$— is a well defined map from M to $\mathcal{E}'(M)$. The difference $[X, Y] = XY - YX$ —the *Lie bracket* of X and Y— is also a map from M to $\mathcal{E}'(M)$, that happens to take values in TM, and is in fact a vector field.

On the other hand, $\mathcal{E}'(M)$ is a topological linear space, so linear operations and limiting processes that in principle appear not to make intrinsic sense on M can be meaningful in $\mathcal{E}'(M)$. For example, if $\gamma : [0, \varepsilon] \to M$ is a curve, and $\gamma(0) = p$, then we would like to define the tangent vector $\dot\gamma(0)$ by just letting

$$\dot\gamma(0) \stackrel{\text{def}}{=} \lim_{h\to 0} h^{-1}(\gamma(h) - p)\,. \tag{1}$$

This may look unacceptable, since M is not a linear space, and $\gamma(h) - p$ does not make sense. However, Formula (1) is perfectly meaningful, and gives the right answer, if we regard it as an identity in $\mathcal{E}'(M)$, since $\lim_{h\to 0} h^{-1}(\gamma(h)-p)$ is the distribution that maps $\varphi \in \mathcal{E}(M)$ to the number $\lim_{h\to 0} h^{-1}(\varphi(\gamma(h)) - \varphi(p))$, i.e. directional differentiation at p in the direction of $\dot\gamma(0)$.

To make the formalism work well, it is convenient to write $\pi\varphi$ —rather than $\pi(\varphi)$— for the value of the linear functional $\pi \in \mathcal{E}'(M)$ at the function $\varphi \in \mathcal{E}(M)$. With this notation, when $p \in M$ and $\varphi \in \mathcal{E}(M)$ the value $\varphi(p)$ is now written $p\varphi$. If f is a smooth vector field and $\varphi \in \mathcal{E}(M)$, then $f\varphi$ is in $\mathcal{E}(M)$, and the number usually written as $(f\varphi)(p)$ now becomes $p(f\varphi)$. On the other hand, if we also write pf rather than the more common $f(p)$, the number $p(f\varphi)$ is also equal to $(pf)\varphi$, so we can just write $pf\varphi$ without parentheses.

More generally, maps Φ from M to any manifold N should act on points of M on the right, so we will write $p\Phi$ rather than $\Phi(p)$. A smooth map $\Phi : M \to N$ gives rise to a map $\tilde\Phi : \mathcal{E}'(M) \to \mathcal{E}'(N)$, defined by letting $\tilde\Phi(D)$, for $D \in \mathcal{E}'(M)$, be the distribution on N given by $\tilde\Phi(D)(\psi) = D(\psi \circ \Phi)$, i.e., in our formalism, $(D\tilde\Phi)\psi \stackrel{\text{def}}{=} D(\Phi\psi)$, where, of course, $\Phi\psi$ is the function usually referred to as the "pullback" $\Phi^*(\psi)$ of ψ to M via Φ. (Naturally, since maps act on the right, $\Phi\psi$ is the composite map obtained by applying Φ first, and then ψ.) When D is actually a point $p \in M$, then $(D\tilde\Phi)\psi$ is what is normally written as $\psi(\Phi(p))$, so $p\tilde\Phi = p\Phi$. So $\tilde\Phi$ is an extension to $\mathcal{E}'(M)$ of the map Φ, originally defined on the subset M of $\mathcal{E}'(M)$. It is convenient to just use Φ for this extension. With this notation, if $p \in M$ and v is a tangent vector at p, then $v\Phi$ is the tangent vector at $p\Phi$ usually written as $\Phi_*(v)$, or $D\Phi(p).v$.

In particular, we will use e^{tf} to denote the time t flow map arising from a vector field f, and will write it as acting on the right. So $t \to pe^{tf}$ is the integral curve $\xi_{f,p}$ of f that goes through p at time 0. This curve satisfies the equation

$$(d/dt)(pe^{tf}) = pe^{tf}f\,, \tag{2}$$

which is a much more elegant way of writing the formula $(d/dt)(\xi_{f,p}(t)) = f(\xi_{f,p}(t))$, and amply justifies the use of the exponential notation.

With this formalism, several important formulas involving Lie brackets become completely trivial formally, and the formal calculations can be rigorously justified by working in $\mathcal{E}'(M)$. For example, let f, g be vector fields, and let $p \in M$. Then

$$\begin{aligned} & (d/dt)(pe^{tf}e^{tg}e^{-tf}e^{-tg}) \\ = \;& pe^{tf}fe^{tg}e^{-tf}e^{-tg} + pe^{tf}e^{tg}ge^{-tf}e^{-tg} - pe^{tf}e^{tg}e^{-tf}fe^{-tg} - pe^{tf}e^{tg}e^{-tf}e^{-tg}g\,, \end{aligned}$$

whose value for $t = 0$ is $pf + pg - pf - pg$, i.e. 0.

3 Lie brackets of locally Lipschitz vector fields

The ordinary Lie bracket of two vector fields f, g, will be denoted by $[f, g]_{ord}$. We shall use both $[f, g]_{ord}(x)$ and $x[f, g]_{ord}$ to indicate its evaluation at x That is, we set

$$[f, g]_{ord}(x) = x[f, g]_{ord} \doteq x(fg - gf)\,.$$

This means that, if we write $x\sigma$, xV, rather than $\sigma(x)$, $V(x)$, for the value at x of a function σ or a vector field V, then $x[f, g]_{ord}\varphi = x(fg - gf)\varphi$ for every test function φ.

If f is a locally Lipschitz continuous vector field on a manifold M, we let $DIFF(f)$ denote the set of points $x \in M$ such that f is differentiable at x. Then Theorem 2.2 implies that $M \backslash DIFF(f)$ is a null subset of M.

Definition 3.1 If f, g are locally Lipschitz continuous vector fields on a manifold M, we define the *Lie bracket* $[f, g]$ as follows. For every $x \in M$ we define $[f, g](x)$—the Lie bracket of f and g at x—to be the convex hull of the set of all vectors

$$v = \lim_{j\to\infty} [f, g]_{ord}(x_j), \tag{3}$$

for all sequences $\{x_j\}_{j\in\mathbb{N}}$ such that

1. $x_j \in DIFF(f) \cap DIFF(g)$ for all j,
2. $\lim_{j\to\infty} x_j = x$,
3. the limit v of (3) exists.

For every $x \in M$, it is clear that $[f, g](x)$ is a *convex*, *compact*, and *nonempty* subset of the tangent space T_xM of M at x. Moreover, the *skew-symmetry identity*

$$[f, g](x) = -[g, f](x) \tag{4}$$

clearly holds for all $x \in M$. (This means that $[f, g](x) = \{w : -w \in [g, f](x))$. In addition, each locally Lipschitz continuous vector field g satisfies the identity

$$[g, g](x) = \{0\} \;\text{ for every }\; x \in M\,. \tag{5}$$

(Formula (5) is *not* an immediate consequence of (4), but follows easily from Definition 3.1, since $[g, g]_{ord}(x) = 0$ for every $x \in DIFF(g)$.)

Remark 3.2 If $x \in DIFF(f) \cap DIFF(g)$ the bracket $[f, g](x)$ we have just defined *does not coincide, in general, with the singleton* $\{[f, g]_{ord}(x)\}$, although it always contains the vector $[f, g]_{ord}(x)$. For example, let $M = \mathbb{R}$ and let us consider the vector fields

$$f(x) = \frac{\partial}{\partial x} \quad\text{and}\qquad g(x) = \begin{cases} x^2 \sin(1/x)\frac{\partial}{\partial x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then, for $x = 0$

$$[f,g]_{ord}(x) = 0, \qquad \text{whereas } [f,g](x) = \left\{ a\frac{\partial}{\partial x} : a \in [-1,1] \right\}$$

In the special case when f and g are of class C^1, the set $[f,g](x)$ is indeed the singleton $\{[f,g]_{ord}(x)\}$ for each $x \in M$.

3.1 Set-valued iterated brackets

Definition 3.3 If F is a family of vector fields, let us say that its elements are *(ordinary) Lie brackets of degree one.* If $h \in F$ and $k \in F$ are differentiable at any point, let us say that $[h,k]$ is a *a (ordinary) Lie bracket of degree 2.* Moreover, if l and m are differentiable Lie brackets of degree i and j, respectively, let us say that $[h,k]_{ord}$ is *a Lie bracket of degree* $i+j$. So, it remains inductively defined what a *(ordinary) Lie bracket of degree* r is, for any natural number r.

Definition 3.4 For any natural number $q \geq 1$, we say that a vector field is of class $C^{q-1,1}_{loc}$ if it is of class C^{q-1} and the $q-1$-th order derivatives are locally Lipschitz continuous.

Definition 3.5 If p, r, q are non-negative integers such that $p + r = q$, $F = \{h_1, \dots, h_{q+1}\}$ is a family of $C^{q-1,1}_{loc}$ vector fields, and k and l are (ordinary) brackets of degree p and r, respectively, we say that $[k,l]$ is a (set-valued) bracket of degree $q+1$. Notice that Definition 3.1 applies, in that all brackets having a degree less than $q+1$ turn out to be locally Lipschitz continuous.

For instance, if $h_1, \dots, h_{q+1}$ are of class $C^{2,1}_{loc}$, the bracket $[[[h_1,h_2],h_3],h_4]$ is well defined, for both vector fields $[[h_1,h_2],h_3]$ and h_4 are locally Lipschitz continuous.

Actually in some cases the regularity assumptions on the vector fields h_i can be weakened. For instance, in order to define degree 2 bracket $[[h_1,h_2],h_3]$ we need h_1 and h_2 to be of class $C^{1,1}_{loc}$, while it is sufficient for h_3 of being just locally Lipschitz continuous. It also can happen that none of the $q+1$ vector fields h_i need to be of class $C^{q-1,1}_{loc}$, as in the case of the degree 4 bracket $[[h_1,h_2],[h_3.h_4]]$, where $h_1.\dots,h_4$ do not need to be of class $C^{2,1}_{loc}$, and can well be of class $C^{1,1}_{loc}$.

Jacobi identity. If h_1, h_2, h_3 are of class $C^{q-1,1}$, with $q \geq 2$, by exploiting the standard Jacobi identity along sequences $\{x_j\}_{j\in\mathbb{N}}$ where the h_i's are twice differentiable, one can trivially prove the following *extended version of the Jacobi identity*:

$$[[h_1,h_2],h_3](x) + [[h_3,h_1],h_2](x) + [[h_2,h_3],h_1](x) \supseteq \{0\}$$

for all $x \in M$. Of course this is nothing but the usual Jacobi identity as soon as $q > 2$, while in the case when $q = 2$ it says that for every x there is a triple

$$(a_1,a_2,a_3) \in [[h_1,h_2],h_3](x) \times [[h_3,h_1],h_2](x) \times [[h_2,h_3],h_1](x)$$

such that

$$a_1 + a_2 + a_3 = 0.$$

Linear dependence. Let us recall a property of Lie brackets (of smooth vector fields) which is an easy consequence of Jacobi's identity:

Proposition 3.6 *If $h_1, \ldots, h_{q+1}$ are q times differentiable, then any $(q+1)$-th degree Lie bracket is a $\mathbb{R}$-linear combinations of ($(q+1)$-th degree) Lie brackets of the form*

$$[[\ldots[h_{i_1}, h_{i_2}], h_{i_3}, \ldots,]h_{i_{q+1}}] \quad (i_1, \ldots, i_{q+1}) \in \{1, \ldots, q+1\}^{q+1}.$$

(Of course, in view of the skew-symmetry, some of these Lie brackets are identically equal to zero)

As an example, it is straightforward to verify that, if h_1, h_2, h_3, h_4 are of class C^3, then

$$[[h_1, h_2], [h_3, h_4]] = -[[[h_3, h_4], h_1], h_2] + [[[h_3, h_4], h_2], h_1]$$

By arguing as in the case of the Jacobi's identity one easily obtains the following *nonsmooth* extension of the previous proposition, which will be exploited later.

Proposition 3.7 *If $h_1, \ldots, h_{q+1}$ are of class $C^{q-1,1}_{loc}$, then any (set-valued) $(q+1)$-th degree Lie bracket* is contained *in a $\mathbb{R}$-linear combination of the (set-valued) $(q+1)$-th degree Lie brackets*

$$[[\ldots[h_{i_1}, h_{i_2}], h_{i_3}, \ldots,]h_{i_{q+1}}], \quad (i_1, \ldots, i_{q+1}) \in \{1, \ldots, q+1\}^{q+1}.$$

(Again, in view of the skew-symmetry, many of these Lie brackets are identically equal to singleton $\{0\}$).

Notice that the nonsmooth version of the example above reads as follows: if h_1, h_2, h_3, h_4 are of class $C^{3,1}_{loc}$

$$[[h_1, h_2], [h_3, h_4]](x) \subseteq -[[[h_3, h_4], h_1], h_2](x) + [[[h_3, h_4], h_2], h_1](x).$$

4 Commutativity of locally Lipschitz vector fields

It is well known that if f, g are vector fields of class C^1 then the flows of f and g commute for small times if and only if $[f, g]_{ord} = 0$ identically. In this section we show that an analogous result holds true in the case when f and g are just locally Lipschitz vector fields.

To begin with, let us make precise what we mean by saying that the flows of two vector fields *commute*.

Definition 4.1 Let M be a manifold, and let f, g be locally Lipschitz continuous vector fields defined on M. We say that *the flows of f and g commute* if $xe^{sg}e^{tf}$ is defined and

$$xe^{tf}e^{sg} = xe^{sg}e^{tf} \tag{6}$$

whenever $x \in M$, $t, s \in \mathbb{R}$ are such that $xe^{\tau f}e^{\sigma g}$ is defined whenever $\min(t, 0) \leq \tau \leq \max(t, 0)$ and $\min(s, 0) \leq \sigma \leq \max(s, 0)$

There is also a "local" version of the above concept, useful to prove results on manifolds by reducing them to the case when the manifold is $\mathbb{R}^n$:

Definition 4.2 Let M be a manifold, and let f, g be locally Lipschitz continuous vector fields defined on M. Let $x \in M$. We say that *the flows of f and g commute for small times near x* if there exist a neighborhood U of x and a positive number ε such that ($x'e^{tf}e^{sg}$ and $x'e^{sg}e^{tf}$ are defined and)

$$x'e^{tf}e^{sg} = x'e^{sg}e^{tf} \tag{7}$$

for all $x' \in M$, $t, s \in \mathbb{R}$ such that $|t| \leq \varepsilon$ and $|s| \leq \varepsilon$.

Theorem 4.3 *Let M be a manifold, and let f, g be locally Lipschitz continuous vector fields on M. Let $\mathcal{N}$ be a null subset of M. Then the following conditions are equivalent:*

i) $[f, g]_{ord}(x) = 0$ *for every* $x \in \Big(DIFF(f) \cap DIFF(g)\Big) \backslash \mathcal{N}$*;*

ii) $[f, g]_{ord}(x) = 0$ *for every* $x \in DIFF(f) \cap DIFF(g)$*;*

iii) $[f, g](x) = \{0\}$ *for every* $x \in M$*;*

iv) the flows of f and g commute for small times near x for every $x \in M$;

v) the flows of f and g commute.

In order to prove Theorem 4.3, let us first establish an exact integral representation of the quantity $x\, e^{tf}e^{sg}e^{-tf}e^{-sg} - x$ in the case when f and g are C^1. A slightly different version of this representation have been already proved in [3].

Lemma 4.4 *Let f and g be vector fields of class C^1 on a manifold M. Then, for all $x \in M$, $t, s \in \mathbb{R}$ such that $xe^{\tau f}e^{\sigma g}e^{\tau f}e^{-\sigma g}$ is defined whenever* $\min(t, 0) \leq \tau \leq \max(t, 0)$ *and* $\min(s, 0) \leq \sigma \leq \max(s, 0)$*, one has*

$$x\, e^{tf}e^{sg}e^{-tf}e^{-sg} - x = \int_0^t \int_0^s \left(x\, e^{\tau f}e^{(s-\sigma)g}[f, g]e^{\sigma g}e^{-\tau f}e^{-sg}\right) d\tau\, d\sigma \tag{8}$$

holds. Moreover, if w is any vector field, then

$$xw\, e^{tf}e^{sg}e^{-tf}e^{-sg} - xw = \int_0^t \int_0^s \left(xw\, e^{\tau f}e^{(s-\sigma)g}[f, g]e^{\sigma g}e^{-\tau f}e^{-sg}\right) d\tau\, d\sigma \tag{9}$$

(See Subsection 2.2 for the meaning of the subtraction on manifolds.)

Remark 4.5 Under the hypotheses of Lemma 4.4, the vector field $[f, g]$ is continuous, so the integrand $x\, e^{\tau f}e^{(s-\sigma)g}[f, g]e^{\sigma g}e^{-\tau f}e^{-sg}$ is $[f, g](x) + o(1)$ as $(s, t) \to (0, 0)$. Therefore (8) implies the usual second order estimate

$$x\, e^{tf}e^{sg}e^{-tf}e^{-sg} - x = st[f, g](x) + o(st)\,, \tag{10}$$

from which it follows, for instance, that $[f, g]$ vanishes if the flows of f and g commute. Of course an *exact integral formula* is much more interesting than (10). As an instance, the fact that the condition $[f, g] = 0$ implies that the flows of f and g commute is a straightforward consequence of (8), while the standard proof using (10) is nothing but trivial.

Proof. The proof of (8) reduces to the following chain of equalities:

$$
\begin{aligned}
x\, e^{tf} e^{sg} e^{-tf} e^{-sg} - x &= \int_0^t \Big(x\, e^{\tau f} f e^{sg} e^{-\tau f} e^{-sg} - x\, e^{\tau f} e^{sg} f e^{-\tau f} e^{-sg} \Big)\, d\tau \\
&= \int_0^t x\, e^{\tau f} e^{sg} (e^{-sg} f e^{sg} - f) e^{-\tau f} e^{-sg}\, d\tau \\
&= \int_0^t \int_0^s x\, e^{\tau f} e^{sg} (e^{-\sigma g} [f,g] e^{\sigma g}) e^{-\tau f} e^{-sg}\, d\sigma d\tau \\
&= \int_0^t \int_0^s x\, e^{\tau f} e^{(s-\sigma) g} [f,g] e^{\sigma g} e^{-\tau f} e^{-sg}\, d\sigma d\tau
\end{aligned}
$$

where we have used the identities:

$$
\frac{d}{d\tau}\Big(x\, e^{\tau f} e^{sg} e^{-\tau f} e^{-sg} \Big) = x\, e^{\tau f} f e^{sg} e^{-\tau f} e^{-sg} - x\, e^{\tau f} e^{sg} f e^{-\tau f} e^{-sg}
$$

and

$$
\frac{d}{d\sigma}\Big(y (e^{-\sigma g} f e^{\sigma g} - f) \Big) = y e^{-\sigma g} [f,g] e^{\sigma g}\, .
$$

As for (9), it follows straightforwardly from (8). Indeed, one has

$$
\begin{aligned}
& xw\, e^{tf} e^{sg} e^{-tf} e^{-sg} - xw \\
= & \lim_{\epsilon\to 0} \tfrac{1}{\epsilon} \Big[\Big((x + \epsilon(xw))\, e^{tf} e^{sg} e^{-tf} e^{-sg} - x\, e^{tf} e^{sg} e^{-tf} e^{-sg} \Big) - \Big(x + \epsilon(xw) - x \Big) \Big] \\
= & \lim_{\epsilon\to 0} \tfrac{1}{\epsilon} \textstyle\int_0^t \int_0^s \Big[\Big(x + \epsilon(xw)(e^{\tau f} e^{(s-\sigma) g} [f,g] e^{\sigma g} e^{-\tau f} e^{-sg} \Big) - \\
& \Big[\Big(x\, e^{\tau f} e^{(s-\sigma) g} [f,g] e^{\sigma g} e^{-\tau f} e^{-sg} \Big) \Big]\, d\tau\, d\sigma \\
= & \textstyle\int_0^t \int_0^s \Big(xw\, e^{\tau f} e^{(s-\sigma) g} [f,g] e^{\sigma g} e^{-\tau f} e^{-sg} \Big)\, d\tau\, d\sigma.
\end{aligned}
$$

In order to prove Theorem 4.3 we need the following result as well:

Lemma 4.6 *Let f and g be locally Lipschitz continuous vector fields on a manifold M. Suppose that the flows of f and g commute for small times near x for every $x \in M$. Then the flows of f and g commute.*

Proof. Let $x \in M$, $t, s \in \mathbb{R}$ be such that $xe^{\tau f} e^{\sigma g}$ is defined whenever $\min(t,0) \leq \tau \leq \max(t,0)$ and $\min(s,0) \leq \sigma \leq \max(s,0)$. We want to prove that $xe^{tf}e^{sg} = xe^{sg}e^{tf}$.

Let I_t, I_s be the compact intervals $[\min(t,0), \max(t,0)]$ $[\min(s,0), \max(s,0)]$, respectively. Write $I_t = [t_-, t_+]$, $I_s = [s_-, s_+]$.

Let $K = \{xe^{\tau f} e^{\sigma g} : \tau \in I_t,\ \sigma \in I_s\}$. Then K is a compact subset of M, because $xe^{\tau f} e^{\sigma g}$ exists whenever $\tau \in I_t$, $\sigma \in I_s$. So there exist a positive ε such that $x' e^{\tau f} e^{\sigma g}$ is defined and equal to $x' e^{\sigma g} e^{\tau f}$ whenever $x' \in K$, $|\tau| \leq \varepsilon$ and $|\sigma| \leq \varepsilon$.

Let N be a positive integer such that $\frac{|t|}{N} < \varepsilon$ and $\frac{|s|}{N} < \varepsilon$. Let $t_j = \frac{jt}{N}$, $s_k = \frac{kt}{N}$, for $j = 0, \ldots, N$, $k = 0, \ldots, N$. We claim that

(*) *If $j, k \in \{0, \ldots, N\}$, then $xe^{t_j f} e^{s_k g}$ is defined and is equal to $xe^{s_k g} e^{t_j f}$*

To prove (*), we first show that

(#) *If $j \in \{0, \ldots, N\}$, $x' \in K$, and $x' e^{t_i f} \in K$ for all $i \in \{0, \ldots, j\}$, then $x' e^{t_j f} e^{s_1 g}$ is defined and equal to $x' e^{s_1 g} e^{t_j f}$.*

To prove (#), we use induction with respect to j. We first observe that the case when $j=0$ is trivial. Assume that our conclusion is known to be true for a j such that $0 \le j < N$, and let $x' \in K$ be such that $x'e^{t_i f} \in K$ for all $i \in \{0,\ldots,j+1\}$. Then in particular $x'e^{t_i f} \in K$ for all $i \in \{0,\ldots,j\}$, so the inductive assumption implies that $x'e^{t_j f}e^{s_1 g}$ is defined and equal to $x'e^{s_1 g}e^{t_j f}$. Since $x'e^{t_j f} \in K$, $|t_1| \le \varepsilon$, and $|s_1| \le \varepsilon$, we can conclude that $x'e^{t_j f}e^{t_1 f}e^{s_1 g}$ (which is equal to $x'e^{t_{j+1} f}e^{s_1 g}$) is defined and equal to $x'e^{t_j f}e^{s_1 g}e^{t_1 f}$. But $x'e^{t_j f}e^{s_1 g} = x'e^{s_1 g}e^{t_j f}$. Therefore $x'e^{t_j f}e^{s_1 g}e^{t_1 f} = x'e^{s_1 g}e^{t_j f}e^{t_1 f} = x'e^{s_1 g}e^{t_{j+1} f}$. It follows that $x'e^{t_{j+1} f}e^{s_1 f}$ is defined and equal to $x'e^{s_1 g}e^{t_{j+1} f}$, completing the proof of (#).

We now prove (*) by (strong) induction on j. The case $j=0$ is trivial. Suppose that $0 \le j^* < N$, and the conclusion of (*) has been proved for all k and for all j such that $0 \le j \le j^*$. We prove, by induction on k, that then $xe^{t_{j^*+1} f}e^{s_k g}$ is defined and is equal to $xe^{s_k g}e^{t_{j^*+1} f}$ for all $k \in \{0,\ldots,N\}$. Once again, the conclusion is trivial if $k=0$. Suppose that $0 \le k < N$, and we know that $xe^{t_{j^*+1} f}e^{s_k g}$ is defined and is equal to $xe^{s_k g}e^{t_{j^*+1} f}$. Then $xe^{t_{j^*+1} f}e^{s_{k+1} g}$ is defined, because $t_{j^*+1} \in I_t$ and $s_{k+1} \in I_s$. Furthermore,

$$
\begin{aligned}
xe^{t_{j^*+1} f}e^{s_{k+1} g} &= xe^{t_{j^*+1} f}e^{s_k g}e^{s_1 g} && (11)\\
&= xe^{s_k g}e^{t_{j^*+1} f}e^{s_1 g} && (12)\\
&= xe^{s_k g}e^{t_{j^*} f}e^{t_1 f}e^{s_1 g} && (13)\\
&= xe^{s_k g}e^{t_{j^*} f}e^{s_1 g}e^{t_1 f} && (14)\\
&= xe^{s_k g}e^{s_1 g}e^{t_{j^*} f}e^{t_1 f} && (15)\\
&= xe^{s_{k+1} g}e^{t_{j^*+1} f}, && (16)
\end{aligned}
$$

where

1. (11) follows because $s_{k+1} = s_k + s_1$ and the numbers s_k and s_1 have the same sign;
2. (12) follows from the inductive hypothesis of the k-induction;
3. (13) follows because $t_{j^*+1} = t_{j^*} + t_1$ and the numbers t_{j^*} and t_1 have the same sign;
4. (14) follows because $xe^{s_k g}e^{t_{j^*} f} \in K$ (since $xe^{s_k g}e^{t_{j^*} f} = xe^{t_{j^*} f}e^{s_k g}$ by the inductive hypothesis of the j-induction), $|t_1| \le \varepsilon$, and $|s_1| \le \varepsilon$;
5. (15) follows from (#), which can be applied—with $xe^{s_k g}$ in the role of x'—since $xe^{s_k g}e^{t_i f} \in K$ for $i = 0,1,\ldots,j^*$, because our inductive assumption tells us that $xe^{s_k g}e^{t_i f} = xe^{t_i f}e^{s_k g}$ for $i = 0,1,\ldots,j^*$, and $xe^{t_i f}e^{s_k g} \in K$ by the definition of K;
6. (16) follows because $t_{j^*+1} = t_{j^*} + t_1$ and $s_{k+1} = s_k + s_1$.

Having proved (*), we can take $k=N$, $j=N$, and conclude that $xe^{tf}e^{sg} = xe^{sg}e^{tf}$. This completes our proof.

Here and in the next sections we will consider *regularizations* of vector fields on $\mathbb{R}^n$:

Definition 4.7 Let φ is a nonnegative real-valued function on $\mathbb{R}^n$ of class C^∞ such that $\int_{\mathbb{R}^n} \varphi = 1$ and $\varphi(h) = 0$ whenever $\|h\| > 1$. For any continuous vector field k on $\mathbb{R}^n$ consider the *regularizations* k^ζ by setting

$$k^\zeta(x) = \int_{\mathbb{R}^n} \varphi(h)k(x+\zeta h)\,dh\,, \qquad \zeta > 0\,, \tag{17}$$

where φ is a nonnegative real-valued function on $\mathbb{R}^n$ of class C^∞ such that $\int_{\mathbb{R}^n} \varphi = 1$ and $\varphi(h) = 0$ whenever $\|h\| > 1$.

In order to prove Theorem 4.3 we need the following technical result:

Lemma 4.8 *Let f and g be locally Lipschitz continuous vector fields on $\mathbb{R}^n$ and let $\mathcal{M} \subset \mathbb{R}^n$ be any subset having measure equal to zero. If*

$$x[f,g]_{ord} = 0 \qquad \forall x \in \Big(DIFF(f) \cap DIFF(g)\Big) \backslash \mathcal{M}\,,$$

then

$$[f^\zeta, g^\zeta] \to 0$$

uniformly on compact subsets of $\mathbb{R}^n$.

Proof. Let us fix a compact subset $K \subset \mathbb{R}^n$ and let $x \in K$. Since in any compact neighborhood of K the derivatives Df and Dg are bounded (in any metric), it is easy to see that

$$[f^\zeta, g^\zeta](x) =$$
$$\int_{M\backslash\mathcal{N}} \Big[\varphi(z)[Dg(x+\zeta z)f(x+\zeta z) - Df(x+\zeta z)g(x+\zeta z)]\Big]\, dz + \psi(\zeta) = \psi(\zeta)$$

where ψ is infinitesimal as ζ tends to zero. This yields the thesis.

Proof of Theorem 4.3. Let us show that $i)$ implies $iv)$. Let $\bar{x} \in \mathbb{R}^n$, let U be a neighborhood of $\bar{x}$, and let

$$\phi : U \mapsto \phi(U) \subset \mathbb{R}^n$$

be a coordinate chart near $\bar{x}$. Let $(y_1, \ldots, y_n) = \phi(x)$ denote the coordinates of a generic point x. Let $f_{|U}$ and $g_{|U}$ denote the restrictions of f and g to U, respectively, so that

$$\begin{array}{l} f_{|U}(x) = \sum_{i=1}^n f_i(x)\frac{\partial}{\partial y_i} \\ g_{|U}(x) = \sum_{i=1}^n g_i(x)\frac{\partial}{\partial y_i} \end{array} \tag{18}$$

for suitable functions $f_i(x), g_i(x)$, $i = 1, \ldots, n$.

Let K be a C^∞ map on $\phi(U)$ such that $K = 1$ in a compact neighborhood Q_1 of $\phi(\bar{x})$ and $K = 0$ on $\phi(U)\backslash Q_2$, where $Q_2 \subset \phi(U)$ is a compact neighborhood of $\phi(\bar{x})$ containing Q_1 in its interior. Then the vector fields

$$\begin{array}{l} \tilde{f}(y) = K(y)\Big(f_1(\phi^{-1}(y)), \ldots, f_n(\phi^{-1}(y))\Big) \\ \tilde{g}(y) = K(y)\Big(g_1(\phi^{-1}(y)), \ldots, g_n(\phi^{-1}(y))\Big) \end{array} \tag{19}$$

which are defined on the open subset $\phi(U)$, can be extended to vector fields f' and g' on $\mathbb{R}^n$, respectively, by setting

$$f'(y) = \left\{ \begin{array}{ll} \tilde{f}(y) & \forall y \in \phi(U) \\ 0 & \forall y \in \mathbb{R}^n\backslash\phi(U) \end{array} \right. \qquad g'(y) = \left\{ \begin{array}{ll} \tilde{g}(y) & \forall y \in \phi(U) \\ 0 & \forall y \in \mathbb{R}^n\backslash\phi(U) \end{array} \right. \tag{20}$$

Notice that f' and g' are (globally) bounded and Lipschitz continuous. Moreover, by i) they verify

$$[f', g']_{ord} = 0 \qquad \forall y \in \mathbb{R}^n\backslash\phi(\mathcal{N} \cap U)$$

In particular, denoting the $\zeta-$regularizations of f' and g' by f'^ζ and g'^ζ, respectively,the flows $(y,\tau) \mapsto ye^{\tau f'^\zeta}$ and $(z,\sigma) \mapsto ze^{\sigma g'^\zeta}$ can be defined for every $\tau, \sigma \in \mathbb{R}$ and $y, z \in \mathbb{R}^n$ and for any $\zeta \geq 0$. Moreover it is trivial to verify that there exist $\epsilon > 0$ and a compact neighborhood $Q_0 \subset Q_1$ of $\phi(\bar{x})$ such that

$$ye^{\tau f'^{\zeta}} e^{\sigma g'^{\zeta}}, \; ye^{\sigma g'^{\zeta}} e^{\tau f'^{\zeta}}, \; ye^{\tau f'^{\zeta}} e^{\sigma g'^{\zeta}} e^{-\tau f'^{\zeta}} e^{-\sigma g'^{\zeta}} \in Q_1 \tag{21}$$

for all $y \in Q_0$ and τ, σ, ζ such that $|\tau| \leq \epsilon$, $|\sigma| \leq \epsilon$, and $|\zeta| \leq \epsilon$

By applying Lemma 4.4 to the regularizations of f'^{ζ} and g'^{ζ} we obtain

$$ye^{tf'^{\zeta}} e^{sg'^{\zeta}} e^{-tf'^{\zeta}} e^{-sg'^{\zeta}} - y = \int_0^t \int_0^s \left(y\, e^{\tau f'^{\zeta}} e^{(s-\sigma) g'^{\zeta}} [f'^{\zeta}, g'^{\zeta}] e^{\sigma g'^{\zeta}} e^{-\tau f'^{\zeta}} e^{-sg'^{\zeta}} \right) d\tau\, d\sigma \tag{22}$$

for every $y \in Q_0$ and $|t|$, $|s|$, $|\zeta| \leq \epsilon$ Then, letting ζ tend to zero, from Lemma 4.8 and from the trivial fact that the left-hand side of (22) tends to $y\, e^{tf} e^{sg} e^{-tf} e^{-sg} - y$ we obtain

$$ye^{tf'} e^{sg'} e^{-tf'} e^{-sg'} = y$$

In turn, this implies

$$ye^{tf'} e^{sg'} = ye^{sg'} e^{tf'}$$

which, by (21), yields

$$xe^{tf} e^{sg} = xe^{sg} e^{tf}$$

for all $x \in \phi^{-1}(Q_0)$.

In order to show that *ii*) follows from *iv*), let us interpret it in local coordinates —which is made possible by the same kind of arguments as in the previous point. Then, the fact that *ii*) follows from *iv*) is a straightforward consequence of the classic asymptotic formula

$$x\, e^{tf} e^{tg} e^{-tf} e^{-tg} - x = t^2 [f, g](x) + o(t^2) \quad \text{as } t \downarrow 0 \tag{23}$$

which is easily seen to hold true at each point $x \in DIFF(f) \cap DIFF(g)$.

Of course *ii*) implies *i*) and is equivalent to *iii*).

Finally, by Lemma 4.6 iv) is equivalent to v), so the theorem is proved.

5 A Chow's theorem with non-smooth vector fields

Chow's theorem is perhaps the most classic higher order controllability result. In this section we shall state a version of Chow' s theorem which generalizes the classic one in two directions. The first, major generalization, consists in considering (set-valued) q-th degree Lie brackets of vector fields of class $C^{q-1,1}_{loc}$. The second, minor, generalization consists in allowing some of the vector fields to be just *continuous*, so that, in general, the corresponding flows turn out to be set-valued. As is well-known the proof of Chow's theorem relies on the application of an open mapping theorem to suitable compositions of the involved flows. Of course, this requires the application of some chain rule. Similarly, the version of Chow's theorem we are going to state are based on an open mapping theorem. As a matter of fact we shall rely on the theory of Generalized Differential Quotients, which has been introduced by one of the authors in [5]. This differentiation theory, which will be here briefly recalled, apply to non-smooth, set-valued maps, so, in particular, set-valued flows of continuous vector fields can be considered. Furthermore, the set-valued Lie brackets defined in the sections above allow one to build Generalized Differential Quotients of suitable multi-flows and to apply an open mapping theorem.

In order to make the statement of this controllability result clearer let us introduce some additional notation. Let q be an integer greater than 2. Corresponding to a set of vector fields $g_1, ..., g_r$ on M of class $C^{q-1,1}_{loc}$, for every $p = 2, ..., q-1$ let V_p denote the set of all (classic) p-th degree Lie brackets of $g_1, \ldots, g_r$. Let W_q denote the family of all the (set-valued) q-th degree Lie brackets (at x) of the vector fields $g_1, ..., g_r$.

Notice that the elements of V_p are vector fields of class $C^{q-p,1}_{loc}$, while W_q is a family of set-valued vector fields. Since W_q is finite, say it contains s elements, we can order it and write $W_q = \{Z_1, \ldots, Z_s\}$.

For every $x \in M$ and every $p = 2, ...q$, let $V_p(x) \subset T_x M$ denote the set of the evaluations of the elements of V_p at x, and let $Z_1(x), \ldots, Z_s(x)$ denote the (set-valued) images of x of $Z_1, \ldots, Z_s$, respectively. Finally let us set

$$W_q(x) = \{Z_1(x), \ldots, Z_s(x)\}$$

Theorem 5.1 *Let $f_1, \ldots, f_m, g_1, \ldots, g_r$, $m, r \geq 0$, vector fields on M. Let $f_1, \ldots, f_m$ be continuous, and let $g_1, \ldots, g_r$ be of class $C^{q-1,1}_{loc}$, for an integer number $q \geq 1$.*

Let Σ be the driftless control system on M

$$\Sigma: \qquad \dot{x} = \sum_{i=1}^{m} u_i f_i(x) + \sum_{j=1}^{r} v_j g_j(x),$$

with control constraints $|u_i| \leq 1$ for $i = 1, \ldots, m$ and $|v_j| \leq 1$ for $j = 1, \ldots, r$. Let x_* be a point of M such that, for all choices

$$(z_1, ..., z_s) \in Z_1(x_*) \times \ldots \times Z_s(x_*),$$

the set of vectors

$$\{f_1(x_*), .., f_m(x_*)\} \cup \{g_1(x_*), .., g_r(x_*)\} \cup V_1(x_*) \cup .. \cup V_{q-1}(x_*) \cup \{z_1, .., z_s\}$$

linearly spans $T_{x_*}M$.

Then Σ is locally controllable from x_* in small time. More precisely, there exists a positive constant A having the property that for every sufficiently small ε it is possible to reach every point within a distance $\leq \varepsilon$ from x_* in time not exceeding $A\varepsilon^{\frac{1}{q}}$ by means of a piecewise constant bang-bang controls such that at each time t only one of the numbers

$$u_1(t), \ldots, u_m(t), v_1(t), \ldots, v_r(t)$$

is nonzero.

Remark 5.2 The statement might appear meaningless, for a notion of distance is used without having specified which Riemannian structure we are referring to. However the statement is meaningful because it has a local character, and all Riemannian metrics are locally equivalent (actually, only the constant A will depend on the chosen Riemannian metric).

The remaining part of this paper will be devoted to set up some tools to prove Theorem 5.1. Actually, this proof, which will be given in the last section, will rely on a classic scheme: i) first, one constructs suitable compositions of local flows; ii) secondly one differentiates these composed maps; iii) finally, one applies an open mapping theorem. Though this scheme is standard, its pursue requires new tools and some technical, intermediate results. To begin with, in the next section we briefly recall some basic facts concerning *Generalized Differential Quotients*. Next (see Sections 7 and 8) we shall devote our attention to the construction of Generalized Differential Quotients of suitable compositions of local flows. Also, we shall find Generalized Differential Quotients for the flows of the continuous vector fields $f_1, ..f_m$. By applying chain rules and a suitable open mapping theorem, in the last section we shall give the actual proof of Theorem 5.1.

6 Generalized Differential Quotients

In [] one of the authors has introduced the notion of Generalized Differential Quotient. Let us recall some facts concerning this concept.

A *set-valued map* (abbr. SVM) is a triple $F = (A, B, G)$ such that A and B are sets and G is a subset of $A \times B$. The sets A, B, G are, respectively, the *source*, *target*, and *graph* of F, and we write $A = \mathrm{So}(F)$, $B = \mathrm{Ta}(F)$, $G = \mathrm{Gr}(F)$. If x is any object, we write $F(x) = \{y : (x, y) \in \mathrm{Gr}(F)\}$. (Hence $F(x) = \emptyset$ unless $x \in \mathrm{So}(F)$.) The sets $\mathrm{Do}(F) = \{x \in \mathrm{So}(F) : F(x) \neq \emptyset\}$, $\mathrm{Im}(F) = \bigcup_{x\in\mathrm{So}(F)} F(x)$, are, respectively, the *domain* and *image* of F. If $F = (A, B, G)$ is an SVM, we say that F is an *SVM from A to B*, and write $F : A\, q \mapsto\!\!\!\to B$. We use $SVM(A, B)$ to denote the set of all SVMs from A to B.

If X, Y are metric spaces, then $SVM_{comp}(X, Y)$ will denote the subset of $SVM(X, Y)$ whose members are the set-valued maps from X to Y that have a compact graph. We say that a sequence $\{F_j\}_{j\in\mathbb{N}}$ of members of $SVM_{comp}(X, Y)$ *inward graph-converges* to an $F \in SVM_{comp}(X, Y)$—and write $F_j \xrightarrow{\mathrm{igr}} F$—if for every open subset Ω of $X \times Y$ such that $\mathrm{Gr}(F) \subseteq \Omega$ there exists a $j_\Omega \in \mathbb{N}$ such that $\mathrm{Gr}(F_j) \subseteq \Omega$ whenever $j \geq j_\Omega$.

Definition 6.1 Assume that X, Y are metric spaces. A *regular set-valued map* from X to Y is a set-valued map $F \in SVM(X, Y)$ such that

- for every compact subset K of X, the restriction $F\lceil K$ of F to K belongs to $SVM_{comp}(K, Y)$ and is a limit—in the sense of inward graph-convergence—of a sequence of continuous single-valued maps from K to Y.

We use $\mathrm{REG}(X; Y)$ to denote the set of all regular set-valued maps from X to Y.

It is easy to see that if $F : X \mapsto Y$ is a single-valued and everywhere defined map, then F belongs to $\mathrm{REG}(X; Y)$ if and only if F is continuous.

Moreover, one has

Theorem 6.2 *Assume that X, Y, Z are metric spaces. Let $F \in \mathrm{REG}(X; Y)$, $G \in \mathrm{REG}(Y; Z)$. Then the composite map $G \circ F$ belongs to $\mathrm{REG}(X; Z)$.*

Definition 6.3 Let $m, n \in \mathbb{Z}_+$, let $F : \mathbb{R}^m\, q \mapsto\!\!\!\to \mathbb{R}^n$ be a set-valued map, and let Λ be a nonempty compact subset of $\mathbb{R}^{n\times m}$. Let S be a subset of $\mathbb{R}^m$. We say that Λ is a *generalized differential quotient* (abbreviated "GDQ") *of F at* $(0, 0)$ *in the direction of S*, and write $\Lambda \in GDQ(F; 0, 0; S)$, if for every positive real number δ there exist U, G such that

1. U is a compact neighborhood of 0 in $\mathbb{R}^m$ and $U \cap S$ is compact;

2. G is a regular set-valued map from $U \cap S$ to the δ-neighborhood Λ^δ of Λ in $\mathbb{R}^{n\times m}$;
3. $G(x) \cdot x \subseteq F(x)$ for every $x \in U \cap S$.

If X, Y are real linear spaces, we use $\mathrm{Lin}(X, Y)$ to denote the set of all linear maps from X to Y. A *linear multimap* from X to Y is a subset of $\mathrm{Lin}(X, Y)$. A *linear multimultimap* from X to Y is a set of linear multimaps from X to Y.

If M, N are C^1 manifolds, $\bar{x} \in M$, $\bar{y} \in N$, $S \subseteq M$, and $F : M \, q \mapsto\!\!\to N$, then we can define a linear multimultimap $GDQ(F; \bar{x}, \bar{y}; S)$ from $T_{\bar{x}}M$ to $T_{\bar{y}}N$ by picking coordinate charts $M \ni x \to \xi(x) \in \mathbb{R}^m$, $N \ni y \to \eta(y) \in \mathbb{R}^n$—where $m = \dim M$, $n = \dim N$—defined near $\bar{x}$, $\bar{y}$ and such that $\xi(\bar{x}) = 0$, $\eta(\bar{y}) = 0$, and declaring a subset Λ of $\mathrm{Lin}(T_{\bar{x}}M, T_{\bar{y}}N)$ to belong to $GDQ(F; \bar{x}, \bar{y}; S)$ if the multimap $D\eta(\bar{y}) \circ \Lambda \circ D\xi(\bar{x})^{-1}$ is in $GDQ(\eta \circ F \circ \xi^{-1}; 0, 0; \xi(S))$. It turns out that, with this definition, *the set $GDQ(F; \bar{x}, \bar{y}; S)$ does not depend on the choice of the charts ξ, η.* Moreover, the following results hold true:

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WHICH ARE THE EXACT REFERENCES FOR THE FOLLOWING 3 THEOREMS?

Theorem 6.4 *If $F : \mathbb{R}^m \mapsto \mathbb{R}^n$ is a continuous map, $x \in \mathbb{R}^m$, and F is classically differentiable at x, then $\{DF(x)\} \in GDQ(F; x, F(x); \mathbb{R}^m)$.*

Theorem 6.5 *If $F : \mathbb{R}^m \mapsto \mathbb{R}^n$ is Lipschitz-continuous, and $x \in \mathbb{R}^m$, then the Clarke generalized Jacobian $\partial F(x)$ belongs to $GDQ(F; x, F(x); \mathbb{R}^m)$.*

Theorem 6.6 *GDQ has the strong directional open mapping property with linear rate (SDOMPLR).*

The meaning of the SDOMPLR is as follows:

(SDOMPLR) *Assume that n, m are nonnegative integers, $F : \mathbb{R}^m \, q \mapsto\!\!\to \mathbb{R}^n$, $v \in \mathbb{R}^n$, C is a closed convex cone in $\mathbb{R}^m$, Λ belongs to $GDQ(F; 0, 0; C)$, and $v \in \bigcap_{L\in\Lambda} \mathrm{Int}(LC)$. Then there exist a closed convex cone D in $\mathbb{R}^n$ and a function*

$$]\,0, +\infty[\, \ni \delta \mapsto \varepsilon(\delta) \in \,]\,0, +\infty[$$

such that $v \in \mathrm{Int}(D)$, the function $\delta \mapsto \varepsilon(\delta)$ is linear for small enough δ, and

(*) *if $y \in D$ and $\|y\| \leq \varepsilon(\delta)$ then there exists a compact connected subset Z_y of $\{x \in C : \|x\| \leq \delta\} \times [0, 1]$ such that $(0, 0) \in Z_y$, $(x, 1) \in Z_y$ for some $x \in C$ such that $\|x\| \leq \delta$, and $ry \in F(x)$ whenever $0 \leq r \leq 1$ and $(x, r) \in Z_y$.*

7 Generalized Differential Quotients of flows' compositions

7.1 Continuous vector fields

The occurrence of the continuous vector fields $f_1, \ldots, f_m$ in Theorem 5.1 poses the question of building a GDQ for the set-valued flow of a continuous vector field.

Let us begin by introducing suitable linear maps, which will be utilized in the next session as well:

For a given $w \in \mathbb{R}^n$, let Q^w denote the linear map

$$\mathbb{R}^{n+1} \sim \mathbb{R}^n \times \mathbb{R} \ni (v,r) \mapsto v + rw \stackrel{\text{def}}{=} Q^w(v,r) \in \mathbb{R}^n .$$

If W is a subset of $\mathbb{R}^n$, write

$$\mathcal{Q}^W \stackrel{\text{def}}{=} \{Q^w : w \in W\}$$

Let f be a continuous vector field on M, and let us consider its (local) flow

$$(x,t) \to xe^{tf} \tag{24}$$

For x in a compact neighborhood K of a given point x_*, this flow turns out to be defined on $K \times]-\varepsilon, \varepsilon[$, for a suitable $\varepsilon > 0$. Since f is merely continuous, in general this flow is a *set-valued* map. More precisely, we set

$$xe^{tf} = \{x = x(t) \mid x(\cdot) \quad \text{solves} \quad \dot{x} = f \quad x(0) = x\}$$

Let us notice that if f were a C^1 map its flow would be continuously differentiable and the map $Q^{f(x_*)}$ would be nothing but its differential at $(x,t) = (x_*, 0)$. It is remarkable that a similar fact occurs when f is just continuous. In fact, the following theorem was proved in [3].

Theorem 7.1 *The singleton $\mathcal{Q}^{f(x_*)}$ is a Generalized Differential Quotient of the flow xe^{tf} at $((x_*,0),x_*)$ in the direction of M.*

Remark 7.2 Despite the fact that at each point the flow $(x,t) \mapsto xe^{tX}$ possesses a Generalized Differential Quotient consisting of a (continuous) singleton, this map is not differentiable, actually it is not even a single-valued map. On the other hand, it can be proved that a *single-valued* map F having a singleton $\{v\}$ as GDQ at a point $(y, F(y)$ is in fact differentiable at y, and $F'(y) = v$.

7.2 Higher degree Lie brackets as GDQ's of flows' compositions

We are now going to address an issue connected with GDQ's of compositions of the g_i's flows.

Let $h_1, \dots, h_{q+1}$ be vector fields on M of class $C^{q-1,1}$. We wish to show that for every $x_* \in M$ and for every Lie bracket L of the fields $h_1, \dots, h_{q+1}$ of degree $s \leq q+1$ there is a suitable composition of flows having $\mathcal{Q}^{L(x_*)}$ as a Generalized Differential Quotient at $((x_*,0),x_*)$. Notice that $L(x_*)$ is a singleton as soon as $s < q+1$.

In view of Proposition 3.7, we shall limit ourselves to Lie brackets of the form

$$[[..[h_{i_1}, h_{i_2}], ..], h_{i_s}],$$

for all $s \leq q$, where $i_1, ...i_s$ are positive integers less than or equal to $q+1$. In fact, these are the only brackets we shall need in the proof of Theorem 5.1 (see Section 9)

Definition 7.3 Let $s \geq 2$ be an integer and let $h_1, \ldots, h_s$ be locally Lipschitz continuous vector fields on M. We recursively define the map

$$(x, t_1, \ldots, t_s) \to x\psi^{h_1,\ldots,h_s}(t_1, \ldots, t_s)$$

as follows:

i) for $s = 2$, let us set,for any $x \in \mathbb{R}^n$ and for all (t_1, t_2) in a sufficiently small neighborhood of $(0, 0)$,

$$x\psi^{h_1,h_2}(t_1, t_2) = xe^{-t_2h_2}e^{t_1h_1}e^{t_2h_2}e^{-t_1h_1}.$$

ii) for $s > 2$ we set

$$x\psi^{h_1,\ldots,h_s}(t_1, \ldots, t_s) =$$
$$xe^{-t_sh_s}\psi^{h_1,..,h_{s-1}}(t_1, .., t_{s-1})e^{t_sh_{q-1}}\left(\psi^{h_1,..,h_{s-1}}(t_1, .., t_{s-1})\right)^{-1},$$

where

$$y \mapsto y\left(\psi^{h_1,..,h_{s-1}}(t_1, .., t_{s-1})\right)^{-1}$$

denotes the inverse of the map

$$x \mapsto x\psi^{h_1,..,h_{s-1}}(t_1, .., t_{s-1}).$$

Let us also define the map $(x, \epsilon) \to x\Xi^{h_1,\ldots,h_s}(\epsilon)$ by letting

$$x\Xi^{h_1,\ldots,h_s}(\epsilon) = \begin{cases} x\psi^{h_1,\ldots,h_s}(\varepsilon^{\frac{1}{s}}, \ldots, \varepsilon^{\frac{1}{s}}) & \text{if} \quad \varepsilon \geq 0\,, \\ x\left(\psi^{h_1,\ldots,h_s}((-\varepsilon)^{\frac{1}{s}}, \ldots, (-\varepsilon)^{\frac{1}{s}})\right)^{-1} & \text{if} \quad \varepsilon \leq 0\,. \end{cases}$$

For every $x \in \Omega$ the domain of $\epsilon \to x\Xi^{h_1,\ldots,h_s}(\epsilon)$ is an x-dependent neighborhood of zero. Of course, the map $(x, \epsilon) \to x\Xi^{h_1,\ldots,h_s}(\epsilon)$ is continuous.

The following result puts in evidence the strict relation occurring between the flows $(x, \epsilon) \to x\Xi^{h_1,\ldots,h_s}(\epsilon)$ and the (possibly set-valued) Lie brackets introduced above.

Theorem 7.4 *Let $q \in \mathbb{Z}$, $q > 0$,let $h_1, \ldots, h_{q+1}$ be vector fields belonging to $C^{q-1,1}_{loc}(\Omega)$, and let $x_* \in \Omega$. Let $s \in \mathbb{Z}$, $0 < s \leq q+1$. Then, for every $(i_1, \ldots, i_s) \in \{1, \ldots, q+1\}^s$, the set $\mathcal{Q}^{[..[h_{i_1},h_{i_2}],..],h_{i_s}](x_*)}$ is a Generalized Differential Quotient of the map $(x, \epsilon) \to x\Xi^{h_1,\ldots,h_s}(\epsilon)$ at $((x_*, 0), x_*)$ in the direction of $\mathbb{R}^n$.*

Remark 7.5 Notice that $\mathcal{Q}^{[..[h_{i_1},h_{i_2}],..],h_{i_s}]}$ is a singleton as soon as $s < q+1$. Since in this case the map $\Xi^{h_{i_1},\ldots,h_{i_s}}$ is univalued, it follows that –see Remark 7.2– it is differentiable at $(x_*, 0)$ and its differential coincides with the map

$$\mathbb{R}^n \times \mathbb{R} \ni (v, r) \mapsto v + r[..[h_{i_1}, h_{i_2}], ..], h_{i_s}](x_*).$$

8 Proof of Theorem 7.4

Let us observe that it is sufficient to prove Theorem 7.4 *just the case when $s = q + 1$ and $(i_1, \ldots, i_{q+1}) = (1, \ldots, q + 1)$.* Moreover, in view of the local character of the thesis, we are allowed to prove the theorem under the following, simplifying, extra assumption:

Hypothesis XA. *The manifold M coincides with $\mathbb{R}^n$, and the vector fields $h_1, \ldots, h_{q+1}$ are of class $C_c^{q-1,1}$. This means that they have compact supports, they are of class C^{q-1}, and there exists a real number $L > 0$ such that their derivatives up to the $(q-1)$-th order are bounded by $L > 0$, and the $(q-1)$-th order derivatives are globally Lipschitz continuous with L as a Lipschitz constant.*

Remark 8.1 The fact that this hypothesis can be assumed *without loss of generality* follows from the following, standard cut-off function arguments. To begin with, consider the local expression of the vector fields h_i in a neighborhood of x_* in a system of coordinates whose range is an open subset $U \subset \mathbb{R}^n$. Let y_* be the image of x_* in this system of coordinates and let $\eta > 0$ be such that the closed ball $B[y_*, \eta]$ is contained in U. Let ξ be a C^∞ map on U such that $\xi = 1$ on $B[y_*, \frac{\eta}{2}]$ and $\xi(y) = 0$ for any $y \in U \backslash B[y_*, \eta]$. Let us consider the vector fields

$$\tilde{h}_i = \xi h_i \qquad i = 1, \ldots, q+1$$

(where, with abuse of notation, we have written h_i to denote what in fact is the local expression of h_i) and let us extend each of them to all of $\mathbb{R}^n$ by setting $\tilde{h}_i = 0$ outside Ω. Then the maps $\tilde{h}_1, \ldots, \tilde{h}_{q+1}$ verify **(XA)**. Moreover, for any $i = 1, \ldots, q+1$, the flow of h_i coincides with the flow of $\tilde{h}_i$ in a neighborhood of $(y_*, 0)$. So, in view of the local character of the definition of Generalized Differential Quotient, it is straightforward to check that proving Theorem 7.4 is equivalent to proving the same theorem with the $\tilde{h}_i$'s in place of the $h_i' s$. Hence we can definitely assume hypothesis **(XA)**.

The proof of Theorem 7.4 will consist of three essential steps. We shall begin by proving the result in the smooth case. Secondly, we shall consider regularizations and establish some estimates of the distances between Lie brackets made of the given vector fields and those made of the regularized vector fields. Finally, we shall exploit these estimates and a general approximation property of GDQ's to prove the thesis of Theorem 7.4. 8.2.

8.1 Step 1: smooth vector fields

Theorem 8.2 *For every $q \in \mathbb{N}$ there exists a function $\phi = \phi(K, t_1, \ldots, t_{q+1})$ defined on $[0, +\infty) \times \mathbb{R}^{q+1}$ such that the following hold:*

1. *for every $(t_1, \ldots, t_{q+1}) \in \mathbb{R}^{q+1}$ $\phi(\cdot, t_1, \ldots, t_{q+1})$ is non decreasing, and*

$$\lim_{(t_1,\ldots,t_{q+1}) \to 0} \frac{\phi(K, t_1, \ldots, t_{q+1})}{|(t_1, \ldots, t_{q+1})|^{q+1}} = 0 \qquad \forall K \in [0, +\infty);$$

2. *If $h_1, \ldots, h_{q+1}$ are vector fields of class C^q and K is a bound for the vector fields $h_1, \ldots, h_{q+1}$ together with their derivatives up to the q-th order, then for every $x \in \mathbb{R}^n$ one has*

$$\begin{aligned}\Big\| x\psi^{h_1,\ldots,h_{q+1}}(t_1, \ldots, t_{q+1}) - x - t_1 \cdot \ldots \cdot t_{q+1}[\ldots[[h_1, h_2], h_3], \ldots, h_{q+1}](x) \Big\| \\ \leq \phi(K, t_1, \ldots, t_{q+1}) \end{aligned} \quad .$$

HECTOR: this was the point where I had found a problem, but now it seems to me that it can be easily proved by Taylor's expansion

8.2 Step 2: regularizations

Let $h_1, \dots, h_{q+1}$ be vector fields verifying hypothesis **(XA)**, and let us consider the regularizations h_i^ζ introduced in Definition 4.7.

Then, for every $i = 1, \dots, q$, and for all positive ζ, h_i^ζ is of class C^∞ Moreover, by the elementary properties of the convolution product and by hypothesis **(XA)**, all vector fields h_i^ζ and their derivatives *up to the* (q)*-th order* are bounded by the constant $L > 0$. Furthermore, we have Proposition 8.3 below, where ϕ denotes any derivative of order less than or equal to $q - 1$ of a vector field h_i and ϕ^ζ stand for the corresponding derivative of the ζ-regularization h_i^ζ. (By elementary properties of convolutions ϕ^ζ coincides with the ζ-regularization of ϕ.)

Proposition 8.3 *If* **(XA)** *is assumed, then (ϕ is continuous and) there exists a constant C such that*

$$\|\phi^\zeta(x) - \phi(x)\| \le C\zeta \tag{25}$$

for any $x \in \mathbb{R}^n$ and $\zeta > 0$.

As a basic step to prove Theorem 7.4, the next proposition provides an estimate of the distance

$$d\Big([\dots[h_1^\zeta, h_2^\zeta], \dots], h_{q+1}^\zeta]_{ord}(x)\ ,\ [\dots[h_1, h_2], \dots], h_{q+1}](y)\Big).$$

Proposition 8.4 *Let us assume hypothesis* **(XA)**. *Then there is a constant $M > 0$ such that for all $x, y \in \mathbb{R}^n$ such that $\|x - y\| \le 1$ and any positive ζ one has*

$$d\Big([[\dots[h_1^\zeta, h_2^\zeta], \dots], h_{q+1}^\zeta]_{ord}(x)\ ,\ [[\dots[h_1, h_2], \dots,]h_{q+1}](y)\Big) \tag{26}$$

$$\le \gamma_y(\|x - y| + \zeta) + M(\zeta + \zeta^q) \tag{27}$$

In order to prove this proposition we need to introduce the functions A^{q-1} below.

Let i be a positive integer, and let P a multi-linear map from $(\mathbb{R}^n)^i$ into $\mathbb{R}^n$. Let us denote the image of an element $(v_1, \dots, v_i) \in (\mathbb{R}^n)^i$ by

$$P\Big\langle v_1, \dots, v_i\Big\rangle.$$

Let X be a vector field differentiable up to the i-th order at a point $x \in \mathbb{R}^n$, let $D^iX(x)$ designate the $i - th$ order differential of X at x. In particular, $D^iX(x)$ is a multi-linear map on $\mathbb{R}^i$ whose entries are suitable i-th order partial derivatives of X at x.

By Rademacher's theorem, the vector fields $h_1, \dots, h_{q+1}$ have derivatives up to the q-th order on the complement of a null set E.

For any $x \in \mathbb{R}^n \backslash E$ let us set

$$\begin{aligned}
& A^{q-1}(h_1, \dots, h_{q+1})(x) = \\
& [\dots[[h_1, h_2], ...,]h_{q+1}] \\
- \quad & (-1)^q D^q h_1(x)\Big\langle h_2(x), h_3(x), \dots, h_{q+1}(x)\Big\rangle \\
- \quad & (-1)^{q-1} D^q h_2(x)\Big\langle h_1(x), h_3(x), \dots, h_{q+1}(x)\Big\rangle
\end{aligned}$$

It is just a matter of elementary computation to prove the following lemma.

Lemma 8.5 *Let us assume hypothesis* **(XA)**. *For any $x \in \mathbb{R}^n \backslash E$,*

$$A^{q-1}(h_1, \dots, h_{q+1})(x)$$

is a polynomial of the variables

$$h_1(x), .., h_{q+1}(x), D^1 h_1(x), .., D^1 h_{q+1}(x), .., D^{q-1} h_1(x), .., D^{q-1} h_{q+1}(x).$$

In particular, since the vector fields h_i are of class C^{q-1} , we obtain:

Corollary 8.6 The map

$$x \to A^{q-1}(h_1, \dots, h_{q+1})(x)$$

has a unique continuous extension to all of $\mathbb{R}^n$, which will be still denoted by $A^{q-1}(h_1, \dots, h_{q+1})(\cdot)$

Next convergence result follows trivially by Lemma 8.5 and Proposition 8.3:

Lemma 8.7 *Let us assume hypothesis* **(XA)**. *Then, there exists a constant C' such that*

$$\|A^{q-1}(h_1^\zeta, \dots, h_{q+1}^\zeta)(x) - A^{q-1}(h_1, \dots, h_{q+1})(x)\| \le C'\zeta$$

for any $x \in \mathbb{R}^n$ and $\zeta > 0$.

For any $y \in \mathbb{R}^n$ let us set

$$\gamma_y(\rho) = \sup_x d\Big([\dots[h_1, h_2], \dots], h_{q+1}](y) \ , \ [\dots[h_1, h_2], \dots], h_{q+1}]_{ord}(x)\Big)$$

where the supremum is taken over the set $(\mathbb{R}^n \backslash E) \cap B[y, \rho]$.

It is easy to see that

$$\lim_{\rho \downarrow 0} \gamma_y(\rho) = 0 \,.$$

Proof of Proposition 8.4. Let $E \subset \mathbb{R}^n$ be a subset of measure zero such that the vector fields $h_1, ..., h_{q+1}$ are differentiable up to the q-th order. For any $x \in \mathbb{R}^n$ and $\zeta > 0$, let us define the zero-measure subset

$$E_{x,\zeta} = \{z \in \mathbb{R}^n \ : \ x + \zeta z \in E\}.$$

Then

$$\begin{aligned}
&[[\dots[h_1^\zeta, h_2^\zeta], \dots,]h_{q+1}^\zeta](x) = A^q(h_1^\zeta, \dots, h_{q+1}^\zeta)(x) + \\
&\int_{\mathbb{R}^n \backslash E_{x,\zeta}} \Big(\varphi(z)(-1)^q D^q h_1(x + \zeta z)\Big\langle(h_2^\zeta(x), h_3\zeta(x), \dots, h_{q+1}^\zeta(x)\Big\rangle\Big) dz + \\
&\int_{\mathbb{R}^n \backslash E_{x,\zeta}} \Big(\varphi(z)(-1)^{q-1} D^q h_2(x + \zeta z)\Big\langle h_1^\zeta(x), h_3^\zeta(x), \dots, h_{q+1}^\zeta(x)\Big\rangle\Big) dz \\
&= (I)_\zeta(x) + (II)_\zeta(x) + (III)_\zeta(x) \,,
\end{aligned}$$

where

$$(I)_\zeta = \int_{\mathbb{R}^n \setminus E_{x,\zeta}} \varphi(z)[[\dots[h_1,h_2],\dots,]h_{q+1}]_{ord}(x+\zeta z)dz\,,$$

$$(II)_\zeta(x) = \int_{\mathbb{R}^n \setminus E_{x,\zeta}} \varphi(z)\Big(A^q(h_1^\zeta,\dots,h_{q+1}^\zeta)(x) - A^q(h_1,\dots,h_{q+1})(x+\zeta z)\Big)dz\,,$$

and

$$\begin{aligned}(III)_\zeta(x) = \int_{\mathbb{R}^n \setminus E_{x,\zeta}} &\Big(\varphi(z)(-1)^q D^q h_1(x+\zeta z)\Big\langle(h_2^\zeta(x),h_3^\zeta(x),\dots,h_{q+1}^\zeta(x)) - \\ &\qquad (h_2(x+\zeta z),h_3(x+\zeta z),\dots,h_{q+1}(x+\zeta z))\Big\rangle\Big)dz + \\ \int_{\mathbb{R}^n \setminus E_{x,\zeta}} &\Big(\varphi(z)(-1)^{q-1} D^q h_2(x+\zeta z)\Big\langle(h_1^\zeta(x),h_3^\zeta(x),\dots,h_{q+1}^\zeta(x)) - \\ &\qquad (h_1(x+\zeta z),h_3(x+\zeta z),\dots,h_{q+1}(x+\zeta z))\Big\rangle\Big)dz\,.\end{aligned}$$

Hence

$$\begin{aligned}&d\Big([[\dots[h_1^\zeta,h_2^\zeta],\dots],h_{q+1}^\zeta](x),[[\dots[h_1,h_2],\dots],h_{q+1}](y)\Big)\\ &\le d\Big((I)_\zeta\,,\,[[\dots[h_1,h_2],\dots],h_{q+1}](y)\Big) + \|(II)_\zeta\| + \|(III)_\zeta\|\end{aligned}$$

Now $(I)_\zeta$ is a convex combination of vectors $[[\dots[h_1,h_2],\dots,]h_{q+1}](z)$ with $z \in B[x,\zeta]\setminus E$, therefore

$$d\Big((I)_\zeta\,,\,[[\dots[h_1,h_2],\dots],h_{q+1}](y)\Big) \le \gamma_y(\zeta + \|x-y\|).$$

Moreover, by hypothesis **(XA)**, the functions $A^q(h_1,\dots,h_{q+1})(\cdot)$, $h_1(\cdot)$, ..., $h_{q+1}(\cdot)$ turn out to be Lipschitz continuous, say with a Lipschitz constant G. Finally, since the $(q-1)$-th order derivatives of the fields h_i are Lipschitz continuous (with constant L), it follows that there exists a constant G' such that

$$\|D^q h_i(z)\Big\langle w_1,\dots,w_q\Big\rangle\| \le G'\|w_1,\dots,w_q\|^q$$

for all $z \in \mathbb{R}^n\setminus E$ and all $(w_1,\dots,w_q) \in \mathbb{R}^q$. Hence, in view of Lemma 8.7 and of (25), one has

$$\|(II)_\zeta\| \le \Big(G + C'\Big)\zeta$$

and

$$\|(III)_\zeta\| \le G"\zeta^q,$$

for a suitable G'', Hence, (26) is proved, with $M = \sup\{G", G' + C\}$. ⋄

8.3 Proving Theorem 7.4

Proof of Theorem 7.4. Notice that if the vector fields h_i verify **(XA)**, then the same holds true for their regularizations h_i^ζ with the same constant L. Hence we can apply Theorem 8.2 to the set of vector fields $\{h_1^\zeta,\dots,h_{q+1}^\zeta\}$, for all $\zeta > 0$.

Therefore, if $t_1 \cdot \ldots \cdot t_{q+1} > 0$, we have

$$
\begin{aligned}
& d\Big((t_1 \cdot \ldots \cdot t_{q+1})^{-1} \cdot \Big(x\psi^{h_1^\zeta,\ldots,h_{q+1}^\zeta}(t_1,\ldots,t_{q+1}) - x\Big)\ ,\ [[\ldots[h_1,h_2]\ldots,h_{q+1}](x_*)\Big) \\
& \leq d\Big((t_1 \cdot \ldots \cdot t_{q+1})^{-1} \cdot \Big(x\psi^{h_1^\zeta,\ldots,h_{q+1}^\zeta}(t_1,\ldots,t_{q+1}) - x\Big)\ ,\ [[\ldots[h_1^\zeta,h_2^\zeta]\ldots,h_{q+1}^\zeta]\Big) + \\
& \qquad d\Big([[\ldots[h_1^\zeta,h_2^\zeta],\ldots],h_{q+1}^\zeta](x)\ ,\ [[\ldots[h_1,h_2],\ldots,]h_{q+1}](x_*)\Big) \\
& \qquad \leq (t_1 \cdot \ldots \cdot t_{q+1})^{-1}\phi(L,t_1,\ldots,t_{q+1}) \quad + \gamma_{x_*}(\|x-x_*| + \zeta) + M(\zeta+\zeta^q), \quad (28)
\end{aligned}
$$

so, letting ζ go to zero, we obtain

$$
\begin{aligned}
& d\Big((t_1 \cdot \ldots \cdot t_{q+1})^{-1} \cdot \Big(x\psi^{h_1,\ldots,h_{q+1}}(t_1,\ldots,t_{q+1}) - x\Big)\ ,\ [[\ldots[h_1,h_2]\ldots,h_{q+1}](x_*)\Big) \\
& \qquad \leq \gamma_{x_*}(\|x-x_*\|) + (t_1 \cdot \ldots \cdot t_{q+1})^{-1}\phi(L,t_1,\ldots,t_{q+1}). \quad (29)
\end{aligned}
$$

In turn, (29) implies

$$
\begin{aligned}
& \inf\Big\{\ \|x\Xi^{h_1,\ldots,h_{q+1}}(\varepsilon) - x - \varepsilon v\| : v \in [[\ldots[h_1,h_2]\ldots,h_{q+1}](x_*)\ \Big\} \\
& \quad = o(\varepsilon + \|x-x_*\|)\,. \qquad (30)
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
& \inf\Big\{\ \|x\Xi^{h_1,\ldots,h_{q+1}}(\varepsilon) - \Xi^{h_1,\ldots,h_{q+1}}(x_*,0) - M(x-x_*,\varepsilon)\| \\
& \quad : M \in \mathcal{Q}^{[[\ldots[h_1,h_2]\ldots,h_{q+1}](x_*)}\ \Big\} = o(\varepsilon + \|x-x_*\|)\,. \qquad (31)
\end{aligned}
$$

Hence the thesis follows from the general sufficient condition given in [6] for a set to be a GDQ.

HECTOR
Perhaps it is better to write down the sufficient condition of [6] we have mentioned in the last line of this proof

9 Proof of Theorem 5.1

Proof of Theorem 5.1. If $q > 2$, for any integer p such that $2 \leq p \leq q-1$ let us choose a subsets $V'_p \subset V_p$ such that the following hold:

1. *Each element of V'_p is of the form*

$$[..[g_{i_1},[g_{i_2},[......],g_{i_p}]$$

for some $(i_1,\ldots,i_p) \in \{1,\ldots r\}^p$.

2. *The elements of V'_p are linearly independent.*
3. *Every element of V_p is a linear combination of elements of V'_p .*

Moreover, if $q > 1$ let us choose a subset $W'_q = \{Z'_1,\ldots,Z'_{s'}\} \subset W_q$ such that the following hold:

1_q. *Each element of Z'_q is of the form*

$$[..[g_{i_1},[g_{i_2},[......],g_{i_q}]$$

for some $(i_1,\ldots,i_q) \in \{1,\ldots r\}^q$.

2_q. *The elements of W'_q are linearly independent. This means that there exists a selection $(\zeta_1,\ldots,\zeta_s) \in Z_1\times\ldots\times Z_s$ such that if $a_1\zeta_1+\ldots+a_s\zeta_s = 0$ then all coefficients a_i have to be zero.*

3_q. *Every element Z_i of W_q is a linear combination of elements of W'_q. This means that if $z\in Z_i$ then there exist real coefficients $a_1,\ldots,a_{s'}$ and a selection $(\zeta'_1,\ldots,\zeta'_{s'}) \in Z'_1\times\ldots\times Z'_{s'}$ such that*

$$z = a_1{\zeta'}_1 + \ldots + a_{s'}{\zeta'}_{s'}$$

.

In view of Proposition 3.6 and 3.7 such subsets V'_p and W'_q do exist, though the choice is not unique.

It is trivial to verify that proving Theorem 5.1 is equivalent to proving the result below.

Theorem 9.1 *Under the same assumptions on the vector fields $f_1,\ldots,f_m$, $g_1,\ldots,g_r$ as in Theorem 5.1, let x_* be a point of M such that,* for all choices

$$(z'_1,...,z'_{s'}) \in Z'_1(x_*)\times\ldots\times Z'_{s'}(x_*),$$

the set of vectors

$$\{f_1(x_*),..,f_m(x_*)\}\cup\{g_1(x_*),..,g_r(x_*)\}\cup V'_1(x_*)\cup..\cup V'_{q-1}(x_*)\cup\{z_1,..,z_{s'}\}$$

linearly spans $T_{x_*}M$.

Then Σ is locally controllable from x_* in small time. More precisely, there exists a positive constant A having the property that for every sufficiently small ε it is possible to reach every point within a distance $\le\varepsilon$ from x_* in time not exceeding $A\varepsilon^{\frac{1}{q}}$ by means of a piecewise constant bang-bang controls such that at each time t only one of the numbers

$$u_1(t),\ldots,u_m(t),v_1(t),\ldots,v_r(t)$$

is nonzero.

Proof. cardinality of V'_p and let $n(q)$ ${V'}_1,\ldots,{V'}_{q-1},{W'}_q$. For elements of vector Let $\omega(\in\mathbb{Z})$ the cardinality of the set

$$\mathcal{W} = {V'}_2\cup\ldots\cup{V'}_{q-1}\cup{W'}_q$$

Let us choose an order on $\mathcal{W}$ in such a way we can write

$$\mathcal{W} = \{Y_1,\ldots,Y_\omega\}$$

and

$${W'}_q = \{Z'_1,\ldots,Z'_{s'}\} = \{Y_{\omega-s'+1},\ldots,Y_\omega\},$$

Hence there is a one-to-one function which maps any index $j=1,...,\omega$ into

$$(p(j),g^j_{i_1},\ldots,g^j_{i_{p(j)}}),$$

where $2\le p(j)\le q$ and $(g^j_{i_1},\ldots,g^j_{i_{p(j)}})\in\{g_1,\ldots,g_r\}^{p(j)}$, and such that

$$Y_j = [..[g^j_{i_1},\ldots],g_{i^j_{p(j)}}]$$

For every integer j such that $1 \le j \le \omega$ let us also consider the map

$$F_j : (x,\epsilon) \to xF_j(\epsilon) = x\Xi^{g^j_{i_1},\ldots\ldots,g^j_{i_{p(j)}}}(\epsilon)$$

Furthermore, let us define the map $\mathcal{F}$ from $\mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^\omega$ into $\mathbb{R}^n$ by setting

$$\mathcal{F}(\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_r,\gamma_1,\ldots,\gamma_\omega) =$$

$$x_* e^{\alpha_1 f_1}\ldots e^{\alpha_1 f_m} e^{\beta_1 g_1}\ldots e^{\beta_r g_r} F_1(\gamma_1)\ldots F_\omega(\gamma_\omega)$$

Finally, for any choice

$$(z'_1,\ldots,z'_{s'}) \in Y_{\omega-s'+1}\times\ldots\times Y_\omega,$$

let $L^{(z'_1,\ldots,z'_{s'})}$ be the linear functional from $\mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^\omega$ into $\mathbb{R}^n$ defined by

$$L^{(z'_1,\ldots,z'_{s'})}(\alpha,\beta,\gamma) =$$
$$\sum_{i=1}^m \alpha_i f_i(x_*) + \sum_{j=1}^r \beta_j g_j(x_*) + \sum_{l=1}^{\omega-s'} \gamma_l Y_l(x_*) + \sum_{p=1}^{s'} \gamma_{p+\omega-s'} z'_p$$

Now, by Theorem 7.1 the singletons $\mathcal{Q}^{\{f_i(x)\}}$, $i = 1,\ldots,m$, and $\mathcal{Q}^{\{g_j(x)\}}$, $j = 1,\ldots,r$, are Generalized Differential Quotients (at $((x,0),0)$) of the maps

$$(x,\epsilon) \mapsto xe^{\epsilon f_i} \qquad \text{and} \qquad (x,\epsilon) \mapsto xe^{\epsilon g_j},$$

respectively. Moreover, Theorem 7.4 implies that for any $j = 1,\ldots,\omega$ $\mathcal{Q}^{F_s(x)}$ is a Generalized Differential Quotient (at $((x,0),0)$) of the map

$$(x,\epsilon) \mapsto xF_j(\epsilon)$$

Therefore the chain rule for GDQ's (see [**?**])implies that the family of linear functionals

$$\left\{L^{(z'_1,\ldots,z'_{s'})},\ (z'_1,\ldots,z'_{s'}) \in Y_{\omega-s'+1}\times\ldots\times Y_\omega\right\}$$

is a GDQ of $\mathcal{F}$ at $((0,0,0),x_*)$ in the direction of $\mathbb{R}^n$. Hence the desired conclusion follows from the open mapping result stated in Theorem 6.6.

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