MATHEMATICS 300 — FALL 2001

Introduction to Mathematical Reasoning
(640 - 300, Section 04)

H. J. Sussmann

INSTRUCTOR'S NOTES, PART 1

(September 5, 2001)

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1 Information on the course

Course schedule:

We meet on **Mondays and Wednesdays, 7th period (6:10 to 7:30 pm)**, in HHA4 (College Avenue Campus).

About the instructor:

My *name* is **H.J. Sussmann.** My *office* is **Hill 538**.
My *Rutgers phone extension* is 5-5407.
My *e-mail address* is **sussmann@math.rutgers.edu**.

Web page. I have set up a *Web page* for our Math 300 section:

http://www.math.rutgers.edu/~sussmann/math300page.html

All the instructor's notes will be available there.

Office hours. My office hours will be:

- Tuesday and Thursday, 5th period (2:50 to 4:10) in my office, Hill 538;
- any other time, by appointment, in my office, **Hill 538**.

You should let me know that you are coming to see me *even if you are planning to do so during my regular office hours*. REASON: I don't want to have to stay in my office waiting just in case someone might show up. If I know that you are coming then I will be there, but if you don't tell me that I should expect you then I will feel free to go away and do other things.

The Fall 2001 semester: The semester starts on *Tuesday, September 4* and ends on *Wednesday, Dec. 12*. The *Thanksgiving recess* goes from *Thursday, November 22* to *Sunday, November 25.* (In addition, there are the following *changes in designation of class days:* Tuesday November 20 has Thursday Classes; Wednesday November 21 has Friday Classes. The only effect of these changes on our Math 300 section is that we do not have a class on Wednesday November 21.

Our Math 300 calendar: We will have **28 meetings**, on the following dates:

- September: W5, M10, W12, M17, W19, M24, W26 (7 meetings)
- October: M1, W3, M8, W10, M15, W17, M22, W24, M29, W31 (10 meetings)
- November: M5, W7, M12, W14, M19, M26, W28 (7 meetings)
- December: M3, W5, M10, W12 (4 meetings)

2 Homework, midterms, etc.

Homework and quizzes will count for about one third of your grade. There will be *two midterms*, which will count—together—for another third of your grade. The *final exam* will count for the remaining third.

Midterm dates: Monday October 15 and **Wednesday November 28**.

Please read and study carefully all the instructor's notes. The material they contain is as much a part of the course as the material in the book. Furthermore, the notes contain all kinds of important information. For example, in this set of notes there are lots of things you need to know in order to do your homework.

3 The textbook and the instructor's notes

We will be using:

- the book *Mathematics, a discrete introduction*, by **Edward R. Scheinerman** (Brooks/Cole Publishing Co., Pacific Grove, CA, 2000),
- the notes written by the instructor.

The material of the instructor's notes is an integral part of the course, as much as that of the book.

4 Readings for the first 3 days (Sept. 5, 10, 12)

- the book's note "To the Student,"
- the book's Chapter 1 (**all of it!**,
- the instructor's notes, No. 1.

The instructor's Handout No. 2 may be added to this list, if it becomes available on time.

5 Some important remarks

5.1 Integers and natural numbers

The *integers* are the numbers $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$. The *natural numbers* (also called "nonnegative integers") are the numbers $0, 1, 2, 3, \ldots$. In particular, **the number 0 is a natural number.**

Later on, after we have started to work with set notation, we will use $\mathbb{N}$ as a name for the set of all natural numbers, so

$$\mathbb{N} = \{\, 0, 1, 2, 3, 4, \ldots \}\,. \tag{5.1.1}$$

Therefore the *members* of $\mathbb{N}$ are the natural numbers. For example, 3 is a member of $\mathbb{N}$ (or, equivalently, "3 belongs to $\mathbb{N}$"). We indicate this by writing

$$3 \in \mathbb{N}\,. \tag{5.1.2}$$

Similarly, $5 \in \mathbb{N}$, $22 \in \mathbb{N}$, $60 \in \mathbb{N}$, $0 \in \mathbb{N}$. But $-1 \notin \mathbb{N}$ (that is, "-1 does not belong to $\mathbb{N}$."

We will use $\mathbb{Z}$ to denote the set of *all integers*. That is,

$$\mathbb{Z} = \{\, 0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots \}\,.$$

Also, we will write $\mathbb{R}$ to denote the set of *real numbers*.

A real number (and, in particular, an integer) is "positive" if it is > 0. A number is "nonnegative" if it is ≥ 0.

In particular, 0 **is not positive**, but 0 **is nonnegative.** 0 **is a nonnegative integer, but** 0 **is not a positive integer.**

The **positive integers** are $1, 2, 3, 4, \ldots$.

5.2 Reading formulas precisely

It is important that you acquire from the very beginning the habit of reading mathematical formulas precisely and correctly. For example, the correct way to read Formula (5.1.2) aloud is

$$3 \text{ belongs to } \mathbb{N}$$

or

$$3 \text{ is a member of } \mathbb{N}$$

or

$$3 \text{ is in } \mathbb{N}.$$

But you **cannot** read (5.1.2) as, for example, "3 is contained in $\mathbb{N}$." "Is contained" means something else!

Here is another example: Formula (5.1.1) is read as

$$\mathbb{N} \text{ is the set consisting of 0, 1, 2, 3, 4, etc.}$$

It would be **wrong** to read it as "$\mathbb{N}$ is 0, 1, 2, 3, 4, etc." (Do you see the difference? The best way to understand it is to think of more familiar examples.

For example, you cannot say that "the U.S. Supreme Court is Chief Justice Rehnquist and Justice Scalia and Justice Thomas and Justice O'Connor and Justice Souter and Justice Kennedy and Justice Ginsburg and Justice Breyer and Justice Stevens." Why not? Because the U.S. Supreme Court is **one thing, not nine things!** What you *can* say is:

> The U.S. Supreme Court is a court whose members are Chief Justice Rehnquist and Justice Scalia and Justice Thomas and Justice O'Connor and Justice Souter and Justice Kennedy and Justice Ginsburg and Justice Breyer and Justice Stevens.

Similarly, the set $\mathbb{N}$ of natural numbers is **one object, not many objects**, so you can say "$\mathbb{N}$ is the set consisting of 0, 1, 2, 3, 4, etc." but you cannot say "$\mathbb{N}$ is 0, 1, 2, 3, 4, etc."

Also, you cannot say "$\mathbb{N}$ is the natural numbers," or "$\mathbb{N}$ is natural numbers." (Actually, "$\mathbb{N}$ is the natural numbers" is bad, but "$\mathbb{N}$ is natural numbers" is even worse. Why?) What you should say is "$\mathbb{N}$ is the set of natural numbers," or "$\mathbb{N}$ is the set of all natural numbers."

5.3 Answering questions

In this course, whenever you are given a problem where you are asked to do something, your answer should be either:

(a) to do what you were asked to do,

or

(b) to show—that is *prove*—that it cannot be done.

(See, for example, Problems 2 and 4 below, where the correct answer is "what you asked me to do cannot be done.")

Notice that, when the answer is that "it cannot be done," it is not enough for you to *say* that it cannot be done. *You have to tell me why.* In other words, you have to *prove* that it cannot be done.

This remark is very important, and will apply throughout the semester, not just during the first week. And it applies to all your work, to the homework, the quizzes, the midtemrs and the final exam. So please read it and reread it until you are sure you got the point. ♢

5.4 Some examples of problems with correct answers

Here are some problems with correct solutions:

PROBLEM 1: Express the number 26 as a sum of two odd natural numbers.

ANSWER: $26 = 3 + 23$. ♢

REMARK: *There are lots of other solutions, of course! For example,* $26 = 7 + 19$*,* $26 = 13 + 13$*.*

PROBLEM 2: How many solutions does this Problem 1 have?

ANSWER: This question cannot be answered in a unique way, because it is not clear what is meant by a "solution of Problem 1," and, depending on how you define solutions, the answers to our question may be different.

Here are two equally legitimate ways to define "solution of Problem 1":

First way: A *solution of Problem 1* is an ordered pair (n, m) of odd natural numbers such that $n + m = 26$. (What this means is that a solution consists of specifying a first odd natural number n and a second odd natural number m such that $n + m = 26$. So "$n = 3$ and $m = 23$" is a solution, and "$n = 23$ and $m = 3$" is a *different* solution.)

According to this definition of "solution of Problem 1," there are exactly **13 solutions**, namely, the ordered pairs $(1, 25)$, $(3, 23)$, $(5, 21)$, $(7, 19)$, $(9, 17)$, $(11, 15)$, $(13, 13)$, $(15, 11)$, $(17, 9)$, $(19, 7)$, $(21, 5)$, $(23, 3)$, $(25, 1)$.

Second way: A *solution of Problem 1* is an unordered pair (n, m) of odd natural numbers such that $n + m = 26$.

According to this definition of "solution of Problem 1," there are exactly **7 solutions**, namely, the ordered pairs $\{1, 25\}$, $\{3, 23\}$, $\{5, 21\}$, $\{7, 19\}$, $\{9, 17\}$, $\{11, 15\}$, $\{13, 13\}$). (What this means is that a solution consists of specifying two odd natural numbers whose sum is 26, without singling out one of them as the "first" one. So in this case "$n = 3$ and $m = 23$" is a solution, and "$n = 23$ and $m = 3$" is *the same* solution.) ♢

PROBLEM 3: Express the number 27 as a sum of two odd natural numbers.

SOLUTION: This is impossible. REASON: the sum of two odd numbers is always even. Since 27 is odd, it cannot be the sum of two odd numbers. ♢

PROBLEM 4: Prove that $2 + 2 = 5$.

SOLUTION: This cannot be done. REASON: The statement "$2 + 2 = 5$" is false, and false statements cannot be proved.

PROBLEM 5: Prove that for every positive integer x there is another positive integer y such that $0 < y < x$.

SOLUTION: This cannot be done. REASON: The statement that we are being asked to prove is false, and a false statement cannot be proved.

The reason that the statement is false is as follows: 1 is a positive integer, and there is no positive integer y such that $0 < y < 1$, because 1 is the smallest positive integer. So if we take x to be 1 then the conclusion is not true. So it is not true that "for *every* positive integer x there is another positive integer y such that $0 < y < x$." ♢

PROBLEM 6: Express the number 12 as the difference of the squares of two natural numbers.

SOLUTION: $12 = 16 - 4$, so $12 = 4^2 - 2^2$. ♢

PROBLEM 7: Express the number 20 as the sum of the squares of two natural numbers.

SOLUTION: $20 = 4 + 16 = 2^2 + 4^2$. ♢

PROBLEM 8: Express the number 4 as the sum of the squares of two positive integers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 4 as a sum of two positive integers:

$$4 = 1 + 3\,, \qquad 4 = 2 + 2\,.$$

In neither of these two equalities are the two terms on the right-hand side squares.

PROBLEM 9: Express the number 4 as the sum of the squares of two natural numbers.

SOLUTION: $4 = 0 + 4 = 0^2 + 2^2$. ♢

PROBLEM 10: Express the number 12 as the sum of the squares of two natural numbers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 12 as a sum of two natural numbers:

$12 = 0{+}12$, $12 = 1{+}11$, $12 = 2{+}10$, $12 = 3{+}9$, $12 = 4{+}8$, $12 = 5{+}7$, $12 = 6{+}6$.

In none of these expressions are the two terms on the right-hand side squares of natural numbers. ♢

PROBLEM 11: Express the number 12 as the sum of the squares of three natural numbers.

SOLUTION: $12 = 4 + 4 + 4$, so $12 = 2^2 + 2^2 + 2^2$. ♢

PROBLEM 12: Express the number 15 as the sum of the squares of three natural numbers.

SOLUTION: This cannot be done. REASON: If we are going to write 15 as the sum of three squares, then we should be able to write it as the sum of two natural numbers, one of which is a square, while the other is the sum of two squares.

So let us look at all the ways to write 15 as a sum of two natural numbers, see in which cases one of these numbers is square, and then see if the other number is the sum of two squares.

Here are all the ways to express 15 as a sum of two natural numbers:

$$\begin{array}{llll} 15 = 0 + 15, & 15 = 1 + 14, & 15 = 2 + 13, & 15 = 3 + 12, \\ 15 = 4 + 11, & 15 = 5 + 10, & 15 = 6 + 9, & 15 = 7 + 8\,. \end{array}$$

There are four expressions in the above list where one of the numbers is a square:

$$15 = 0 + 15, \quad 15 = 1 + 14, \quad 15 = 4 + 11, 15 = 6 + 9\,.$$

In each case, we must ask whether the other number is a sum of two squares.

For "15=0+15," we have to see if 15 is the sum of two squares. The answer is "no," because we already have the list of all (eight) ways of writing 15 as a sum of two natural numbers, and in no case are both those numbers suqares.

For "15=1+14," we have to see if 14 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 14 are 0, 1, 4, and 9. So if are goint to write 14 as a sum of two squares, the only possibilities are $14 = 0 + 14$, $14 = 1 + 13$, $14 = 4 + 10$, $14 = 9 + 6$. But in all these cases the other number (that is, 14, 13, 10, or 6) is not a square.)

For "15=4+11," we have to see if 11 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 11 are 0, 1, 4, and 9. So if are goint to write 11 as a sum of two squares, the only possibilities are $11 = 0 + 11$, $11 = 1 + 10$, $11 = 4 + 7$, $14 = 9 + 5$. But in all these cases the other number (that is, 11, 10, 7, or 5) is not a square.)

Finally, for "15=6+9," we have to see if 6 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 6 are 0, 1, and 4. So if are goint to write 6 as a sum of two squares, the only possibilities are $6 = 0 + 6$, $6 = 1 + 5$, $6 = 4 + 2$. But in all these cases the other number (that is, 6, 5, or 2) is not a square.) ♢

PROBLEM 13: Express the number 15 as the sum of the squares of four natural numbers.

SOLUTION: $15 = 1 + 1 + 4 + 9 = 1^2 + 1^2 + 2^2 + 3^2$. ♢

6 Writing Mathematics

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements* with a *clear meaning* that can can be understood by anyone reading what you wrote. For example:
  - If you tell me that "she is very smart," but you haven't told me who "she" is, then I don't know what you are talking about, so you haven't made a statement with a clear meaning.
  - If you write "$x > 0$," but you haven't told me who "x" is, then I don't know what you are talking about, so you haven't made a statement.
  - If I ask you to state Pythagoras' theorem and your answer only says "$a^2 + b^2 = c^2$," then nobody will know what you are talking about, because you have not said what "a," "b," and "c" are supposed to be.
- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in detail in class, and is also explained in §7.
- Provide proper connectives between equations as well as between ideas.

- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.
- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.* If you have to memorize a definition or a statement, then that is not a good sign, because it indicates lack of understanding.
- Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you've tried to read any complicated document.
- *Do not assume that the people reading your paper can read your mind. Do assume that they are intelligent, but also assume that they are busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.*

Effective written exposition will be worth at least 50% of your grade. Conversely, bad or unclear exposition may be penalized as much as 50% of the grade or even more.

- The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

Pay attention to presentation, especially for the homework:

- A nicely typed homework (e.g., using a word processor) is preferable to handwritten work. Handwritten work is acceptable too, but in that case:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**. If you are not willing to spend some of your time doing this; if what you hand in shows that you were in a hurry and that you did not make the effort to write things neatly and properly, then there is no reason for me to spend any of my time reading what you wrote, and I will not do it.
  - You should use a pen. Never use a pencil.

  - Use any color other than red (for example, black, blue, or green), but DO NOT USE RED. (Reason: The use of red is reserved for the instructor's comments.)

- Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more than one sheet then the sheets are **stapled** and the pages are **numbered**.

- If you tear off the sheets from a spiral notebook, please make sure before you hand them in that there are none of those ugly hanging shreds of paper at the margins. Use scissors, or a cutter, if necessary.

Homework assignments that fail to meet the above requirements will be marked "unacceptable" and returned unread.

Keep in mind that

If you hand in a homework assignment that

- **is written carelessly or in a hurry,**

or

- **has lots of words crossed out,**

or

- **has unreadable handwriting,**

or

- **has pages that are unstapled or unnumbered or fail to show your name,**

or

- **has shreds of paper at the margins**

(in other words, anything that looks ugly, or anything that you would be ashamed to hand in, say, if you were writing a letter to apply for a job), then

- **this is a form of disrespect towards your instructor, and that's exactly how you should expect that your instructor will treat it.**

7 Using letters in mathematical writing

When we talk about mathematical objects, we often use *letters* (or other symbols such as $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, 0, 1, 2, etc.). The use of letters in mathematical discourse may often appear mysterious to the students. The first thing you need to do is to realize that it is not mysterious at all. It is, in fact, quite simple!

Letters are only used in mathematics in order to shorten certain statements that would otherwise be too long.

7.1 Some examples

Example 1. Suppose you want to define the word "cousin." You could say:

> A person is said to be a *cousin* of another person if one of the first person's parents is a sibling of one of the second person's parents. (7.1.1)

This is O.K., but having to talk about the "first person" and the "second person" and having to repeat the word "person" so many times makes it awkward. So what we can do is give names to those persons, for example a and b, and say the following:

> Let x and y be persons. Then x is said to be a *cousin* of y if one of x's parents is a sibling of one of y's parents. (7.1.2)

Example 2. Suppose you want to say

> If you take two real numbers, add them, and square the sum, then you get the same result as if you had squared each number separately, added the squares, and added to the result twice the product of the two numbers. (7.1.3)

This is awfully complicated, awkward, and ugly. You can make it much shorter by saying

> Let x, y be two real numbers. Then $(x+y)^2 = x^2 + y^2 + 2xy$. (7.1.4)

> For all real numbers x, y, $(x+y)^2 = x^2 + y^2 + 2xy$. (7.1.5)

We can say (7.1.5) even more concisely, by writing

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \Big((x+y)^2 = x^2 + y^2 + 2xy \Big) . \tag{7.1.6}$$

Here the symbol $\forall$ is the **universal quantifier**. Statement (7.1.6) is read as follows:

> For all real numbers x and for all real numbers y, the square of x plus y is equal to the sum of the square of x, the square of y, and twice the product of x and y. (7.1.7)

Actually **you can even read (7.1.6) without any letter variables at all.** How? Well, that's precisely what (7.1.3) does! ♢

Example 3. Suppose you want to say

If you take the number $\frac{\sqrt{5}-1}{2}$, and add it to its square, you get 1. (7.1.8)

You could say this as follows:

$$\left(\frac{\sqrt{5}-1}{2}\right)^2 + \frac{\sqrt{5}-1}{2} = 1\,. \tag{7.1.9}$$

Or you could give a name to the number, say a, and write

Let $a = \frac{\sqrt{5}-1}{2}$. Then $a^2 + a = 1$. (7.1.10)

Example 4. Suppose you want to say

It is possible to find a real number whose cube, added to its square, is equal to the number itself. (7.1.11)

Again, this looks horribly complicated and ugly, and you can make it much shorter by saying

There exists a real number x such that $x^2 + x^3 = x\,.$ (7.1.12)

or

There is a real number x such that $x^2 + x^3 = x\,.$ (7.1.13)

or

$x^2 + x^3 = x$ for some real number x. (7.1.14)

We can say this thing even more concisely, by writing

$$\exists x \in \mathbb{R}, (x^2 + x^3 = x)\,. \tag{7.1.15}$$

Here the symbol $\exists$ is the **existential quantifier**. Statement (7.1.15) is read as follows:

There exists a real number x such that $x^2 + x^3 = x$. (7.1.16)

There exists a real number x such that the sum of the square of x and the cube of x cube equals x. (7.1.17)

or even, if you don't want to use any letter variables at all

There exists a real number such that the sum of the square of the number plus the cube of the number equals the number itself. (7.1.18)

or, if you prefer, you can read it as statement (7.1.11) above. ♢

7.2 A very important remark on reading

The correct way to read a statement of the form "$\exists x \in \mathbb{R}$, something about x" is

There exists a real number x such that something about x. (7.2.1)

or

There exists an x belonging to $\mathbb{R}$ such that something about x. (7.2.2)

It is **wrong** to read "$\exists x \in \mathbb{R}$, something about x" as

There exists a real number x and something about x. (7.2.3)

If you read "$\exists x \in \mathbb{R}$, something about x" as in (7.2.3) then you are completely changing the meaning of the statement, and you are not conveying what it actually means.

EXAMPLE: Suppose you want to explain what it means for two people to be "siblings or half-siblings" using an existential quantifier. You could say:

Let x and y be persons. We say that x *and* y *are siblings or half-siblings* if $x \neq y$ and $\exists z$, z is a parent of x and a parent of y. (7.2.4)

How would you read this in plain English? You would certainly **not** say "there exists a z and z is a parent of x and a parent of y."!

What you would say (somewhat awkwardly, I know!) is "there exists a z who is parent of x and a parent of y," or (even more awkwardly, but still correctly) "there exists a z such that z is a parent of x and a parent of y."

It is important that you acquire as soon as possible the habit of using "such that" correctly. Please start right now! ♢

RECOMMENDED QUESTION FOR YOU TO THINK ABOUT: If I were to say "There exists a real number such that $x^2 + x^3 = x$." This would **not** be acceptable. Why?

ANOTHER RECOMMENDED QUESTION FOR YOU TO THINK ABOUT: Is (7.1.15) true? Why?

7.3 More examples

Example 5. Suppose you want to say

For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20. (7.3.1)

Once again, this is awful, so you can shorten the whole thing and just say

$$\text{Let } N = \textstyle\sum_{k=2}^{25} \frac{1}{k^2+5} \text{ . Then } N < 20. \tag{7.3.2}$$

Or, if you prefer, you could just say:

$$\textstyle\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,. \tag{7.3.3}$$

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: Is it true or false that $\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$? How about $\sum_{k=2}^{25} \frac{1}{k^2} < 20$? And $\sum_{k=2}^{25} 1 < 20$? And what would you say if I ask you whether it's true that $\sum_{n=2}^{25} \frac{1}{k^2+5} < 20$?

Example 6. Suppose you want to say

> If you take the function that to each real number assigns its cube, and integrate it from 0 to 4, then the result is 64. (7.3.4)

Again, this can be made much shorter by saying

$$\text{Let } N = \textstyle\int_0^4 x^3\,dx\,\text{ . Then } N = 64. \tag{7.3.5}$$

Or you could just say:

$$\textstyle\int_0^4 x^3\,dx = 64\,. \tag{7.3.6}$$

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: Is it true or false that $\int_0^4 x^3\,dx = 64$? Is it true or false that $\int_0^4 y^3\,dx = 64$?

Example 7. Suppose you want to say:

> The set of all real numbers whose square is larger than the cube of the number has at least two different members. (7.3.7)

We can make this shorter by saying

> Let S be the set of all real numbers x such that $x^2 > x^3$. Then S has at least two members. (7.3.8)

or

$$\text{Let } S = \{x \in \mathbb{R} : x^2 > x^3\}\,\text{ . Then } S \text{ has at least two members.} \tag{7.3.9}$$

Or you could just say:

$$\text{Let } S = \{x \in \mathbb{R} : x^2 > x^3\}. \text{ Then } \exists x \exists y, (x \in S, y \in S, \text{ and } x \neq y). \quad (7.3.10)$$

Example 8. Suppose you want to say that

> I am older than 20 and less than 70, and my brother is younger than I am, but his age is larger than half my age. (7.3.11)

Here too we can be more concise, and say

> Let x be my age and let y be my brother's age. Then $20 < x < 70$ and $\frac{x}{2} < y < x$. (7.3.12)

7.4 Constants and variables

In all the examples we have discussed so far, we have used **special symbols** (most commonly letters, such as x, y, a, k, N, n, S) as labels, so that we can avoid writing clumsy phrases such as "the number that we have been talking about." Sometimes we use other special symbols, such as Greek letters α, β, ε, etc.), letters modified in a particular way (for example $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$), numerals (0, 1, 2, . . .)

These special symbols stand for *objects*, or *things* of various kinds (numbers, human beings, sets), so we are going to call them *individual symbols*, or *thing-symbols*. (There are other kinds of mathematical symbols that do not stand for things. For example: the universal quantifier $\forall$, the existential quantifier $\exists$, the symbol + for the addition operation. We will talk about many of these symbols later.)

All the thing-symbols we have used are either

- **constants**

or

- **dummy variables**.

A *constants* is a symbol that stands for a thing that stays fixed throughout an explanation or argument.

For instance, in Example 3, the letter a stands for the number $\frac{\sqrt{5}-1}{2}$. We made this clear by **introducing** the letter a by means of the **declaration**

$$\text{Let } a = \frac{\sqrt{5}-1}{2}.$$

That is, we **declared** the value of a to be $\frac{\sqrt{5}-1}{2}$, so from that moment on we know that a stands for "$\frac{\sqrt{5}-1}{2}$."

Variables (also called "dummy variables") are letters used as an *internal label* used within a statement, but which do not have a value outside this statement.

This may look complicated, but it is in fact quite simple. Actually, it is something you are probably used to if you write computer programs in a language like Pascal. You often introduce a variable to keep track of some number with which you perform several operations. For example, a list of 20 numbers, called list[1], list[2], ..., up to list[20]. Suppose you want to add the 20 numbers, and end up with a number call "sum" as the result. You could write the following program:

```
begin
    sum := 0;
    For i := 1 to 20 do sum := sum + list[i];
end.
```

In this program, the letter i is just a label used to keep track of the variable index that keeps changing all the time. You could have written instead

```
begin
    sum := 0;
    For k := 1 to 20 do sum := sum + list[k];
end.
```

and you would have achieved the same thing. *Within the program*, the letter i (or k in our second version) has a value that keeps changing all the time. *Outside the program*, it does not make to ask "what is i?"

Notice that what we are doing with this program is computing the sum

$$\text{Sum} = \sum_{i=1}^{20} \text{list}[i]\,,$$

which is exactly the same as saying

$$\text{Sum} = \sum_{k=1}^{20} \text{list}[k]\,,$$

or

$$\text{Sum} = \sum_{\alpha=1}^{20} \text{list}[\alpha]\,,$$

or

$$\text{Sum} = \sum_{\clubsuit=1}^{20} \text{list}[\clubsuit]\,.$$

The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing. For example, if you write

$$\text{For all real numbers } x,\ y,\ (x+y)^2 = x^2+y^2+2xy\,, \tag{7.4.1}$$

or

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \left((x+y)^2 = x^2+y^2+2xy\right), \tag{7.4.2}$$

then that has exactly the same meaning as you had written

$$\text{For all real numbers } a,\ b,\ (a+b)^2 = a^2+b^2+2ab\,, \tag{7.4.3}$$

or

$$\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \left((a+b)^2 = a^2+b^2+2ab\right), \tag{7.4.4}$$

or

$$\text{For all real numbers } \alpha,\ \beta,\ (\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta\,, \tag{7.4.5}$$

or

$$\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \left((\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta\right), \tag{7.4.6}$$

or

$$\text{For all real numbers } m,\ n,\ (m+n)^2 = m^2+n^2+2mn\,, \tag{7.4.7}$$

or

$$\forall m \in \mathbb{R}, \forall n \in \mathbb{R}, \left((m+n)^2 = m^2+n^2+2mn\right), \tag{7.4.8}$$

or

$$\text{For all real numbers } (\diamond+\#)^2 = \diamond^2+\#^2+2\diamond\#\,. \tag{7.4.9}$$

or

$$\forall \diamond \in \mathbb{R}, \forall \# \in \mathbb{R}, (\diamond+\#)^2 = \diamond^2+\#^2+2\diamond\#\,. \tag{7.4.10}$$

Similarly, instead of $\exists x \in \mathbb{R}, \left(x^2+x^3=x\right),$

you could have said $\exists z \in \mathbb{R}, \left(z^2+z^3=z\right),$

or $\exists a \in \mathbb{R}, \left(a^2+a^3=a\right),$

or $\exists \omega \in \mathbb{R}, \left(\omega^2+\omega^3=\omega\right),$

or $\exists \aleph \in \mathbb{R}, \left(\aleph^2+\aleph^3=\aleph\right),$

or $\exists \diamond \in \mathbb{R}, \left(\diamond^2+\diamond^3=\diamond\right).$

Just one more example:

Instead of $\displaystyle\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$ you could have said $\displaystyle\sum_{n=2}^{25} \frac{1}{n^2+5} < 20\,,$

or $\displaystyle\sum_{\rho=2}^{25} \frac{1}{\rho^2+5} < 20\,,$ or $\displaystyle\sum_{GNU=2}^{25} \frac{1}{GNU^2+5} < 20\,.$

The "dummy variables" in the above examples (that is, i, k, α, $\clubsuit$, x, y, a, b, β, m, n, $\diamond$, $\#$, z, ω, $\aleph$, ρ, GNU) are really just **internal tags**, using to keep track of numbers **within a particular statement**. Outside this text, they variables do not have any meaning. For example, after you have written

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$$

it would not make sense for me to ask you "what is the value of k?" The letter k is just a tag to specify a *range of values*, as part of a set of instructions on how to compute a sum. Any other letter could be used instead, so in fact "k isn't really there." You can persuade yourself of this in another way, by remembering that

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$$

is just another way of saying

- For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20.

As you can see, there is no value of k there!

Similarly, when we say $\int_0^4 x^3\,dx = 64$, this is just a way of saying "if you take the function that to each real number assigns its cube, and integrate it from 0 to 4, you get 64." Once again, there is no value of x there! You could equally well have said $\int_0^4 y^3\,dy = 64$, or $\int_0^4 u^3\,du = 64$, or $\int_0^4 \aleph^3\,d\aleph = 64$.

And when we say $\exists x \in \mathbb{R}, x^2 = 2$, this is just a way of saying "there is a real number whose square is 2" (or, even better, "2 has a square root"). This statement does not talk about x, AND DOES

if you take the function that to each real number assigns its cube, and integrate it from 0 to 4, you get 64."

This way of speaking and writing, using dummy variables, is very useful, especially when what we are trying to say is quite long. But **you have to be careful** when writing this way, and have to make sure that you are using the letters correctly.

THE FIRST FUNDAMENTAL RULE

A letter symbol can only be used if either its value has been declared before or it is a dummy variable.

For example: if you write

$$x^2 + x = 90$$

just like that, without any words before or after explaining what x is, or making it a dummy variable, then what you wrote is **meaningless**. We have no way to tell if what you wrote is true or false, because we have no idea what you are talking about.

If you write this, then I will write in red **what is x?**

For comparison, here are some perfectly meaningful things you could have said:

1. Let $x = 9$. Then $x^2 + x = 90$.
2. Let $x = 6$. Then $x^2 + x = 90$.
3. Let x be the number of planets in the solar system. Then $x^2 + x = 90$.
4. $x^2 + x = 90$ for every real number x.
5. $x^2 + x = 90$ for some real number x.
6. $(\forall x \in \mathbb{R})(x^2 + x = 90)$.
7. $(\exists x \in \mathbb{R})(x^2 + x = 90)$.
8. The equation $x^2 + x = 90$ has a solution $x \in \mathbb{R}$. (Or "the equation $x^2 + x = 90$ has a real solution.")

Some of these are true, while the others are false. (Precisely: 1,3,5,7,8 are true; 2,4,6 are false. Moreover: 6 is just another way of saying the same thing as 4, and 7 and 8 are just alternative ways of stating 5.)

Notice that in 1, 2 and 3 the letter x has a value that was declared before. In 4, 5, 6, 7 and 8 the letter x is a dummy variable.

Again for comparison, let us now write some things that do **not** have a precise meaning:

A. The equation $x^2 = 2$ has a solution.

B. The equation $x^2 + 1 = 0$ has a solution.

C. The equation $3x = 5$ has a solution.

Why are these things bad? Because they do not specify the **range** of values within which the solution x is to be sought. For example, A is true if we look for solutions $x \in \mathbb{R}$, but false if we look for solutions $x \in \mathbb{Q}$. B is true if we look for solutions $x \in \mathbf{C}$ (the set of complex numbers) but false if we look for solutions $x \in \mathbb{R}$. C is true if we look for solutions $x \in \mathbb{Q}$ (or $x \in \mathbb{R}$) but false if we look for solutions $x \in \mathbb{Z}$.

On the other hand, the following are perfectly meaningful statements:

- A1. The equation $x^2 = 2$ has a solution $x \in \mathbb{R}$. (That is: $(\exists x \in \mathbb{R})(x^2 = 2)$.)
- A2. The equation $x^2 = 2$ has a solution $x \in \mathbb{Q}$. (That is: $(\exists x \in \mathbb{Q})(x^2 = 2)$.)
- B1. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbb{Z}$. (That is: $(\exists x \in \mathbb{Z})(x^2 + 1 = 0)$.)
- B2. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbb{R}$. (That is: $(\exists x \in \mathbb{R})(x^2 + 1 = 0)$.)
- B3. The equation $x^2 + 1 = 0$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(x^2 + 1 = 0)$.)
- C1. The equation $3x = 5$ has a solution $x \in \mathbb{Z}$. (That is: $(\exists x \in \mathbb{Z})(3x = 5)$.)
- C2. The equation $3x = 5$ has a solution $x \in \mathbb{Q}$. (That is: $(\exists x \in \mathbb{Q})(3x = 5)$.)
- C3. The equation $3x = 5$ has a solution $x \in \mathbb{R}$. (That is: $(\exists x \in \mathbb{R})(3x = 5)$.)
- C4. The equation $3x = 5$ has a solution $x \in \mathbf{C}$. (That is: $(\exists x \in \mathbf{C})(3x = 5)$.)

(Actually, A1, B3, C2, C3 and C4 are true, whereas A2, B1, B2 and C1 are false. Of course, in each case you could say the same thing in lots of different ways. For example, instead of C2 you could say "the equation $3x = 5$ has a rational solution," or "the equation $3x = 5$ can be solved in the set of rational numbers," or "there is a rational number that satisfies the equation $3x = 5$," or "$3x = 5$ is solvable in the set of rational numbers.")

A sentence whose meaning is imprecise, because it can be interpreted in more than one way, is said to be *ambiguous*. For example, the sentences

Flying planes can be dangerous

College students are turning to vegetables

The University president promised to stop drinking on campus

are ambiguous. Why? Think of other examples of ambiguous sentences.

Similarly, A, B, C above are ambiguous. Another example of ambiguous sentence is a sentence where a letter does not have a declared value and is not a dummy variable either. For example, if I tell you to "read chapter x," then that's ambiguous —unless I have told you before what "x" stands for, or given you clues so you can find out, or perhaps I have said "read Chapter x for all x" (that is "read all the chapters") or "read Chapter x for some x" (that is "read one chapter")— because you don't know what x is supposed to be.

THE SECOND FUNDAMENTAL RULE

Avoid ambigous sentences!

THE THIRD FUNDAMENTAL RULE

Formulas without words are usually meaningless and should be avoided like the plague.

It's the **words** that glue the formulas together and give a structure and a meaning to mathematical writing. If you just write a bunch of formulas then you probably do not know what you are saying, and if you don't know then you should not expect that the instructor or the grader will know.

Look, for example, at Statements 1, 2, 4 and 5 above. In all cases, the "formula" $x^2 + x = 90$ is the same, but the statements are quite different and have totally different meanings. That's because the words are different!

The three "fundamental rules" can in fact be subsumed into a single principle:

BE PRECISE!

7.5 Global (universal) constants

The simplest example of the use of letters in mathematics is provided by the *global constants*, or *universal constants*, as we now explain.

Example 9. The ratio of the length of a circle to the length of its diameter, whose value is approximately 3.141592, is a very important number. Since mathematicians do want to have to say "the ratio of the length of a circle to the length of its diameter" every time wish to refer to this number, they have agreed to use the Greek letter π ("pi") as a name (or, if you prefer, as an abbreviation) for it. ♢

Example 10. Another important number is the basis of natural logarithms, whose value is approximately 2.718. This is a number that shows up all the time in calculus, so we give it a name and call it e. ♢

The symbols π and e are two examples of *global constants*, that is, symbols that mathematicians agreed once and for all to use for a fixed mathematical object. In this case, the constants are *numerical*, because they stand for numbers (real numbers, to be precise).

Here are some other examples of global numerical constants: 0 ("zero"), 1 ("one"), 2 ("two"), 3 ("three").

Actually, mathematicians want to be able to write a name for *every* natural number. For this purpose, it not possible to choose a different single symbol like 0 or 1 (or perhaps letters) name for every number, because there are infinitely many numbers. So we need to use *strings of symbols* to name numbers. (Such strings are called "numerals.") Moreover, to decide which string corresponds to which number, it would be silly to do it on a one-by-one basis, because then we would have to memorize an enormous "dictionary" translating numbers into strings of symbols and viceversa[1].

It is much better to have a systematic approach, enabling us to build the names of complicated numbers from some basic symbols according to simple *rules*. Since ancient times, many such approaches have been devised. (For example, the Roman numeral system use the letters I, V, X, L, C, and M. In the Roman system, the numerals III, IX, and $MMCCCLXIX$ represent the

[1]It's actually much worse! First of all, the "dictionary" would have to be infinitely long, because there are infinitely many natural numbers. Second, what would the entries of such a dictionary look like? For each entry, you would show a number and then its corresponding name. But, how can you "show a number"? Only by writing down its name! So, all a "dictionary" can give you is a *correspondence* between two different kinds of *names*. (For example, something "the number three-hundred and sixty-five is represented by the numeral 365.") You cannot really have "dictionary entries" consisting of a number and a numeral, any more than you can have a dictionary of animal names where, say, one of the entries consists of a lion (a true lion, not the name of a lion) together with the word "lion." In other words, *you can only introduce a new way of naming natural numbers if you already have some other other way to begin with.*

numbers that we would call today 3, 9, and 2,369.) The one in use today is the system of so-called *decimal Arabic numerals*, that is, the numerals that you are undoubtedly familiar with: 0, 1, 2, 3, ..., 235, ... , 34,607, etc.

7.6 Local (temporary) constants

Often, a number shows up repeatedly in a particular computation or argument, so we want to introduce a letter symbol to serve as a name for that number *for the duration of this particular computation or argument only.* Such letter symbols are called *local constants.*

7.6.1 Constants with a declared known value

We can introduce a local constant by *declaring its value.* For example, we can say:

$$\boxed{\text{Let } a = 35. \text{ Then } a \text{ is divisible by } 5. \qquad (7.6.11)}$$

Here, the words "let $a = 35$" effectively declare the symbol a to be a constant, representing the number 35. So the above statement is just a roundabout way to say "35 is divisible by 5."

Once this particular use of the letter a is over, you could, if you want to, use the same letter to represent some other number. But in that case it would have to be very clear that the old declaration that $a = 35$ no longer applies. You could do this, for example, by saying something like

> Let $a = 35$. Then a is divisible by 5.
> $\vdots$
> Now let $a = 36$. Then a is not divisible by 5.

The word "now" serves the purpose of telling the reader that "we are starting all over again, and the old declared value of a no longer applies."

Example 11. Your bank uses a computer program to prepare your monthly statements. Within that computer program, various numbers related to your bank account are represented by symbols or strings of symbols. Those representations are *local.* This means, quite simply, that those symbols represent what they do *only within the particular computation that the bank's computer is carrying out to prepare your statement.* Say, for example, that the bank's program uses the string "DEP" to represent the sum total of your deposits. Then "DEP" will stand for a particular number that the program will compute, but once that computation is over then the string "DEP" will no longer have a value, and the computer will be free to start another computation, for another customer, and use "DEP" again, this time to refer to the sum total of the new customer's deposits. Outside the bank's computations for your bank statement, the symbol "DEP" has no value. It only has a value *internally*, within the computation. ♢

Example 12. Local constants are used, for example, in contracts, or in trials. If you are buying a house, then the contract will probably contain a clause at the beginning declaring the words "the buyer" to stand for you for that particular contract. This means that the letter string "the buyer" is functioning as a local constant, whose value is you for this contract. Once your attorney goes to the signing for the next house sale, where the buyer is a different person, then he will present this person with a different contract, in which the string "the buyer" has a totally different value. So the value of the string "the buyer" is fixed only within a specific contract, and changes when yo go another contract.

A similar situation occurs in trials. At the beginning of a trial, the parties will agree that the words "the defendant" stand for a specific person. But this declaration of value is only applicable within a particular trial. Once we move on to another case, the string "the defendant" with be redeclared to have a different value. ♢

7.6.2 Constants with a declared initially unknown value

The declared value of a constant could be *unknown* to us when we start writing, even if it is fixed in some way. It may happen that the value is eventually fully determined, or that later on we find out more about it without fully determining it, or that at the end of our discussion we know as little about the value as we knew at the beginning.

For example, we could write

$$\boxed{\begin{array}{l}\text{Let } a \text{ be the number such that } 3a+5=17.\\ \text{Then } 3a=17-5=12.\ \text{Therefore } a=4.\end{array}} \tag{7.6.12}$$

In this case, when we declare a to be a constant, we are not explicitly stating what its value is. We are just declaring that a must satisfy a certain condition, namely, the equation $3a+5=17$. Then we go on and solve the equation, so we end up knowing that $a=4$, but our knowledge of the value of a is a *consequence* of our original declaration together with some reasoning.

Here is another example:

$$\boxed{\begin{array}{l}\text{Let}\\ \qquad a=\begin{cases}1 & \text{if Goldbach's conjecture is true},\\ 0 & \text{if Goldbach's conjecture is false}.\end{cases}\\ \text{Then } a^2=a, \text{ because } 0^2=0,\ 1^2=1, \text{ and } a \text{ is either 0 or 1.}\end{array}} \tag{7.6.13}$$

Here we do not know what the value of a is. We know that $a=0$ or $a=1$, but we do not know which one it is. Knowing the value of a would be tantamount to knowing if the Golbach conjecture is true. And as of this writing (September 2001) no mathematician knows.

7.6.3 Arbitrary constants

Another way in which a constant can be fixed but unknown is by declaring it to be an *arbitrary* object, and stipulating certain conditions that it must satisfy. (See Box 1 for a detailed explanation of how this works.)

For example, suppose you want to argue that the sum of two even integers is even. (See the proof in the book, page 16.) other than 2 is odd. Then you could write the following:

1. Let x and y be arbitrary even integers.
2. Since x is even, we know from the definition of "even number" (book, page 1) that x is divisible by 2.
3. Since y is even, we know from the definition of "even number" (book, page 1) that y is divisible by 2.
4. Since x is divisible by 2, we know from the definition of "divisible" (book, page 2) that there is an integer c such that $2c = x$.
5. Since y is divisible by 2, we know from the definition of "divisible" (book, page 2) that there is an integer d such that $2d = y$.
6. Since $2c = x$ and $2d = y$, it follows that $2c + 2d = x + y$.
7. Since $2c + 2d = 2(c + d)$, it follows that $2(c + d) = x + y$.
8. Therefore there exists an integer e (namely, $c + d$)) such that $2e = x + y$.
9. Since $2e = x + y$ and e is an integer, it follows from the definition of "divisible" that $x + y$ is divisible by 2.
10. Since $x + y$ is divisible by 2, it follows from the definition of "even number" (book, page 1) that $x + y$ is even.

(7.6.14)

The above proof uses five letters, namely, x, y, c, d, and e. Notice that **each letter is properly introduced by declaring its value**:

- x and y are declared to be "arbitrary even integers;"
- c is declared to be a "an integer such that $2c = x$;
- d is declared to be a "an integer such that $2d = y$;
- e is declared to be $c + d$.

In principle, the introduction of an arbitrary object must take the form:

Let a be an arbitrary object of such and such kind.

For example, one could write

Let a be an arbitrary integer.

or

Let x be an arbitrary even integer.

or

Let p be an arbitrary prime number.

or

Let x be an arbitrary real number.

or

Let ε be an arbitrary positive real number.

The word "arbitrary" is often is omitted, and one writes things like

- Let a be an integer.
- Let x be an even integer.
- Let p be a prime number.
- Let x be a real number.
- Let ε be a positive real number.

It is strongly recommended that you acquire the habit, whenever you see a statement like

Let x be an even integer.

to see it as really another way of saying

Let x be an arbitrary even integer.

This way of thinking will turn out to make things much easier for you.

BOX NO. 1: ARBITRARY OBJECTS.

An *arbitrary object* is a fixed object about which we know nothing. Arbitrary objects can be subjected to certain conditions; for example, an "arbitrary integer" is an arbitrary object that cannot be anything whatsoever (it cannot be, for instance, the State of New Jersey, or a cow, or the number 3/5; it has to be an integer), but other than that it is an integer we know nothing whatsoever about it.
The way you should think about "arbitrary objects" is as follows: imagine that there is somebody (a friend, or a computer, or an alien from the planet of the Kligons) whose job is, every time you write "let a be an arbitrary object," to pick one such object, write down what it is on a piece of paper, and then puts the paper in an envelope and seal the envelope.
We will call such a person (or computer, or alien) the **CAT** ("creator of arbitrary things"). So, for example, if you say "let a be arbitrary; suppose a is an integer" then the **CAT** will pick an integer—say 23—and write "$a = 23$" on the paper that will go inside the envelope. (Actually, the **CAT** is also allowed pick other objects such as Mount Everest or the king of Norway, which are not integers, but, as we shall soon see, a reasonable **CAT** would never do that.)
The crucial facts are:

1. the **CAT** is allowed to pick any object whatsoever and assign it to be the value of your constant;
2. only the **CAT** knows which object he has picked;
3. the **CAT** wants to prove you wrong;
4. the **CAT** is allowed to wait up to the last moment before disclosure time before choosing the values of your arbitrary objects;
5. at the end of your argument, the **CAT** will disclose the chosen values of all the arbitrary constants that you had introduced, and *if these values satisfy all the conditions that were imposed on them but fail to satisfy at least one of the conclusions that you drew about them in your argument the* **CAT** *wins and you lose.* (If the values do not satisfy all the conditions, the **CAT** loses. If the values satisfy all the conclusions, the **CAT** loses.)

(Rule 5 is temporary, and will later be replaced by the more precise Rule 5#.)
So, for example, suppose you say:

Let a, b, c be a arbitrary integers such that $ab = ac$. Then $b = c$.

Then the **CAT** is obliged to choose a, b and c to be integers, and to pick them so that $ab = ac$, because if he does not choose them that way then he will lose for sure, because he will fail to satisfy your conditions. So he could pick $a = 3$, $b = 4$, $c = 4$, for example. But, since he is allowed to wait until you have said all you were going to say, he will wait, and when he hears that you said "then $b = c$," he will pick $a = 0$, $b = 3$, $c = 5$, for example, because *with these choices all the conditions you imposed (a, b, c are integers, and $ab = ac$) are satisfied, but your conclusion is not, so you lose and the* **CAT** *wins.*

In (7.6.11), (7.6.12), (7.6.13), and (7.6.14), we have seen four seemingly different examples of "introducing a constant with a possibly unknown value." In (7.6.14) the value is really unknown, and will remain so forever, or until the **CAT** discloses it. In (7.6.12) the value is initially unknown, but the requirements that were imposed on a are such that we end up finding out the exact value of a. In (7.6.11), the value of a is completely known, and in (7.6.13) it is partially unknown.

Actually, these four situations are really examples of the same thing: **we introduce a local constant by declaring its value to be arbitrary, and then impose certain conditions on it**. (We can even regard (7.6.11) as fitting this pattern: when we say "Let $a = 35$," we really saying "Let a be arbitrary; assume that $a = 35$.") The conditions may be such as to allow a lot of freedom to the **CAT** (as in (7.6.14)), or they may be so constraining that the **CAT**'s choice is limited to only one possibility (as in (7.6.11) and (7.6.12)). But all four cases our analysis of the arguments and the possible **CAT**'s choices works in the same way:

1. In (7.6.14), we end up with the conclusion that "p is odd." Here the **CAT** could have chosen p to be prime and different from 2, or not. If not, then he would have failed to satisfy our conditions, and would have lost for sure. If he did choose p to be prime and different from 2, then he may have chosen p to be 3, or 5, or 93, or 101. But, whatever his choice was, when the sealed envelope is opened, it will show an odd number, so we win and the **CAT** loses.

2. In (7.6.11), we ended up with the conclusion that a is divisible by 5, but our original declaration left the **CAT** no choice. When we open the envelope the **CAT** will lose no matter what, for if he picked anything other than 35 then he will not be satisfying out requirements, while if he picked 35 then our conclusion will hold.

3. In (7.6.12), we ended up with the conclusion that $a = 4$. On the other hand, here the **CAT** had to pick a number a, and the condition was that $3a + 5 = 17$. If the **CAT** picked a to be anything other than 4, then he would lose because he is not satisfying our conditions. If he picked $a = 4$, then when we open the envelope we will see that $a = 4$, which is our conclusion, so once again we win and the **CAT** loses.

4. In (7.6.13), what could the **CAT** have picked? This may seem unclear to you, because whether he picks 0 or 1 then he will not be sure (and in fact nobody will be sure!) that our conditions have been met. But the point is this: the **CAT** *has* to pick something. To win, he has to (a) satisfy our conditions, *and* (b) violate at least one of our conclusions. If he picks a to be anything other than 0 or 1 then he loses, because we can prove to him that the conditions we imposed are not satisfied, because these conditions clearly say that a must be 0 or 1. If he picked a to be 0, then $a^2 = a$, so our conclusions hold (even though we do not know if the conditions on a

hold), so the **CAT** loses because he has failed to satisfy (b). If he picked a to be 1, then he loses for the same reason.

Now, here is a different example.

Let
$$a = \begin{cases} 1 & \text{if Goldbach's conjecture is true,} \\ 0 & \text{if Goldbach's conjecture is false.} \end{cases}$$
Then $a^2 = a$, because $0^2 = 0$, $1^2 = 1$, and a is either 0 or 1. Therefore, dividing both sides by a, we find $a = 1$. Therefore the Goldbach conjecture is true.

(7.6.15)

This argument is clearly wrong, as it should be, because nobody knows if the Goldbach conjecture is true, and it would be extremely suprising if, after mathematicians have struggled with this question for more than 2000 years, someone could prove it in such a cheap way.

Moreover, you can see right away where the mistake is. You are not allowed to divide both sides of an identity by a unless you know that $a \neq 0$. In our case, knowing that $a \neq 0$ would be knowing that $a = 1$, since $a = 0$ or $a = 1$. But knowing that $a = 1$ amounts precisely to knowing that the Goldbach conjecture is true, which is what we are trying to prove. So the only way to justify the unjustified step in the "proof" would to assume that the Goldbach conjecture is true, in which case we would end up with a circular proof.

Now, how would our **CAT** analysis of this purported argument work? At first sight, you might think that the **CAT** loses, and we win, because that **CAT** cannot produce a value of a for which he can guarantee that our conditions are true. (If he picks $a = 0$, then he would have to prove to us that the Goldbach conjecture is false. If he picks $a = 1$, then he would have to prove to us that the Goldbach conjecture is true. And he almost certainly does not know how to do either. If he picks another a, say $a = 2$, then he loses for sure, since in this case a definitely fails to meet our conditions.)

So it would appear that the **CAT** loses, we win, and either we have proved the Goldbach conjecture or there is something wrong with our rules about the **CAT**. Which one is it?

The answer is, of course, that *there is something wrong with our rules about the* **CAT**. What is wrong with our rules, as laid out in Box 1, is that *the rules are not precise enough.* The problem occurs in Rule 5, which says that

5. at the end of your argument, the **CAT** will disclose the chosen values of all the arbitrary constants that you had introduced, and *if these values satisfy all the conditions that were imposed on them but fail to satisfy at least one of the conclusions that you drew about them in your argument the* **CAT** *wins and you lose.* (If the values do not satisfy all the conditions, the **CAT** loses. If the values satisfy all the conclusions, the **CAT** loses.)

This rule is problematic because it does not tell us what to do in a situation where the **CAT** has produced values for the constants and we cannot decide whether those values satisfy the conditions, or whether they satisfy the conclusions. For you to win, it would have to be established that the **CAT**'s chosen values for the constants either violate the conditions or satisfy the conclusions. For the **CAT** to win, it would have to be established that the **CAT**'s chosen values for the constants both satisfy the conditions and violate one of the conclusions. It would appear that an intermediate possibility arises, if we cannot decide if we are in the case where you win or in the case where the **CAT** wins.

What do we do in this situation? The answer is that *the burden of the proof* is on you, not on the **CAT**. So *if you cannot prove that the* **CAT** *loses then the* **CAT** *is declared the winner.* In other words, we have to replace Rule 5 of Box 1 by the following.

5#. At the end of your argument, the **CAT** will disclose the chosen values of all the arbitrary constants that you had introduced; *if you can prove that these values violate at least one of the conditions imposed on them, or that they satisfy all the conclusions that you drew about them in your argument, then you win and the* **CAT** *loses. Otherwise the* **CAT** *wins.*

Let us see how this works for (7.6.15). The **CAT** can pick $a = 0$. To win, you would have to prove either that $a = 0$ violates the conditions, or that it satisfies the conclusions. To prove that it satisfies the conclusions you would have to prove the Goldbach conjecture, which is one of the conclusions. To prove that it violates the conditions, then once again you would have to prove the Goldbach conjecture, for if the conjecture was false then $a = 0$ would satisfy the conditions. So there is no way to win unless you prove the Goldbach conjecture[2]! Since you do not know how to do that, the **CAT** wins.

Here is another example of the same situation.

Let a be arbitrary. Suppose that $a = 0$ and $a = 1$. Then $0 = 1$ (because $0 = a$ and $a = 1$).

(7.6.16)

Is this correct? Have we proved that $0 = 1$? Let us take a closer look.

No matter what value the **CAT** picks for a, the condition "$a = 0$ and $a = 1$" will *never* be true, and you will always be able to prove so to the **CAT**. Therefore the **CAT** loses! You win!

Does that mean that you have correctly proved that $0 = 1$? The answer is a resounding NO!, as it should be, because **we should never be able to prove a false statement.**

[2]Naturally, you would have to prove it *in some other way.* (7.6.15) purports to be a proof, but it surely does not count until we know it is valid, which precisely what we are trying to determine here.

In order to understand why we have not proved that $0 = 1$, we have to digress and discuss the issue of *assumptions* and what it means to "prove a statement S under certain assumptions," and how that differs from just "proving S."

7.7 Introducing assumptions

In a proof you can introduce any asumption you want, at any time. It does not matter if the assumption is true, of if you know whether it is true.

Students often find this statement shocking, and ask questions such as "how can I assume something if I do not know that it is true?"

The answer is that you can, and in fact you do precisely that most of the time, but for some mysterious reason when students write about mathematics they do not believe that the same thing that they are used to doing in their everyday life is O.K. and works the same way in mathematics.

In you everyday life, you often imagine some possibility in order to figure out what would happen if that was the case. For example, before the last presidential election yoy probably tried to imagine what would happen if Candidate X won, and what would happen if Candidate Y won. Then you compared the results and came to the conclusion that you liked better what would happen if one of those two candidates won, and decided for that reason to vote for that candidate.

To imagine what would happen if Candidate X won, you almost certainly started thinking along the following lines: "let me suppose that X been elected president; then I guess he would appoint so-and-so as secretary of state, and this person would probably push our foreign policy in sushc-and-such direction, given what I know about his views." And you would have continued like that.

In other words, **you made the assumption that X won without knowing that it was true, and then you explored its consequences**.

You do the same kind of thing all the time. For example, to decide if you will go to Restaurant R for dinner, you would try to imagine what you would eat if you were there, and you might very well end up concluding that none of the dishes that you can imagine yourself ordering looks appealing, so in fact you will not go to Restaurant R. In other words: **you made the assumption that you had gone to Restaurant R and then concluded that you did not like its consequences, so in the end you made the assumption false by not going to the restaurant.**

In mathematics we do the same thing. We introduce assumptions, and then explore their consequences. We do not care if the assumptions are true, because all we are trying to find out is "what else would have to be the case if these assumptions were true?"

BOX NO. 2: HYPOTHESES (A.K.A. ASSUMPTIONS).

Hypotheses, or **assumptions**, are statements that we **assume** to be true in a particular situation.

When we discuss proofs later on, we will see that that there are certain statements that you are free to bring in any time you want, because you know them to be true. (Such statements are called "axioms," or "postulates.") In additions, *definitions* function exactly like axioms, the only difference being that an axiom is something you *accept* as true (say, because you are convinced it is true but you do not know how to prove it) whereas a definition (say, for example, $2 = 1 + 1$) is something that you *declare*, or *decide* to be true. **Within a particular proof, hypotheses/assumptions function exactly like axioms and definitions: they are statements that you can bring in any time.**

There is, however, a big difference. Axioms and definitions are things that we take to be true forever, because we believe they are true. Hypotheses/assumptions are taken to be true not because we believe they are true, but **because we want to know the consequences of their being true.**

For example: suppose we to find out what the world would be like if pigs had wings. To do this, we first make the assumption "pigs have wings," by saying

(...) Assume that pigs have wings.

Then we would set out to explore the consequences of this assumption. That is, if you prefer, we would set out to study an **imaginary world** in which pigs have wings, and try to understand what that world would have to be like. We might then be able to "prove" all kind of "theorems," such as

THEOREM 1: *Pigs can fly.*

THEOREM 2: *If a pigsty does not have a roof, then the pigs will fly away.*

(Proving these theorems might not be easy. For example, it is not true that "every animal that has wings can fly," since, for example, ostriches, penguins and dodos do not fly. But you might notice that all the exceptions to the general rules seems to be birds, so perhaps you could make a convincing case that "every animal that has wings and is not a bird can fly." If you manage to do that, then you might go on to argue that pigs can fly, since pigs clearly are not birds.)

All these theorems would be proved **under the hypothesis** that pigs have wings. That is, we would never be saying that we have proved that pigs can fly. All we would be saying is that **IF** pigs had wings **THEN** pigs could fly. In other words, all the facts we would "prove" about pigs under the "pigs have wings" hypothesis would be valid **in the imaginary world of pigs with wings**, which, in principle, is not our real world.

BOX NO. 3: ENTERING AND LEAVING IMAGINARY WORLDS.

As we explained in Box 2, when you assume something you are setting out to explore an **imaginary world**, namely, a world in which that thing you are assuming is true. When can you leave that world and go back to the real world, taking with you the things you have discovered in your imaginary world? The answer is simple: if the thing you assumed is actually true in the real world, then everything you proved **under that assumption** is true, because in that case the imaginary world really is the real world.
Our rules of inference capture this very obvious idea as follows:

- You are always allowed to introduce anything you want as an asumption, any time you want.
- When you do that, whatever you have proved will have been proved **under your assumption**.
- If, assuming a statement P, you prove some other statement Q, then you will be allowed to go back to "the real world" and assert there that $P \longrightarrow Q$.
- If, in addition, you can prove P, then the Modus Ponens rule allows you to go to Q.

In simpler terms: whatever you prove under the assumption P will be valid unrestrictedly (that is, without having to assume that you are under the assumption P) if you can also prove P.

EXAMPLE: If you prove that "pigs can fly" under the assumption that "pigs have wings," then the conclusion that "pigs have wings" will have been established in a world where pigs have wings. You can also go back from such an imaginary world to the real world by stating that "if pigs have wings then pigs can fly." This, however, is not enough to be able to conclude that in the real world pigs can fly, because in the real world you do not know that pigs have wings, and **if you do not know that pigs have wings then you cannot go from "if pigs have wings then pigs can fly" to "pigs can fly."**

7.8 Did we prove that $0 = 1$?

Now that we have talked about making completely arbitrary assumptions, we can go back to the question at the end of subsection 7.6. Does the argument of (7.6.16) correctly prove that $0 = 1$?

The answer is "of course not." What does it prove? All it proves is that in an imaginary world in which there is a number a such that $a = 0$ and $a = 1$ it would also have to be the case that $0 = 1$. We can go from this to the the real world, and conclude that, for an arbitrary a, "if $a = 0$ and $a = 1$ then $0 = 1$." But, since we cannot find any a for which $a = 0$ and $a = 1$, we cannot conclude that$0 = 1$.

7.9 Introducing things that do not exist

So far, it would appear that, if we introduce an arbitrary object a and stipulate that a satisfies certain conditions, then whatever we have proved is only proved "under the assumption that these conditions hold," that is, "in an imaginary world world in which the conditions are true." There is, however, an important case where you can guarantee that what we have proved is true "in the real world." What is needed is that the following conditions be met.

1. The proven conclusion does not involve a.
2. We know that an object satisfying all the conditions imposed on a exists.

Here is an example. Let us prove

Theorem 1. *Let*

AN EXAMPLE

Here is an example of the kind of thing that you should *not* do when you write the solution of a homework problem.

QUESTION: describe the set $\{x \in \mathbb{R} : 1 < x^2 < 2\}$ as a union of intervals.

A TRULY AWFUL SOLUTION (*the final answer is correct, so this would get some partial credit, say 40%*):

$$1 < x^2 < 2$$
$$1 < x < \sqrt{2} \quad \text{or} \quad -\sqrt{2} < x < -1$$
$$A = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$$

WHAT MAKES THIS BAD? Well, lots of things. For example:

- 1. To begin with, I do not see any *statements* here. Remember that a *statement* is supposed to be something that is either true or false. When the student writes the inequality "$1 < x^2 < 2$," what is the student asserting? What statement is the student making? (I know, you are probably thinking that what the student is saying is that "x^2 is between 1 and 2." But that does not do me any good. I still do not know what this means, because I have no idea what x is supposed to be. Do you? And please don't tell me "x is supposed to be a real number." What good does that do to me? I still don't know whether $1 < x^2 < 2$ is true or false, unless you tell me *which* number it is.) Is this true or is it false? If you cannot tell, it means that there is no statement there. (There is a standard convention used very often in mathematics: when you say something involving variables, and you don't specify the values of

the variables or bound the variables with a quantifier, it means that you are saying that your formula is true for *all* values of the variables. In this case: could the student be trying to say that $1 < x^2 < 2$ is true for *all* values of x? I sure hope not! Because if that was it, then it would be **false**! For example, $1 < x^2 < 2$ is not true for $x = 3$.)

- 2. The letters x and A are used, but we are never told what they are supposed to stand for. (I know: you will tell me that A is the set $\{x \in \mathbb{R} : 1 < x^2 < 2\}$. But the student did not say that anywhere!)
- 3. Throughout this proposed "solution," there are no words explaining what is being done. And there is not a single punctuation mark!
- 4. There is nothing connecting each line to the next one.

Now, here is an example of a good solution of the same problem. (The words in italics are my own comments. They are not part of the student's solution.)

A GOOD SOLUTION (*which gets full credit*):
Let $A = \{x \in \mathbb{R} : 1 < x^2 < 2\}$. (*Comment: the student is announcing what the letter A is going to mean in this solution.*)
Let $x \in \mathbb{R}$. (*Comment: the student is announcing that x is going to stand for an arbitrary real number. This means that, from now on, whatever assertion the student makes about x should be true for* **all** *real numbers x. For example, if in the next line the student said "*$1 < x^2 < 2$*," then that would be bad, because "*$1 < x^2 < 2$*" is* **not** *true for all x.*)
Then "$x \in A$" is equivalent to "$1 < x^2 < 2$." (*Comment: the student is* **not** *saying that* $x \in A$*, or that* $1 < x^2 < 2$*. He/she is just saying that those two statements are equivalent, that is, that they are either both true or both false. So the statement being made here is perfectly true.*)
And "$1 < x^2 < 2$" is equivalent to "either $1 < x < \sqrt{2}$ or $-\sqrt{2} < x < -1$." (*Comment: again, a true statement.*)
Therefore "$x \in A$" is equivalent to "either $x \in (1, \sqrt{2})$ or $x \in (-\sqrt{2}, -1)$." So, finally,

$$A = (1, \sqrt{2}) \cup (-\sqrt{2}, -1), \text{ and this is the desired answer.}$$

GENERAL COMMENT: notice the use of the words "so", "therefore," "then." These words give continuity to the whole thing. They tell you how each paragraph is related to the previous one. English is a very rich language. Take advantage of this to write well! Notice also the nice touch of putting the word "finally," to announce that we are getting to the end. Why not use "then" all the time? **Because it's nicer when you don't repeat the same words over and over again.** *It is not nice to give the impression that you have a very poor vocabulary.*

And here is another example of a correct solution to the same problem.

ANOTHER PERFECTLY FINE SOLUTION (*which also gets full credit*):

A real number belongs to the set under discussion if and only if the square of this number is larger than one and less than two. This will happen if either the number itself is strictly between one and the square root of two (*Comment: why "strictly"? Because I am saying that it cannot be equal to one or square root of two.*), or its negative is strictly between minus the square root of two and minus one. So our set is $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$. *General comment: notice that the student did everything with words, without using any letter variables. Is that bad? Not at all! There is nothing wrong with using words rather than math symbols. The reason that we usually don't do it is that math symbols make it* **easier** *to say things, as we saw before.*

A FINAL COMMENT. The main point of mathematical writing is not the use of weird symbols. It is the use of **precise language.** You can be precise with words or with symbols. But symbols are just a way to say more concisely certain things that would take too long to say with words. As a rule, **if you say something with mathematical symbols, then you ought to know how to say exactly the same thing using only words.**

ANOTHER EXAMPLE

Let us look at the definition of "interval," and let us compare some correct ways of stating it and some incorrect ones.

The definition says:

- (D1) an *interval* is a set I of real numbers such that, whenever a, $b \in I$ and $c \in \mathbb{R}$ are such that $a < c < b$, it follows that $c \in I$.

Again, as in our previous examples, all this can be said without letters. For example, here is one possible way:

- (D2) an *interval* is a set of real numbers with the property that any time you take two elements of the set then every real number lying between those two elements is also in the set.

It should be clear to you that (D2) and (D1) say exactly the same thing. (If it isn't clear, then you should go on thinking and rethinking this whole thing until it becomes clear.)

Now let us look at some bad definitions of "interval":

EXAMPLE B1: trying to define "interval," a student wrote:

- (W1) for a set to be an interval I, $(\forall a \in \mathbb{R}), (\forall b \in \mathbb{R}), (\forall c \in \mathbb{R})$ If $(a \in I), (b \in I)$, there exists a c such that $a \leq c \leq b$, If $c \in I$, then the set I is an interval.

(NOTE: *This a real example. I actually took it from a midterm exam written by a Math 300 student.*) Can you figure out what this means? Can you say what this means without letters? I cannot figure it out. When I grade, this definition of interval is worth 0. (Can you make sense of things like "for all c there exists a c"? I cannot.)

EXAMPLE B2: also trying to define "interval," another student writes:

- (W2) an *interval*, I, exists when real numbers a, b are in I, and there is a real number c in the *interval* I if and only if $a < c < b$.

Again, see if you can figure out what this means. As before, ask yourself: how would you say this without letters? I cannot figure it out. When I grade, this definition of interval is also worth 0.

EXAMPLE B3: here is another answer:

- (W3) an *interval* I is defined by the following rule where for all real numbers a and b such that $a \in I$ and $b \in I$, there exists a real number c such that $a < c < b$.

Again, let's see if we can figure out what this means. How can we say this without letters? Well, in this case I am afraid we *can* say it. (There are some very wrong things that the student is doing here, like for instance using the expression "the following rule where ..." which is completely ungrammatical English. Perhaps "namely" would have been better than "where"? Also, the student never said that an interval is supposed to be a set of real numbers, so for all we know the set of all apples could be an interval. But let's ignore all this.)

What this student is saying is this: *for a set to be an interval, what has to happen is that whenever you pick two members of this set then there has to be a third number strictly between these two members.* (Why do I say "strictly"? Because the student wrote $a < c < b$, so c cannot be equal to one of the two numbers you picked.)

We could also say what the student seems to be saying using the symbols that we introduced earlier. An interval, according to the student, is a set I such that

$$(\forall a \in I)(\forall b \in I)(\exists c \in \mathbb{R})(a < c < b)\,.$$

(Look at this formula, think it over, and see whether or not it is an adequate translation of what the student wrote. If you *don't* agree, tell us why and we can discuss this in class.)

Now, let us see what sets turn out to be "intervals" according to this definition. Suppose I is a set of real numbers. Suppose I is not the empty set. So I has at least one member. Now, nothing prevents me from picking a to be one

of the members of I, and picking b to be exactly that very same element. (The student said "for all $a \in I$ and $b \in I$" so this certainly includes the possibility of taking a and b to be equal.) But if I pick a and b this way, then of course there cannot exist a c such that $a < c < b$. So the definition of interval is not satisfied. So: **according to this student's definition of "interval," the only set of real numbers that is a true interval is the empty set!**

So this definition of "interval" also gets a 0, but this time my reason is different. This time the definition does say something. But what it says is wrong! (Probably, the student did not ask himself/herself the question: "what am I saying here? What does this say in plain English?" You should always ask yourself that question!)

I am sure the student does not believe that the only set of real numbers that is an interval is the empty set. So what we have got here is not failure to know, but **failure to communicate**. This needs serious work, and it's you, the students, who have to do the work. I cannot teach you how to express your thoughts. I can help you, but most of the work has to be done by you. And it will require **hard work** on your part.

Students have been using letter symbols in ways that clearly violate the rules. **Please be careful and try not to do this again!**

Actually, the rules are quite simple, so you should not have any difficulty with them.

The first two rules are:

1. A letter symbol has to be either a constant or a variable and it cannot be both.

2. If a letter symbol is a constant, then it has to be introduced by declaring its value[3]. The value has to be an object that will remain fixed from then on. The object need not be known, but it has to be a definite object.

For example, here are some things you could say making a perfectly legitimate use of the letter symbols a, b:

Let $a = 5$. Then $a^2 = 25$. (7.9.17)

Let $a = 5$. Then $a^2 = 20$. (7.9.18)

Let a be an arbitrary real number. Then $(a+1)^2 = a^2 + 2a + 1$. (7.9.19)

Let a be an arbitrary real number. Then $(a+1)^2 = a^2 + 1$. (7.9.20)

Let a be a rational number such that $a^2 = 2$. Then $a^4 = 4$. (7.9.21)

[3]The "value" of a constant is the object for which it stands.

Let a be a rational number such that $a^2 = 2$. Then $a^4 = 3$. (7.9.22)

Let a be a real number such that $a^2 = 2$. Then $a^4 = 4$. (7.9.23)

Let a be a real number such that $a^2 = 2$. Then $a^4 = 3$. (7.9.24)

Let a and b be arbitrary real numbers. Then $(a + b)^2 = a^2 + 2ab + b^2$. (7.9.25)

Let a and b be arbitrary real numbers. Then $(a + b)^2 = a^2 + b^2$. (7.9.26)

(Here we are only discussing *grammatical correctness*, that is, agreement with the rules of mathematical grammar. The above examples are "right" in the same way as the sentence "Clinton was the fifth president of the United States" is "right," that is, in the sense of being grammatically right, even though it is clearly false. This sentence should be contrasted with something like "Clinton president fifth was," which is grammatically unacceptable. All we are saying here is that our seven examples are written with correct mathematical grammar. Whether what they say is true is a different problem altogether. In (7.9.17), (7.9.19), (7.9.21), (7.9.22), (7.9.23), and (7.9.25), the second sentence is true under the assumption made in the first sentence. On the other hand, in (7.9.18), (7.9.20), (7.9.24) and (7.9.26), the second sentence can be false[4] under the assumption made in the first sentence[5].)

Now that we know how to use constants, we have to explain how to use *variables*. Once again, the rules are quite simple:

3. Letter variables can be used in a statement only if they are "dummy variables."

We will discuss dummy variables in great detail later, but the basic meaning of the concept is this: a *dummy variable* within a statement is a letter used as a label (or "tag") to keep track of an object that occurs several times. The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing.

Here are seven examples of statements involving dummy variables.

For every x, if x is a natural number then $x \geq 0$. (7.9.27)

For some natural number x, $x^2 = 4$. (7.9.28)

[4]Why did I write "can be false" rather than "is false"?

[5]Make sure this is clear to you. You may find it strange that (7.9.22) is correct. The reason for this, however, is quite simple: *there are no rational numbers whose square is equal to* 2.

For some natural number x, $x^2 = 2$. (7.9.29)

If you take all the natural numbers n between 1 and 25, compute $(n+1)^2 - n^2$ for each such n, and add all the results, what you get is 676. (7.9.30)

If you take all the natural numbers n between 1 and 5, compute n^2 for each such n, and multiply all the results, what you get is $14,400$. (7.9.31)

If you take the function that to each x assigns x^2, and integrate it from 0 to 1, the result is $\frac{1}{3}$. (7.9.32)

The set of all the natural numbers n such that n is prime and $n > 2$ consists entirely of odd numbers. (7.9.33)

The above statements can be rewritten in a more formal way as follows:

$$\forall x, (x \in \mathbb{N} \longrightarrow x \geq 0). \tag{7.9.27.\#}$$

$$\exists x, (x \in \mathbb{N} \wedge x^2 = 4). \tag{7.9.28.\#}$$

$$\exists x, (x \in \mathbb{N} \wedge x^2 = 2). \tag{7.9.29.\#}$$

$$\sum_{n=1}^{25} \left((n+1)^2 - n^2\right) = 676\,. \tag{7.9.30.\#}$$

$$\prod_{n=1}^{5} n^2 = 14,400\,. \tag{7.9.31.\#}$$

$$\int_0^1 x^2\,dx = \tfrac{1}{3}\,. \tag{7.9.32.\#}$$

$$\{n : n \in \mathbb{N} \wedge n \text{ is prime} \wedge n > 2\} \subseteq \{n : n \in \mathbb{N} \wedge n \text{ is odd}\}\,. \tag{7.9.33.\#}$$

In addition, they could also be written using other letter variables. For example, we could say:

$$\forall m, (m \in \mathbb{N} \longrightarrow m \geq 0). \tag{7.9.27.$*$}$$

$$\exists q, (q \in \mathbb{N} \wedge q^2 = 4). \tag{7.9.28.$*$}$$

$$\exists n, (n \in \mathbb{N} \wedge n^2 = 2). \tag{7.9.29.$*$}$$

$$\sum_{k=1}^{25} \left((k+1)^2 - k^2\right) = 676\,. \tag{7.9.30.$*$}$$

$$\prod_{j=1}^{5} j^2 = 14,400\,. \qquad (7.9.31.*)$$

$$\int_0^1 z^2\,dz = \tfrac{1}{3}\,. \qquad (7.9.32.*)$$

$$\{r : r \in \mathbb{N} \wedge r \text{ is prime} \wedge r > 2\} \subseteq \{u : u \in \mathbb{N} \wedge u \text{ is odd}\}\,. \qquad (7.9.33.*)$$

Or we could go to the other extreme and say all these things without using letter symbols at all:

The square of every natural number is nonnegative. (7.9.27.&)

There exists a natural number whose square is equal to 4. (7.9.28.&)

There exists a natural number whose square is equal to 2. (7.9.29.&)

If you take all the natural numbers from 1 to 25, then compute for each such number the sum of the number plus 1, square this sum and subtract from the resulting square the square of the number itself, and, finally, add all the differences of squares obtained in this way, the result is 676. (7.9.30.&)

The product of the squares of all the natural numbers from 1 to 5 is equal to 14,400. (7.9.31.&)

The integral from 0 to 1 of the function that assigns to each real number its square is equal to $\frac{1}{3}$. (7.9.32.&)

The set of all natural numbers that are prime and greater than 2 is a subset of the set of all odd natural numbers. (7.9.33.&)

We have now finished listing all the ways in which letter symbols can be used. Nothing else is permitted.

So now you know. If you write something like "let x =cows," then you know for sure that it is not acceptable. (Why? Well, because when you say "let x ...," then the only possibility is that the letter x will be a constant. And if it is a constant then it has to stand for a definite object. And "cows" is not a definite object. A particular cow is a definite object, but "cows" isn't. The set of all cows

On the other hand, "Let x be a cow" would be acceptable, and "Let x be the set of all cows" would be acceptable.

)

Let us look at a second example. A student begins his solution of the "giraffes" problem by writing: "C: cow, S: sheep." This is completely unacceptable, and I am sure the student does not quite mean it, really. Does the student mean to say that he is going to use the letter C as an abbreviation for "cow"'? If

so, then the student, to be consistent, would have to write things like "x is a C" instead of "x is a cow." But that is not what the student did. What the student did was write expressions like "Cx". Now, if we take seriously the student's assertion that "C means "cow," then when the student wrote "Cx" what he is actually saying is "cow x." But I doubt very much that the student was trying to say that. I bet that what he was trying to say was "x is a cow." But, if this is the case, then why did he write "Cx" rather than "x is a C"?

8 Homework assignments nos. 1, 2 and 3

Before you start writing your homework, read carefully the rest of this handout, in particular §5 and §6, and §8, on "writing mathematics."

8.1 Homework assignment no. 1, due on Wednesday, September 12.

- Problems 1,2, 3, 4, 5, 6 in page 5 of the book.
- Is the number π even or odd? Justify your answer.
- Is ''$\mathbb{N}$ is a set of natural numbers" a correct statement?
- Is "$\mathbb{N}$ is a set of natural numbers" a correct answer to the question "What is $\mathbb{N}$?" t?

Here are some supplementary problems. Don't worry if some of these problems seem too hard. Just do as much as you can. One of the main purposes of this set of questions is to get you to experience what it is like to think about problems that could be quite hard (even if they look easy!), so that you will appreciate what it means to find a **truly good solution**. For you to see this, you have to think about the problems first, and if you cannot solve them **after you have tried to** and then you see the solution you will understand much better what a **real proof** is.

- Prove that every year must have a Friday the 13th.
- Prove that every year must have a Friday the 13th.
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- Express the number 28 as the difference of the squares of two natural numbers.
- Express the number 29 as the difference of the squares of two natural numbers.
- Express the number 30 as the difference of the squares of two natural numbers.

- (*Optional. This could be quite hard! Do not worry if you cannot do it, but at least try to* **think** *about it.*) Express the number $4,582,623$ as the sum of the squares of two natural numbers.

8.2 Assignment no. 2, due on Wednesday, September 19

- Section 2 of the book, Problems 1(b)(c), 2, 4, 7.
- Section 3 of the book, Problems 2, 5, 6, 10, 12, 13.
- Section 4 of the book, Problems 3, 5, 6, 7, 9.
- PROBLEM (N): A student gave the following "definition of 'composite number'": we say that a natural number n is *composite* if there exists a natural number b such that $1 < b < n$ and $n|b$. **Prove** that with this definition of "composite number" there do not exist any composite numbers.
- (This problem is harder.) Suppose two integers are sums of two squares of integers. Prove that the product of two integers must also be the sum of two squares of integers. (NOTE: here is an example of the kind of situation we are talking about. The number 13 is the sum of two squares, because $13 = 4 + 9$. The number 8 is the sum of two squares, because $8 = 4 + 4$. So, if our statement is true, it ought to be the case that the number 104—that is, 13×8—is the sum of two squares. And, indeed, it is, because $104 = 100 + 4$.)

8.3 Assignment no. 3, due on Wednesday, September 26

- Section 5 of the book, Problems 1(a)(d)(e), 3, 4, 8, 11(b)(c)(e), 12(b), 18.
- Section 6 of the book, Problems 2, 6, 7, 9(d)(f)(g), 12, 15.
- Here are five sentences about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than."

  (1) Cows are taller than sheep. (That is, "every cow is taller than every sheep," or, if you prefer, "for every x and every y, if x is a cow and y is a sheep, then x is taller than y.")

  (2) Giraffes are taller than cows.

  (3) If something is taller than something else which is in turn taller than a third thing, then the first one is taller than the third one. (That is "for every x, every y and every z, if x taller than y and y is taller than z, then x is taller than z.")

  (4) Olivia is a giraffe.

  (5) Dolly is a sheep.

Using sentences (1), (2), (3), (4), (5), *and no other fact about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than,"* prove that Olivia is taller than Dolly.

Here are four answers given by students to the question "express 30 as the difference of the square of two natural numbers":

1. This cannot be done because in all the ways of expressing 30 as the difference of two numbers none of them hold.
2. This cannot be done because there do not exist two natural numbers n and m such that $n^2 - m^2 = 30$.
3. This cannot be done because 30 is divisible by 6, and a number that is divisible by 6 cannot be expressed as a difference of two squares of natural numbers.
4. I have tried to express 30 as the difference of the squares of two natural numbers, but I wasn't able to figure out any way to do it. So I suspect that it cannot be done, but I do not know how to prove it.

Discuss, criticize, and compare these four answers. (In your critical analysis of these answers, do not hesitate to be harsh if you think harshness is appropriate. But do not criticize anything without giving *good arguments* for your criticism.) Which one (or which ones) of the four should in your opinion get the highest grade? Which one (or which ones) of the four should in your opinion get the lowest grade?

- To answer the question "Prove that every year must have a Friday the 13th," a student wrote: "There are 52 weeks in a year. Let x be the number of Fridays in a year. Then $x = 52$. Since $x > 13$, there has to be at least a Friday the 13th in every year." Discuss and criticize this answer. (Hint: imagine that the Earth has slowed down a little bit so that the year has 364 days rather than 365; imagine that a new calendar—the "Sussmannian calendar"—has been adopted, in which the year is divided into 13 months rather than 12, each month having exactly 28 days. In the Sussmannian calendar there are 52 weeks in each year, and 52 is of course > 13. Would it still be true in the Sussmannian calendar that every year has a Friday the 13th? If not, why not? Does the student's purported "argument" apply to the Sussmannian calendar? If not, then why would it apply for the "true" calendar but not for the Sussmannian calendar? Of course, you may object that the Sussmannian calendar is just a figment of our imagination, and has nothing to do with reality, because for the true calendar the year has 365 days, not 364, there are 12 months, not 13, and the months do not have 28 days but have variable lengths (31, 28, 31, 30, 31, 30, 31, 31, 30, 31, 30 ,31). In other words, the real calendar is **not** the Sussmannian calendar. But, where in the "proof" does the author **use** the fact that the real calendar is not the Sussmannian calendar?)