Needle Variations and Almost Lower Semicontinuous Differential Inclusions[1]

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1 Introduction and overview

The classical version of the "Pontryagin Maximum Principle," presented in the book [8], relies on the construction of "needle variations" at various

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times τ in the domain $[a,b]$ of the given reference trajectory ξ_*. The effects of these variations are then propagated to the terminal time b by means of the differentials $D\Phi_{b,\tau}(\xi_*(\tau))$ of the reference flow maps $\Phi_{b,\tau}$. This produces a convex cone (the "Pontryagin cone") which in some sense is a first-order approximation to the attainable set near the point $\xi_*(b)$. The proof is then concluded by means of a topological argument, based on some variant of the Brower fixed point theorem, to pass from the separation of two sets to the separation of their approximating cones.

The purpose of this note is to explain how the construction of needle variations can be carried out for *differential inclusions* rather than for the case of vector fields of class C^1 considered in [8]. This is part of a broader research program whose ultimate goal is to unify and generalize the various existing versions of the finite-dimensional maximum principle by developing a general "primal" technique based on mimicking the proof of [8], but modifying it by using *flows* instead of vector fields, *generalized differentials* instead of ordinary ones, and *abstract variations* instead of classical needle variations (cf. Sussmann [9, 10, 11, 12, 13, 14, 15, 16]).

A needle variation involves a *time-varying vector field* (abbr. TVVF) $\mathbb{R}^n \times \mathbb{R} \ni (x,t) \mapsto f(x,t) \in \mathbb{R}^n$. The "effect" of the variation at a point $(\bar{x},\bar{t})$ is a vector related to the differential $D\Phi^f(\bar{q})$ of the flow map Φ^f (cf. Definition 8 in §2.9 for the meaning of Φ^f) at the point $\bar{q} = (\bar{x},\bar{t},\bar{t})$. More precisely, it is easy to see that, if f is regular enough (for example, of class C^1 as a function of x and t) then Φ^f is differentiable at every point $\bar{q} = (\bar{x},\bar{t},\bar{s})$ of its domain. If $\bar{q}$ is of the special form $(\bar{x},\bar{t},\bar{t})$, then the differential $D\Phi^f(\bar{q})$ is the linear map

$$\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \ni (v,h,k) \mapsto \Lambda^w(v,h,k) = v + (h-k)w \in \mathbb{R}^n ,$$

where $w = f(\bar{x},\bar{t})$, because $\Phi^f(\bar{x}+v,\bar{t}+h,\bar{t}+k) \sim \bar{x}+v+(h-k)f(\bar{x},\bar{t})$ and $\bar{x} = \Phi^f(\bar{x},\bar{t},\bar{t})$, so that $\Phi^f(\bar{x}+v,\bar{t}+h,\bar{t}+k) - \Phi^f(\bar{x},\bar{t},\bar{t}) \sim v+(h-k)f(\bar{x},\bar{t})$. The vector w is then the "effect of the variation at $(\bar{x},\bar{t})$." (A detailed explanation of the notion of a needle variation, and why it is important to differentiate the flow at points of the form $(\bar{x},\bar{t},\bar{t})$, is given in §1.2 below.)

For the optimal control of a system $\dot{x} = f(x,u,t)$, $x \in \mathbb{R}^n$, $u \in U$, the TVVFs that are used to construct needle variations are of the special form $\mathbb{R}^n \times \mathbb{R} \ni (x,t) \mapsto f(x,\eta(t),t) \stackrel{\text{def}}{=} f_\eta(x,t)$, where $t \mapsto \eta(t)$ is some open-loop control, which is usually either the reference control or a constant control. For such vector fields, the argument given above to establish that $D\Phi^{f_\eta}(\bar{x},\bar{t},\bar{t}) = \Lambda^{f_\eta(\bar{x},\bar{t})}$ is not rigorous, even if the map $(x,u,t) \mapsto f(x,u,t)$ is very smooth, because η could be the reference control, which will typically not be continuous. It is possible, however, to render the argument rigorous for almost all $\bar{t}$, by invoking the Scorza-Dragoni theorem and using a notion of approximate continuity analogous to the concept of a Lebesgue point.

The question discussed here is how to carry out the construction of needle variations—and obtain flows that are differentiable at $(\bar{x}, \bar{t}, \bar{t})$ and have a prescribed effect w for various points $(\bar{x}, \bar{t})$—when, instead of using solutions of a differential equation $\dot{x} = f(x, u, t)$, we want to work with solutions of *differential inclusions* $\dot{x} \in F(x, t)$. A natural idea would be to use single-valued selections f of the set-valued map F, and then study the differentiability of the flows Φ^f. Ideally, one might hope that for every $(\bar{x}, \bar{t})$ and every $w \in F(\bar{x}, \bar{t})$ there exists a good enough selection $f = f_{\bar{x},\bar{t},w}$ such that $f(\bar{x}, \bar{t}) = w$. However, this line of attack leads to dead end, because

- set-valued maps $(x, t) \mapsto F(x, t)$ with nonconvex values typically do not admit continuous selections, even if they are very regular (for example continuous, or Lipschitz),

while, on the other hand,

- the application of the Scorza-Dragoni theorem requires that one deal with vector fields $(x, t) \mapsto f(x, t)$ that are continuous with respect to x and measurable with respect to t.

It turns out, remarkably, that this difficulty can be overcome by adopting a more sophisticated perspective and no longer requiring the selections to be continuous with respect to x. The first evidence that this approach might be feasible appeared in papers by Cambini and Querci [6], and Pucci [7], who studied some discontinuous "directionally continuous" vector fields that are as good as continuous ones as far as existence of solutions goes, and in addition have other desirable related properties such as upper semicontinuity of the flow. Subsequently, A. Bressan proved that lower semicontinuous set-valued maps with nonempty closed values admit selections in this class (cf. Bressan [1, 2, 3, 4, 5]).

In this paper, we pursue these ideas further, by exhibiting a natural class of discontinuous time-varying vector fields $(x, t) \mapsto f(x, t)$—that we call "admissible vector fields"—larger than that studied by Bressan, that has all the properties needed to make needle variations. The defining property of these vector fields is that the map that assigns to each curve $[a, b] \ni t \mapsto \xi(t)$ the indefinite integral $[a, b] \ni t \mapsto \int_a^t f(\xi(s), s)\, ds$ is continuous on a suitably large space $\mathcal{A}$ of curves. This condition is exactly what is required to make it possible to apply the Schauder fixed point theorem to establish existence of trajectories on small time intervals. (We prove existence in Theorem 3.) These vector fields have other useful properties, that we study in detail.

We prove that Bressan's directional continuity implies admissibility, but admissibility is a much more general property. In particular, admissible vector fields have the "measurable intertwining property" (cf. Theorem 1 below). This makes it possible to construct plenty of admissible selections for set-valued maps F that are "almost lower semicontinuous" (abbr. ALSC) by first constructing conically continuous selections of lower semicontinuous maps, and then intertwining them.

In order to be able to differentiate the flow Φ^f of an admissible vector field f at a point $(\bar{x}, \bar{t}, \bar{t})$, one needs f to be "approximately continuous" at $(\bar{x}, \bar{t})$. We prove that for ALSC maps there always exist many admissible selections with this extra property

Finally, it is well-known that differentiability of the flow in the most obvious sense (that is, existence of a linear map that approximates the map to first order) is not enough to make the topological arguments applicable. For example, the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = \frac{1}{n}$ if $n \in \mathbb{N}$, $n > 0$, and $\frac{1}{n} \leq x < \frac{1}{n-1}$, and $f(x) = x$ if $x \leq 0$, is differentiable at 0, and its derivative at 0 is equal to 1. However, it is not true that f maps neighborhoods of 0 to neighborhoods of 0. This illustrates the fact that in the open mapping theorem—that is, the statement that if $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ maps 0 to 0 and $Df(0)$ exists and is surjective then f maps neighborhoods of 0 to neighborhoods of 0—it is essential to assume that f is continuous near 0. Since this is already true for single-valued maps, something similar must be true for set-valued maps. That is, the set-valued analogue of the open mapping theorem, if it exists, cannot just involve the existence and surjectivity of the differential at 0, and must contain a supplementary condition, playing a similar role in the set-valued case as continuity does in the single-valued case. It turns out that the set-valued version of the open mapping theorem does exist, and the supplementary condition that plays the role of continuity is "regularity." (For a detailed account of this, cf., for example, Sussmann [11].)

Hence the differentiability property of the flow needed for the maximum principle for differential inclusions is *regular differentiability*, that is, the fact that the flow is both differentiable at $(\bar{x}, \bar{t}, \bar{t})$ and regular in a neighborhood of $(\bar{x}, \bar{t}, \bar{t})$. Our main result (Theorem 10 in §4.5) says precisely that *if a differential inclusion $\dot{x} \in F(x,t)$ in $\mathbb{R}^n$ has a right-hand side F with nonempty closed values, then for almost every $\bar{t}$ it is possible to make a needle variation using F at any point $\bar{x}$ generating the direction of any vector $\bar{y} \in F(\bar{x}, \bar{t})$, provided that F is almost lower semicontinuous and satisfies in addition a technical "locally integrable lower boundedness" condition.*

The use of the results of this paper to prove a maximum principle for differential inclusions is outlined in Sussmann [10]. The work carried out here is essentially that of filling in all the details for the half of the results of [10] that has to do with the needle variations. The other part—not considered here—is the one dealing with the differentiability of the reference flow along the reference trajectory. This requires a different set of tools, and will be the subject of another paper.

1.1 A brief outline of the logical structure of the paper

- As indicated above, our main result is Theorem 10 in §4.5, in which the set-valued map F is required to be "almost lower semicontinuous" and "locally integrably lower bounded."
- "Almost lower semicontinuity" is, essentially, the property that would follow from the condition that the set-valued map $x \mapsto F(x,t)$ is lower semicontinuous for each t, and the set-valued map $t \mapsto F(x,t)$ is measurable for each x, if the Scorza-Dragoni theorem was true for lower semicontinuous set-valued maps. Since the Scorza-Dragoni theorem is *not* true in this situation, its conclusion is turned into the definition of a new concept. This is done in Definition 19 in §4.3.
- "Local integrable lower boundedness" means that, locally, there exists an integrable function $t \mapsto \varphi(t)$ such that $\{y \in F(x,t) : \|y\| \leq \varphi(t)\}$ is nonempty, as explained in Definition 23 in §4.5.
- The conclusion of Theorem 10 is that for almost every $\bar{t}$ there exists, for each $\bar{x}$ and each $\bar{y} \in F(\bar{x},\bar{t})$, a single-valued selection f of F which is such that the flow map $(x,t,s) \mapsto \Phi^f(x,t,s)$ of the vector field f is "regularly differentiable" at $(\bar{x},\bar{t},\bar{t})$ and its differential is the linear map $(v,h,k) \mapsto v + (h-k)\bar{y} \stackrel{\text{def}}{=} \Lambda^{\bar{y}}(v,h,k)$.
- Flow maps are introduced in Definition 8 in §2.9.
- The meaning of "regular differentiability" is explained in Definition 4 in §2.5. Basically, a set-valued map is regularly differentiable at a point q if it is "regular" q and differentiable at q in the usual sense of admitting a first-order linear approximation.
- The notion of a "regular map" is reviewed in Definition 3 in §2.4. As explained above, it is a natural set-valued analogue of continuity of a single-valued map. (This is also discussed sketchily in §2.4—cf. especially Remark 3 in §2.4—and in much greater detail in, for example, Sussmann [11, 15, 16].) A regular map is a set-valued map that can be approximated in a particular way by ordinary continuous maps. The precise sense of the approximation is that of "inward graph convergence," introduced in Definition 2 in §2.5.
- The differentiability conclusion of Theorem 10 is derived from the other conclusions in conjunction with Theorem 6 of §3.9.

- The other conclusions of Theorem 10 assert the existence of an "admissible selection" f that has $(\bar{x},\bar{t})$ as a "point of approximate continuity" (abbr. PAC) and is such that $f(\bar{x},\bar{t})=\bar{y}$.
- Theorem 6 says that, if a vector field $(x,t)\mapsto f(x,t)$ is admissible and $(\bar{x},\bar{t})$ is a PAC of f, then the flow map $(x,t,s)\mapsto\Phi^f(x,t,s)$ is regularly differentiable at $(\bar{x},\bar{t},\bar{t})$ and its differential is $\Lambda^{\bar{y}}$.
- Theorem 6 arises from combining Theorem 3 of §3.5 with Theorem 4 of §3.7 and Theorem 5 of §3.8. Theorem 3 gives the local existence of solutions and upper semicontinuity of the flow, Theorem 4 gives the differentiability and Theorem 5 gives the regularity.
- The concept of a "point of approximate continuity" is presented in Definition 17 in §3.6. It is, roughly, a generalization to maps $(x,t)\mapsto f(x,t)$ of the usual notion of a Lebesgue point of a function $t\mapsto f(t)$. For a Lebesgue point $\bar{t}$ of a function $t\mapsto f(t)$, the bound $\int_{\bar{t}-h}^{\bar{t}+h}\|f(t)-f(\bar{t})\|\,dt=o(h)$ implies that the integral $t\mapsto\int_a^t f(s)\,ds$ is differentiable at $\bar{t}$ with derivative $f(\bar{t})$. Theorem 4 is proved by using a similar argument to establish differentiability of the flow at a point of approximate continuity and compute the differential of the flow.
- "Admissibility" is introduced in Definition 14 in §3.5. As explained above, it is exactly what is required to be able to apply a fixed point argument to establish existence of trajectories on small time intervals. The existence result is Theorem 3 (cf. also Remark 7 in §3.5).
- Remarkably, the admissibility condition has several other important consequences besides the existence of trajectories. One of them is the result of Theorem 5, which says that if $(x,t)\mapsto f(x,t)$ is an admissible vector field, then the flow map $(x,t,s)\mapsto\Phi^f(x,t,s)$ is regular on a sufficiently small neighborhood of any point $(\bar{x},\bar{t},\bar{t})$. This is proved by showing that the regularized vector fields f_ρ (obtained in the usual way, by convolution with smooth functions $x\mapsto\varphi_\rho(x)$ that converge to a Delta function as $\rho\downarrow 0$) give rise to flows Φ^{f_ρ} (which are continuous single-valued maps) that converge to the flow Φ^f (which in general is set-valued, because the equation $\dot{x}=f(x,t)$ need not have uniqueness of trajectories) in the sense of inward graph convergence.
- Theorem 10, on the existence of an admissible selection that has a given point $(\bar{x},\bar{t})$ as a PAC and takes a prescribed value $\bar{y}$ at $(\bar{x},\bar{t})$, is derived from Theorem 9 of §4.4, which gives the existence of selections that are "integrally continuous" on suitably large sets and satisfy the prescribed conditions involving $\bar{x}$, $\bar{t}$, and $\bar{y}$.
- Theorem 9 is derived from Theorem 8 in §4.4, together with the "measurable intertwining " Theorem 1 of §3.1 and Theorem 2 of §3.3.

- Theorem 8 gives the existence, for a sequence $\{J_k\}_{k=1}^{\infty}$ of pairwise disjoint compact subsets of $\mathbb{R}$ such that $\text{meas}\Big(\mathbb{R}\backslash(\cup_{k=1}^{\infty} J_k)\Big) = 0$, of a selection f that has, when t varies on each of the J_k, a "conic continuity" property. Using this, the argument leading from Theorem 8 to Theorem 9 proceeds as follows: conic continuity implies integral continuity by Theorem 2, and integral continuity on the individual sets J_k implies full integral continuity by Theorem 1.
- Theorem 8 is derived from Theorem 7 of §4.2, on the existence of conically continuous single-valued selections for lower semicontinuous set-valued maps. This result is essentially due to A. Bressan. The main technical point where we depart from Bressan's work is that we prove existence of conically continuous selections that in addition are *continuous* (rather than just directionally continuous) at a given point $(\bar{x},\bar{t})$, and have a prescribed value there. This technical improvement is the crucial step enabling us to get selections having a given point as a PAC, and prove differentiability of the flow.

1.2 Needle variations

One constructs *variations* for a triple (f,η_*,ξ_*) consisting of a control system $\dot{x}=f(x,u,t)$, $x\in\mathbb{R}^n$, $u\in U$, a "reference control" $[a,b]\ni t\mapsto\eta_*(t)\in U$, and a corresponding "reference trajectory" $[a,b]\ni t\mapsto\xi_*(t)\in\mathbb{R}^n$. This is done as follows. Starting with the flow $\Phi^*=\{\Phi^*_{t,s}\}_{a\le s\le t\le b}$ of the reference TVVF $(x,t)\mapsto f(x,\eta_*(t),t)$, one constructs a one-parameter family $\{\Phi^\varepsilon\}_{0\le\varepsilon\le\bar{\varepsilon}}$ of flows $\Phi^\varepsilon=\{\Phi^\varepsilon_{t,s}\}_{a\le s\le t\le b}$, defined for small nonnegative ε, such that $\Phi^0=\Phi^*$. (That is, one "draws a curve $\varepsilon\mapsto\Phi^\varepsilon$ in the space of flows, starting at Φ^* for $\varepsilon=0$.") *Needle variations* are obtained by doing this in a particular way: Φ^ε is the flow of the control η_ε obtained from η_* by replacing the value $\eta_*(t)$ by a constant control $\bar{u}$ on the interval $[\tau,\tau+\varepsilon]$, for some time $\tau\in[a,b[$. In other words, we replace the reference flow by the flow $\Psi=\{\Psi_{t,s}\}_{a\le s\le t\le b}$ of the time-varying vector field $(x,t)\mapsto f(x,\bar{u},t)$ on the interval $[\tau,\tau+\varepsilon]$. Equivalently, we define $\Phi^\varepsilon_{t,s}$ by

$$\Phi^\varepsilon_{t,s}=\begin{cases}\Phi^*_{t,s} & \text{if}\quad a\le s\le t\le\tau \text{ or } \tau+\varepsilon\le s\le t\le b\,,\\ \Psi_{t,s} & \text{if}\quad \tau\le s\le t\le\tau+\varepsilon\,,\\ \Psi_{t,\tau}\circ\Phi^*_{\tau,s} & \text{if}\quad a\le s\le\tau\le t\le\tau+\varepsilon\,,\\ \Phi^*_{t,\tau+\varepsilon}\circ\Psi_{\tau+\varepsilon,s} & \text{if}\quad \tau\le s\le\tau+\varepsilon\le t\le b\,,\\ \Phi^*_{t,\tau+\varepsilon}\circ\Psi_{\tau+\varepsilon,\tau}\circ\Phi^*_{\tau,t} & \text{if}\quad a\le s\le\tau \text{ and } \tau+\varepsilon\le t\le b\,.\end{cases}$$

As explained before, the differential at a point $(\bar{x},\bar{t})\in\mathbb{R}^n\times\mathbb{R}$ of the flow Φ^g of a vector field $\mathbb{R}^n\times\mathbb{R}\ni(x,t)\mapsto g(x,t)\in\mathbb{R}^n$ is the linear map

$\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \ni (v,h,k) \mapsto v + (h-k)g(\bar{x},\bar{t})$, if g is sufficiently regular. Applying this with $g(x,t) = f(x,\bar{u},t)$, and with $g(x,t) = f(x,\eta_*(t),t)$, we find that $\Psi_{\tau+\varepsilon,\tau}(\bar{x}) = \bar{x} + \varepsilon f(\xi_*(\tau),\bar{u},\tau) + o(\varepsilon + \|\bar{x}-\xi_*(\tau)\|)$ and also that $\Phi^*_{\tau+\varepsilon,\tau}(\bar{x}) = \bar{x} + \varepsilon f(\xi_*(\tau),\eta_*(\tau),\tau) + o(\varepsilon + \|\bar{x}-\xi_*(\tau)\|)$, so that

$$\begin{array}{rcl} \Phi^\varepsilon_{b,\tau}(\bar{x}) - \Phi^*_{b,\tau}(\bar{x}) & = & \Phi^*_{b,\tau+\varepsilon} \circ \Psi_{\tau+\varepsilon,\tau}(\bar{x}) - \Phi^*_{b,\tau+\varepsilon} \circ \Phi^*_{\tau+\varepsilon,\tau}(\bar{x}) \\ & \sim & D\Phi^*_{b,\tau+\varepsilon}(\bar{x}) \cdot \Big(\Psi_{\tau+\varepsilon,\tau}(\bar{x}) - \Phi^*_{\tau+\varepsilon,\tau}(\bar{x})\Big) \\ & \sim & D\Phi^*_{b,\tau}(\bar{x}) \cdot \Big(\Psi_{\tau+\varepsilon,\tau}(\bar{x}) - \Phi^*_{\tau+\varepsilon,\tau}(\bar{x})\Big) \\ & \sim & \varepsilon D\Phi^*_{b,\tau}(\bar{x}) \cdot w \,, \end{array}$$

where $w = f(\xi_*(\tau),\bar{u},\tau) - f(\xi_*(\tau),\eta_*(\tau),\tau)$. This gives the expansion

$$\Phi^\varepsilon_{b,\tau}(\bar{x}) \sim \Phi^*_{b,\tau}(\bar{x}) + \varepsilon D\Phi^*_{b,\tau}(\bar{x}) \cdot w \,, \tag{1.2.1}$$

showing that the difference between "following the modified flow starting at $\bar{x}$ from time τ to time b minus doing the same thing for the unmodified flow" is, to first order, equal to ε times the vector w propagated forward by the differential $D\Phi^*_{b,\tau}(\bar{x})$.

In the proof of the maximum principle, $\bar{x}$ might be $\xi_*(\tau)$, if we are applying our variation directly to the reference trajectory. But $\bar{x}$ might also be some other point close to $\xi_*(\tau)$ if, for example, another variation—at an earlier time $\tau' < \tau$, corresponding to a constant control $\bar{u}'$—has already been applied, and we want to study the combined effect of both variations.

In that case, we really have to deal with a two-parameter variation, and the terminal point map that we want to analyze is the map

$$x \mapsto \Phi^{\varepsilon_2,\varepsilon_1}_{b,\tau'}(x) = \Phi^*_{b,\tau+\varepsilon_2} \circ \Psi^{\bar{u}}_{\tau+\varepsilon_2,\tau} \circ \Phi^*_{\tau,\tau'+\varepsilon_1} \circ \Psi^{\bar{u}'}_{\tau'+\varepsilon_1,\tau'}(x) \,,$$

where, naturally, $\Psi^{\bar{u}}$ and $\Psi^{\bar{u}'}$ are the flows of the TVVFs $(x,t) \mapsto f(x,\bar{u},t)$ and $(x,t) \mapsto f(x,\bar{u}',t)$, respectively.

To get a linear approximation for $\Phi^{\varepsilon_2,\varepsilon_1}_{b,\tau'}(x)$, we use (1.2.1) taking $\varepsilon = \varepsilon_2$ and $\bar{x} = \Phi^{\varepsilon_2,\varepsilon_1}_{\tau,\tau'}(x) = \Phi^*_{\tau,\tau'+\varepsilon_1} \circ \Psi^{\bar{u}'}_{\tau'+\varepsilon_1,\tau'}(x)$. This gives

$$\Phi^{\varepsilon_2,\varepsilon_1}_{b,\tau'}(x) = \Phi^{\varepsilon_2}_{b,\tau}(\bar{x}) \sim \Phi^*_{b,\tau}(\bar{x}) + \varepsilon_2 D\Phi^*_{b,\tau}(\bar{x}) \cdot w \,, \tag{1.2.2}$$

while on the other hand, using (1.2.1) again, we find

$$\bar{x} \sim \Phi^*_{\tau,\tau'}(x) + \varepsilon_1 D\Phi^*_{\tau,\tau'}(x)(w') \,,$$

where $w' = f(\xi_*(\tau'), \bar{u}', \tau') - f(\xi_*(\tau'), \eta_*(\tau'), \tau')$. Therefore

$$\begin{aligned}\Phi^*_{b,\tau}(\bar{x}) &\sim \Phi^*_{b,\tau}\Big(\Phi^*_{\tau,\tau'}(x) + \varepsilon_1 D\Phi^*_{\tau,\tau'}(x)(w')\Big)\\ &\sim \Phi^*_{b,\tau}\Big(\Phi^*_{\tau,\tau'}(x)\Big) + \varepsilon_1 D\Phi^*_{b,\tau}\Big(\Phi^*_{\tau,\tau'}(x)\Big)\Big(D\Phi^*_{\tau,\tau'}(x)(w')\Big)\\ &\sim \Phi^*_{b,\tau'}(x) + \varepsilon_1 D\Phi^*_{b,\tau'}(x)(w')\,,\end{aligned}$$

using the flow identity $\Phi^*_{b,\tau}\circ = \Phi^*_{\tau,\tau'}\Phi^*_{b,\tau'}$ as well as the chain rule

$$D\Phi^*_{b,\tau}\Big(\Phi^*_{\tau,\tau'}(x)\Big)\circ D\Phi^*_{\tau,\tau'}(x) = D\Big(\Phi^*_{b,\tau}\circ\Phi^*_{\tau,\tau'}\Big)(x) = D\Phi^*_{b,\tau'}(x)\,.$$

If we substitute our approximate expression for $\Phi^*_{b,\tau}(\bar{x})$ into (1.2.2), observe that $D\Phi^*_{b,\tau}(\bar{x}) \sim D\Phi^*_{b,\tau}(\xi_*(\tau))$ (because $\bar{x}$ is close to $\xi_*(\tau)$), write $\tau_2 = \tau$, $\tau_1 = \tau'$, $x_2 = \xi_*(\tau_2)$, $x_1 = \xi_*(\tau_1)$, $u_2 = \bar{u}$, $u_1 = \bar{u}'$, and then define $w_i = f(x_i, u_i, \tau_i) - f(x_i, \eta_*(\tau_i), \tau_i)$ for $i = 1, 2$, we find that

$$\Phi^{\varepsilon_2,\varepsilon_1}_{b,\tau_1}(x_1) \sim \Phi^*_{b,\tau}(x_1) + \varepsilon_1 D\Phi^*_{b,\tau_1}(x_1)\cdot w_2 + \varepsilon_2 D\Phi^*_{b,\tau_2}(x_2)\cdot w_2\,. \qquad (1.2.3)$$

It is clear that (1.2.3) can be generalized to a formula expressing the combined effect of several variations. For our purposes in this paper, the crucial observation is that *for the flows Ψ^{u_i} that are used to define the variations, one needs approximate expressions not only for quantities such as $\Psi^{u_i}_{\tau_i+\varepsilon_i,\tau_i}(\xi_*(\tau_i))$, but also for $\Psi^{u_i}_{\tau_i+\varepsilon_i,\tau_i}(x)$ where x is close to $\xi_*(\tau_i)$*. The reason for this, as has been explained above, is that the point x to which the approximation will be applied will not in general lie on the reference trajectory ξ_* itself, due to the fact that x will already contain the cumulative effect of previously applied variations. In other words, *it is not enough to be able to differentiate flows $(x, t, s) \mapsto \Phi_{t,s}(x)$ with respect to the time variables t and s, for a fixed x. One has to be able to differentiate the flow at a point with respect to x, t, and s.*

Remark 1. The preceding explanation may suggest that one only needs to differentiate flow maps $(x, t, s) \mapsto \Phi_{t,s}(x)$ with respect to (x, t) for fixed s, since in our discussion of needle variations s (called τ or τ') was fixed. However, there are several reasons for wanting to consider the joint differentiability of the flow map with respect to x, t, and s. The simplest one is the fact that one may want to combine several "needle variations applied at the same time τ." For this purpose, it turns out to be convenient to perturb the reference flow by inserting different constant controls on non-overlapping intervals close to a point τ. For example, one may want to consider a two-parameter variation such that $\Phi^{\varepsilon_1,\varepsilon_2}_{b,\tau} = \Phi_{b,\tau+\varepsilon_1+\varepsilon_2}\circ\Psi^{u_2}_{\tau+\varepsilon_1+\varepsilon_2,\tau+\varepsilon_1}\circ\Psi^{u_1}_{\tau+\varepsilon_1,\tau}$.

It is easy to see that, to compute the effect of such a variation to first order, the flow map $(x,t,s) \mapsto \Psi^{u_2}_{t,s}(x)$ has to be differentiated jointly with respect to x, t, and s, at the point $(\xi_*(\tau),\tau,\tau)$. $\diamondsuit$

2 Notations and preliminary definitions

2.1 Numbers, intervals, metric spaces, neighborhoods

We use $\mathbb{N}$ to denote the set $\{0,1,2,\ldots\}$ of all nonnegative integers, and write $\overline{\mathbb{N}}$ to denote the set $\mathbb{N}\cup\{+\infty\}$. We use $\mathbb{R}$ to denote the set of real numbers. A *real interval* is an arbitrary connected subset of $\mathbb{R}$. We use the notations $[a,b]$, $]a,b[$, $[a,b[$, $]a,b]$ for the closed, open, left-closed right-open and right-closed left-open intervals with endpoints a,b. The expression (a,b) will always denote the ordered pair whose components are a and b. An interval is *nontrivial* if it contains more than one point. We write $\mathbb{R}_+ = [0,+\infty[$, $\overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty,+\infty\}$, $\overline{\mathbb{R}}_+ = \mathbb{R}_+\cup\{+\infty\} = [0,+\infty]$.

If X is a metric space, we use d_X to denote the distance function on X, and write $\mathbb{B}_X(x,r)$, $\overline{\mathbb{B}}_X(x,r)$, to denote, respectively, the open and closed balls with center x and radius r. We use $\mathrm{Comp}(X)$ to denote the set of all nonempty compact subsets of X. If $X=\mathbb{R}^n$, we write $\mathbb{B}^n(x,r)$, $\overline{\mathbb{B}}^n(x,r)$, rather than $\mathbb{B}_{\mathbb{R}^n}(x,r)$, $\overline{\mathbb{B}}_{\mathbb{R}^n}(x,r)$.

The word "neighborhood" will always be used in the standard sense of point-set topology: a *neighborhood* of a point a (resp. of a set S) in a topological space A is any set $U\subseteq A$ such that a is an interior point of U (resp. such that $S\subseteq \mathrm{Int}(U)$).

If X and Y are metric spaces, the product $X\times Y$ is equipped with the *product metric* $d_{X\times Y}$, given by

$$d_{X\times Y}\Big((x,y),(x',y')\Big) = d_X(x,x') + d_Y(y,y') \quad \text{if} \quad x,x'\in X\,,\ y,y'\in Y\,.$$

2.2 Set-valued maps

A *set-valued map* is a triple $F=(X,Y,G)$ such that X and Y are sets and G is a subset of $X\times Y$. If $F=(X,Y,G)$ is a set-valued map, then we refer to X, Y and G as the *source*, *target* and *graph* of F, and write $X=\mathrm{So}(F)$, $Y=\mathrm{Ta}(F)$, $G=\mathrm{Gr}(F)$. We write $F: X\mapsto\!\!\!\to Y$ to indicate that F is a set-valued map from X to Y, and we use $SVM(X,Y)$ to denote the set of all such set-valued maps, so "$F\in SVM(X,Y)$" is equivalent to "$F: X\mapsto\!\!\!\to Y$."

If $F=(X,Y,G)$ is a set-valued map, and x is any object, we write $F(x)\stackrel{\mathrm{def}}{=}\{y : (x,y)\in G\}$. More generally, if S is any set, we define

$F(S) = \bigcup_{x \in S} F(x)$. The *domain* $\mathrm{Dom}(F)$ is the set $\{x : F(x) \neq \emptyset\}$, and the *image* $\mathrm{Im}(F)$ of F is the set $F(\mathrm{So}(F))$. We call F *everywhere defined* if $\mathrm{Dom}(F) = \mathrm{So}(F)$, and say that F is *surjective* if $\mathrm{Im}(F) = \mathrm{Ta}(F)$. Then $\mathrm{Dom}(F) \subseteq \mathrm{So}(F)$ and $\mathrm{Im}(F) \subseteq \mathrm{Ta}(F)$, and F is everywhere defined iff $\mathrm{Dom}(F) = \mathrm{So}(F)$.

If X is a set, we use id_X to denote the identity map of X, i.e. the set-valued map $(X, X, \mathrm{Diag}(X))$, where $\mathrm{Diag}(X) = \{(x,x) : x \in X\}$.

If $F = (X, Y, G)$ is a set-valued map, and S is a subset of X, then the *restriction* $F\lceil S$ is the set-valued map $(X \cap S, Y, G \cap (S \times Y))$. Therefore $\mathrm{So}(F\lceil S) = X \cap S$, $\mathrm{Ta}(F\lceil S) = \mathrm{Ta}(F)$, and $(F\lceil S)(x) = F(x)$ when $x \in S$, while $(F\lceil S)(x) = \emptyset$ if $x \notin S$.

The *composite* $F = F_2 \circ F_1$ of two set-valued maps $F_i = (X_i, Y_i, G_i)$, $i = 1, 2$, is defined if and only if $Y_1 = X_2$. In that case, $F \stackrel{\mathrm{def}}{=} (X_1, Y_2, G_2 \circ G_1)$, where $G_2 \circ G_1 \stackrel{\mathrm{def}}{=} \{(x,z) : (\exists y)((x,y) \in G_1 \wedge (y,z) \in G_2\}$.

A set-valued map F is *single-valued* if $F(x)$ consists of a single point of $\mathrm{Ta}(F)$ for every $x \in \mathrm{Dom}(F)$. An *ordinary map* is a set-valued map that is single-valued and everywhere defined. We write $F : X \mapsto Y$ to indicate that F is an ordinary map with source X (so that $\mathrm{Dom}(F) = X$ as well) and target Y. Notice that according to these conventions the notation $F : X \mapsto\!\!\!\to Y$ allows for the possibility that F be multivalued, or partially defined, or both, but the notation $F : X \mapsto Y$ automatically implies that F is both single-valued and defined on the whole set X. A *selection* of a set-valued map $F : X \mapsto\!\!\!\to Y$ is a single-valued map $f : X \mapsto Y$ such that $f(x) \in F(x)$ for all $x \in X$. With our definitions, the Axiom of Choice implies that a set-valued map has selections iff it is everywhere defined.

2.3 Upper semicontinuous set-valued maps

If X and Y are topological spaces, a set-valued map $F : X \mapsto\!\!\!\to Y$ is *upper semicontinuous* (abbr. USC) if the inverse image under F of every closed subset of Y is closed in X. The following observation is then easily verified.

Fact 1. *Assume that X and Y are metric spaces, X is compact, and $F : X \mapsto\!\!\!\to Y$. Then* $\mathrm{Gr}(F)$ *is compact if and only if F is USC and $F(x)$ is compact for each $x \in X$.* $\diamondsuit$

Remark 2. For ordinary real-valued functions, the words "upper semicontinuous" will always refer to upper semicontinuity in the classical sense of real function theory. (That is, f is upper semicontinuous if the set $\{x : f(x) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$.) If we want to say that such an f

is USC in the set-valued sense, we will say that f is "upper semicontinuous as a set-valued map." $\diamondsuit$

2.4 Regular Set-Valued Maps

As was explained in the introduction, the classical proof of the maximum principle involves a topological argument, based on the Brouwer fixed point theorem. In order to carry out similar arguments for set-valued maps, one needs a good set-valued analogue of the notion of a continuous single-valued map. This is provided by the concept of a "regular map," which we now proceed to define. (It will be explained in Remark 3 why this notion is the right one.)

If X is a metric space and $L_1, L_2 \in \mathrm{Comp}(X)$, we define the "Hausdorff semidistance" $\Delta_X(L_1, L_2)$ from L_1 to L_2 by letting

$$\Delta_X(L_1, L_2) = \sup\Big\{\,\mathrm{dist}(x, L_2) : x \in L_1\,\Big\}\,. \tag{2.4.1}$$

This function is not in general symmetric. (For example, if $L_1 \subseteq L_2$ and $L_1 \neq L_2$, then $\Delta_X(L_1, L_2) = 0$ but $\Delta_X(L_2, L_1) > 0$.) On the other hand, Δ_X satisfies the "triangle inequality":

$$\Delta_X(L_1, L_3) \leq \Delta_X(L_1, L_2) + \Delta_X(L_2, L_3) \ \text{ if } \ L_1, L_2, L_3 \in \mathrm{Comp}(X)\,. \tag{2.4.2}$$

Definition 1. Let X be a metric space, let $L \in \mathrm{Comp}(X)$, and let $\{L_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathrm{Comp}(X)$. We say that L_j *inward converges* to L if $\lim_{j\to\infty} \Delta_X(L_j, L) = 0$. $\diamondsuit$

If K, Y are metric spaces, and K is compact, we write $SVM_{comp}(K, Y)$ to denote the set of all set-valued maps $F : K \mapsto\!\!\!\!\rightarrow Y$ whose graph is compact and nonempty. In view of our definitions, it is clear that $SVM_{comp}(K, Y)$ is exactly the same as the set $\{K\} \times \{Y\} \times \mathrm{Comp}(K \times Y)$, so $SVM_{comp}(K, Y)$ can be canonically identified with $\mathrm{Comp}(K \times Y)$.

Definition 2. Let K, Y be metric spaces such that K is compact. Let $F \in SVM_{comp}(K, Y)$, and let $\{F_j\}_{j\in\mathbb{N}}$ be a sequence in $SVM_{comp}(K, Y)$. We say that $\{F_j\}_{j\in\mathbb{N}}$ *inward graph converges to F as $j \to \infty$* if the graphs $\mathrm{Gr}(F_j)$ inward converge to $\mathrm{Gr}(F)$ in the space $\mathrm{Comp}(K \times Y)$, i.e., if $\lim_{j\to\infty} \Delta_{K\times Y}(\mathrm{Gr}(F_j), \mathrm{Gr}(F)) = 0$. We write $F_j \xrightarrow{\mathrm{igr}} F$ to indicate that the sequence $\{F_j\}_{j\in\mathbb{N}}$ inward graph converges to F as $j \to \infty$. $\diamondsuit$

Definition 3. Let K, Y be metric spaces such that K is compact. A *regular set-valued map from K to Y* is a set-valued map $F \in SVM_{comp}(K, Y)$ which is a limit in $SVM_{comp}(K,Y)$ of a sequence $\{F_j\}_{j\in\mathbb{N}}$ of ordinary—i.e., single-valued and everywhere defined—continuous maps from K to Y.

We use $\mathrm{REG}(K,Y)$ to denote the set of all regular set-valued maps from K to Y. $\diamondsuit$

Remark 3. Regular maps have good fixed point properties analogous to those of single-valued continuous maps. For example, *a regular set-valued map* $F:\mathbb{B}^n(0,1)\mapsto\mathbb{B}^n(0,1)$ *must have a fixed point.* (Proof: let $\{F_j\}_{j\in\mathbb{N}}$ be a sequence of continuous maps from $\mathbb{B}^n(0,1)$ to $\mathbb{B}^n(0,1)$ such that $F_j\stackrel{\mathrm{igr}}{\to}F$. For each j pick x_j such that $F_j(x_j)=x_j$. Then $(x_j,x_j)\in\mathrm{Gr}(F_j)$. Assume, after passing to a subsequence if necessary, that $x=\lim x_j$ exists. Pick $(x_j',y_j)\in Gr(F)$ closest to (x_j,x_j). Then $\|x_j'-x_j\|+\|y_j-x_j\|\to 0$. So $(x_j',y_j)\to(x,x)$. Therefore $(x,x)\in\mathrm{Gr}(F)$, because $Gr(F)$ is compact.) $\diamondsuit$

2.5 Regular differentiability

Definition 4. Assume that $n,m\in\mathbb{N}$, $F:\mathbb{R}^n\mapsto\!\!\!\to\mathbb{R}^m$, C is a closed convex cone in $\mathbb{R}^n$, and $\bar{x}\in\mathbb{R}^n$. We say that F is *regularly differentiable at $\bar{x}$ in the direction of C* if

(D4.1) there exists a compact neighborhood N of $\bar{x}$ in $\mathbb{R}^n$ such that the restriction $F\lceil(N\cap(\bar{x}+C))$ is a regular set-valued map from $N\cap(\bar{x}+C)$ to $\mathbb{R}^m$;

(D4.2) $F(\bar{x})$ consists of a single point $\bar{y}\in\mathbb{R}^m$;

(D4.3) there exists a linear map $L:\mathbb{R}^n\mapsto\mathbb{R}^m$ such that

$$\lim_{h\to 0\,,\,h\in C}\frac{\sup\Big\{\|y-\bar{y}-L\cdot h\|:y\in F(\bar{x}+h)\Big\}}{\|h\|}=0\,. \qquad (2.5.1)$$

If L is a linear map such that (2.5.1), then L is said to be a *differential of F at $\bar{x}$ in the direction of C.* $\diamondsuit$

2.6 Essential measurability

If $n\in\mathbb{N}$, we use $\mathrm{Bor}(\mathbb{R}^n)$, $\mathrm{Leb}(\mathbb{R}^n)$ to denote, respectively, the σ-algebras of Borel and Lebesgue measurable subsets of $\mathbb{R}^n$. We use $\mathcal{BL}(\mathbb{R}^n,\mathbb{R})$ to denote the σ-algebra whose members are the Borel$\otimes$Lebesgue-measurable subsets of $\mathbb{R}^n\times\mathbb{R}$, that is, the σ-algebra of subsets of $\mathbb{R}^n\times\mathbb{R}$ generated by all the products $A\times B$ such that $A\in\mathrm{Bor}(\mathbb{R}^n)$ and $B\in\mathrm{Leb}(\mathbb{R})$. If $S\subseteq\mathbb{R}^n\times\mathbb{R}$, we use $\mathcal{BL}(\mathbb{R}^n,\mathbb{R};S)$ to denote the class of subsets of S that belong to $\mathcal{BL}(\mathbb{R}^n, \mathbb{R})$. Also, we use $\mathrm{Bor}(\mathbb{R}^n \times \mathbb{R}) \cap S$, $\mathcal{BL}(\mathbb{R}^n, \mathbb{R}) \cap S$, to denote, respectively, the σ-algebras $\{E \cap S : E \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})\}$ and $\{E \cap S : E \in \mathcal{BL}(\mathbb{R}^n, \mathbb{R})\}$.

We use $\mathrm{meas}(E)$ to denote the Lebesgue measure of a measurable subset E of $\mathbb{R}$. A *null subset of* $\mathbb{R}$ is a subset E of $\mathbb{R}$ such that $\mathrm{meas}(E) = 0$. A *full subset of* $\mathbb{R}$ is a subset F of $\mathbb{R}$ such that $\mathbb{R} \backslash F$ is a null set.

If $S \subseteq X \times \mathbb{R}$ for some set X, and $J \subseteq \mathbb{R}$, we write

$$S_J \stackrel{\mathrm{def}}{=} S \cap (X \times J) = \{(s,t) \in S : t \in J\}\,. \tag{2.6.1}$$

Definition 5. Let $n, m \in \mathbb{N}$, and let S be a subset of $\mathbb{R}^n \times \mathbb{R}$. We say that S is *essentially measurable* if there exists a full subset J of $\mathbb{R}$ such that $S_J \in \mathcal{BL}(\mathbb{R}^n, \mathbb{R})$. We use $\mathcal{EM}(\mathbb{R}^n, \mathbb{R})$ to denote the set of essentially measurable subsets of $\mathbb{R}^n \times \mathbb{R}$. If $S \subseteq \mathbb{R}^n \times \mathbb{R}$, we use $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}; S)$ to denote the set of all subsets of S that belong to $\mathcal{EM}(\mathbb{R}^n, \mathbb{R})$, and write $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$ to denote the set of all sets of the form $E \cap S$, $E \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$. ♢

It is clear that $\mathcal{EM}(\mathbb{R}^n, \mathbb{R})$ is a σ-algebra, and that, if $S \subseteq \mathbb{R}^n \times \mathbb{R}$, then $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}; S)$ is a σ-algebra of subsets of S if and only if $S \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$, in which case $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}; S) = \mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$. Moreover, the following inclusions hold

$$\mathrm{Bor}(\mathbb{R}^n \times \mathbb{R}) \subseteq \mathcal{BL}(\mathbb{R}^n, \mathbb{R}) \subseteq \mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \subseteq \mathrm{Leb}(\mathbb{R}^n \times \mathbb{R})\,,$$

and they are all strict if $n > 0$.

Remark 4. *A subset S of $\mathbb{R}^n \times \mathbb{R}$ is essentially measurable if and only if there exists a full Borel subset J of $\mathbb{R}$ such that* $S_J \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$. (In other words, "essential Borel⊗Lebesgue-measurability is equivalent to essential Borel⊗Borel-measurability.") Indeed, let Σ be the set of all subsets S of $\mathbb{R}^n \times \mathbb{R}$ such that $S_J \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$ for some full $J \in \mathrm{Bor}(\mathbb{R})$. Then Σ is obviously a σ-algebra, and $\Sigma \subseteq \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$. Clearly, every set S of the form $E \times F$, with $E \in \mathrm{Bor}(\mathbb{R}^n)$ and $F \in \mathrm{Leb}(\mathbb{R})$ belongs to Σ. Therefore $\mathcal{BL}(\mathbb{R}^n, \mathbb{R}) \subseteq \Sigma$. If $S \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R}; S)$, then we can pick a full subset J of $\mathbb{R}$ such that $S \cap (\mathbb{R}^n \times J) \in \mathcal{BL}(\mathbb{R}^n, \mathbb{R})$. Then $S \cap (\mathbb{R}^n \times J) \in \Sigma$, from which it follows that $\big(S \cap (\mathbb{R}^n \times J)\big) \cap (\mathbb{R}^n \times L) \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$ for some full $L \in \mathrm{Bor}(\mathbb{R})$. If $M \in \mathrm{Bor}(\mathbb{R})$ is such that $M \subseteq J$ and $\mathrm{meas}(J \backslash M) = 0$, and we let $N = M \cap L$, then N is a full Borel subset of $\mathbb{R}$, and

$$S_N = \bigg(\Big(S \cap (\mathbb{R}^n \times J)\Big) \cap (\mathbb{R}^n \times L)\bigg) \cap (\mathbb{R}^n \times M) \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})\,,$$

so $S \in \Sigma$. Therefore $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \subseteq \Sigma$. So $\Sigma = \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$. ♢

Definition 6. Let $n, m \in \mathbb{N}$, and let S be a subset of $\mathbb{R}^n \times \mathbb{R}$. A map $f : S \mapsto \mathbb{R}^m$ is said to be *essentially measurable on* S if $f^{-1}(B)$ belongs to $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}; S)$ for every Borel subset of $\mathbb{R}^m$. ♢

Fact 2. *If $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, and $f : S \mapsto \mathbb{R}^m$, then f is essentially measurable if and only if there exists a full Borel subset J of $\mathbb{R}$ such that S_J belongs to* $\mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$ *and the restriction $f\lceil S_J$ of f to S_J is Borel measurable.*

Proof. Assume that f is essentially measurable. Let $\mathcal{A}$ be a countable set of Borel subsets of $\mathbb{R}^m$ that generates the σ-algebra $\mathrm{Bor}(\mathbb{R}^m)$. For each $A \in \mathcal{A}$, the fact that $f^{-1}(A) \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$, together with Remark 4, imply that we can choose a full Borel subset J_A of $\mathbb{R}$ such that $f^{-1}(A) \cap (\mathbb{R}^n \times J_A)$ belongs to $\mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$. Let $J = \bigcap_{A \in \mathcal{A}} J_A$. Then J is a full Borel subset of $\mathbb{R}$, and

$$f^{-1}(A) \cap (\mathbb{R}^n \times J) = \Big(f^{-1}(A) \cap (\mathbb{R}^n \times J_A)\Big) \cap (\mathbb{R}^n \times J) \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$$

for every $A \in \mathcal{A}$. Therefore $f^{-1}(B) \cap (\mathbb{R}^n \times J) \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$ for every $B \in \mathrm{Bor}(\mathbb{R}^m)$. So $f\lceil S_J$ is Borel measurable. This proves one of the two implications of our statement. The other implication is trivial. ♢

2.7 Curves, arcs, and spaces of arcs

Definition 7. Let $n \in \mathbb{N}$. A *curve* in $\mathbb{R}^n$ is a continuous map $\xi : I \mapsto \mathbb{R}^n$ whose domain $I = \mathrm{Dom}(\xi)$ is a nonempty real interval. An *arc* in $\mathbb{R}^n$ is a curve ξ in $\mathbb{R}^n$ whose domain is a compact subinterval $[a_\xi, b_\xi]$ of $\mathbb{R}$. We use $ARC(\mathbb{R}^n)$ to denote the set of all arcs in $\mathbb{R}^n$, regarded as a metric space with the distance $d_{ARC} : ARC(\mathbb{R}^n) \times ARC(\mathbb{R}^n) \mapsto \mathbb{R}_+$ given by

$$d_{ARC}(\xi, \eta) = |a_\xi - a_\eta| + |b_\xi - b_\eta| + \sup\Big\{\|\tilde{\xi}(t) - \tilde{\eta}(t)\| : t \in \mathbb{R}\Big\},$$

where, for $\zeta \in ARC(\mathbb{R}^n)$, $\tilde{\zeta}$ denotes the extension of the arc ζ to a map $\tilde{\zeta} : \mathbb{R} \mapsto \mathbb{R}^n$ given by $\tilde{\zeta}(t) = \zeta\Big(\min(\max(t, a_\zeta), b_\zeta)\Big)$ for $t \in \mathbb{R}$.

If S is a subset of $\mathbb{R}^n \times \mathbb{R}$, an *arc in* S is a $\xi \in ARC(\mathbb{R}^n)$ such that $(\xi(t), t) \in S$ for all $t \in \mathrm{Dom}(\xi)$. We use $ARC(\mathbb{R}^n; S)$ to denote the set of all arcs in S, regarded as a subspace of the metric space $ARC(\mathbb{R}^n)$. ♢

It is clear that convergence in $ARC(\mathbb{R}^n)$ is *uniform convergence*, in the following precise sense:

(C_{ARC}) *If $\xi \in ARC(\mathbb{R}^n)$, then a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ of arcs belonging to $ARC(\mathbb{R}^n)$ converges to ξ in $ARC(\mathbb{R}^n)$ if and only if*

(i) $a_{\xi_j} \to a_\xi$ and $b_{\xi_j} \to b_\xi$ *as* $j \to \infty$,

and

(ii) whenever $\{t_j\}_{j\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that $a_{\xi_j} \le t_j \le b_{\xi_j}$ for every $j \in \mathbb{N}$, and $t \in \mathbb{R}$ is such that $t_j \to t$ as $j \to \infty$ (so that $a_\xi \le t \le b_\xi$), it follows that $\xi_j(t_j) \to \xi(t)$ as $j \to \infty$.

It is also clear that $ARC(\mathbb{R}^n)$ is a complete metric space, and $ARC(\mathbb{R}^n; S)$ is a closed subset of $ARC(\mathbb{R}^n)$ if and only if S is a closed subset of $\mathbb{R}^n \times \mathbb{R}$.

If $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$, and $\xi \in ARC(\mathbb{R}^n; S)$, then $f \circ \xi$ denotes the map $[a_\xi, b_\xi] \ni t \to f(\xi(t), t) \in \mathbb{R}^m$.

Fact 3. *If $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, and f is $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$-measurable, then $f \circ \xi$ is measurable for every member ξ of $ARC(\mathbb{R}^n; S)$.*

Proof. Let $\xi : [a, b] \mapsto \mathbb{R}^n$ be continuous and such that $(\xi(t), t) \in S$ for each $t \in [a, b]$. Let $K = \{(\xi(t), t) : a \le t \le b\}$, so K is a compact subset of S.

If B is a Borel subset of $\mathbb{R}^m$, then $f^{-1}(B) \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$, so $f^{-1}(B) = A \cap S$ for some $A \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$. Then

$$\begin{aligned}(f \circ \xi)^{-1}(B) &= \{t \in [a, b] : (\xi(t), t) \in f^{-1}(B)\}\\ &= \{t \in [a, b] : (\xi(t), t) \in f^{-1}(B) \cap K\}\\ &= \Xi^{-1}\Big(f^{-1}(B) \cap K\Big),\end{aligned}$$

where Ξ is the map $[a, b] \ni t \mapsto (\xi(t), t) \in \mathbb{R}^n \times \mathbb{R}$. But

$$\Xi^{-1}(f^{-1}(B) \cap K) = \Xi^{-1}(A \cap S \cap K) = \Xi^{-1}(A \cap K)\,,$$

since $K \subseteq S$. Clearly, $A \cap K \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$. So it suffices to show that $\Xi^{-1}(E)$ is a Lebesgue measurable subset of $[a, b]$ whenever E belongs to $\mathcal{EM}(\mathbb{R}^n, \mathbb{R})$.

Now, if $E \in \mathcal{EM}(\mathbb{R}^n, \mathbb{R})$, then there exists a full Borel subset J of $\mathbb{R}$ such that E_J is Borel measurable. Then

$$\Xi^{-1}(E) = \Xi^{-1}(E_J) \cup \{t \in J \cap [a, b] : \Xi(t) \in E\}\,,$$

so $\Xi^{-1}(E)$ is Lebesgue measurable if $\Xi^{-1}(E_J)$ is Lebesgue measurable. So it suffices to show that the set $\Xi^{-1}(E)$ is Lebesgue measurable whenever $E \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$. This conclusion will be true for all $E \in \mathrm{Bor}(\mathbb{R}^n \times \mathbb{R})$ provided it is true for all E of the form $F \times G$, $F \in \mathrm{Bor}(\mathbb{R}^n)$, $G \in \mathrm{Bor}(\mathbb{R})$. But, if $F \in \mathrm{Bor}(\mathbb{R}^n)$ and $G \in \mathrm{Bor}(\mathbb{R})$, then $\Xi^{-1}(F \times G) = \xi^{-1}(F) \cap G$, so $\Xi^{-1}(F \times G)$ is actually Borel measurable, completing our proof. $\diamondsuit$

2.8 Points of density and Lebesgue points

If $\varphi : \mathbb{R} \mapsto \overline{\mathbb{R}}$ is a measurable function, a *Lebesgue point* of φ is a point $t \in \mathbb{R}$ such that $|\varphi(t)| < \infty$ and

$$\lim_{h\to 0+} \frac{1}{h}\int_{[t-h,t+h]} \Big|\varphi(s) - \varphi(t)\Big|\, ds = 0\,. \tag{2.8.1}$$

A *point of density* of a measurable subset $E \subseteq \mathbb{R}$ is a point of E that is a Lebesgue point of the indicator function[5] χ_E of E. Equivalently, a real number t is a point of density of E if and only if $t \in E$ and

$$\lim_{h\downarrow 0} \frac{1}{2h}\mathrm{meas}\Big(E \cap [t-h, t+h]\Big) = 1\,. \tag{2.8.2}$$

It is well known that if $\varphi : \mathbb{R} \mapsto \overline{\mathbb{R}}$ is locally integrable then almost every point of $\mathbb{R}$ is a Lebesgue point of φ. In particular, almost every point of a measurable subset E of $\mathbb{R}$ is a point of density of E. It follows that if $E \subseteq \mathbb{R}$ is measurable and $\mathrm{meas}(E) > 0$, then E has a point of density.

2.9 Trajectories and flows

Definition 8. Let $n \in \mathbb{N}$, and let $F : \mathbb{R}^n \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}^n$ be a set-valued map.

- (D8.i) A *trajectory* of F (or a "trajectory of the differential inclusion $\dot{x} \in F(x,t)$") is a locally absolutely continuous map $\xi : I \mapsto \mathbb{R}^n$, whose domain $\mathrm{Dom}(\xi)$ is a nonempty subinterval I of $\mathbb{R}$, such that $\dot{\xi}(t) \in F(\xi(t), t)$ for almost every $t \in I$.
- (D8.ii) We use $\mathrm{Traj}\,(F)$ to denote the set of all trajectories of F, and $\mathrm{Traj}_c(F)$ to denote the set of all $\xi \in \mathrm{Traj}\,(F)$ whose domain is a compact interval.
- (D8.iii) If V is a subset of $\mathbb{R}^n$, we write $\mathrm{Traj}_c(F, V)$ to denote the set of all $\xi \in \mathrm{Traj}_c(F)$ such that $(\xi(t), t) \in V$ for all $t \in \mathrm{Dom}(\xi)$.
- (D8.iv) A *maximal trajectory* of F is a trajectory $\xi : I \mapsto \mathbb{R}^n$ that cannot be extended to a trajectory of F defined on a strictly larger interval.
- (D8.v) For $(x, t, s) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, we define

$$\Phi^F(x, t, s) = \{\xi(t) : \xi \in \mathrm{Traj}\,(F)\,,\ \xi(s) = x\}\,. \tag{2.9.1}$$

The set-valued map $\Phi^F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}^n$ is called the *flow* of F.

[5] The *indicator function* of E is the function whose value is 1 on E and 0 outside E.

(D8.vi) For $(t,s) \in \mathbb{R} \times \mathbb{R}$, we define a set-valued map $\Phi^F_{t,s} : \mathbb{R}^n \mapsto\!\!\!\to \mathbb{R}^n$ by letting

$$\Phi^F_{t,s}(x) \stackrel{\text{def}}{=} \Phi^F(x,t,s) \qquad \text{for} \quad x \in \mathbb{R}^n\,. \tag{2.9.2}$$

The set-valued maps $\Phi^F_{t,s} : \mathbb{R}^n \mapsto\!\!\!\to \mathbb{R}^n$ are the *flow maps* of F. ♢

The sets $\mathrm{Traj}_c(F)$ are subsets of the metric space $ARC(\mathbb{R}^n)$ defined in §2.7.

With our definitions, if $F : \mathbb{R}^n \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}^n$, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $I = \{t\}$, and $\xi : I \mapsto \mathbb{R}^n$ is given by $\xi(t) = x$, then ξ is a trajectory of F, even if $(x,t) \notin \mathrm{Dom}(F)$. Therefore the identity

$$\Phi^F_{t,t} = \mathrm{id}_{\mathbb{R}^n} \tag{2.9.3}$$

holds. In addition, the flow maps $\Phi^F_{t,s}$ also satisfy the *flow identity*

$$\Phi^F_{t,s} \circ \Phi^F_{s,r} = \Phi^F_{t,r} \qquad \text{if} \quad r \le s \le t\,. \tag{2.9.4}$$

Remark 5. *The equality of (2.9.4) may fail to be true if it is not true that* $r \le s \le t$. For example, if F is a time-varying vector field on $\mathbb{R}^n$ (i.e., a single-valued map $F : \mathbb{R}^n \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}^n$) that does not have unique trajectories, $a < b$, and $\xi_1, \xi_2 : [a,b] \mapsto \mathbb{R}^n$ are two trajectories of $\dot{x} = F(x,t)$ that satisfy $\xi_1(a) = \xi_2(a)$ and $\xi_1(b) \ne \xi_2(b)$, then $\xi_2(b)$ belongs to $(\Phi^F_{b,a} \circ \Phi^F_{a,b})(\xi_1(b))$, so $\Phi^F_{b,a} \circ \Phi^F_{a,b} \ne \mathrm{id}_{\mathbb{R}^n} = \Phi^F_{b,b}$. ♢

Zorn's Lemma easily implies that

Fact 4. *If $n \in \mathbb{N}$ and $F : \mathbb{R}^n \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}^n$, then every trajectory of the differential inclusion $\dot{x} \in F(x,t)$ can be extended to a maximal trajectory.* ♢

The following is an immediate consequence of Fact 4.

Fact 5. *If $n \in \mathbb{N}$ and $F : \mathbb{R}^n \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}^n$, then for every $(x,t) \in \mathbb{R}^n \times \mathbb{R}^n$ there exists a maximal trajectory ξ of $\dot{x} \in F(x,t)$ such that $\xi(t) = x$.* ♢

Naturally, it may turn out that $\mathrm{Dom}(\xi)$ is just $\{t\}$.

3 Discontinuous vector fields and their flows

3.1 Integrally continuous time-varying maps

Definition 9. Assume that $n, m \in \mathbb{N}$, S is a subset of $\mathbb{R}^n \times \mathbb{R}$, and $f : S \mapsto \mathbb{R}^m$. An *integral bound* for f is a Lebesgue integrable function $\mathbb{R} \ni t \mapsto \varphi(t) \in [0, +\infty]$ such that

$$\|f(x,t)\| \le \varphi(t) \ \text{ for all } \ (x,t) \in S\,. \tag{3.1.1}$$

We say that f is *integrably bounded* if there exists an integral bound for f. We say that f is *locally integrably bounded* (abbr. LIB) if for every compact subset K of S the restriction of f to K is integrably bounded.

A set $\mathcal{F}$ of maps from S to $\mathbb{R}^m$ is *uniformly LIB* if every $f \in \mathcal{F}$ is LIB and, in addition, the integral bounds for the restriction of f to K can be chosen, for each compact subset K of S, independently of f. ♢

Definition 10. Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$, and $\mathcal{A} \subseteq ARC(\mathbb{R}^n; S)$. We say that f is *integrable along arcs on* $\mathcal{A}$, or *arc-integrable on* $\mathcal{A}$, if $f \circ \xi$ is Lebesgue integrable for every $\xi \in \mathcal{A}$. ♢

Fact 6. *If $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$, and f is $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$-measurable and LIB, then f is arc-integrable on $ARC(\mathbb{R}^n; S)$.*

Proof. If $\xi \in ARC(\mathbb{R}^n; S)$, then Fact 3 implies that $f \circ \xi$ is measurable, and the integral bound (3.1.1) implies that $f \circ \xi$ is integrable. ♢

If f is arc-integrable on $\mathcal{A}$, we can define a map $\mathcal{T}_f^{\mathcal{A}} : \mathcal{A} \mapsto ARC(\mathbb{R}^m)$—the *integral operator associated to f on $\mathcal{A}$*—by letting $\mathcal{T}_f^{\mathcal{A}}(\xi)$ be, for each $\xi \in \mathcal{A}$, the arc $\eta : [a_\xi, b_\xi] \mapsto \mathbb{R}^m$ given by

$$\eta(t) = \int_{a_\xi}^{t} f(\xi(s), s)\, ds\,. \tag{3.1.2}$$

Definition 11. Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$, and $\mathcal{A} \subseteq ARC(\mathbb{R}^n; S)$. We say that f is *integrally continuous on* $\mathcal{A}$ if

(IC.1) f is $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$-measurable,
(IC.2) f is arc-integrable on $\mathcal{A}$,
(IC.3) the map $\mathcal{T}_f^{\mathcal{A}} : \mathcal{A} \mapsto ARC(\mathbb{R}^m)$ is continuous. ♢

We also define maps $\mathcal{T}_{f,E}^{\mathcal{A}} : \mathcal{A} \mapsto \mathbb{R}^m$, if E is a measurable subset of $\mathbb{R}$, by

$$\mathcal{T}_{f,E}^{\mathcal{A}}(\xi) \stackrel{\text{def}}{=} \int_{E \cap \mathrm{Dom}(\xi)} f(\xi(s), s)\, ds \quad \text{for} \quad \xi \in \mathcal{A}\,, \tag{3.1.3}$$

and maps $\mathcal{T}_{f,a,b}^{\mathcal{A}} : \mathcal{A} \mapsto \mathbb{R}^m$, if $a, b \in \mathbb{R}$ and $a \le b$, by

$$\mathcal{T}_{f,a,b}^{\mathcal{A}}(\xi) \stackrel{\text{def}}{=} \int_{[a,b] \cap \mathrm{Dom}(\xi)} f(\xi(s), s)\, ds \quad \text{for} \quad \xi \in \mathcal{A}\,. \tag{3.1.4}$$

We observe that

$$\mathcal{T}_{f,a,b}^{\mathcal{A}}(\xi) = \mathcal{T}_f^{\mathcal{A}}(\xi)\Big(\lambda(\xi, b)\Big) - \mathcal{T}_f^{\mathcal{A}}(\xi)\Big(\lambda(\xi, a)\Big) \quad \text{for} \quad \xi \in \mathcal{A}\,, \tag{3.1.5}$$

where $\lambda(\xi, s) \stackrel{\text{def}}{=} \min\Big(\, \max(a_\xi, s)\,,\, b_\xi\,\Big)\,.$

Proposition 1. *Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $\mathcal{A}$ is a subset of $ARC(\mathbb{R}^n; S)$, and f is $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$-measurable and LIB. Then Condition (IC.3) is equivalent to the following "strong integral continuity condition":*

(SIC) *for every $E \in \mathrm{Leb}(\mathbb{R})$, the map $\mathcal{T}^{\mathcal{A}}_{f,E} : \mathcal{A} \mapsto \mathbb{R}^m$ is continuous,*

as well as to the "weak integral continuity condition":

(WIC) *the map $\mathcal{T}^{\mathcal{A}}_{f,a,b} : \mathcal{A} \mapsto \mathbb{R}^m$ is continuous whenever $a, b \in \mathbb{R}$ and $a \le b$.*

Proof. To prove the equivalence of (IC.3), (SIC) and (WIC), we first observe that the implication (SIC)⇒(WIC) is trivial, and the implication (IC.3)⇒(WIC) follows immediately from (3.1.5).

Next, we consider a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ of arcs ξ_j that belong to $\mathcal{A}$ and converge to a $\xi_\infty \in \mathcal{A}$. Then $a_{\xi_j} \to a_{\xi_\infty}$ and $b_{\xi_j} \to b_{\xi_\infty}$ as $j \to \infty$. In addition, it is easy to see that the set

$$K = \Big\{ (x,t) : (\exists j \in \overline{\mathbb{N}})\Big(t \in \mathrm{Dom}(\xi_j) \wedge x = \xi_j(t) \Big) \Big\}$$

is a compact subset of S. Therefore there exists an integrable function $\varphi_K : \mathbb{R} \mapsto [0,\infty]$ such that the bound $\|\dot{\xi}_j(t)\| \le \varphi_K(t)$ holds for every pair (j,t) such that $j \in \overline{\mathbb{N}}$ and $t \in \mathrm{Dom}(\xi_j)$.

Fix real numbers α, β such that $\alpha \le a_{\xi_j}$ and $\beta \ge b_{\xi_j}$ for all $j \in \overline{\mathbb{N}}$. Define maps $\theta_j, \eta_j : [\alpha,\beta] \mapsto \mathbb{R}^m$ by letting

$$\begin{aligned}
\theta_j(t) &= \begin{cases} f(\xi_j(t),t) & \text{if} \quad t \in [a_{\xi_j}, b_{\xi_j}]\,, \\ 0 & \text{if} \quad t \in [\alpha,\beta] \setminus [a_{\xi_j}, b_{\xi_j}]\,, \end{cases} \\
\eta_j(t) &= \int_\alpha^t \theta_j(s)\,ds\,.
\end{aligned}$$

Then the θ_j satisfy $\|\theta_j(t)\| \le \varphi_K(t)$ for all t, j, and therefore the η_j are absolutely continuous and satisfy the bound $\|\dot{\eta}_j(t)\| \le \varphi_K(t)$ for all t, j. Therefore the sequence $\{\eta_j\}_{j\in\mathbb{N}}$ is uniformly bounded and equicontinuous. Moreover, $\eta_j(b) - \eta_j(a) = \mathcal{T}^{\mathcal{A}}_{f,a,b}(\xi_j)$ for all j and all $a, b \in [\alpha,\beta]$ such that $a \le b$. If (WIC) holds, it follows that the η_j converge pointwise to η_∞, and the equicontinuity implies that the convergence is uniform on $[\alpha,\beta]$. In particular, if $a_{\xi_j} \le t_j \le b_{\xi_j}$ and $t_j \to t$ as $j \to \infty$, then $\eta_j(t_j) \to \eta_\infty(t)$, showing that $\mathcal{T}^{\mathcal{A}}_f(\xi_j) \to \mathcal{T}^{\mathcal{A}}_f(\xi_\infty)$ in $ARC(\mathbb{R}^m)$. This proves the implication (WIC)⇒(IC.3).

Furthermore, given any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that $\int_G \varphi_K < \varepsilon$ whenever G is a measurable subset of $[\alpha,\beta]$ of measure $< \delta$. If E is an arbitrary measurable subset of $\mathbb{R}$, and $\varepsilon > 0$, then we can let $\tilde{E} = E \cap [\alpha, \beta]$, choose $\delta = \delta(\frac{\varepsilon}{3})$, and find a subset F of $[\alpha, \beta]$ which is a finite union of pairwise disjoint intervals and is such that the symmetric difference $\tilde{E}\Delta F$ has measure $< \delta$. Then $|\int_{\tilde{E}} \theta_j - \int_F \theta_j| \le \frac{\varepsilon}{3}$ for all j. If (WIC) holds, then $\int_F \theta_j \to \int_F \theta_\infty$, so we can find a $j_0 \in \mathbb{N}$ such that $|\int_F \theta_j - \int_F \theta_\infty| \le \frac{\varepsilon}{3}$ for $j \ge j_0$, and then $|\int_{\tilde{E}} \theta_j - \int_{\tilde{E}} \theta_\infty| \le \varepsilon$ for $j \ge j_0$, from which it clearly follows that $\mathcal{T}^{\mathcal{A}}_{f,E}(\xi_j) - \mathcal{T}^{\mathcal{A}}_{f,E}(\xi_\infty) \le \varepsilon$ for $j \ge j_0$. This proves the implication (WIC)$\Rightarrow$(SIC).

We have thus established that (WIC)$\Rightarrow$(SIC)$\Rightarrow$(WIC)$\Rightarrow$(IC.3)$\Rightarrow$(WIC), which completes the proof of the equivalence of (IC.3), (SIC) and (WIC).$\diamondsuit$

The equivalence of (SIC) and (IC.3) implies the following *measurable intertwining property*:

Theorem 1. *Assume that $n, m \in \mathbb{N}$ and $S \subseteq \mathbb{R}^n \times \mathbb{R}$. Let $\{f_j\}_{j\in\mathbb{N}}$ be a uniformly LIB sequence of $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$-measurable maps $S \mapsto \mathbb{R}^m$. Let $\mathcal{A} \subseteq ARC(\mathbb{R}^n; S)$ be such that the f_j are integrally continuous on $\mathcal{A}$. Let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint Lebesgue measurable subsets of $\mathbb{R}$ such that $\mathbb{R} = \bigcup_{j\in\mathbb{N}} E_j$. Define a map $f : S \mapsto \mathbb{R}^m$ by letting*

$$f(x,t) = f_j(x,t) \quad \text{for} \quad j \in \mathbb{N}\,,\ (x,t) \in S\,,\ t \in E_j\,. \tag{3.1.6}$$

Then f is integrally continuous on $\mathcal{A}$.

Proof. The $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap S$-measurability of f is obvious, and it is clear that f is LIB, so f is arc-integrable on $ARC(\mathbb{R}^n; S)$ (cf. Fact 6). Let $\{\xi_\ell\}_{\ell\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ that converges to a $\xi_\infty \in \mathcal{A}$. Let E be a measurable subset of $\mathbb{R}$. Let K be a compact subset of S such that $(\xi_\ell(t), t) \in K$ for all $\ell \in \overline{\mathbb{N}}$ and all $t \in \mathrm{Dom}(\xi_\ell)$. Let α, β be real numbers such that $\alpha \le t \le \beta$ whenever $(x,t) \in K$. Let $\varphi : \mathbb{R} \mapsto [0, +\infty]$ be integrable and such that $\|f_j(x,t)\| \le \varphi(t)$ whenever $(x,t) \in K$ and $j \in \mathbb{N}$, and $\|f(x,t)\| \le \varphi(t)$ whenever $(x,t) \in K$.

Let $\varepsilon > 0$. Let $\delta > 0$ be such that $\int_H \varphi \le \varepsilon$ whenever $H \subseteq \mathbb{R}$ is measurable and $\mathrm{meas}(H) \le \delta$. Let j^* be such that

$$\sum_{j>j^*} \mathrm{meas}(E \cap E_j \cap [\alpha, \beta]) \le \delta\,.$$

Using (SIC), we find that for each j

$$\int_{E\cap E_j\cap \mathrm{Dom}(\xi_\ell)} f \circ \xi_\ell \to \int_{E\cap E_j \cap \mathrm{Dom}(\xi_\infty)} f \circ \xi_\infty \ \text{ as } \ \ell \to \infty\,. \tag{3.1.7}$$

Therefore, if we let $\tilde{E} = E \cap (E_0 \cup E_1 \cup \ldots \cup E_{j^*})$, we see that

$$\int_{\tilde{E}\cap \mathrm{Dom}(\xi_\ell)} f\circ\xi_\ell \to \int_{\tilde{E}\cap \mathrm{Dom}(\xi_\infty)} f\circ\xi_\infty \ \text{ as } \ \ell\to\infty\,, \tag{3.1.8}$$

since $\tilde{E} = \cup_{j\le j^*}(E\cap E_j)$ and the sets E_j are pairwise disjoint.

On the other hand, $\mathrm{meas}((E\backslash\tilde{E})\cap[\alpha,\beta]) \le \delta$, and $\mathrm{Dom}(\xi_\ell)\subseteq[\alpha,\beta]$. Therefore

$$\left\|\int_{(E\backslash\tilde{E})\cap \mathrm{Dom}(\xi_\ell)} f\circ\xi_\ell\right\| \le \varepsilon \ \text{ for } \ \ell\in\overline{\mathbb{N}}\,. \tag{3.1.9}$$

It follows that

$$\limsup_{\ell\to\infty}\left\|\int_{E\cap \mathrm{Dom}(\xi_\ell)} f\circ\xi_\ell - \int_{E\cap \mathrm{Dom}(\xi_\infty)} f\circ\xi_\infty\right\| \le 2\varepsilon\,. \tag{3.1.10}$$

Since this is true for every $\varepsilon>0$, we see that

$$\int_{E\cap \mathrm{Dom}(\xi_\ell)} f\circ\xi_\ell \to \int_{E\cap \mathrm{Dom}(\xi_\infty)} f\circ\xi_\infty\,,$$

and our proof is complete. ♢

3.2 Conic continuity

Let $n\in\mathbb{N}$, $S\subseteq\mathbb{R}^n\times\mathbb{R}$. If $\psi:S\mapsto\overline{\mathbb{R}}_+$ is a function, we use $ARC_\psi(\mathbb{R}^n;S)$ to denote the set of all absolutely continuous $\xi\in ARC(\mathbb{R}^n;S)$ such that $\|\dot{\xi}(t)\|\le\psi(\xi(t),t)$ for almost all $t\in\mathrm{Dom}(\xi)$. Also, we write $\widetilde{ARC}_\psi(\mathbb{R}^n;S)$ to denote the (much larger) set of all $\xi\in ARC(\mathbb{R}^n;S)$ such that

$$\limsup_{t\downarrow\bar{t}}\frac{\|\xi(t)-\xi(\bar{t})\|}{t-\bar{t}} \le \psi(\xi(\bar{t}),\bar{t}) \ \text{ for a.e. } \ \bar{t}\in\mathrm{Dom}(\xi)\,. \tag{3.2.1}$$

If $C>0$, and $(\bar{x},\bar{t})\in\mathbb{R}^n\times\mathbb{R}$, we write

$$\begin{aligned}
\Gamma_C(\bar{x},\bar{t}) &\stackrel{\mathrm{def}}{=} \Big\{(x,t)\in\mathbb{R}^n\times\mathbb{R} : t\ge\bar{t}\,,\ \|x-\bar{x}\|\le C(t-\bar{t})\Big\}\\
\Gamma_C^2(\bar{x},\bar{t}) &\stackrel{\mathrm{def}}{=} \Big\{(x,x',t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}: (x,t)\in\Gamma_C(\bar{x},\bar{t}),(x',t)\in\Gamma_C(\bar{x},\bar{t})\Big\}\,.
\end{aligned}$$

Definition 12. Let $n,m\in\mathbb{N}$, $S\subseteq\mathbb{R}^n\times\mathbb{R}$, $f:S\mapsto\mathbb{R}^m$. Let $C>0$, and let $(\bar{x},\bar{t})$ be a point of S.

1. We say that f is *forward* Γ_C*-continuous* at $(\bar{x},\bar{t})$ if

$$\lim_{(x,t)\to(\bar{x},\bar{t})\,,\,(x,t)\in\Gamma_C(\bar{x},\bar{t})\cap S} f(x,t) = f(\bar{x},\bar{t})\,. \tag{3.2.2}$$

2. We call f *weakly forward* Γ_C*-continuous* at $(\bar{x},\bar{t})$ if

$$\lim_{(x,x',t)\to(\bar{x},\bar{x},\bar{t})\,,\,(x,x',t)\in\Gamma^2_C(\bar{x},\bar{t})\cap\tilde{S}} \Big(f(x,t) - f(x',t)\Big) = 0\,, \tag{3.2.3}$$

   where $\tilde{S} = \{(x,x',t) : (x,t) \in S \wedge (x',t) \in S\}$.

If $S \ni (x,t) \mapsto C(x,t) \in \overline{\mathbb{R}}_+$ is a function, we say that the map f is *forward* Γ_C*-continuous* (resp. *weakly forward* Γ_C*-continuous*) if it is forward (resp. weakly forward) $\Gamma_{C(\bar{x},\bar{t})}$-continuous at every point $(\bar{x},\bar{t}) \in S$ such that $C(\bar{x},\bar{t}) < \infty$. ♢

Remark 6. The forward Γ_C-continuity property of Definition 12 is a "conic continuity" condition, because it says that $f(x,t)$ approaches $f(\bar{x},\bar{t})$ when (x,t) approaches (x,t) along the cone $\Gamma_C(\bar{x},\bar{t})$ in $\mathbb{R}^n \times \mathbb{R}$. ♢

3.3 Conic continuity implies integral continuity

If $S \subseteq \mathbb{R}^n \times \mathbb{R}$, and $t \in \mathbb{R}$, we write S_t to denote the set

$$S_t \stackrel{\text{def}}{=} \{\, x \in \mathbb{R}^n : (x,t) \in S\}\,.$$

Theorem 2. *Let $n,m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$. Let C, ψ be $\mathbb{R}_+$-valued functions on S such that,*

$$\text{for a.e. } t \in \mathbb{R}\,,\ \ \psi(x,t) < C(x,t) < \infty \ \text{ for all } x \in S_t\,. \tag{3.3.1}$$

Assume that f is $\mathcal{EM}(\mathbb{R}^n,\mathbb{R}) \cap S$-measurable, locally integrably bounded, and weakly forward Γ_C-continuous. Then f is integrally continuous on the set $\widetilde{ARC}_\psi(\mathbb{R}^n;S)$.

Proof. It follows from Fact 6 that f is arc-integrable on $ARC(\mathbb{R}^n;S)$, since f is LIB and $\mathcal{EM}(\mathbb{R}^n,\mathbb{R}) \cap S$-measurable.

To prove integral continuity, we use Proposition 1, and show that (SIC) holds. We will actually prove the stronger fact that

(#) *if $\{\xi_j\}_{j\in\mathbb{N}}$ is a sequence in $ARC(\mathbb{R}^n;S)$ that converges in $ARC(\mathbb{R}^n)$ to a limit $\xi_\infty \in \widetilde{ARC}_\psi(\mathbb{R}^n;S)$, then*

$$\int_{E\cap\mathrm{Dom}(\xi_j)} f(\xi_j(s),s)\,ds \to \int_{E\cap\mathrm{Dom}(\xi_\infty)} f(\xi_\infty(s),s)\,ds \tag{3.3.2}$$

for every measurable subset E of $\mathbb{R}$.

For this purpose, we fix a sequence $\{\xi_j\}_{j\in\mathbb{N}}$ in $ARC(\mathbb{R}^n;S)$ that converges in $ARC(\mathbb{R}^n)$ to a $\xi_\infty \in \widetilde{ARC}_\psi(\mathbb{R}^n;S)$. We write $a_j = a_{\xi_j}$, $b_j = b_{\xi_j}$, $I_j = \mathrm{Dom}(\xi_j) = [a_j, b_j]$, for $j \in \overline{\mathbb{N}}$.

Since $\xi_j \to \xi_\infty$ in $ARC(\mathbb{R}^n;S)$, the sequences $\{a_j\}_{j\in\mathbb{N}}$ and $\{b_j\}_{j\in\mathbb{N}}$ converge to the limits a_∞, b_∞, respectively. So we can pick a compact interval $J = [a,b]$ such that $a \le a_j \le b_j \le b$ for all $j \in \overline{\mathbb{N}}$. It is then clear that it suffices to prove that (3.3.2) holds for all measurable subsets of J, since $\int_{E\cap I_j} f(\xi_j(s),s)\,ds = \int_{(E\cap J)\cap I_j} f(\xi_j(s),s)\,ds$ for all $j \in \overline{\mathbb{N}}$.

For each ε such that $\varepsilon > 0$, let $\mathcal{E}(\varepsilon)$ be the set of all measurable $E \subseteq J$ having the property that there exists an $N = N(\varepsilon, E) \in \mathbb{N}$ such that

$$\Big\| \int_{E\cap I_j} f\circ\xi_j - \int_{E\cap I_\infty} f\circ\xi_\infty \Big\| \le \varepsilon\,\mathrm{meas}(E)$$

for $j \ge N$. It is clear that $\mathcal{E}(\varepsilon)$ is closed under finite disjoint unions.

Now fix ε and a measurable subset E of J, and let $\mathcal{H}(\varepsilon,E)$ denote the collection of all subsets $\mathcal{U}$ of $\mathcal{E}(\varepsilon)$ such that

(i) every member U of $\mathcal{U}$ is a measurable subset of J such that $U \subseteq E$ and $\mathrm{meas}(U) > 0$,

(ii) $U_1 \cap U_2 = \emptyset$ whenever $U_1, U_2 \in \mathcal{U}$ and $U_1 \ne U_2$.

It is clear that every $\mathcal{U} \in \mathcal{H}(\varepsilon,E)$ is finite or countable, since the members of $\mathcal{U}$ are pairwise disjoint measurable subsets of J of strictly positive measure. Moreover, it is clear that every increasing union of members of $\mathcal{H}(\varepsilon,E)$ is in $\mathcal{H}(\varepsilon,E)$. Finally, $\mathcal{H}(\varepsilon,E) \ne \emptyset$, since $\emptyset \in \mathcal{H}(\varepsilon,E)$. It then follows from Zorn's Lemma that there is a $\mathcal{U} \in \mathcal{H}(\varepsilon,E)$ which is maximal with respect to inclusion.

We now show that

$$\sum_{U\in\mathcal{U}} \mathrm{meas}(U) = \mathrm{meas}(E)\,. \tag{3.3.3}$$

Suppose that (3.3.3) does not hold. Let

$$\hat{U} = \bigcup\{U : U \in \mathcal{U}\}\,.$$

Then $\hat{U}$ is measurable and satisfies

$$\hat{U} \subseteq E \qquad \text{and} \qquad \mathrm{meas}(\hat{U}) = \sum_{U\in\mathcal{U}} \mathrm{meas}(U) < \mathrm{meas}(E)\,, \tag{3.3.4}$$

because the members of $\mathcal{U}$ are pairwise disjoint measurable subsets of E.

Let $V = E \backslash \hat{U}$. Then $\mathrm{meas}(V) > 0$ and $V \subseteq E$. Let $V' = V \backslash I_\infty$. We claim that

$$\mathrm{meas}(V') = 0\,. \tag{3.3.5}$$

Indeed, assume that $\mathrm{meas}(V') > 0$. Then there exist a compact interval L such that $L \cap I_\infty = \emptyset$ and $\mathrm{meas}(V' \cap L) > 0$. Since $a_j \to a_\infty$ and $b_j \to b_\infty$, there must exist a $j^* \in \mathbb{N}$ such that $L \cap I_j = \emptyset$ whenever $j \in \overline{\mathbb{N}}$ and $j \geq j^*$. Then $\int_{V' \cap L \cap I_j} f \circ \xi_j = 0$ if $j \in \overline{\mathbb{N}}$ and $j \geq j^*$. Therefore

$$\lim_{j \to \infty} \int_{V' \cap L \cap I_j} f \circ \xi_j = \int_{V' \cap L \cap I_\infty} f \circ \xi_\infty\,.$$

Hence $V' \cap L \in \mathcal{E}(\varepsilon)$. Since $V' \cap U = \emptyset$ whenever $U \in \mathcal{U}$, $V' \cap L \subseteq E$, and $\mathrm{meas}(V' \cap L) > 0$, the set $\mathcal{U} \cup \{V' \cap L\}$ also belongs to $\mathcal{H}(\varepsilon, E)$, contradicting the maximality of $\mathcal{U}$. This contradiction shows that (3.3.5) holds.

Now let $V_1 = V \cap I_\infty$. Then $V = V' \cup V_1$, so (3.3.5) implies that $\mathrm{meas}(V_1) > 0$. Let V_2 be the set of all $t \in \mathbb{R}$ such that

$$a_\infty < t < b_\infty\,,$$

$$\psi(x,t) < C(x,t) < \infty \qquad \text{for all} \quad x \in S_t$$

and

$$\limsup_{t' \to t+} \frac{\|\xi_\infty(t') - \xi_\infty(t)\|}{t' - t} \leq \psi(\xi_\infty(t), t)\,.$$

It then follows from (3.3.1) and the fact that $\xi_\infty \in \widetilde{ARC}_\psi(\mathbb{R}^n; S)$ that $\mathrm{meas}(I_\infty \backslash V_2) = 0$, so $\mathrm{meas}(V_1 \cap V_2) = \mathrm{meas}(V_1) > 0$. We can therefore pick a point of density $\bar{t}$ of $V_1 \cap V_2$.

Let $\bar{x} = \xi_\infty(\bar{t})$. Set $\bar{C} = C(\bar{x}, \bar{t})$, $\bar{B} = \psi(\bar{x}, \bar{t})$, and choose D such that $\bar{B} < D < \bar{C}$. Using the weak $\Gamma_{\bar{C}}$-continuity of f at $(\bar{x}, \bar{t})$, find a $\delta > 0$ such that

$$\begin{aligned} &\Big(\bar{t} < t \leq \bar{t} + \delta \wedge \|x - \bar{x}\| \leq C(t - \bar{t}) \wedge \|x' - \bar{x}\| \leq C(t - \bar{t})\Big) \\ &\qquad \Longrightarrow \quad \|f(x,t) - f(x',t)\| \leq \varepsilon\,. \end{aligned} \tag{3.3.6}$$

By making δ smaller, if necessary, we may assume that

$$a_\infty < \bar{t} - \delta < \bar{t} + \delta < b_\infty\,, \tag{3.3.7}$$

$$\mathrm{meas}\Big(V_1 \cap V_2 \cap [\bar{t} - \delta, \bar{t} + \delta]\Big) \geq \frac{3}{2}\delta\,, \tag{3.3.8}$$

and

$$\|\xi_\infty(t) - \xi_\infty(\bar t)\| \le D(t-\bar t) \ \text{ whenever } \ \bar t < t \le \bar t + \delta\,. \tag{3.3.9}$$

Let $\rho = \frac{\delta}{3}$. Pick $N \in \mathbb{N}$ such that

$$\|\xi_j(t) - \xi_\infty(t)\| \le (C-D)\rho \qquad \text{for} \quad \bar t \le t \le \bar t + \delta\,,\ j \in \mathbb{N}\,,\ j \ge N\,.$$

Then, if $j \in \overline{\mathbb{N}}$, $j \ge N$ and $\bar t + \rho \le t \le \bar t + \delta$, we have

$$\begin{aligned} \|\xi_j(t) - \bar x\| &\le \|\xi_j(t) - \xi_\infty(t)\| + \|\xi_\infty(t) - \bar x\| \\ &\le (C-D)\rho + D(t-\bar t) \\ &\le C(t-\bar t)\,. \end{aligned}$$

Therefore

$$\|f(\xi_j(t),t) - f(\xi_\infty(t),t)\| \le \varepsilon \ \text{ whenever } \bar t + \rho \le t \le \bar t + \delta\,,\ j \in \mathbb{N}\,,\ j \ge N\,.$$

So, if $j \ge N$, the inequality

$$\int_{V_1\cap V_2\cap[\bar t+\rho,\bar t+\delta]} \|f(\xi_j(t),t) - f(\xi_\infty(t),t)\|\,dt \le \varepsilon\,\mathrm{meas}(V_1\cap V_2\cap[\bar t+\rho,\bar t+\delta])\,.$$

holds. On the other hand,

$$\mathrm{meas}(V_1\cap V_2\cap[\bar t+\rho,\bar t+\delta]) > 0\,, \tag{3.3.10}$$

because if $\mathrm{meas}(V_1\cap V_2\cap[\bar t+\rho,\bar t+\delta]) = 0$ then we would have

$$\begin{aligned} \mathrm{meas}(V_1\cap V_2\cap[\bar t-\delta,\bar t+\delta]) &= \mathrm{meas}(V_1\cap V_2\cap[\bar t-\delta,\bar t+\rho]) \\ &\le \delta+\rho \\ &= \frac{4\delta}{3}\,, \end{aligned}$$

contradicting (3.3.8). So, if we define $W = V_1\cap V_2\cap[\bar t+\rho,\bar t+\delta]$, we see that $W \in \mathcal{E}(\varepsilon)$ and $\mathrm{meas}(W) > 0$. Moreover, $W\cap U = \emptyset$ whenever $U \in \mathcal{U}$. Therefore $\mathcal{U}\cup\{W\} \in \mathcal{H}(\varepsilon,E)$, contradicting the maximality of $\mathcal{U}$. So (3.3.3) is proved.

Given any $\delta > 0$, (3.3.3) implies that we can find a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that

$$\sum_{U\in\mathcal{V}} \mathrm{meas}(U) \ge \mathrm{meas}(E) - \delta\,. \tag{3.3.11}$$

If we let $V = \bigcup\{U : U \in \mathcal{V}\}$, we see that $V \in \mathcal{E}(\varepsilon)$. So we have shown that

(A) *If E is a measurable subset of J and $\varepsilon > 0$, $\delta > 0$, then there is a measurable subset V of E such that* $\mathrm{meas}(E\backslash V) \leq \delta$ *and*

$$\Big\| \int_V f \circ \xi_j - \int_V f \circ \xi_\infty \Big\| \leq \varepsilon \,\mathrm{meas}(V)$$

for all sufficiently large j.

Now let

$$K = \bigcup_{j \in \overline{\mathbb{N}}} \Big\{ (\xi_j(t), t) : a_j \leq t \leq b_j \Big\} .$$

Then K is a compact subset of S. Using the fact that f is LIB, choose an integrable function $\varphi : \mathbb{R} \mapsto [0, +\infty]$ such that $\|f(x,t)\| \leq \varphi(t)$ whenever $(x,t) \in K$. Given ε, choose δ such that $\int_L \varphi < \varepsilon$ whenever L is a measurable subset of $\mathbb{R}$ such that $\mathrm{meas}(L) \leq \delta$. If we apply (A) with this choice of δ, we see that

$$\Big\| \int_E f \circ \xi_j - \int_E f \circ \xi_\infty \Big\| \leq \varepsilon (2 + \mathrm{meas}(E))$$

for all sufficiently large j. Since ε is arbitrary, we can conclude that

$$\Big\| \int_E f \circ \xi_j - \int_E f \circ \xi_\infty \Big\| \to 0 \ \text{ as } \ j \to \infty \,, \tag{3.3.12}$$

and the proof is complete. ♢

3.4 Time-varying vector fields

Definition 13. Let $n \in \mathbb{N}$. A *time-varying vector field* (abbr. "TVVF") on $\mathbb{R}^n$ is a—possibly partially defined—single-valued map from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n$. We use $TVVF(\mathbb{R}^n)$ to denote the set of all TVVFs on $\mathbb{R}^n$. If S is a subset of $\mathbb{R}^n \times \mathbb{R}$, we use $TVVF(\mathbb{R}^n; S)$ to denote the set of all TVVFs on $\mathbb{R}^n$ whose domain is S. ♢

A TVVF on $\mathbb{R}^n$ is therefore a set-valued function $f : \mathbb{R}^n \times \mathbb{R} \mapsto\!\!\!\to \mathbb{R}^n$ that happens to be single-valued. It follows that all the definitions introduced in §2.9 for differential inclusions $\dot{x} \in F(x,t)$, and all the results stated there, apply to TVVFs. So we have a well defined concept of "trajectory of a TVVF f," or "trajectory of the ordinary differential equation $\dot{x} = f(x,t)$," and every TVVF f has a *flow* Φ^f.

We now prove some elementary estimates for LIB TVVFs.

If $\bar{x} \in \mathbb{R}^n$, $\bar{t} \in \mathbb{R}$, and $\delta > 0$, we write

$$\begin{aligned}\mathcal{N}_\delta(\bar{x},\bar{t}) &\stackrel{\text{def}}{=} \Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x-\bar{x}\| \le \delta \,\wedge\, \bar{t}-\delta \le t \le \bar{t}+\delta\Big\},\\ \mathcal{N}^{(2)}_\delta(\bar{x},\bar{t}) &\stackrel{\text{def}}{=} \Big\{(x,t,s) : (x,t) \in \mathcal{N}_\delta(\bar{x},\bar{t}) \wedge (x,s) \in \mathcal{N}_\delta(\bar{x},\bar{t})\Big\}.\end{aligned}$$

Lemma 1. *Let $\varphi : \mathbb{R} \mapsto [0,+\infty]$ be an integrable function. Let $n \in \mathbb{N}$, and let $f \in TVVF(\mathbb{R}^n)$. Let δ_1, δ be positive numbers such that*

$$\delta + \int_{\bar{t}-\delta}^{\bar{t}+\delta} \varphi \;<\; \delta_1\,, \tag{3.4.1}$$

and

$$\|f(x,t)\| \le \varphi(t) \qquad \text{whenever} \quad (x,t) \in \mathcal{N}_{\delta_1}(\bar{x},\bar{t}) \cap \mathrm{Dom}(f)\,. \tag{3.4.2}$$

Then

$$\|y-\bar{x}\| \le \delta_1 \;\text{ whenever }\; y \in \Phi^f(x,t,s) \;\text{ for some }\; (x,t,s) \in \mathcal{N}^{(2)}_\delta(\bar{x},\bar{t})\,. \tag{3.4.3}$$

Proof. Pick x,t,s,y such that $(x,t,s) \in \mathcal{N}^{(2)}_\delta(\bar{x},\bar{t})$ and $y \in \Phi^f(x,t,s)$. Let ζ be a trajectory of f such that $\zeta(s) = x$ and $\zeta(t) = y$. Let I be the compact interval $[\min(s,t), \max(s,t)]$. Let ξ be the restriction of ζ to I. Then $I \subseteq [\bar{t}-\delta, \bar{t}+\delta]$, and $\xi \in C^0(I,\mathbb{R}^n)$.

Let L be the set of those $\tau \in I$ such that $\|\xi(\sigma)-\bar{x}\| \le \delta_1$ for all σ that lie between s and τ. Then L is a compact interval, and $s \in L \subseteq I$. So $L = [\tau,\sigma]$, where $\tau \in I$, $\sigma \in I$, and $s \in \{\tau,\sigma\}$.

Clearly, $(\xi(r),r) \in \mathcal{N}_{\delta_1}(\bar{x},\bar{t})$ whenever $r \in L$. Moreover, $(\xi(r),r)$ belongs to $\mathrm{Dom}(f)$ and $\dot{\xi}(r) = f(\xi(r),r)$ for almost all $r \in L$.

It follows that, if $r \in L$, then

$$\begin{aligned}\|\xi(r)-\bar{x}\| &\le \|\xi(r)-x\| + \|x-\bar{x}\|\\ &= \|\xi(r)-\xi(s)\| + \|x-\bar{x}\|\\ &= \Big\|\int_s^r f(\xi(u),u)\,du\Big\| + \|x-\bar{x}\|\\ &\le \int_{\bar{t}-\delta}^{\bar{t}+\delta} \varphi(u)\,du + \delta\\ &< \delta_1\,.\end{aligned}$$

We know that one of the endpoints of L is s. Let ρ be the other endpoint. Assume that $\rho \ne t$. Then ρ is an interior point of I. But $\|\xi(\rho)-\bar{x}\| < \delta_1$, because $\rho \in L$. Therefore there exists a number ε such that $\varepsilon > 0$, $[\rho-\varepsilon, \rho+\varepsilon] \subseteq I$, and $\|\xi(t) - \bar{x}\| < \delta_1$ whenever $t \in [\rho-\varepsilon, \rho+\varepsilon]$. But then the interval $[\rho-\varepsilon, \rho+\varepsilon]$ must be entirely contained in L, contradicting the fact that ρ was a boundary point of L. This contradiction shows that the assumption that $\rho \neq t$ is false. So $\rho = t$, and then $L = I$. Then $t \in L$, so $\|y - \bar{x}\| = \|\xi(t) - \bar{x}\| \leq \delta_1$. ♢

The following continuity property of the flow is a trivial consequence of Lemma 1:

Corollary 1. *Let $n \in \mathbb{N}$, $f \in TVVF(\mathbb{R}^n)$. Assume that f is LIB. Assume also that for some positive number δ the restriction of f to $\mathcal{N}_\delta(\bar{x}, \bar{t}) \cap \mathrm{Dom}(f)$ is integrably bounded. Then*

$$\lim_{(x,t,s)\to(\bar{x},\bar{t},\bar{t})\,,\,\Phi^f(x,t,s)\neq\emptyset} \sup\Big\{\|y - \bar{x}\| : y \in \Phi^f(x,t,s)\Big\} = 0\,. \qquad (3.4.4)$$

Proof. Let $\varepsilon > 0$. Pick δ_1 such that $0 < \delta_1 \leq \varepsilon$ and the restriction of f to $\mathcal{N}_{\delta_1}(\bar{x}, \bar{t}) \cap \mathrm{Dom}(f)$ is integrably bounded. Then choose an integrable $\varphi : \mathbb{R} \mapsto [0, +\infty]$ such that (3.4.2) is true, and pick δ for which (3.4.1) is satisfied. Then Lemma 1 implies that $\|y - \bar{x}\| \leq \varepsilon$ whenever $\|x - \bar{x}\| \leq \delta$, $\|t - \bar{t}\| \leq \delta$, $\|s - \bar{t}\| \leq \delta$, and $y \in \Phi^f(x,t,s)$. So (3.4.4) is true. ♢

3.5 Admissible time-varying vector fields

We will be interested in time-varying vector fields $f \in TVVF(\mathbb{R}^n)$ that satisfy the following property:

(AD) *The map f is $\mathcal{EM}(\mathbb{R}^n, \mathbb{R}) \cap \mathrm{Dom}(f)$-measurable, and for every compact subset K of $\mathrm{Dom}(f)$ there exists an integrable function $\varphi : \mathbb{R} \mapsto [0, +\infty]$ such that*

(AD.i) $\|f(x,t)\| \leq \varphi(t)$ *for all* $(x,t) \in K$,

(AD.ii) f *is integrally continuous on* $ARC_\varphi(\mathbb{R}^n; K)$.

Definition 14. A time-varying vector field on $\mathbb{R}^n$ for which (AD) holds will be called *admissible*. We use $ADM(\mathbb{R}^n)$ to denote the class of all admissible time-varying vector fields on $\mathbb{R}^n$. If S is a subset of $\mathbb{R}^n \times \mathbb{R}$, we use $ADM(\mathbb{R}^n; S)$ to denote the class of all $f \in ADM(\mathbb{R}^n)$ such that $\mathrm{Dom}(f) = S$. ♢

Definition 15. Let $n \in \mathbb{N}$, $f \in TVVF(\mathbb{R}^n)$. We call f *trajectory-compact* if the set $\mathrm{Traj}_c(f, K)$ is compact for every compact subset K of $\mathrm{Dom}(f)$.♢

In addition to the sets $\mathcal{N}_\delta(\bar{x},\bar{t})$, $\mathcal{N}_\delta^{(2)}(\bar{x},\bar{t})$ defined earlier, we introduce their "one-sided" versions:

$$\begin{aligned}
\mathcal{N}_{\delta,+}(\bar{x},\bar{t}) &\stackrel{\text{def}}{=} \Big\{(x,t)\in\mathbb{R}^n\times\mathbb{R} : \|x-\bar{x}\|\le\delta \,\wedge\, \bar{t}\le t\le\bar{t}+\delta\Big\}\,,\\
\mathcal{N}_{\delta,-}(\bar{x},\bar{t}) &\stackrel{\text{def}}{=} \Big\{(x,t)\in\mathbb{R}^n\times\mathbb{R} : \|x-\bar{x}\|\le\delta \,\wedge\, \bar{t}-\delta\le t\le\bar{t}\Big\}\,,\\
\mathcal{N}_{\delta,+}^{(2)}(\bar{x},\bar{t}) &\stackrel{\text{def}}{=} \Big\{(x,t,s) : (x,t)\in\mathcal{N}_{\delta,+}(\bar{x},\bar{t})\wedge(x,s)\in\mathcal{N}_{\delta,+}(\bar{x},\bar{t})\Big\}\,,\\
\mathcal{N}_{\delta,-}^{(2)}(\bar{x},\bar{t}) &\stackrel{\text{def}}{=} \Big\{(x,t,s) : (x,t)\in\mathcal{N}_{\delta,-}(\bar{x},\bar{t})\wedge(x,s)\in\mathcal{N}_{\delta,-}(\bar{x},\bar{t})\Big\}\,.
\end{aligned}$$

Definition 16. Assume that $n\in\mathbb{N}$, $f\in TVVF(\mathbb{R}^n)$, $S=\mathrm{Dom}(f)$, and $(\bar{x},\bar{t})\in S$. We say that f is *flow-upper semicontinuous* (resp. *forward flow-upper semicontinuous*, *backward flow-upper semicontinuous*) near $(\bar{x},\bar{t})$ if there exists a $\delta\in\mathbb{R}$ such that $\delta>0$ and the restriction to the set $\mathcal{N}_\delta^{(2)}(\bar{x},\bar{t})$ (resp. $\mathcal{N}_{\delta,+}^{(2)}(\bar{x},\bar{t})$, $\mathcal{N}_{\delta,-}^{(2)}(\bar{x},\bar{t})$) of the set-valued map Φ^f is upper semicontinuous and has nonempty compact values. ♢

Theorem 3. *Let $n\in\mathbb{N}$, $f\in ADM(\mathbb{R}^n)$, $S=\mathrm{Dom}(f)$. Then f is trajectory-compact. Moreover,*

- *(T3.i) f is flow-upper semicontinuous at every interior point $(\bar{x},\bar{t})$ of S,*
- *(T3.ii) f is forward flow-upper semicontinuous at every point $(\bar{x},\bar{t})$ such that $\mathcal{N}_{\delta,+}(\bar{x},\bar{t})\subseteq S$ for some positive δ,*
- *(T3.iii) f is backward flow-upper semicontinuous at every point $(\bar{x},\bar{t})$ such that $\mathcal{N}_{\delta,-}(\bar{x},\bar{t})\subseteq S$ for some positive δ.*

Proof. We first prove trajectory-compactness. Let $K\subseteq S$ be compact, and let $\{\xi_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathrm{Traj}_c(f,K)$. Let $\varphi:\mathbb{R}\to[0,+\infty]$ be integrable and such that $\|f(x,t)\|\le\varphi(t)$ for all $(x,t)\in K$ and the integral map $\mathcal{T}_f^{ARC_\varphi(\mathbb{R}^n;K)} : ARC_\varphi(\mathbb{R}^n;K)\mapsto ARC(\mathbb{R}^n)$ is continuous.

Then the arcs ξ_j are entirely contained in K. Moreover, the ξ_j are absolutely continuous and satisfy $\|\dot{\xi}_j(t)\|\le\varphi(t)$ for almost all $t\in\mathrm{Dom}(f)$. Therefore the sequence $\{\xi_j\}_{j\in\mathbb{N}}$ is uniformly bounded and equicontinuous, so there is a subsequence $\{\xi_{j(\ell)}\}_{\ell\in\mathbb{N}}$ that converges in $ARC(\mathbb{R}^n)$ to an arc ξ_∞. Then $\xi_\infty\in ARC(\mathbb{R}^n;K)$, since K is compact. Moreover, if we write $\mathrm{Dom}(\xi_j)=[a_j,b_j]$ for $j\in\overline{\mathbb{N}}$, then $a_{j(\ell)}\to a_\infty$, $b_{j(\ell)}\to b_\infty$, and $\|\xi_{j(\ell)}(t)-\xi_{j(\ell)}(s)\|\le\int_s^t\varphi$ whenever $j\in\mathbb{N}$ and $a_j\le s\le t\le b_j$. It follows that $\|\xi_\infty(t)-\xi_\infty(s)\|\le\int_s^t\varphi$ whenever $a_\infty\le s\le t\le b_\infty$. Therefore $\xi_\infty\in ARC_\varphi(\mathbb{R}^n;K)$.

Let $\eta_j = \mathcal{T}_f^{ARC_\varphi(\mathbb{R}^n;K)}$ for $j \in \overline{\mathbb{N}}$. Then $\eta_j \in ARC(\mathbb{R}^n)$, and the continuity of $\mathcal{T}_f^{ARC_\varphi(\mathbb{R}^n;K)}$ on $ARC_\varphi(\mathbb{R}^n;K)$ implies that $\eta_{j(\ell)} \to \eta_\infty$ as $\ell \to \infty$. Since $\xi_j \in \mathrm{Traj}_c(f)$ for $j \in \mathbb{N}$, the equalities

$$\xi_j(t) = \xi_j(a_j) + \int_{a_j}^t f(\xi_j(s), s)\, ds = \xi_j(a_j) + \eta_j(t)$$

hold if $j \in \mathbb{N}$, $a_j \le t \le b_j$. Taking $j = j(\ell)$ and letting $\ell \to \infty$, we see that

$$\xi_\infty(t) = \xi_\infty(a_\infty) + \eta_\infty(t) = \xi_\infty(a_\infty) + \int_{a_\infty}^t f(\xi_\infty(s), s)\, ds$$

if $a_\infty \le t \le b_\infty$. So $\xi_\infty \in \mathrm{Traj}_c(f, K)$, proving that f is trajectory-compact.

We now prove Conclusion (T3.i). Pick an interior point $(\bar{x}, \bar{t})$ of S. For $\delta \in \mathbb{R}$, $\delta > 0$, write $\mathcal{K}(\delta) \stackrel{\text{def}}{=} \mathcal{N}_\delta(\bar{x}, \bar{t})$. Let δ_1 be such that $\delta_1 > 0$ and $\mathcal{K}(\delta_1) \subseteq S$. Let φ be an integrable function on $\mathbb{R}$ such that $\|f(x,t)\| \le \varphi(t)$ whenever $(x,t) \in \mathcal{K}(\delta_1)$ and f is integrally continuous on $ARC_\varphi(\mathbb{R}^n; \mathcal{K}(\delta_1))$. Let δ be such that $\delta > 0$ and (3.4.1) holds. Let G be the graph of the restriction to $\mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t})$ of the set-valued map Φ^f, so that

$$G = \Big\{ (x,t,s,y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : y \in \Phi^f(x,t,s),\ (x,t,s) \in \mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t}) \Big\}.$$

We will prove that G is compact. For this purpose, we pick a sequence $\{(x_j, t_j, s_j, y_j)\}_{j\in\mathbb{N}}$ of points of G, and show that it has a subsequence that converges to a point $(x,t,s,y) \in G$. It is clear that the sequences $\{x_j\}_{j\in\mathbb{N}}$, $\{t_j\}_{j\in\mathbb{N}}$, $\{s_j\}_{j\in\mathbb{N}}$, are bounded. We may therefore assume that the limits $x_\infty = \lim_{j\to\infty} x_j$, $t_\infty = \lim_{j\to\infty} t_j$, $s_\infty = \lim_{j\to\infty} s_j$, exist.

For each $j \in \mathbb{N}$, let ζ_j be a trajectory of f such that $\zeta_j(s_j) = x_j$ and $\zeta_j(t_j) = y_j$. Let I_j be the compact interval $[\min(s_j, t_j), \max(s_j, t_j)]$, and let ξ_j be the restriction of ζ_j to I_j. Then ξ_j is a trajectory of f, $\xi_j(s_j) = x_j$, $\xi_j(t_j) = y_j$, and $\mathrm{Dom}(\xi_j) = I_j$. If $t \in I_j$, then $(x_j, t, s_j) \in \mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t})$, and Lemma 1 implies that $\|\xi_j(t) - \bar{x}\| \le \delta_1$, so $(\xi_j(t), t) \in \mathcal{K}(\delta_1)$.

It follows that $\xi_j \in \mathrm{Traj}_c(f, \mathcal{K}(\delta_1))$ for every $j \in \mathbb{N}$. Since f is trajectory-compact, there is a subsequence $\{\xi_{j(\ell)}\}_{\ell\in\mathbb{N}}$ of $\{\xi_j\}_{j\in\mathbb{N}}$ that converges in $ARC(\mathbb{R}^n)$ to a limit $\xi_\infty \in \mathrm{Traj}_c(f, \mathcal{K}(\delta_1))$. Let $y_\infty = \xi_\infty(t_\infty)$. Then

$$\lim_{\ell\to\infty} y_{j(\ell)} = \lim_{\ell\to\infty} \xi_{j(\ell)}(t_{j(\ell)}) = \xi_\infty(t_\infty) = y_\infty\,,$$

so $\lim_{\ell\to\infty}(x_{j(\ell)}, t_{j(\ell)}, s_{j(\ell)}, y_{j(\ell)}) = (x_\infty, t_\infty, s_\infty, y_\infty)$. Moreover,

$$\xi_\infty(s_\infty) = \lim_{\ell\to\infty} \xi_{j(\ell)}(s_{j(\ell)}) = \lim_{\ell\to\infty} x_{j(\ell)} = x_\infty\,.$$

Therefore $y_\infty \in \Phi^f(x_\infty, t_\infty, s_\infty)$. Since $(x_j, t_j, s_j) \in \mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t})$ for every $j \in \mathbb{N}$, it is clear that $(x_\infty, t_\infty, s_\infty) \in \mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t})$. Therefore $(x_\infty, t_\infty, s_\infty, y_\infty) \in G$. This completes the proof that G is compact. Equivalently, we have shown that the set-valued map Φ^f has compact values and is upper semicontinuous on the set $\mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t})$.

To conclude our proof, we have to show (cf. Remark 7) that

$$\Phi^f(x, t, s) \neq \emptyset \qquad \text{whenever} \quad (x, t, s) \in \mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t})\,. \tag{3.5.1}$$

Fix a point (x, t, s) of $\mathcal{N}_\delta^{(2)}(\bar{x}, \bar{t})$. Let $J = [\bar{t} - \delta, \bar{t} + \delta]$. Let $\mathcal{X}$ be the Banach space of all continuous maps $\xi : J \mapsto \mathbb{R}^n$, endowed with the sup norm. Let $\mathcal{C}$ be the subset of $\mathcal{X}$ consisting of those $\xi \in \mathcal{X}$ that are absolutely continuous and satisfy $\|\xi(r) - \bar{x}\| \leq \delta_1$ for all $r \in J$ and $\|\dot{\xi}(r)\| \leq \varphi(r)$ for a.e. $r \in J$. Then $\mathcal{C}$ is a compact convex subset of $\mathcal{X}$. Define a map $\mu : \mathcal{C} \mapsto \mathcal{X}$ by letting

$$\mu(\xi)(r) = x + \int_s^r f(\xi(u), u)\, du \qquad \text{for} \quad \xi \in \mathcal{C}\,,\ r \in J\,. \tag{3.5.2}$$

(The map μ is clearly well defined, because (i) if $\xi \in \mathcal{C}$ and $u \in J$ then $(\xi(u), u) \in \mathcal{K}(\delta_1)$, so $f(\xi(u), u)$ is defined; (ii) the essential measurability of f implies that the map $J \ni u \mapsto f(\xi(u), u) \in \mathbb{R}^n$ is measurable, and (iii) the integrable bound

$$\|f(\xi(u), u)\| \leq \varphi(u) \tag{3.5.3}$$

implies that $J \ni u \mapsto f(\xi(u), u) \in \mathbb{R}^n$ is integrable.)

If $\xi \in \mathcal{C}$, and $\eta = \mu(\xi)$, then it follows from (3.5.2) and (3.5.3) that η is absolutely continuous and $\|\dot{\eta}(r)\| \leq \varphi(r)$ for all $r \in J$. Moreover, since $\|x - \bar{x}\| \leq \delta$, (3.4.1) implies that $\|\eta(r) - \bar{x}\| \leq \delta_1$ for all $r \in J$. Therefore $\mu(\xi) \in \mathcal{C}$. So μ is a map from $\mathcal{C}$ to $\mathcal{C}$.

We now prove that the map μ is continuous. To see this, we write $\mathcal{A} = ARC_\varphi(\mathbb{R}^n; \mathcal{K}(\delta_1))$, and observe that $\mathcal{C} \subseteq \mathcal{A}$. The definition of μ implies the identity

$$\mu(\xi)(r) = x + \mathcal{T}_f^{\mathcal{A}}(\xi)(r) - \mathcal{T}_f^{\mathcal{A}}(\xi)(s) \quad \text{whenever} \ \xi \in \mathcal{C}\,,\ r \in J\,. \tag{3.5.4}$$

Since $\mathcal{T}_f^{\mathcal{A}}$ is continuous on $\mathcal{A}$, the continuity of μ follows.

By the Schauder Fixed Point Theorem, μ has a fixed point. Clearly, if ξ is a fixed point of μ, then

$$\xi(r) = x + \int_s^t f(\xi(u), u)\, du \ \text{ for } \ r \in J\,. \tag{3.5.5}$$

Then ξ is a trajectory of f defined on J, and $\xi(s) = x$. If we let $y = \xi(t)$, then $y \in \Phi^f(x, t, s)$, so $\Phi^f(x, t, s) \neq \emptyset$. This completes the proof of (T3.i).

The proofs of (T3.ii) and (T3.iii) are identical, with obvious trivial modifications. Alternatively, one can derive (T3.ii) and (T3.iii) directly from (T3.i) by applying (T3.i) to suitably chosen TVVFs. For example, to derive (T3.ii) we pick $(\bar{x}, \bar{t})$ and a positive δ such that $\mathcal{N}_{\delta,+}(\bar{x}, \bar{t}) \subseteq S$, write

$$\hat{S} = \Big(\mathbb{R}^n \times \,]-\infty, \bar{t}\,[\Big) \bigcup \Big(S \cap \Big(\mathbb{R}^n \times [\bar{t}, +\infty\,[\Big)\Big),$$

and define a map $g : \hat{S} \mapsto \mathbb{R}^n$ by letting

$$g(x,t) = \begin{cases} f(x,t) & \text{if} \quad (x,t) \in S \ \text{ and } \ t \geq \bar{t}\,, \\ f(\bar{x},\bar{t}) & \text{if} \quad x \in \mathbb{R}^n \ \text{ and } \ t < \bar{t}\,. \end{cases}$$

Then $g \in ADM(\mathbb{R}^n)$, $\mathrm{Dom}(g) = \hat{S}$, and $(\bar{x}, \bar{t})$ is an interior point of $\hat{S}$. So we can apply (T3.i) and conclude that Φ^g is upper semicontinuous and has compact nonempty values on $\mathcal{N}^{(2)}_{\delta}(\bar{x}, \bar{t})$ for some positive δ. If we restrict Φ^g to $\mathcal{N}^{(2)}_{\delta,+}(\bar{x}, \bar{t})$, (T3.ii) follows. The proof of (T3.iii) is similar. $\diamondsuit$

Remark 7. The assertion that the values of the flow are nonempty (that is, Statement 3.5.1), is the *existence of solutions theorem* for differential equations $\dot{x} = f(x, t)$ with $f \in ADM(\mathbb{R}^n)$. It is therefore not surprising that the proof of this statement depends on a fixed point argument. $\diamondsuit$

A special case of Theorem 3 is the usual Carathéodory existence theorem:

Corollary 2. *Assume that $n \in \mathbb{N}$, Ω is open in $\mathbb{R}^n$, and I is a nonempty subinterval of $\mathbb{R}$. Let $S = \Omega \times I$. Let $f : S \mapsto \mathbb{R}^n$ be a LIB time-varying vector field such that the map $\Omega \ni x \mapsto f(x, t) \in \mathbb{R}^n$ is continuous for almost every $t \in I$, and the map $I \ni t \mapsto f(x, t) \in \mathbb{R}^n$ is measurable for every $x \in \Omega$. Then f is integrally continuous on the space $ARC(\mathbb{R}^n; S)$, so $f \in ADM(\mathbb{R}^n; S)$. In particular,*

- *(C2.i) f is trajectory-compact,*
- *(C2.ii) f is flow-upper semicontinuous at every $(\bar{x}, \bar{t}) \in \Omega \times \mathrm{Int}(I)$,*
- *(C2.iii) f is forward flow-upper semicontinuous at every point $(\bar{x}, \bar{t})$ such that $\bar{x} \in \Omega$ and $[\bar{t}, \bar{t} + \delta] \subseteq I$ for some positive δ,*
- *(C2.iv) f is backward flow-upper semicontinuous at every point $(\bar{x}, \bar{t})$ such that $\bar{x} \in \Omega$ and $[\bar{t} - \delta, \bar{t}] \subseteq I$ for some positive δ.*

Proof. It follows easily from the hypotheses that f is essentially measurable. Let $\mathcal{A} = ARC(\mathbb{R}^n; S)$. Then Fact 6 tells us that f is arc-integrable on $\mathcal{A}$.

Let $\{\xi_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ that converges in $ARC(\mathbb{R}^n)$ to a $\xi_\infty \in \mathcal{A}$. For $j \in \overline{\mathbb{N}}$, let $L_j = \mathrm{Dom}(\xi_j) = [a_j, b_j]$, and write $\eta_j = \mathcal{T}_f^{\mathcal{A}}(\xi_j)$. Then $\lim_{j\to\infty} a_j = a_\infty$ and $\lim_{j\to\infty} b_j = b_\infty$.

Let K be a compact subset of S such that $(\xi_j(t), t) \in K$ whenever $j \in \overline{\mathbb{N}}$ and $a_j \le t \le b_j$. Choose an integrable function φ on $\mathbb{R}$ such that $\|f(x,t)\| \le \varphi(t)$ whenever $(x,t) \in K$. Then, if $\{t_j\}_{j\in\mathbb{N}}$ is a sequence such that $a_j \le t_j \le b_j$ for every $j \in \mathbb{N}$, and $t_j \to t$ as $j \to \infty$, we have

$$\eta_j(t_j) = \int_{a_j}^{t_j} f(\xi_j(s), s)\, ds \to \int_a^t f(\xi_\infty(s), s)\, ds = \mathcal{T}_f^{\mathcal{A}}(a, t, \xi) \tag{3.5.6}$$

by the Lebesgue Dominated Convergence Theorem, since $\xi_j(s) \to \xi_\infty(s)$ for each s, and $\|f(\xi_j(s), s)\| \le \varphi(s)$ for each j and each $s \in L_j]$.

This proves the integral continuity of f on $\mathcal{A}$. Therefore f is admissible, and then all the other conclusions are consequences of Theorem 3. $\diamondsuit$

3.6 Points of approximate continuity

Definition 17. Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$, and $(\bar{x}, \bar{t}) \in S$. We say that $(\bar{x}, \bar{t})$ is a *point of approximate continuity* (abbreviated PAC) of f if there exist positive numbers $\bar{h}, \bar{\delta}$ such that

- (D17.1) $\mathbb{B}^n(\bar{x}, \bar{\delta}) \times [\bar{t} - \bar{h}, \bar{t} + \bar{h}] \subseteq S$,
- (D17.2) there exist measurable functions $\sigma_\delta : [\bar{t} - \bar{h}, \bar{t} + \bar{h}] \mapsto [0, \infty]$, for $0 < \delta < \bar{\delta}$, such that
  - (D17.2.1) the bound $\|f(x,t) - f(\bar{x}, \bar{t})\| \le \sigma_\delta(t)$ holds whenever (x,t) belongs to $\mathbb{B}^n(\bar{x}, \delta) \times [\bar{t} - \bar{h}, \bar{t} + \bar{h}]$
  - (D17.2.2) $\lim_{\delta\downarrow 0\,,\, h\downarrow 0} \frac{1}{h} \int_{\bar{t}-h}^{\bar{t}+h} \sigma_\delta(t)\, dt = 0$.

The concepts of a *point of forward approximate continuity* and a *point of backward approximate continuity* are defined similarly, with the interval $[\bar{t} - \bar{h}, \bar{t} + \bar{h}]$ replaced by $[\bar{t}, \bar{t} + \bar{h}]$ and $[\bar{t} - \bar{h}, \bar{t}]$, respectively. $\diamondsuit$

Proposition 2. *Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$, and $(\bar{x}, \bar{t}) \in S$. Assume that there exist K, E, φ such that*

- *(P2.a)* $E \in \mathrm{Leb}(\mathbb{R})$, *$K$ is a compact subset of S, $\varphi : \mathbb{R} \mapsto [0, +\infty]$ is an integrable function, $(\bar{x}, \bar{t})$ is an interior point of K, and $\bar{t}$ is a point of density of E and a Lebesgue point of φ,* *(P2.b)* $\lim_{(x,t)\to(\bar{x},\bar{t})\,,\,(x,t)\in K\,,\,t\in E} f(x,t) = f(\bar{x},\bar{t})$*;*

(P2.c) $\|f(x,t)\| \le \varphi(t)$ *for all* $(x,t) \in K$*.*

Then $(\bar{x},\bar{t})$ *is a point of approximate continuity of* f*.* ♢

Proof. Pick positive numbers $\bar{\delta}$, $\bar{h}$ such that $\mathbb{B}^n(\bar{x},\bar{\delta}) \times [\bar{t}-\bar{h},\bar{t}+\bar{h}] \subseteq K$. For $0 < \delta \le \bar{\delta}$, $\bar{t}-\bar{h} \le t \le \bar{t}+\bar{h}$, define $\sigma_\delta(t)$ by

$$\sigma_\delta(t) = \begin{cases} \sup\Big\{\|f(x,s)-f(\bar{x},\bar{t})\| : \|x-\bar{x}\|\le\delta, |s-\bar{t}|\le|t-\bar{t}|, s\in E\Big\} & \text{if } t\in E, \\ \varphi(t)+\varphi(\bar{t}) & \text{if } t\notin E. \end{cases}$$

Then σ_δ is measurable, because it is monotonically nondecreasing on the set $E\cap[\bar{t},\bar{t}+\bar{h}]$ and monotonically nonincreasing on $E\cap[\bar{t}-\bar{h},\bar{t}]$, and coincides with the integrable function $t\mapsto\varphi(t)+\varphi(\bar{t})$ on $[\bar{t}-\bar{h},\bar{t}+\bar{h}]\backslash E$. It is clear that the bound of (D17.2.1) holds. Moreover,

$$\begin{aligned} \int_{\bar{t}-h}^{\bar{t}+h} \sigma_\delta(t)dt &= \int_{[\bar{t}-h,\bar{t}+h]\cap E} \sigma_\delta(t)\,dt + \int_{[\bar{t}-h,\bar{t}+h]\backslash E} \sigma_\delta(t)\,dt \\ &\le \int_{[\bar{t}-h,\bar{t}+h]\cap E} \omega(\delta,h)\,dt + \int_{[\bar{t}-h,\bar{t}+h]\backslash E} (\varphi(t)+\varphi(\bar{t}))\,dt \\ &\le 2\,h\,\omega(\delta,h) + \int_{[\bar{t}-h,\bar{t}+h]\backslash E} (\varphi(t)-\varphi(\bar{t})+2\varphi(\bar{t}))\,dt \\ &\le 2\,h\,\omega(\delta,h) + \int_{\bar{t}-h}^{\bar{t}+h} |\varphi(t)-\varphi(\bar{t})|\,dt + \int_{[\bar{t}-h,\bar{t}+h]\backslash E} 2\,\varphi(\bar{t})\,dt \\ &\le 2\,h\,\omega(\delta,h) + \int_{\bar{t}-h}^{\bar{t}+h} |\varphi(t)-\varphi(\bar{t})|\,dt + 2\,\varphi(\bar{t})\,\mu(h)\,, \end{aligned}$$

where $\omega(\delta,h) \stackrel{\text{def}}{=} \sup\{\|f(x,t)-f(\bar{x},\bar{t})\| : \|x-\bar{x}\| \le \delta, \|t-\bar{t}\| \le h, t\in E\}$ and $\mu(h) \stackrel{\text{def}}{=} \text{meas}\Big([\bar{t}-h,\bar{t}+h]\backslash E\Big)$.

It follows from (P2.b) that $\lim_{\delta\downarrow 0,h\downarrow 0}\omega(\delta,h) = 0$. The fact that $\bar{t}$ is a Lebesgue point of φ implies that $\lim_{h\downarrow 0}\frac{1}{h}\int_{\bar{t}-h}^{\bar{t}+h}|\varphi(t)-\varphi(\bar{t})|\,dt = 0$. Finally, $\lim_{h\downarrow 0}\frac{1}{h}\mu(h) = 0$, because $\bar{t}$ is a point of density of E. Therefore (D17.2.2) holds, and our proof is complete. ♢

3.7 Differentiability of the flow

The differentiation theorem will say, roughly, that if $(\bar{x},\bar{t})$ is a PAC of f, then

$$\Phi^f(x,t,s) = x + (t-s)f(\bar{x},\bar{t}) + o\Big(\|x-\bar{x}\|+|t-\bar{t}|+|s-\bar{t}|\Big) \qquad (3.7.1)$$

as $(x,t,s) \to (\bar{x},\bar{t},\bar{t})$.

Since Φ^f is set-valued, Equation (3.7.1) requires interpretation. The precise meaning of (3.7.1) is that, if we use Λ^w to denote, for a given $w \in \mathbb{R}^n$, the linear map $\Lambda^w : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^n$ given by

$$\Lambda^w(v,h,k) = v + (h-k)w \qquad \text{for } v \in \mathbb{R}^n\,,\ h,k \in \mathbb{R}\,, \tag{3.7.2}$$

then $\Lambda^{f(\bar{x},\bar{t})}(v,h,k)$ is a "first-order approximation" to the set-valued map

$$(x,t,s) \mapsto \Phi^f(x,t,s) - \Phi^f(\bar{x},\bar{t},\bar{t}) = \Phi^f(x,t,s) - \bar{x}$$

near $(\bar{x},\bar{t},\bar{t})$. Precisely, this means that

$$\lim \frac{\sup\Big\{\|y - \bar{x} - \Lambda^{f(\bar{x},\bar{t})}(x-\bar{x}, t-\bar{t}, s-\bar{t})\| : y \in \Phi^f(x,s,t)\Big\}}{\|x-\bar{x}\| + |t-\bar{t}| + |s-\bar{t}|} = 0 \tag{3.7.3}$$

as $(x,s,t) \to (\bar{x},\bar{t},\bar{t})$. If we set $v = x - \bar{x}$, $h = t - \bar{t}$, $k = s - \bar{t}$ in (3.7.2), we see that (3.7.3) says that

$$\lim_{(x,s,t)\to(\bar{x},\bar{t},\bar{t})} \frac{\sup\Big\{\|y - x - (t-s)f(\bar{x},\bar{t})\| : y \in \Phi^f(x,s,t)\Big\}}{\|x-\bar{x}\| + |t-\bar{t}| + |s-\bar{t}|} = 0 \tag{3.7.4}$$

When this happens, we say that Φ^f *is differentiable at* $(\bar{x},\bar{t},\bar{t})$ *and the differential of* Φ^f *at* $(\bar{x},\bar{t},\bar{t})$ *is the linear map* $\Lambda^{f(\bar{x},\bar{t})}$.

The one-sided concepts of *forward and backward differentiability* are defined similarly, replacing "$(x,t,s) \to (\bar{x},\bar{t},\bar{t})$" by "$(x,t,s) \to (\bar{x},\bar{t},\bar{t})$ via values of t,s such that $t \geq \bar{t}$ and $s \geq \bar{t}$" and "$(x,t,s) \to (\bar{x},\bar{t},\bar{t})$ via values of t,s such that $t \leq \bar{t}$ and $s \leq \bar{t}$," respectively.

Theorem 4. *Assume that* $n \in \mathbb{N}$*,* $S \subseteq \mathbb{R}^n \times \mathbb{R}$*,* $f \in TVVF(\mathbb{R}^n; S)$*, and* $(\bar{x},\bar{t}) \in S$*. Then, if* $(\bar{x},\bar{t})$ *is a point of approximate continuity of* f *it follows that the flow* Φ^f *is differentiable at* $(\bar{x},\bar{t},\bar{t})$*, and the differential of* Φ^f *at* $(\bar{x},\bar{t},\bar{t})$ *is the map* $\Lambda^{f(\bar{x},\bar{t})}$ *defined by (3.7.2).*

Proof. Let $\bar{\delta}$, $\bar{h}$, and the functions $\sigma_\delta : [\bar{t}-\bar{h}, \bar{t}+\bar{h}] \mapsto [0,\infty]$ be as in Definition 17. For $0 < \delta \leq \bar{\delta}$, write

$$\omega(\delta) = \sup\Big\{\|y-\bar{x}\| : y \in \Phi^f(x,s,t) : (x,s,t) \in \mathcal{N}^{(2)}_\delta(\bar{x},\bar{t})\Big\}\,. \tag{3.7.5}$$

Then $\omega(\delta) \to 0$ as $\delta \downarrow 0$ by Corollary 1. Choose δ^* such that $0 < \delta^* \leq \bar{\delta}$ and $\omega(\delta^*) \leq \bar{\delta}$.

Let $0 < \delta \leq \delta^*$. Suppose $y \in \Phi^f(x,s,t)$, $(x,s,t) \in \mathcal{N}_\delta^{(2)}(\bar{x},\bar{t})$. Let ζ be a trajectory of f such that $\zeta(s) = x$ and $\zeta(t) = y$. Let I be the interval with endpoints s,t. Let ξ be the restriction of ζ to I. Then

$$y = x + \int_s^t f(\xi(r),r)\,dr = x + (t-s)f(\bar{x},\bar{t}) + R\,,$$

where the error term R is given by $R = \int_s^t \Big(f(\xi(r),r) - f(\bar{x},\bar{t})\Big)\,dr$.

To estimate R, we first observe that for each $r \in I$ the point $\xi(r)$ belongs to $\Phi^f(x,r,s)$, and $(x,r,s) \in \mathcal{N}_\delta^{(2)}(\bar{x},\bar{t})$, so $\|\xi(r) - \bar{x}\| \leq \omega(\delta)$. Then

$$r \in I \Longrightarrow \|f(\xi(r),r) - f(\bar{x},\bar{t})\| \leq \sigma_{\omega(\delta)}(r)\,.$$

Therefore

$$\|R\| \leq \int_{\bar{t}-h}^{\bar{t}+h} \sigma_{\omega(\delta)}(r)\,dr\,, \tag{3.7.6}$$

where $h = \max(|t-\bar{t}|, |s-\bar{t}|)$.

Inequality (3.7.6) has been proved for arbitrary $(x,t,s) \in \mathcal{N}_\delta^{(2)}(\bar{x},\bar{t})$ and $y \in \Phi^f(x,t,s)$, provided that $0 < \delta \leq \delta^*$. For given (x,t,s), we can take $\delta = \|x - \bar{x}\|$, and define $h = \max(|t-\bar{t}|, |s-\bar{t}|)$ as before. This yields

$$\begin{aligned}
\frac{\|y - x - (t-s)f(\bar{x},\bar{t})\|}{\|x-\bar{x}\| + |t-\bar{t}| + |s-\bar{t}|} &= \frac{\|R\|}{\delta + |t-\bar{t}| + |s-\bar{t}|} \\
&\leq \frac{\|R\|}{h} \\
&\leq \frac{1}{h}\int_{\bar{t}-h}^{\bar{t}+h} \sigma_{\omega(\|x-\bar{x}\|)}(r)\,dr\,,
\end{aligned}$$

so (3.7.3) holds. ♢

Corollary 3. *Assume that $n \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f \in TVVF(\mathbb{R}^n;S)$, and $(\bar{x},\bar{t}) \in S$. Then, if $(\bar{x},\bar{t})$ is a point of forward (resp. backward) approximate continuity of f it follows that the flow Φ^f is forward (resp. backward) differentiable at $(\bar{x},\bar{t},\bar{t})$, and the forward (resp. backward) differential of Φ^f at $(\bar{x},\bar{t},\bar{t})$ is the map $\Lambda^{f(\bar{x},\bar{t})}$ defined by (3.7.2).*

Proof. This can be proved using exactly the same arguments as in the proof of Theorem 4 with obvious one-sided modifications. Alternatively, the result can be derived from Theorem 4 by applying Theorem 4 to the TVVF g considered in the proof of Theorem 3. ♢

3.8 Regularization

Fix an $n \in \mathbb{N}$, a nonempty open subset Ω of $\mathbb{R}^n$, and a subinterval I of $\mathbb{R}$ of positive length. Let $S = \Omega \times I$, and fix an admissible time-varying vector field $f \in ADM(\mathbb{R}^n; S)$. For $\rho > 0$, let Ω_ρ be the set of all $x \in \mathbb{R}^n$ such that the closed ball $\mathbb{B}^n(x,\rho) \stackrel{\text{def}}{=} \{x' \in \mathbb{R}^n : \|x' - x\| \le \rho\}$ is entirely contained in Ω. Then Ω_ρ is open. Let $S_\rho = \Omega_\rho \times I$.

Fix, once and for all, a function $\theta : \mathbb{R}^n \to \mathbb{R}$ of class C^∞, such that $\theta(z) \ge 0$ for all y, $\theta(z) = 0$ whenever $\|z\| \ge 1$, and $\int_{\mathbb{R}^n} \theta = 1$. Also fix, once and for all, locally integrable functions $\varphi_K : I \mapsto [0,\infty]$, for each compact subset K of Ω, such that $\|f(x,t)\| \le \varphi_K(t)$ whenever $x \in K$ and $t \in I$. If K is a compact subset of Ω and $\rho > 0$, we write $K^\rho = \{x \in \mathbb{R}^n : d(x,K) \le \rho\}$, and observe that $K^\rho \subseteq \Omega$ if and only if $K \subseteq \Omega_\rho$.

Define

$$f_\rho(x,t) = \int_{\mathbb{R}^n} f(x - \rho z, t)\theta(z)\, dz \ \text{ for } \ (x,t) \in S_\rho\,. \tag{3.8.1}$$

Then for almost all $t \in I$ the integral is defined for all $x \in \Omega_\rho$, because for a.e. t the map $x \mapsto f(x,t)$ is measurable and bounded on compact sets. Moreover, if K is a compact subset of Ω_ρ then f_ρ satisfies the bound

$$\|f_\rho(x,t)\| \le \varphi_{K_\rho}(t) \quad \text{whenever} \quad x \in K\,,\, t \in I\,.$$

In particular, f_ρ is locally integrably bounded on S_ρ. Since

$$f_\rho(x,t) = \rho^{-n} \int_{\mathbb{R}^n} f(y,t)\theta\Big(\frac{x-y}{\rho}\Big)dy \ \text{ for } \ x \in \Omega_\rho\,,\, t \in I\,, \tag{3.8.2}$$

it is easy to see that $f_\rho(x,t)$ is of class C^∞ as a function of x for almost every t. Moreover, $f_\rho(x,t)$ is clearly measurable with respect to t for each fixed x. Furthermore, using (3.8.2), we see that

$$f_\rho(x,t) - f_\rho(x',t) = \rho^{-n} \int_{\mathbb{R}^n} f(y,t)\left(\theta\Big(\frac{x-y}{\rho}\Big) - \theta\Big(\frac{x'-y}{\rho}\Big)\right)dy \tag{3.8.3}$$

for $x, x' \in \Omega_\rho$, $t \in I$. Therefore, if $K \subseteq \Omega_\rho$ is compact, and $x, x' \in K$, $t \in I$, we have

$$\Big\|f_\rho(x,t) - f_\rho(x',t)\Big\| \le \rho^{-n}\varphi_{K_\rho}(t) \int_{\mathbb{R}^n} \left|\theta\Big(\frac{x-y}{\rho}\Big) - \theta\Big(\frac{x'-y}{\rho}\Big)\right| dy\,. \tag{3.8.4}$$

Then

$$\Big(K \subseteq \Omega_\rho \,\wedge\, K \text{ compact} \,\wedge\, x, x' \in K \,\wedge\, t \in I\Big) \\ \Longrightarrow \Big\| f_\rho(x,t) - f_\rho(x',t) \Big\| \le \kappa \rho^{-n-1} \varphi_{K_\rho}(t) \|x - x'\| \,, \qquad (3.8.5)$$

where

$$\kappa = \sup \Big\{ \|\nabla\theta(z)\| : z \in \mathbb{R}^n \Big\} \,. \qquad (3.8.6)$$

Clearly, (3.8.5) implies that

Fact 7. *If $\rho > 0$ then the map $\Omega_\rho \ni x \mapsto f_\rho(x,t)$ is locally Lipschitz for almost all $t \in I$, and the Lipschitz constant $C_K(t)$ can be chosen, for each compact subset K of Ω_ρ, to be a locally integrable function of t.* $\diamondsuit$

It then follows that f_ρ has the usual uniqueness and continuous dependence properties. In particular:

Fact 8. *The flow Φ^{f_ρ} is single-valued and continuous on its domain of definition $\mathrm{Dom}(\Phi^{f_\rho})$, which is a relatively open subset of $\Omega_\rho \times I \times I$ containing the set $\{(x,t,t) : x \in \Omega_\rho\,,\ t \in I\}$.* $\diamondsuit$

If $(\bar{x}, \bar{t}) \in \mathbb{R}^n \times I$ and $\delta > 0$, we let

$$\tilde{\mathcal{N}}_\delta(\bar{x},\bar{t}) = \begin{cases} \mathcal{N}_\delta(\bar{x},\bar{t}) & \text{if} \quad \bar{t} \in \mathrm{Int}(I)\,, \\ \mathcal{N}_{\delta,+}(\bar{x},\bar{t}) & \text{if} \quad \bar{t} = \min I\,, \\ \mathcal{N}_{\delta,-}(\bar{x},\bar{t}) & \text{if} \quad \bar{t} = \max I\,, \end{cases}$$

$$\tilde{\mathcal{N}}^{(2)}_\delta(\bar{x},\bar{t}) = \begin{cases} \mathcal{N}^{(2)}_\delta(\bar{x},\bar{t}) & \text{if} \quad \bar{t} \in \mathrm{Int}(I)\,, \\ \mathcal{N}^{(2)}_{\delta,+}(\bar{x},\bar{t}) & \text{if} \quad \bar{t} = \min I\,, \\ \mathcal{N}^{(2)}_{\delta,-}(\bar{x},\bar{t}) & \text{if} \quad \bar{t} = \max I\,. \end{cases}$$

Fix $(\bar{x}, \bar{t}) \in \Omega \times I$, and pick δ such that $\delta > 0$ and

(#.1) *$\tilde{\mathcal{N}}_\delta(\bar{x},\bar{t}) \subseteq S$ and the set-valued map Φ^f is upper semicontinuous and has compact nonempty values on $\tilde{\mathcal{N}}^{(2)}_\delta(\bar{x},\bar{t})$.*

Theorem 5. *Let $\delta > 0$ be such that (#.1) holds. Then there exists a positive $\bar{\rho}$ such that*

(#.2) $\mathbb{B}^n(\bar{x},\delta) \subseteq \Omega_{\bar{\rho}}$,

(#.3) $\tilde{\mathcal{N}}^{(2)}_\delta(\bar{x},\bar{t}) \subseteq \mathrm{Dom}(\Phi^{f_\rho})$ *whenever* $0 < \rho \le \bar{\rho}$,

and

(#.4) the single-valued maps $\Phi^{f_\rho}\lceil\tilde{\mathcal{N}}_\delta^{(2)}(\bar{x},\bar{t})$ *satisfy*

$$\Phi^{f_\rho}\lceil\tilde{\mathcal{N}}_\delta^{(2)}(\bar{x},\bar{t}) \overset{\text{igr}}{\to} \Phi^{f}\lceil\tilde{\mathcal{N}}_\delta^{(2)}(\bar{x},\bar{t}) \text{ as } \rho\to 0\,. \tag{3.8.7}$$

In particular, the set-valued map $\Phi^f\lceil\tilde{\mathcal{N}}_\delta^{(2)} : \tilde{\mathcal{N}}_\delta^{(2)} \mapsto\!\!\!\to \mathbb{R}^n$ *is regular.*

Proof. The image $L = \Phi^f(\tilde{\mathcal{N}}_\delta^{(2)}(\bar{x},\bar{t}))$ is a compact subset of Ω. Let K be a compact subset of Ω such that $L \subseteq \mathrm{Int}(K)$. We will show that

(*) *if* $\{(\rho_j, x_j, t_j, s_j)\}_{j\in\mathbb{N}}$ *is a sequence in* $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$ *such that* $\rho_j > 0$, $(x_j,t_j,s_j) \in \tilde{\mathcal{N}}_\delta^{(2)}(\bar{x},\bar{t})$, $\rho_j \to 0$, $(x_j,t_j,s_j)\to(x,t,s)$, *then*

(*.1) $(x_j,t_j,s_j) \in \mathrm{Dom}(\Phi^{f_{\rho_j}})$ *for large enough* j

and

(*.2) *if* $y_j = \Phi^{f_{\rho_j}}(x_j,t_j,s_j)$, *then* $y_j \in K$ *for large enough* j, *and every limit point* y *of the sequence* $\{y_j\}$ *satisfies* $y \in \Phi^f(x,t,s)$.

It is clear that (*) implies our desired conclusion. To prove (*), we first fix a $\bar{\rho} > 0$ such that $K_{\bar{\rho}} \subseteq \Omega$, and write $\varphi = \varphi_{K_{\bar{\rho}}}$.

Assume that the hypothesis of (*) holds. Define a compact interval $\tilde{I}$ by letting $\tilde{I} = [\bar{t}-\delta, \bar{t}+\delta]$ if $t \in \mathrm{Int}(I)$, $\tilde{I} = [\bar{t}, \bar{t}+\delta]$ if $t = \min I$, $\tilde{I} = [\bar{t}-\delta, \bar{t}]$ if $t = \max I$. Let ξ_j denote the maximal trajectory of f_{ρ_j} that goes through x_j at time s_j. Let $J_j = \mathrm{Dom}(\xi_j)$, so that s_j lies in the interior of J_j relative to I. Let $I_j = [\min(s_j,t_j), \max(s_j,t_j)]$, so that $I_j \subseteq \tilde{I}$. Let H_j denote the set of all $\tau \in I_j \cap J_j$ such that $\xi_j(u) \in K$ for all $u \in [\min(s_j,\tau), \max(s_j\tau)]$. Then H_j is a compact interval, $H_j \subseteq J_j \cap I_j$ and s_j is one of the endpoints of H_j. We let τ_j be the other endpoint, so $H_j = [\min(s_j,\tau_j), \max(s_j,\tau_j)]$. Then, if we let $\partial K = K\backslash\mathrm{Int}(K)$, it is easy to verify that

Fact 9. *For each* j, *either* $\tau_j = t_j$ *or* $\xi_j(\tau_j) \in \partial K$.

Let $z_j = \xi_j(\tau_j)$. Then $(x_j,\tau_j,s_j,z_j) \in \tilde{\mathcal{N}}_\delta^{(2)}(\bar{x},\bar{t}) \times K$. Write $\eta_j = \xi_j\lceil H_j$. Then $\eta_j \in \mathrm{Traj}_c(f_{\rho_j})$, and $\eta_j(t) \in K$ whenever $t \in H_J$, so η_j belongs to $ARC(\mathbb{R}^n; K\times\tilde{I})$, and

$$\|\dot{\eta}_j(r)\| \le \varphi(r) \text{ for } r \in H_j\,. \tag{3.8.8}$$

It then follows that the sequence $\boldsymbol{\eta} = \{\eta_j\}_{j\in\mathbb{N}}$ is uniformly bounded and equicontinuous. Therefore every subsequence of $\boldsymbol{\eta}$ has a subsequence that converges to a limit in $ARC(\mathbb{R}^n)$. Equivalently, every infinite subset V

of $\mathbb{N}$ contains an infinite subset W such that the sequence $\boldsymbol{\eta}^W \stackrel{\text{def}}{=} \{\eta_j\}_{j\in W}$ converges in $ARC(\mathbb{R}^n)$ to a limit η^W.

Let $\mathcal{W}$ be the set of all infinite subsets W of $\mathbb{N}$ be such that $\boldsymbol{\eta}^W$ converges. We will prove that

$$W \in \mathcal{W} \Longrightarrow \eta^W \in \mathrm{Traj}\,(f)\,. \tag{3.8.9}$$

We first show that (3.8.9) implies our conclusion. Let

$$W' = \{j \in \mathbb{N} : t_j \notin J_j\}\,,\ \ W'' = \{j \in \mathbb{N} : t_j \neq \tau_j\}\,. \tag{3.8.10}$$

Then

$$j \in W' \Rightarrow j \in W'' \Rightarrow z_j \in \partial K\,, \tag{3.8.11}$$

where the first implication follows because $\tau_j \in J_j$, and the second one is a consequence of Fact 9. If the set W'' was infinite, then we could pick W to be a subset of W'' such that $W \in \mathcal{W}$, and apply (3.8.9) to conclude that $\eta^W \in \mathrm{Traj}\,(f)$. Since $\mathrm{Dom}(\eta_j) = H_j = [\min(s_j,\tau_j),\max(s_j,\tau_j)]$, and the sequence $\{\eta_j\}_{j\in W}$ converges to η^W in $ARC(\mathbb{R}^n)$, it is clear that the limit $\tau^W = \lim_{j\to\infty, j\in W}$ exists, and $\mathrm{Dom}(\eta^W) = H^W = [\min(s,\tau^W),\max(s,\tau^W)]$. Let $z^W = \eta^W(\tau^W)$. Then $z^W = \lim_{j\to\infty\,,\,j\in W} z_j$, so (3.8.11) implies that $z \in \partial K$. Since $z^W = \eta^W(\tau^W)$, (3.8.9) implies that $z^W \in \Phi^f(x,\tau^W,s)$. Since $(x,\tau^W,s) \in \tilde{\mathcal{N}}^{(2)}_\delta(\bar{x},\bar{t})$ (because $(x_j,\tau_j,s_j) \in \tilde{\mathcal{N}}^{(2)}_\delta(\bar{x},\bar{t})$ for every j), $z^W \in L \subseteq \mathrm{Int}\,K$. So we have reached a contradiction, proving that W'' is finite. Then W' is finite as well, and this shows that (*.1) holds. Moreover, the fact that W'' is finite shows that $t_j = \tau_j$ for sufficiently large j, so $z_j = y_j$ for large j, proving that $y_j \in K$ for large enough j. Finally, let y be a limit point of the sequence $\{y_j\}_{j\in\mathbb{N}}$. Pick an infinite subset W' of $\mathbb{N}$ such that $y_j \to y$ as $j \to \infty$ via values in W'. Then pick $W \in \mathcal{W}$ such that $W \subseteq W'$. Then $y = \eta^W(t)$. This shows that $y \in \Phi^f(x,s,t)$, and the proof of (*) is complete.

To conclude our proof, we must establish (3.8.9). Fix $W \in \mathcal{W}$, and let $\tau^W = \lim_{j\to\infty,j\in W}\tau_j$, $H^W = \mathrm{Dom}(\eta^W) = [\min(s,\tau^W),\max(s,\tau^W)]$, $z^W = \eta^W(\tau^W)$. Our conclusion is trivially true if $s = \tau^W$. Assume that $s \neq \tau^W$. Pick a number u such that $\min(s,\tau^W) < u < \max(s,\tau^W)$. Let $N = P \times Q$, where P is a compact neighborhood of $\eta^W(u)$ in Ω, and Q is a compact interval which is contained in the interior of H^W and is such that u is an interior point of Q. Using the admissibility of f, choose an integrable function $\gamma : Q \mapsto [0,\infty]$ such that $\|f(x,t)\| \leq \gamma(t)$ for all $(x,t) \in N$, and f is integrally continuous on $ARC_\gamma(\mathbb{R}^n;N)$. Write $\mathcal{A} = ARC_\gamma(\mathbb{R}^n;N)$. Then there exist an $\varepsilon > 0$ and a $j^* \in \mathbb{N}$ such that the following are true for $j \geq j^*$:

(1) $\rho_j \leq \varepsilon$,

(2) $Q \subseteq H_j$,

(3) $(\eta_j(v) + z, v) \in N$ whenever $|v-u| \leq \varepsilon$, $\|z\| \leq \varepsilon$.

(Indeed, (2) holds if j is large enough because the endpoints of H_j converge to the endpoints of H^W and $Q \subseteq \mathrm{Int}(H^W)$. Given ε, (1) holds for large enough j because $\rho_j \to 0$. So all we need is to show that there exists an ε such that (3) holds for large enough j. If ε did not exist, there would exist, for each $k \in \mathbb{N}$, a $j(k)$ such that $j(k) \geq k$ and a pair (v_k, z_k) such that $\|z_k\| \leq 2^{-k}$, $\|v_k - u\| \leq 2^{-k}$, and $(\eta_{j(k)}(v_k) + z_k, v_k) \notin N$. But then $v_k \to u$, and $z_k \to 0$ as $k \to \infty$, so $(\eta_{j(k)}(v_k) + z_k, v_k) \to (\eta^W(u), u)$. Since $(\eta^W(u), u)$ is an interior point of N, this is a contradiction.)

Then, for $j \geq j^*$, if $v_1, v_2 \in [u-\varepsilon, u+\varepsilon]$ and $v_1 < v_2$, we have

$$\begin{aligned}
\eta_j(v_2) - \eta_j(v_1) &= \int_{v_1}^{v_2} f_{\rho_j}(\eta_j(w), w)dw \\
&= \int_{v_1}^{v_2} \Big(\int_{\mathbb{R}^n} f(\eta_j(w) - \rho_j z, w)\theta(z)\, dz \Big)\, dw \\
&= \int_{\mathbb{R}^n} \Big(\int_{v_1}^{v_2} f(\eta_j(w) - \rho_j z, w)\, dw \Big)\theta(z)\, dz \\
&= \int_{\mathbb{R}^n} \Big(\int_{v_1}^{v_2} f(\eta_j^z(w), w)\, dw \Big)\theta(z)\, dz \\
&= \int_{\mathbb{R}^n} \Big(\mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_2) - \mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_1) \Big)\theta(z)\, dz\,,
\end{aligned}$$

where $\eta_j^z : [u-\varepsilon, u+\varepsilon] \to \Omega$ is defined by

$$\eta_j^z(w) = \eta_j(w) - \rho_j z \ \text{ for } \ |w-u| \leq \varepsilon\,,$$

and the interchange of the integrals is justified by Fubini's Theorem, using the bound $\|f(\xi_j(w) - \rho_j z, w)\| \leq \gamma(w)$, which is valid because the point $(\xi_j(w) - \rho_j z, w)$ belongs to N. It is clear that the curves η_j^z belong to $\mathcal{A}$ whenever $\|z\| \leq 1$ and $j \geq j^*$. Moreover, the η_j^z converge in $ARC(\mathbb{R}^n)$ to the restriction $\eta^W \lceil [u-\varepsilon, u+\varepsilon]$. Since f is integrally continuous on $\mathcal{A}$, it follows that $\mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_2) - \mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_1) \to \int_{v_1}^{v_2} f(\eta^W(w), w)dw$ for each $z \in \mathbb{R}^n$ such that $\|z\| \leq 1$. Since $\|\mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_2) - \mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_1)\| \leq \int_{u-\varepsilon}^{u+\varepsilon} \gamma(w)dw$, and θ is bounded and supported in $\{z : \|z\| \leq 1\}$, we have

$$\begin{aligned}
\eta^W(v_2) - \eta^W(v_1) &= \lim_{j\to\infty} \Big(\eta_j(v_2) - \eta_j(v_1) \Big) \\
&= \lim_{j\to\infty} \int_{z \in \mathbb{R}^n, \|z\| \leq 1} \Big(\mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_2) - \mathcal{T}_f^{\mathcal{A}}(\eta_j^z)(v_1) \Big)\theta(z)\, dz \\
&= \int_{v_1}^{v_2} f(\eta^W(w), w)dw\,.
\end{aligned}$$

So every interior point u of H^W has a neighborhood $[u-\varepsilon, u+\varepsilon]$ with the property that $[u-\varepsilon, u+\varepsilon] \subseteq H^W$ and $\eta^W\lceil[u-\varepsilon, u+\varepsilon]$ is a trajectory of f. This clearly implies that $\eta^W \in \mathrm{Traj}\,(f)$, completing the proof. ♢

3.9 Regular differentiability of the flow

We now combine the results of the two previous subsections into a single statement, asserting the regular differentiability of the flow of an admissible vector field $f \in ADM(\mathbb{R}^n)$ at every point of approximate continuity of f, and explicitly exhibiting a differential.

Theorem 6. *Assume that $n \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f \in ADM(\mathbb{R}^n; S)$. Let $(\bar{x}, \bar{t})$ be a point of approximate continuity of f. Then the flow Φ^f is regularly differentiable at $(\bar{x}, \bar{t}, \bar{t})$, and the linear map $\Lambda^{f(\bar{x},\bar{t})}$ defined by (3.7.2) is a differential of Φ^f at $(\bar{x}, \bar{t}, \bar{t})$.*

Proof. Theorem 5 gives us a neighborhood of $(\bar{x}, \bar{t}, \bar{t})$ where Φ^f is regular, and Theorem 4 says that $\Lambda^{f(\bar{x},\bar{t})}$ is a first-order approximation to Φ^f near $(\bar{x}, \bar{t}, \bar{t})$. ♢

Corollary 4. *Assume that $n \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, and $f \in ADM(\mathbb{R}^n; S)$. Let $(\bar{x}, \bar{t})$ be a point of forward (resp. backward) approximate continuity of f. Then the flow Φ^f is forward (resp. backward) regularly differentiable at $(\bar{x}, \bar{t}, \bar{t})$, and the forward (resp. backward) differential of Φ^f at $(\bar{x}, \bar{t}, \bar{t})$ is the map $\Lambda^{f(\bar{x},\bar{t})}$ defined by (3.7.2).*

Proof. This follows by applying Theorem 6 to the TVVF g considered in the proof of Theorem 3. ♢

4 Almost LSC differential inclusions

4.1 Lower semicontinuous set-valued maps

Definition 18. Assume that X and Y are topological spaces. A set-valued map $F : X \mapsto\!\!\!\to Y$ is *lower semicontinuous* (abbr. LSC) if the inverse image under F of every open subset of Y is open in X. ♢

The following observation is a trivial consequence of the definition.

Fact 10. *If X, Y are topological spaces and $F : X \mapsto Y$ is an ordinary (i.e. single-valued, everywhere defined) map, then F is LSC as a set-valued map if and only if F is a continuous map.* ♢

In addition, the following fact is well known.

Lemma 2. *Let S be a topological space and let Q be a metric space. Suppose that $F : S \mapsto\!\!\!\rightarrow Q$ is a set-valued map with nonempty values. For each ordinary map $f : S \mapsto Q$, define a function $\rho_{f,F} : S \to \mathbb{R}$ by letting*

$$\rho_{f,F}(s) \stackrel{\text{def}}{=} d_Q(f(s), F(s))$$

Then the following three conditions are equivalent:

- *(L2.a) F is lower semicontinuous;*
- *(L2.b) $\rho_{f,F}$ is upper semicontinuous for every continuous f;*
- *(L2.c) $\rho_{f,F}$ is upper semicontinuous for every constant f.*

Proof. We first prove that (L2.a) implies (L2.b). Assume F is LSC and f is continuous. Let $s \in S$, and let $\varepsilon > 0$. Pick $q \in F(s)$ such that $d_Q(f(s), q) < \rho_{f,F}(s) + \frac{\varepsilon}{3}$. Then choose a neighborhood U of s such that, whenever $s' \in U$, (a) $F(s') \cap B_Q(q, \frac{\varepsilon}{3}) \neq \emptyset$, and (b) $d_Q(f(s'), f(s)) < \frac{\varepsilon}{3}$. Then $d_Q(f(s'), F(s')) < \rho_{f,F}(s) + \varepsilon$ whenever $s' \in U$, so $\rho_{f,F}(s') < \rho_{f,F}(s) + \varepsilon$ for all $s' \in U$. So $\rho_{f,F}$ is upper semicontinuous.

It is cleat that (L2.b) implies (L2.c). We conclude by proving that (L2.c) implies (L2.a). Assume that (L2.c) holds. Let $U \subseteq Q$ be open. Let $s \in F^{-1}(U)$, so $F(s) \cap U \neq \emptyset$. Pick $q \in F(s) \cap U$. Let ε be such that $\varepsilon > 0$ and $B_Q(q, \varepsilon) \subseteq U$. Let $f : S \mapsto Q$ be the constant function with value q. Then $\rho_{f,F}$ is continuous. Since $\rho_{f,F}(s) = 0$, because $q \in F(s)$, there is a neighborhood V of s in S such that $\rho_{f,F}(s') < \varepsilon$ for all $s' \in V$. It follows that $F(s') \cap B_Q(q, \varepsilon) \neq \emptyset$ for all $s' \in V$. Since $B_Q(q, \varepsilon) \subseteq U$, we see that $V \subseteq \{s' \in S : F(s') \cap U \neq \emptyset\} = F^{-1}(U)$. So $F^{-1}(U)$ is a neighborhood of s whenever $s \in F^{-1}(U)$. Therefore $F^{-1}(U)$ is open. So F is LSC. $\diamondsuit$

4.2 Conically continuous selections of LSC set-valued maps

Theorem 7. *Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, and $F : S \mapsto\!\!\!\rightarrow \mathbb{R}^m$ is a lower semicontinuous set-valued map with nonempty closed values. Let $g : S \mapsto \mathbb{R}^m$, $\beta : S \mapsto \mathbb{R}$ be continuous maps such that*

$$\beta(x,t) > \rho_{g,F}(x,t) \text{ for all } (x,t) \in S\,. \tag{4.2.1}$$

Let $(\bar{x}, \bar{t}) \in S$, $\bar{y} \in F(\bar{x}, \bar{t})$, be such that

$$\|g(\bar{x}, \bar{t}) - \bar{y}\| < \beta(\bar{x}, \bar{t})\,. \tag{4.2.2}$$

Let $C : S \mapsto]\,0, \infty\,[$ be upper semicontinuous. Then there exists a single-valued selection $f : S \mapsto \mathbb{R}^m$ of F such that

(T7.1) $f(\bar{x},\bar{t}) = \bar{y}$,

(T7.2) f *is continuous at* $(\bar{x},\bar{t})$,

(T7.3) f *is forward* Γ_C*-continuous,*

(T7.4) $\|f(x,t) - g(x,t)\| < \beta(x,t)$ *for all* $(x,t) \in S$,

(T7.5) f *is* $\mathrm{Bor}(\mathbb{R}^n \times \mathbb{R}) \cap S$*-measurable.*

Proof. Let $\tilde{F}$ be the set-valued function given by

$$\tilde{F}(x,t) = \begin{cases} F(x,t) & \text{if} \quad (x,t) \in S\backslash\{(\bar{x},\bar{t})\}\,, \\ \{\bar{y}\} & \text{if} \quad (x,t) = (\bar{x},\bar{t})\,. \end{cases}$$

Then $\tilde{F}$ is also LSC, and

$$\begin{aligned} \rho_{g,\tilde{F}}(x,t) &= \rho_{g,F}(x,t) \quad \text{when} \quad (x,t) \neq (\bar{x},\bar{t})\,, \\ \rho_{g,\tilde{F}}(\bar{x},\bar{t}) &= \|g(\bar{x},\bar{t}) - \bar{y}\|\,. \end{aligned}$$

Therefore $\rho_{g,\tilde{F}}(x,t) < \beta(x,t)$ for all $(x,t) \in S$. Then $\tilde{F}$ satisfies the same hypotheses as F, and in addition $F(\bar{x},\bar{t}) = \{\bar{y}\}$. So we may—and will—assume, without loss of generality, that $F(\bar{x},\bar{t}) = \{\bar{y}\}$.

If $N \in \mathbb{N}$ and $N > 0$, we let $S_N = \{(x,t) \in S : C(x,t) < N\}$. Then the upper semicontinuity of C implies that every S_N is relatively open in S, and the fact that C has finite values implies that and $S = \bigcup_{N=1}^{\infty} S_N$.

We define, for each $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, each $N \in \mathbb{N}\backslash\{0\}$, and each α such that $\alpha > 0$, a set $A(x,t,\alpha,N)$, by

$$A(x,t,\alpha,N) = \Big\{(x',t') \in \mathbb{R}^n \times \mathbb{R} : t \leq t' < t+\alpha,\, \|x'-x\| \leq N(t'-t)\Big\}\,. \tag{4.2.3}$$

We let $\mathbf{A}$ be the set of all sets $A(x,t,\alpha,N)$, for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $\alpha \in \,]0,1]$, $N \in \mathbb{N}\backslash\{0\}$, such that $A(x,t,\alpha,N) \cap S \subseteq S_N$.

We then let $\mathbf{A}^*$ be the set of all those sets $E \in \mathbf{A}$ such that $(\bar{x},\bar{t}) \notin \partial E$. (Here $\partial E \stackrel{\text{def}}{=} \mathrm{Clos}(E)\backslash\mathrm{Int}(E)$, the closure and the interior being taken relative to $\mathbb{R}^n \times \mathbb{R}$.) We observe that

Fact 11. *If* $(x,t) \in S$ *is arbitrary, and* V *is a neighborhood of* (x,t) *in* $\mathbb{R}^n \times \mathbb{R}$*, then* V *contains a set* $E \in \mathbf{A}^*$ *which is a neighborhood of* (x,t) *in* $\mathbb{R}^n \times \mathbb{R}$*.*

Indeed, if suffices to pick $E = A(x, t - \frac{\alpha}{2}, \alpha, N)$ where N is such that $(x,t) \in S_N$, and α is sufficiently small. If this choice happens to result in a set E such that $(\bar{x},\bar{t}) \in \partial E$, then we take instead the smaller set $\tilde{E} = A(x, t - \frac{\alpha}{4}, \frac{\alpha}{2}, N)$, and observe that $\partial E \cap \partial\tilde{E} = \emptyset$, so $\tilde{E}$ has the desired property, and Fact 11 is proved.

We then let $\mathcal{G}$ be the Boolean algebra of subsets of S generated by the sets $E \cap S$, $E \in \mathbf{A}^*$.

Let $\Phi_{\mathcal{G}}$ be the set of all maps $f : S \mapsto \mathbb{R}^n$ such that there exists a partition $\mathcal{P}$ of S into members of $\mathcal{G}$ with the property that the restriction of f to each $G \in \mathcal{P}$ is continuous. Let $\bar{\Phi}_{\mathcal{G}}$ be the closure of $\Phi_{\mathcal{G}}$ with respect to uniform convergence on compact subsets of S. We claim that

Fact 12. *If $f \in \bar{\Phi}_{\mathcal{G}}$, then f is continuous at $(\bar{x}, \bar{t})$ and forward Γ_C-continuous.*

To prove Fact 12, we pick a sequence $\{(x^j, t^j)\}_{j \in \mathbb{N}}$ in S that converges to a limit $(x, t) \in S$, and assume that either

$$x = \bar{x} \ \text{ and } \ t = \bar{t}\,, \tag{4.2.4}$$

or

$$\|x^j - x\| \le C(x,t)(t^j - t) \ \text{ for all } \ j\,. \tag{4.2.5}$$

We must show that $f(x^j, t^j) \to f(x, t)$. Since f is a uniform limit on compact sets of functions in $\Phi_{\mathcal{G}}$, and the set $\{(x,t)\} \cup \{(x^j, t^j) : j \in \mathbb{N}\}$ is compact, it suffices to assume that $f \in \Phi_{\mathcal{G}}$. For this purpose, we show

Fact 13. *Every member H of $\mathcal{G}$ has the following property:*

(P) if $(x,t) \in H$ then $(x^j, t^j) \in H$ for sufficiently large j.

To prove Fact 13, we let $\mathcal{Z}$ be the set of all subsets H of S such that both H and $S \backslash H$ have property (P). Then $\mathcal{Z}$ is clearly closed under complementation (that is, $H \in \mathcal{Z} \Rightarrow S \backslash H \in \mathcal{Z}$) and under the binary operations of union and intersection. So to prove that every $H \in \mathcal{G}$ has (P) it suffices to show that $E \cap S \in \mathcal{Z}$ whenever $E \in \mathbf{A}^*$. That is, we must pick $E \in \mathbf{A}^*$ and show that (P) holds for $H = E \cap S$ and also for $H = S \backslash E$. Let $E = A(x', t', \alpha, N)$.

Assume first that (4.2.4) holds, i.e that $(x,t) = (\bar{x}, \bar{t})$. Then (x,t) does not belong to ∂E, so (x,t) is an interior point of E or of $(\mathbb{R}^n \times \mathbb{R}) \backslash E$. So $(x^j, t^j) \in E$ for large j if $(x,t) \in E$, and $(x^j, t^j) \in (\mathbb{R}^n \times \mathbb{R}) \backslash E$ for large j if $(x,t) \in (\mathbb{R}^n \times \mathbb{R}) \backslash E$. Therefore $(x^j, t^j) \in E \cap S$ for large j if $(x,t) \in E \cap S$, and $(x^j, t^j) \in S \backslash E$ for large j if $(x,t) \in S \backslash E$. So (P) holds both for $H = E \cap S$ and $H = S \backslash E$.

Now consider the case when $(x,t) \neq (\bar{x}, \bar{t})$, so that (4.2.5) holds. Then $t^j \ge t$ and $\|x^j - x\| \le C(x,t)(t^j - t)$ for all j. If (x,t) is an interior point of E or of $(\mathbb{R}^n \times \mathbb{R}) \backslash E$, then the desired conclusion follows as in the previous case. So let us assume that $(x,t) \in \partial E$. Then either $t < t' + \alpha$ or $t = t' + \alpha$. We will consider these two cases separately.

If $t < t' + \alpha$, then $\|x - x'\| = N(t - t')$, and $(x, t) \in E$. But $(x, t) \in S$, and $E \cap S \subseteq S_N$, so (x, t) belongs to S_N, and then $C(x, t) < N$. Therefore $\|x^j - x\| \leq N(t^j - t)$ for all j, from which it follows that

$$\|x^j - x'\| \leq N(t^j - t) + N(t - t') = N(t^j - t') \quad \text{for all } j\,.$$

Since $t < t' + \alpha$ and $t^j \to t$, we have $t^j < t' + \alpha$ for large j. So $(x^j, t^j) \in E$ for large j, and then $(x^j, t^j) \in E \cap S$ for large j, so (P) holds if $H = E \cap S$. Furthermore, (P) also holds, vacuously, if $H = S \backslash E$, because the premiss "$(x, t) \in H$" is false.

If $t = t' + \alpha$, then $(x, t) \in S \backslash E$. Moreover, $(x^j, t^j) \notin E$ for all j, since $t^j \geq t \geq t' + \alpha$. So $(x^j, t^j) \in S \backslash E$ for all j, showing that (P) holds for $H = S \backslash E$. Since $(x, t) \notin E$, (P) holds vacuously for $H = E \cap S$.

We have thus shown that property (P) holds in all possible cases both for $H = E \cap S$ and $H = S \backslash E$, whenever $E \in \mathbf{A}^*$. Therefore (P) holds for all $H \in \mathcal{G}$.

Having shown that (P) holds for all $H \in \mathcal{G}$, Fact 12 follows easily: as explained above, it suffices to show for our given (x, t) and our given sequence $\{(x^j, t^j)\}$ that $f(x^j, t^j) \to f(x, t)$ if $f \in \Phi_{\mathcal{G}}$. But this is trivial, for (x, t) belongs to a $G \in \mathcal{G}$ on which f is continuous, and then $(x^j, t^j) \in G$ for large j in view of (P), so $f(x^j, t^j) \to f(x, t)$. Therefore Fact 12 is proved.

We are now ready to construct the desired selection f of F.

First, we observe that, since the function $\rho_{g,F}$ is upper semicontinuous, (4.2.1) implies that there exists a continuous function $\gamma : S \mapsto \mathbb{R}$ such that

$$\rho_{g,F}(x, t) < \gamma(x, t) < \beta(x, t) \text{ for all } (x, t) \in S\,. \tag{4.2.6}$$

(Proof: for every $q = (x, t) \in S$, pick a relatively open bounded subset U_q of S such that $q \in U_q$ and a constant κ_q such that $\rho_{g,F}(q') < \kappa_q < \beta(q')$ for all $q' \in U_q$. Then $\mathcal{U} = \{U_q : q \in S\}$ is an open covering of S. Since S is metric and hence paracompact, there exists a locally finite set $\mathcal{V}$ of relatively open subsets of S which is a covering of S and a refinement of $\mathcal{U}$. For each $V \in \mathcal{V}$, pick a point q_V of S such that $V \subseteq U_{q_V}$. Define functions $\psi_V : S \mapsto [0, \infty[$ by $\psi_V(v) = \text{dist}(v, S \backslash V)$ for $v \in S$. Then the ψ_V are continuous, nonnegative, and such that $\psi_V(v) > 0$ if and only if $v \in V$. Let $\psi = \sum_{V \in \mathcal{V}} \psi_V$. Define $\varphi_V = \frac{\psi_V}{\psi}$. Then the φ_V are continuous, nonnegative, and such that $\varphi_V(v) > 0$ if and only if $v \in V$, and $\sum_{V \in \mathcal{V}} \varphi_V \equiv 1$. Now define $\gamma(q) = \sum_{V \in \mathcal{V}} \varphi_V(q) \kappa_{q_V}$.)

We then let $\delta = \frac{1}{2}(\beta - \gamma)$, and define $\delta_k = (1 - 2^{-k})\delta$ for $k = 0, 1, \ldots$, so δ and the δ_k are continuous functions from S to $\mathbb{R}$.

Pick, for each $(x,t) \in S$, a point $z_{x,t} \in F(x,t)$ such that

$$\|g(x,t) - z_{x,t}\| < \gamma(x,t)\,. \tag{4.2.7}$$

(This is possible because of (4.2.6)). Then find neighborhoods $G_{x,t}$ of (x,t) in $\mathbb{R}^n \times \mathbb{R}$ such that $G_{x,t} \in \mathbf{A}^*$, with the property that the inequalities

$$d(z_{x,t}, F(x',t')) \quad < \quad \frac{1}{2}\delta(x',t')\,, \tag{4.2.8}$$

$$\|g(x',t') - z_{x,t}\| \quad < \quad \gamma(x',t')\,, \tag{4.2.9}$$

hold for all $(x',t') \in G_{x,t} \cap S$. (The fact that (4.2.8) can be achieved is a consequence of the lower semicontinuity of F, together with the continuity of δ, since $d(z_{x,t}, F(x,t)) = 0$, and $\delta(x,t) > 0$. The fact that (4.2.9) can also be obtained follows from the continuity of γ and g, together with (4.2.7)).

Next, find a sequence $(x_1,t_1), (x_2,t_2), \ldots$ such that the sets $G_{x_j,t_j} \cap S$, for $j = 1, 2, \ldots$ cover S. (This is possible because S is separable metric and hence second countable.) Let $W_1^0 = G_{x_1,t_1} \cap S$, and define inductively

$$W_j^0 = (G_{x_j,t_j} \cap S)\backslash(W_1^0 \cup W_2^0 \cup \ldots W_{j-1}^0) \;\text{ for }\; j = 2, 3, \ldots\,. \tag{4.2.10}$$

Then $\mathcal{P}^0 = \{W_j^0 : j = 1, 2, \ldots\}$ is a partition of S into members of $\mathcal{G}$. Define $f_0(x,t) = z_{x_j,t_j}$ if $(x,t) \in W_j^0$. Then f_0 is constant on each member of $\mathcal{P}^0$, and the inequalities

$$d(f_0(x,t), F(x,t)) < \frac{1}{2}\delta(x,t)\,, \tag{4.2.11}$$

$$\|f_0(x,t) - g(x,t)\| < \gamma(x,t)\,, \tag{4.2.12}$$

hold for all (x,t).

We now suppose that we have constructed maps $f_0, \ldots, f_k : S \mapsto Y$, and finite or countable partitions $\mathcal{P}^0, \ldots, \mathcal{P}^k$ of S into members of $\mathcal{G}$, such that

(i) f_j is constant on each member of $\mathcal{P}^j$ for $j = 0, \ldots, k$,

(ii) the inequalities

$$d(f_j(x,t), F(x,t)) < 2^{-j-1}\delta(x,t)$$

$$\|g(x,t) - f_j(x,t)\| < \gamma(x,t) + \delta_j(x,t)$$

hold for all $(x,t) \in S$, $j \in \{0, \ldots, k\}$,

(iii) $\|f_j(x,t) - f_{j-1}(x,t)\| \le 2^{-j}\gamma(x,t)$ for all $(x,t) \in S$, $j \in \{1, \ldots, k\}$,

(iv) for each $j \in \{1,\ldots,k\}$, $\mathcal{P}^j$ is a refinement of $\mathcal{P}^{j-1}$.

(Notice that all these conditions hold for $k = 0$, since (i) and (ii) have already been established, and (iii) and (iv) are vacuously true.)

We then construct f_{k+1} as follows. Pick a $Q \in \mathcal{P}^k$. Let $y^{k,Q}$ be the constant value of f_k on Q. Then

$$d(y^{k,Q}, F(x,t)) = d(f_k(x,t), F(x,t)) < 2^{-k-1}\delta(x,t) \tag{4.2.13}$$

and

$$\|g(x,t) - y^{k,Q}\| < \gamma(x,t) + \delta_k(x,t) \tag{4.2.14}$$

for all $(x,t) \in Q$. So we can pick, for each $(x,t) \in Q$, a $z^{k,Q}_{x,t} \in F(x,t)$ such that

$$\|y^{k,Q} - z^{k,Q}_{x,t}\| < 2^{-k-1}\delta(x,t)\,. \tag{4.2.15}$$

Then

$$\begin{array}{rcl} \|g(x,t) - z^{k,Q}_{x,t}\| & < & \gamma(x,t) + \delta_k(x,t) + 2^{-k-1}\delta(x,t) \\ & = & \gamma(x,t) + \delta_{k+1}(x,t)\,. \end{array}$$

Using the lower semicontinuity of F and the continuity of δ, γ, g, and δ_{k+1}, we can find for each $(x,t) \in Q$ a relative neighborhood $W^{k,Q}(x,t)$ of (x,t) in S such that the inequalities

$$\begin{array}{rcl} d(z^{k,Q}_{x,t}, F(x',t')) & < & 2^{-k-2}\delta(x',t')\,, \\ \|y^{k,Q} - z^{k,S}_{x,t}\| & < & 2^{-k-1}\delta(x',t')\,, \\ \|g(x',t') - z^{k,Q}_{x,t}\| & < & \gamma(x',t') + \delta_{k+1}(x',t')\,, \end{array}$$

hold for all $(x',t') \in W^{k,Q}(x,t)$. Using Fact 11, we can assume, after shrinking $W^{k,Q}(x,t)$ if necessary, that $W^{k,Q}(x,t) \in \mathcal{G}$.

We then find for each $Q \in \mathcal{P}^k$ a finite or countable subset $M^{k,Q}$ of Q such that

$$Q \subseteq \bigcup \left\{ W^{k,Q}(x,t) : (x,t) \in M^{k,Q} \right\}.$$

We enumerate each $M^{k,Q}$ in an arbitrary fashion as a finite or countably infinite sequence $\left\{(x^{k,Q}_j, t^{k,Q}_j)\right\}_{1\le j<m(k,Q)}$, where $m(k,Q) \in \mathbb{N} \cup \{\infty\}$. We then define

$$\hat{W}^{k,Q}_1 = Q \cap W^{k,Q}(x^{k,Q}_1, t^{k,Q}_1)$$

and, for $1 < \ell < m(k,Q)$, we let

$$\hat{W}^{k,Q}_\ell = \left(Q \cap W^{k,Q}(x^{k,Q}_\ell, t^{k,Q}_\ell)\right)\backslash(\hat{W}^{k,Q}_1 \cup \ldots \cup \hat{W}^{k,Q}_{\ell-1})\,.$$

We then let $\mathcal{P}^{k+1}$ be the collection of all sets $\hat{W}_\ell^{k,Q}$, for all $Q \in \mathcal{P}^k$ and all $\ell \in \mathbb{N}$ such that $1 \le \ell < m(k,Q)$. Then $\mathcal{P}^{k+1}$ is a finite or countable partition of S into members of $\mathcal{G}$, and is a refinement of $\mathcal{P}^k$. If $(x,t) \in S$, we define

$$f_{k+1}(x,t) = z^{k,Q}_{x_\ell^{k,Q},t_\ell^{k,Q}}\,, \tag{4.2.16}$$

where Q is the unique member of $\mathcal{P}^k$ such that $(x,t) \in Q$, and ℓ is the unique integer such that $1 \le \ell < m(k,Q)$ and $(x,t) \in \hat{W}_\ell^{k,Q}$. Then f_{k+1} is constant on each member of $\mathcal{P}^{k+1}$. Moreover, if $(x,t) \in \hat{W}_\ell^{k,Q}$, then $(x,t) \in W_\ell^{k,Q}(x_\ell^{k,Q}, t_\ell^{k,Q})$, so that the inequalities

$$\begin{aligned}
\|f_{k+1}(x,t) - f_k(x,t)\| &= \|z^{k,Q}_{x_\ell^{k,Q},t_\ell^{k,Q}} - y^{k,Q}\| \le 2^{-k-1}\delta(x,t)\,,\\
d(f_{k+1}(x,t), F(x,t)) &= d(z^{k,Q}_{x_\ell^{k,Q},t_\ell^{k,Q}}, F(x,t)) < 2^{-k-2}\delta(x,t)\,,\\
\|g(x,t) - f_{k+1}(x,t)\| &= \|g(x,t) - z^{k,Q}_{x_\ell^{k,Q},t_\ell^{k,Q}}\| < \gamma(x,t) + \delta_{k+1}(x,t)\,.
\end{aligned}$$

are satisfied. It follows that the maps $f_0, \ldots, f_{k+1}$ and the corresponding partitions $\mathcal{P}^0, \ldots, \mathcal{P}^{k+1}$ also satisfy (i),. . .,(iv).

The above inductive construction therefore leads to an infinite sequence $\{f_j\}$ of members of $\Phi_{\mathcal{G}}$. In view of (iii), this sequence converges uniformly on compact subsets of S to a limit f. It then follows from (ii), together with the fact that F has closed values, that f is a selection of F and

$$\|g(x,t) - f(x,t)\| \le \gamma(x,t) + \delta(x,t) < \beta(x,t) \quad \text{for all} \quad (x,t) \in S\,.$$

In view of Fact 12, f is continuous at $(\bar{x},\bar{t})$ and forward Γ_C-continuous on S, since $f \in \bar{\Phi}_{\mathcal{G}}$ by construction.

It is clear that $f(\bar{x},\bar{t}) = \bar{y}$, because the map f is a selection of F and $F(\bar{x},\bar{t}) = \{\bar{y}\}$. In addition, the maps f_j are obviously $\mathrm{Bor}(\mathbb{R}^n \times \mathbb{R}) \cap S$-measurable, because $\mathcal{G} \subseteq \mathrm{Borel}(\mathbb{R}^n \times \mathbb{R}) \cap S$, since $\mathbf{A}^* \subseteq \mathrm{Borel}(\mathbb{R}^n \times \mathbb{R})$. Since $f_j \to f$ pointwise, we see that f is $\mathrm{Bor}(\mathbb{R}^n \times \mathbb{R}) \cap S$-measurable. So f satisfies all our conditions, and our proof is complete. $\diamondsuit$

4.3 Almost lower semicontinuous set-valued maps

The natural analogue of the well known Scorza-Dragoni theorem for continuous set-valued maps is **not** true in general for lower semicontinuous set-valued maps.

Example 1. Let $h : [0,1] \mapsto [0,1]$ be a function which is not Lebesgue-measurable. Define $F : [0,1]\times[0,1] \mapsto\!\!\!\to [0,1]$ by letting $F(x,t) = [0,1]$ if $x \neq t$, $F(x,x) = \{h(x)\}$. Then it is clear that $x \mapsto F(x,t)$ is LSC for each fixed t, and $t \mapsto F(x,t)$ is measurable for each fixed x. If there existed for each positive $n \in \mathbb{N}$ a compact subset K_n of $[0,1]$ such that $\mathrm{meas}([0,1]\backslash K_n) \leq \frac{1}{n}$ with the property that $F\lceil([0,1]\times K_n)$ is LSC, then in particular $F_n = F\lceil\tilde{K}_n$ would be LSC, where $\tilde{K}_n = \{(x,x) : x \in K_n\}$. But then F_n would actually have to be continuous, because it is single-valued. Since $F_n(x,x) = h(x)$, this implies that $h\lceil K_n$ is continuous for each n. But then h is Lebesgue-measurable, and we have reached a contradiction. $\diamondsuit$

In view of this fact, it is reasonable to turn the conclusion of the Scorza-Dragoni theorem into a definition, and introduce the class of "almost lower semicontinuous" set-valued maps.

Recall that, if X is a set and S is a subset of $X \times \mathbb{R}$, then S_J denotes the set $S \cap (X \times J)$. If X, Y are sets, $S \subseteq X \times \mathbb{R}$, $F : S \mapsto\!\!\!\to Y$ is a set-valued map, and $J \subseteq \mathbb{R}$, we write F_J to denote the restriction of F to S_J, so F_J is a set-valued map from S_J to Y $F_J(x,t) = F(x,t)$ whenever $(x,t) \in S_J$, and $\mathrm{Dom}(F_J) = \Big(\mathrm{Dom}(F)\Big)_J$.

Definition 19. Let X, Y be topological spaces, let S be a subset of the product $X \times \mathbb{R}$, and let $F : S \mapsto\!\!\!\to Y$ be a set-valued map. We call F *almost lower semicontinuous* (abbr. ALSC) if

(ALSC) *for every $\varepsilon > 0$ there exists a closed subset J of $\mathbb{R}$ such that $\mathrm{meas}(\mathbb{R}\backslash J) \leq \varepsilon$ and F_J is lower semicontinuous.* $\diamondsuit$

The following characterization of the ALSC property is easily proved.

Fact 14. *If X, Y, S, F are as in Definition 19, then F is ALSC iff*

(ALSC.a) there exists a sequence $\{J_k\}_{k=0}^{\infty}$ such that

(ALSC.a.i) the J_k are pairwise disjoint subsets of $\mathbb{R}$, $\mathbb{R} = \bigcup_{k=0}^{\infty} J_k$, J_0 has measure zero, and the J_k for $k > 0$ are compact,

(ALSC.a.ii) F_{J_k} is LSC whenever $k > 0$.

Proof. It is clear that (ALSC) implies (ALSC.a). To prove the converse, assume that (ALSC.a) holds, and let $\{J_k\}_{k=0}^{\infty}$ be a family of sets that satisfies (ALSC.a.i). Let C be the set of all closed subsets L of $\mathbb{R}$ such that F_L is LSC. Then it is easy to see that C is closed under restrictions (i.e., if $L \in C$, $L' \subseteq S$, and L' is closed, then $L' \in C$) and locally finite unions (i.e., if $\{L^\alpha\}_{\alpha\in A}$ is a locally finite family of members of C, then $L = \cup_\alpha L^\alpha$ also belongs to C). The sets $J_{k,n} = J_k \cap [n, n+1]$, for $k \in \mathbb{N}$, $k > 0$, and n an arbitrary integer, obviously belong to C. Pick $\varepsilon > 0$, and then pick, for each n, a $k^*(n)$ such that

$$\operatorname{meas}\left([n, n+1]\Big\backslash\Big(\bigcup_{k=1}^{k^*(n)} J_{k,n}\Big)\right) < \frac{\varepsilon}{3 \cdot 2^{|n|}}\,.$$

Let $J = \bigcup_n \left(\cup_{k=1}^{k^*(n)} J_{k,n}\right)$. Then J is a locally finite union of members of C, so $J \in C$. In addition, $\operatorname{meas}(\mathbb{R}\backslash J) \le \varepsilon$. So (ALSC) holds. $\diamondsuit$

In the special case when F is single-valued, we know from Fact 10 that the LSC property is equivalent to ordinary continuity. Therefore

Fact 15. *If X, Y, S, F are as in Definition 19, and in addition F is single-valued and* $\mathrm{Dom}(F) = S$*, then F is ALSC as a set-valued map if and only if*

(ALSC.s) there exists a sequence $\{J_k\}_{k=0}^{\infty}$ that satisfies (ALSC.a.i) and is such that $F\lceil S_{J_k}$ is continuous for every $k > 0$. $\diamondsuit$

Definition 20. If X, Y, S, F are as in Fact 15, we call F *almost continuous* if it is ALSC as a set-valued map, i.e., if it satisfies (ALSC.s). $\diamondsuit$

Under more special conditions (for example, if X is a locally compact separable metric space, Y is a separable metric space, and $S = \Omega \times I$ with Ω open in X and I an interval), almost continuity is equivalent, by the Scorza-Dragoni Theorem, to the much simpler statement that $g(x, t)$ is continuous in x for almost every fixed t and measurable in t for all x.

By analogy with Definition 20, we can also introduce a concept of "almost upper semicontinuity" for real-valued functions, related in the obvious way to the notion of an upper semicontinuous function (which is, of course, quite different from that of an "upper semicontinuous set-valued map that happens to be an ordinary map with real values," cf. Remark 2).

Definition 21. Let X, Y be topological spaces, let S be a subset of $X \times \mathbb{R}$, and let $f : S \mapsto \mathbb{R}$ be a real-valued function. We call f *almost upper semicontinuous* if there exists a sequence $\{J_k\}_{k=0}^{\infty}$ that satisfies (ALSC.a.i) and is such that $f\lceil S_{J_k}$ is upper semicontinuous for every $k > 0$. $\diamondsuit$

4.4 Selections of ALSC maps

Definition 22. Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, $f : S \mapsto \mathbb{R}^m$, and $C : S \mapsto]0, \infty[$. We call f *almost forward Γ_C-continuous* if there exists a sequence $\{J_k\}_{k=0}^{\infty}$ for which (ALSC.a.i) holds, such that the restriction $f_{J_k} = f\lceil S_{J_k}$ is forward Γ_C-continuous whenever $k > 0$. $\diamondsuit$

Theorem 8. *Assume that $n, m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, and $F : S \mapsto\!\!\!\!\to \mathbb{R}^m$ is an ALSC set-valued map with nonempty closed values. Let $g : S \mapsto \mathbb{R}^m$ be an almost continuous single-valued map and let $\beta : S \mapsto \mathbb{R}$ be almost continuous and such that*

$$\text{for a.e. } t \in \mathbb{R} : \beta(x,t) > \rho_{g,F}(x,t) \text{ whenever } x \in \mathbb{R}^n\,,\ (x,t) \in S\,. \quad (4.4.1)$$

Assume that both g and β are LIB. Let $C : S \mapsto]0,\infty[$ be almost upper semicontinuous. Then the following is true for almost all $\bar{t} \in \mathbb{R}$:

(T8.#) For every $\bar{x} \in \mathbb{R}^n$ such that $(\bar{x},\bar{t})$ is an interior point of S, and every $\bar{y} \in F(\bar{x},\bar{t})$ such that

$$\|g(\bar{x},\bar{t}) - \bar{y}\| < \beta(\bar{x},\bar{t})\,. \quad (4.4.2)$$

there exists a selection $f : S \mapsto \mathbb{R}^m$ of F such that

- *(T8.1) $f(\bar{x},\bar{t}) = \bar{y}$;*
- *(T8.2) $(\bar{x},\bar{t})$ is a point of approximate continuity of f;*
- *(T8.3) f is almost forward Γ_C-continuous;*
- *(T8.4) for a.e. $t \in \mathbb{R}$, $\|f(x,t) - g(x,t)\| < \beta(x,t)$ for all $x \in \mathbb{R}^n$ such that $(x,t) \in S$,*
- *(T8.5) f is $\mathcal{EM}(\mathbb{R}^n,\mathbb{R}) \cap S$-measurable.*

Proof. Let $\Omega = \text{Int}(S)$. Let $\mathcal{K}$ be a finite or countably infinite set of compact subsets of Ω such that $\Omega = \bigcup\{\text{Int}(K) : K \in \mathcal{K}\}$. For each $K \in \mathcal{K}$, let $\varphi_K : \mathbb{R} \mapsto [0,\infty]$ be integrable and such that $\|g(x,t)\| + \beta(x,t) \le \varphi_K(t)$ whenever $(x,t) \in K$, and let L_K be the set of Lebesgue points of φ_K. Let $L = \bigcap\{L_K : K \in \mathcal{K}\}$. Then L is a full subset of $\mathbb{R}$.

Let $\mathcal{J} = \{J_k\}_{k=0}^{\infty}$ be a sequence for which (ALSC.a.i) holds, such that that the maps $\beta_k = \beta\lceil S_{J_k}$, $g_k = g\lceil S_{J_k}$ are continuous, the functions $C_k = C\lceil S_{J_k}$ are upper semicontinuous, and the set-valued maps $F_k = F\lceil S_{J_k}$ are LSC. (The existence of $\mathcal{J}$ is a trivial consequence of our hypotheses.)

For $k > 0$, let $\tilde{J}_k$ be the set of points of density t of J_k such that $t \in L$. Let $\tilde{J} = \cup_{k=1}^{\infty}\tilde{J}_k$. Then $\text{meas}(\mathbb{R}\backslash\tilde{J}) = 0$. We show that every $\bar{t} \in \tilde{J}$ has Property (T8.#).

Let $\bar{t} \in \tilde{J}$. Let k_0 be the unique $k \in \mathbb{N}$ such that $\bar{t} \in J_k$, so that $\bar{t} \in \tilde{J}_{k_0}$, $k_0 > 0$, and $\bar{t} \in L$. Let $\bar{x} \in \mathbb{R}^n$ be such that $(\bar{x},\bar{t}) \in \text{Int}(S)$.

Theorem 7 implies that for every $k \in \mathbb{N}$ such that $k > 0$ there exists a selection $f_k : S_{J_k} \mapsto \mathbb{R}^m$ of F_k which is forward Γ_{C_k}-continuous and such that the bound $\|g_k(x,t) - f_k(x,t)\| < \beta_k(x,t)$ holds for all $(x,t) \in S_{J_k}$. Moreover, when $k = k_0$ we can choose f_k such that $f_k(\bar{x},\bar{t}) = \bar{y}$ and f_k is continuous at $(\bar{x},\bar{t})$. Define f_0 to be an arbitrary selection of $F\lceil S_{J_0}$. Then define f by letting $f(x,t)=f_k(x,t)$ for $(x,t)\in S_{J_k}$.

Clearly, f satisfies (T8.1), (T8.3), (T8.4), and (T8.5). Moreover, if we let $E=J_{k_0}$, pick a $K\in\mathcal{K}$ such that $(\bar{x},\bar{t})\in\mathrm{Int}(K)$, and define $\varphi\stackrel{\mathrm{def}}{=}\varphi_K$, then conditions (P2.a) and (P2.b) of Proposition 2 hold. Furthermore, since $\|f(x,t)-g(x,t)\|<\beta(x,t)$ for all $(x,t)\in S$, and $\|g(x,t)\|+\beta(x,t)\le\varphi(t)$ for all $(x,t)\in K$, condition (P2.c) holds as well. Therefore Proposition 2 implies that $(\bar{x},\bar{t})$ is a PAC of f. So (T8.2) holds. $\diamondsuit$

Theorem 9. *Assume that $n,m\in\mathbb{N}$ and $S\subseteq\mathbb{R}^n\times\mathbb{R}$. Let $F:S\mapsto\!\!\!\to\mathbb{R}^m$ be an ALSC set-valued map with nonempty closed values. Let $\psi:S\mapsto[0,\infty]$ be an almost continuous single-valued locally integrably bounded function such that* $\min\Big\{\|y\|:y\in F(x,t)\Big\}<\psi(x,t)$ *for every* $(x,t)\in S$. *Let $\zeta:S\mapsto[0,\infty]$ be an almost upper semicontinuous single-valued function.*

Then the following is true for almost all $\bar{t}\in\mathbb{R}$:

- *(T9.#) For every pair $(\bar{x},\bar{y})$ such that $(\bar{x},\bar{t})\in\mathrm{Int}(S)$, $\bar{y}\in F(\bar{x},\bar{t})$, and $\|\bar{y}\|<\psi(\bar{x},\bar{t})$, there exists a selection $f:S\mapsto\mathbb{R}^m$ of F such that*
  - *(T9.1) $f(\bar{x},\bar{t})=\bar{y}$;*
  - *(T9.2) $(\bar{x},\bar{t})$ is a point of approximate continuity of f;*
  - *(T9.3) f is integrally continuous on $\widetilde{ARC}_\zeta(S)$,*
  - *(T9.4) $\|f(x,t)\|<\psi(x,t)$ for all $(x,t)\in S$.*

Proof. We apply Theorem 8 with $g\equiv0$, $\beta=\psi$, $C=1+\zeta$. This yields a selection f that satisfies (T9.1), (T9.2) and (T9.4), and in addition is $\mathcal{EM}(\mathbb{R}^n,\mathbb{R})\cap S$-measurable and almost forward Γ_C-continuous. Then there exists a sequence $\{J_k\}_{k=0}^\infty$ such that (ALSC.a.i) holds, having the property that the restrictions $\psi_k=\psi\lceil S_{J_k}$, $C_k=C\lceil S_{J_k}$, $f_k\stackrel{\mathrm{def}}{=}f\lceil S_{J_k}$ are, respectively, continuous, upper semicontinuous, and forward Γ_{C_k}-continuous. Define maps $\tilde{f}_k(x,t):S\mapsto\mathbb{R}^m$ by:

$$\tilde{f}_k(x,t)=\begin{cases} f_k(x,t) & \text{if} \quad (x,t)\in S\,,\ t\in J_k\,,\\ 0 & \text{if} \quad (x,t)\in S\,,\ t\notin J_k\,.\end{cases}$$

(Naturally, $\tilde{f}_k$ may fail to be a selection of F.) We then have

Fact 16. *$\tilde{f}_k$ is weakly forward Γ_C continuous on S for every $k>0$.*

To see this, we fix a k, and show that

(&) *if the following three conditions are true:*

1. $(x,t) \in S$,
2. $(x_\ell,t_\ell) \in S$, $(x'_\ell,t_\ell) \in S$, $\|x_\ell - x\| \le C(x,t)(t_\ell - t)$, *and* $\|x'_\ell - x\| \le C(x,t)(t_\ell - t)$ *for* $\ell = 1,2,3,\ldots$,
3. $x_\ell \to x$, $x'_\ell \to x$, *and* $t_\ell \to t$, *as* $\ell \to \infty$,

then $\tilde f_k(x_\ell,t_\ell) - \tilde f_k(x'_\ell,t_\ell) \to 0$.

Define $L_1 = \{\ell : t_\ell \in J_k\}$, $L_2 = \{\ell : t_\ell \notin J_k\}$. If L_1 is infinite, then $\lim_{\ell\to\infty,\,\ell\in L_1} t_\ell = t$, so $t \in J_k$, because J_k is closed. Then $C(x,t) = C_k(x,t)$. Clearly,

$$\tilde f_k(x_\ell,t_\ell) - \tilde f_k(x'_\ell,t_\ell) = f_k(x_\ell,t_\ell) - f_k(x'_\ell,t_\ell) \tag{4.4.3}$$

if $\ell \in L_1$. Since f_k is forward Γ_{C_k}-continuous, $f_k(x_\ell,t_\ell) \to f_k(x,t)$ and $f_k(x'_\ell,t_\ell) \to f_k(x,t)$ as $\ell \to \infty$ via values in L_1. So $\tilde f_k(x_\ell,t_\ell) - \tilde f_k(x'_\ell,t_\ell) \to 0$ as $\ell \to \infty$ via L_1. If L_2 is infinite, then $\tilde f_k(x_\ell,t_\ell) - \tilde f_k(x'_\ell,t_\ell) \to 0$ as $\ell \to \infty$ via L_2, because $\tilde f_k(x_\ell,t_\ell) = \tilde f_k(x'_\ell,t_\ell) = 0$ when $\ell \in L_2$. This completes the proof of Fact 16.

Since each $\tilde f_k$ is LIB with integral bound ψ, $\mathcal{EM}(\mathbb{R}^n,\mathbb{R}) \cap S$-measurable, and weakly Γ_C-continuous, Theorem 2 implies that all the $\tilde f_k$ are integrally continuous on $\widetilde{ARC}_\zeta(S)$. Moreover, the sequence $\{\tilde f_k\}_{k=1}^\infty$ is uniformly LIB. It therefore follows from the measurable intertwining property (Theorem 1) that f is integrally continuous on $\widetilde{ARC}_\zeta(S)$. $\diamondsuit$

4.5 The main theorem

Definition 23. Assume that $n,m \in \mathbb{N}$, $S \subseteq \mathbb{R}^n \times \mathbb{R}$, and $F : S \mapsto\!\!\!\to \mathbb{R}^m$ has nonempty values. We say that F is *locally integrably lower bounded* (abbr. LILB) if for every compact subset K of S there exists an integrable function $\varphi : \mathbb{R} \mapsto [0,\infty]$ such that $\inf\Big\{\|y\| : y \in F(x,t)\Big\} \le \varphi(t)$ whenever $(x,t) \in K$. $\diamondsuit$

We recall that a subset S of a Euclidean space $\mathbb{R}^k$ is locally compact if and only if it is the intersection of an open set and a closed set. Moreover, if S is locally compact then there exists a sequence $\{K_i\}_{i=1}^\infty$ of compact subsets of S such that $K_i \subseteq \mathrm{Int}(K_{i+1})$ for every i and $S = \bigcup_{i=1}^\infty K_i$. Such a sequence then has the property that if K is an arbitrary compact subset of S then $K \subseteq \mathrm{Int}(K_i)$ for some i.

Lemma 3. *Assume that* $n,m \in \mathbb{N}$, S *is a locally compact subset of* $\mathbb{R}^n \times \mathbb{R}$, *and* $F : S \mapsto\!\!\!\to \mathbb{R}^m$ *is a set-valued map with closed nonempty values. Then* F

is LILB if and only if there exists an almost continuous single-valued locally integrably bounded function $\psi : S \mapsto [0, \infty]$ *such that*

$$\min\Big\{\, \|y\| : y \in F(x,t) \,\Big\} < \psi(x,t) \quad \text{for every} \quad (x,t) \in S\,. \tag{4.5.1}$$

Proof. If ψ exists then it is clear that F is LILB. Conversely, suppose that F is LILB. Let $\{K_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of S such that $K_i \subseteq \mathrm{Int}(K_{i+1})$ for every i, and $S = \bigcup_{i=1}^{\infty} K_i$. For each i, let $\varphi_i : \mathbb{R} \mapsto [0, \infty]$ be an integrable function such that

$$\rho_{0,F}(x,t) = \min\Big\{\, \|y\| : y \in F(x,t) \,\Big\} < \varphi_i(t) \quad \text{for every} \quad (x,t) \in K_i\,.$$

Let $L_1 = K_1$, $L_i = K_i \backslash \mathrm{Int}(K_{i-1})$ if $i > 1$. Also, let $\Omega_1 = \mathrm{Int}(K_2)$, $\Omega_2 = \mathrm{Int}(K_3)$, and $\Omega_i = \mathrm{Int}(K_{i+1}) \backslash K_{i-2}$ for $i > 2$. Then the L_i are compact, the Ω_i are relatively open subsets of S, and $L_i \subseteq \Omega_i$ for every i. Moreover, if K is an arbitrary compact subset of S, then there is an i such that $K \subseteq \mathrm{Int}(K_i)$, and then $K \cap \Omega_j = \emptyset$ whenever $j \geq i+2$, so the sequence $\{\Omega_i\}_{i=1}^{\infty}$ is locally finite. Finally, it is clear that $S = \bigcup_{i=1}^{\infty} L_i$. For each i, let $\eta_i : S \mapsto \mathbb{R}$ be a nonnegative continuous function such that $\eta_i(x,t) = 1$ whenever $(x,t) \in L_i$ and $\mathrm{Clos}\Big(\{(x,t) : \eta_i(x,t) \neq 0\}\Big) \subseteq \Omega_i$. Let $\eta(x,t) = \sum_{i=1}^{\infty} \eta_i(x,t)$, so η is a continuous strictly positive function on S. Let $\theta_i(x,t) = \frac{\eta_i(x,t)}{\eta(x,t)}$, so the θ_i are nonnegative continuous functions on S such that $\mathrm{Clos}\Big(\{(x,t) : \theta_i(x,t) > 0\}\Big) \subseteq \Omega_i$ for each i, and $\sum_{i=1}^{\infty} \theta_i \equiv 1$.

Define $\psi(x,t) = \sum_{i=1}^{\infty} \varphi_{i+1}(t)\theta_i(x,t)$. Let $(x,t) \in S$. Let $I(x,t)$ be the set of all i such that $\theta_i(x,t) > 0$. If $i \in I(x,t)$, then $(x,t) \in \Omega_i$, so $(x,t) \in K_{i+1}$, and then $\rho_{0,F}(x,t) \leq \varphi_{i+1}(t)$. Therefore $\psi(x,t)$ is a convex combination of real numbers r such that $\rho_{0,F}(x,t) \leq r$. It follows that $\rho_{0,F}(x,t) \leq \psi(x,t)$ for all $(x,t) \in S$, so (4.5.1) holds.

If K is a compact subset of S, and i is such that $K \subseteq \mathrm{Int}(K_i)$, then $\psi(x,t) = \sum_{j=1}^{i+1} \varphi_j(t)\theta_j(x,t)$, because $K \cap \Omega_j = \emptyset$ whenever $j \geq i+2$. So $\psi(x,t)$ is bounded above, for $(x,t) \in K$, by the integrable function $\mathbb{R} \ni t \mapsto \sum_{j=1}^{i+1} \varphi_j(t)$. Therefore ψ is locally integrably bounded.

Finally, we show that ψ is almost continuous. Given a positive ε, we can use Lusin's theorem to find, for each i, a closed subset C_i of $\mathbb{R}$ such that $\mathrm{meas}(\mathbb{R}\backslash C_i) < 2^{-i}\varepsilon$ and $\varphi_i \lceil C_i$ is finite-valued and continuous. Let $C = \bigcap_{i=1}^{\infty} C_i$. Then S is closed, $\mathrm{meas}(\mathbb{R}\backslash C) < \varepsilon$, and $\varphi_i \lceil C$ is continuous. It follows that the locally finite sum $\sum_{i=1}^{\infty} \varphi_{i+1}(t)\theta_i(x,t)$ is continuous as a function on $S \cap (\mathbb{R}^n \times C)$. In other words, the restriction of ψ to S_C is continuous, as desired. ♢

We are now ready to state and prove our main result.

Theorem 10. *Assume that $n \in \mathbb{N}$ and S is a locally compact subset of $\mathbb{R}^n \times \mathbb{R}$. Let $F : S \mapsto\!\!\!\rightarrow \mathbb{R}^n$ be an ALSC LILB set-valued map with nonempty closed values. Then the following is true for almost all $\bar{t} \in \mathbb{R}$:*

(T10.#) For every $\bar{x}$ such that $(\bar{x}, \bar{t}) \in \operatorname{Int}(S)$ and every $\bar{y} \in F(\bar{x}, \bar{t})$ there exist a selection $f : S \mapsto \mathbb{R}^n$ of F and an almost continuous LIB function $\psi : S \mapsto [0, \infty]$ such that

- *(T10.1) $f(\bar{x}, \bar{t}) = \bar{y}$;*
- *(T10.2) $(\bar{x}, \bar{t})$ is a point of approximate continuity of f;*
- *(T10.3) $\|f(x,t)\| \le \psi(x,t)$ for all $(x,t) \in S$;*
- *(T10.4) f is integrally continuous on $\widetilde{ARC}_\psi(\mathbb{R}^n; S)$;*
- *(T10.5) f is an admissible time-varying vector field on some neighborhood of $(\bar{x}, \bar{t})$;*
- *(T10.6) the flow map Φ^f is regularly differentiable at $(\bar{x}, \bar{t}, \bar{t})$ with differential equal to the linear map*

$$\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \ni (v, h, k) \mapsto v + (h - k)\bar{y} \in \mathbb{R}^n\,.$$

Proof. It follows from Lemma 3 that there exists an almost continuous LIB function $\psi : S \mapsto \mathbb{R}$ such that (4.5.1) holds. Let $\Omega = \operatorname{Int}(S)$, and let $\{K_i\}_{i=1}^\infty$ be a sequence of compact subsets of Ω such that $K_i \subseteq \operatorname{Int}(K_{i+1})$ for every i and $\Omega = \bigcup_{i=1}^\infty K_i$. For each i, pick an integrable function $\varphi_i : \mathbb{R} \mapsto [0, \infty]$ such that $\psi(x,t) \le \varphi_i(t)$ whenever $(x,t) \in K_{i+1}$. We define $\zeta_i : S \mapsto [0, \infty]$ by letting

$$\zeta_i(x,t) = \begin{cases} \psi(x,t) & \text{if} \quad (x,t) \in S \backslash K_i\,, \\ \varphi_i(t) & \text{if} \quad (x,t) \in K_i\,, \end{cases}$$

and remark that $\psi(x,t) \le \zeta_i(x,t)$ for all $(x,t) \in S$.

We claim that ζ_i is almost upper semicontinuous for every i. To see this, fix i and a positive number ε, and find a closed subset C of $\mathbb{R}$ such that $\operatorname{meas}(\mathbb{R} \backslash C) < \varepsilon$ and the functions $S \cap (\mathbb{R}^n \times C) \ni (x,t) \mapsto \psi(x,t) \in [0, \infty]$ and $C \ni t \mapsto \varphi_i(t) \in [0, \infty]$ are finite-valued and continuous.

Let us show that ζ_i is upper semicontinuous on $S_C = S \cap (\mathbb{R}^n \times C)$. To prove this, we fix $\alpha \in \mathbb{R}$, let $W = \{(x,t) \in S_C : \zeta_i(x,t) \ge \alpha\}$, and prove that W is relatively closed in S_C. Let $\{(x_j, t_j)\}_{j \in \mathbb{N}}$ be a sequence of members of W that converges to a limit $(x,t) \in S_C$. Then $\psi(x_j, t_j) \to \psi(x,t)$ and $\varphi_i(t_j) \to \varphi_i(t)$ as $j \to \infty$. If $(x,t) \notin K_i$, then $(x_j, t_j) \notin K_i$ for large enough j. Therefore $\zeta_i(x,t) = \psi(x,t)$ and $\zeta_i(x_j, t_j) = \psi(x_j, t_j)$ for large j, so $\zeta_i(x_j, t_j) \to \zeta_i(x,t)$. Since $\zeta_i(x_j, t_j) \ge \alpha$, we find that $\zeta_i(x,t) \ge \alpha$, so $(x,t) \in W$. Now suppose that $(x,t) \in K_i$. Then $(x_j, t_j) \in K_{i+1}$ for large enough j, because $K_i \subseteq \mathrm{Int}(K_{i+1})$. Therefore $\psi(x_j, t_j) \le \varphi_i(t_j)$ for large j, and then the definition of ζ_i shows that $\zeta_i(x_j, t_j) \le \varphi_j(t_j)$ for large j. It follows that $\zeta_i(x,t) = \varphi_i(t) = \lim_{j\to\infty} \varphi_i(t_j) \ge \limsup_{j\to\infty} \zeta_i(x_j, t_j) \ge \alpha$. Therefore (x,t) belongs to W in this case as well, concluding the proof that ζ_i is almost upper semicontinuous.

Clearly, (4.5.1) holds as well if ψ is replaced by ψ_N, where $\psi_N \stackrel{\text{def}}{=} \psi + N$ and N is an arbitrary nonnegative integer. We can therefore apply Theorem 9, given any N and i, with ψ_N in the role of ψ, and ζ_i^N in the role of ζ, where $\zeta_i^N(x,t) = \zeta_i(t) + N$. We can then find full subsets E_N^i of $\mathbb{R}$ such that, whenever $\bar{t} \in E_N^i$, $(\bar{x}, \bar{t}) \in \mathrm{Int}(S)$, $\bar{y} \in F(\bar{x}, \bar{t})$, and $\|\bar{y}\| < \psi_N(\bar{x}, \bar{t})$, there exists a selection f of F that satisfies (T10.1) and (T10.2), as well as the inequality $\|f(x,t)\| \le \psi_N(x,t)$ for all $(x,t) \in S$, and is such that f is integrally continuous on $\widetilde{ARC}_{\zeta_i^N}(\mathbb{R}^n; S)$.

Let $E = \bigcap_{N=0}^{\infty} \bigcap_{i=1}^{\infty} E_N^i$. Then E is a full subset of $\mathbb{R}$. Let us show that (T10.#) is true for every $\bar{t} \in E$. To see this, pick a $\bar{t} \in E$, an $\bar{x}$ such that $(\bar{x}, \bar{t}) \in \mathrm{Int}(S)$, and a $\bar{y} \in F(\bar{x}, \bar{t})$. Then choose an N such that $\|\bar{y}\| < \psi_N(\bar{x}, \bar{t})$ and an i such that $(\bar{x}, \bar{t})$ is an interior point of K_i. Then $t \in E_N^i$, so there exists a selection f of F such that (T10.1) and (T10.2) hold, (T10.3) holds with ψ replaced by ψ_N, and f is integrally continuous on $\widetilde{ARC}_{\zeta_i^N}(\mathbb{R}^n; S)$.

Since $\psi(x,t) \le \zeta_i(x,t)$ for all $(x,t) \in S$, we also have $\psi_N(x,t) \le \zeta_i^N(x,t)$ for all $(x,t) \in S$, so f is integrally continuous on $\widetilde{ARC}_{\psi^N}(\mathbb{R}^n; S)$, showing that (T10.3) holds with ψ replaced by ψ_N.

Now, $(\bar{x}, \bar{t}) \in \mathrm{Int}(K_i)$, and $\|f(x,t)\| \le \psi_N(x,t) \le \varphi_i(t) + N = \zeta_i(x,t)$ when $(x,t) \in K_i$. So f is bounded by the integrable function $t \mapsto \varphi_i(t) + N$ on K_i, and f is integrally continuous on $\widetilde{ARC}_{\varphi_i + N}(\mathbb{R}^n; K_i)$. Therefore f is *a fortiori* integrally continuous on $ARC_{\varphi_i + N}(\mathbb{R}^n; K_i)$. So f is admissible on K_i. This completes the proof that f satisfies (T10.1), (T10.2), (T10.3), (T10.4), and (T10.5), with ψ replaced by ψ_N.

Finally, (T10.6) follows from (T10.5) and Theorem 6. ♢

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