MATHEMATICS 300 — FALL 2001

Introduction to Mathematical Reasoning
(640 - 300, Section 04)

H. J. Sussmann

INSTRUCTOR'S NOTES, PART 1

(September 5, 2001)

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1 Information on the course

1.1 Course schedule

Our class meets on **Mondays and Wednesdays, 7th period (6:10 to 7:30 pm)**, in HHA4 (Hardenbergh Hall A4, College Avenue Campus).

1.2 About the instructor

My *name* is **H.J. Sussmann.** My *office* is **Hill 538**.
My *Rutgers phone extension* is 5-5407.
My *e-mail address* is **sussmann@math.rutgers.edu**.

1.3 Web page

I have set up a *Web page* for our Math 300 section:

http://www.math.rutgers.edu/˜sussmann/math300page.html

All the instructor's notes will be available there.

1.4 Office hours

My office hours will be:

- Tuesday and Thursday, 5th period (2:50 to 4:10) in my office, Hill 538;
- any other time, by appointment, in my office, Hill 538.

You should let me know that you are coming to see me *even if you are planning to do so during my regular office hours.* REASON: I don't want to have to stay in my office waiting just in case someone might show up. If I know that you are coming then I will be there, but if you don't tell me that I should expect you then I will feel free to go away and do other things.

1.5 The Fall 2001 semester

The semester starts on *Tuesday, September 4* and ends on *Wednesday, Dec. 12.* The *Thanksgiving recess* goes from *Thursday, November 22* to *Sunday, November 25.* (In addition, there are the following *changes in designation of class days:* Tuesday November 20 has Thursday Classes; Wednesday November 21 has Friday Classes. The only effect of these changes on our Math 300 section is that we do not have a class on Wednesday November 21.)

1.6 Our Math 300 calendar:

We will have **28 meetings**, on the following dates:

- September: W5, M10, W12, M17, W19, M24, W26 (7 meetings)
- October: M1, W3, M8, W10, M15, W17, M22, W24, M29, W31 (10 meetings)
- November: M5, W7, M12, W14, M19, M26, W28 (7 meetings)
- December: M3, W5, M10, W12 (4 meetings)

2 Homework, midterms, etc.

Homework and quizzes will count for about one third of your grade. There will be *two midterms*, which will count—together—for another third of your grade. The *final exam* will count for the remaining third.

Midterm dates: Monday October 15 and **Wednesday November 28**.

3 The textbook and the instructor's notes

We will be using:

- the book *Mathematics, a discrete introduction*, by **Edward R. Scheinerman** (Brooks/Cole Publishing Co., Pacific Grove, CA, 2000),
- the notes written by the instructor.

> *You should read and study carefully all the instructor's notes.*
> **The material of these notes is an integral part of the course, as much as that of the book.** *Furthermore, the notes contain all kinds of important information. For example, in this set of notes there are lots of things you need to know in order to do your homework.*

4 Readings for the first 5 days (Sept. 5, 10, 12, 17, 19)

- the book's note "To the Student,"
- the book's Chapter 1 (**all of it!**),
- the instructor's notes, Part 1.

5 Homework assignments for September

Before you start writing your homework, read carefully the rest of this handout, in particular §6 on "writing mathematics and submitiing homework," §7.3 on "answering questions in this course," and §8 on "using letters in mathematical writing."

5.1 Homework assignment no. 1, due on Wednesday, September 12.

A. Problems 1,2, 3, 4, 5, 6 in page 5 of the book. (Note: in Problem 4 you are asked to define "less than" and a few other predicates. The question, as stated in the book, is incomplete, because the author is not telling you what you are allowed to use. For example, could you define "less than" by saying "let x, y be integers; we say that x is *less than* y if y is greater than x."? Obviously, the author does not want to allow that, but he is not telling us clearly what he is allowing. So let me tell you: you should write your definition for the benefit of a person who knows what the integers and the natural numbers are, and what addition is, and nothing else[1].)

B. Is the number π even or odd? Justify your answer.

C. Is "$\mathbb{N}$ is a set of natural numbers" a true statement?

D. Is "$\mathbb{N}$ is a set of natural numbers" a satisfactory answer to the question "What is $\mathbb{N}$?"

E. Euler[2] claimed that the sum of the series

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

is equal to $\frac{1}{2}$. Euler's argument was as follows: take the power series (Taylor series) for the function $\frac{1}{1+x}$, given by

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \ .$$

Plug in $x = 1$. Then

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \ldots \ .$$

Was Euler right? Discuss. (Do not just say "yes, he was right," or "no, he was wrong." Whichever way you choose to go, justify your statement by giving at least one good reason.)

5.2 Assignment no. 2, due on Wednesday, September 19

A. Section 2 of the book, Problems 1(b)(c), 2, 4, 7.

B. Section 3 of the book, Problems 2, 5, 6, 10, 12, 13.

C. Section 4 of the book, Problems 3, 5, 6, 7, 9.

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[1] In the Spring 2001 term, a student wrote "let x, y be integers; we say that x is *less than* y if x is to the left of y." This is an example of what is **not** allowed. The person for whom you are writing your definition only knows about integers and natural numbers and addition. So you cannot talk about "left" and "right."

[2] Leonard Euler (1707-1793) was one of the greatest mathematicians who ever lived.

D. Here are eight answers given by students to the question "define 'composite number' ".

1. Composite is when its divisible by two numbers.
2. A positive integer is *composite* if there exists an integer b such that $1 < b < n$ and $b|a$.
3. A positive integer a is *composite* if there exists an integer b such that $1 < b < n$ and $b|a$.
4. A positive integer a is *composite* if there exists an integer b and $1 < b < a$ and $a|b$.
5. A positive integer a is *composite* if there exists an integer b such that $1 < b < a$ and $a|b$.
6. A positive integer n is *composite* if there exist positive integers a and b such that $n = ab$.
7. A positive integer n is *composite* if there exist integers a and b such that $n = ab$ and $1 < a < n$.
8. A positive integer n is *composite* if there exist integers a and b such that $n = ab$, $1 < a$, and $1 < b$.

Grade these answers on a scale from 0 to 10 and justify your grades. (Do not hesitate to be harsh and to make as many negative comments as may be appropriate.)

E. Of the eight answers given in Problem D to the question "define 'composite number' ", one of them is such that, according to that purported definition, there do not exist any composite numbers at all. Determine which of the eight answers is the one having this undesirable feature, and prove that according to that definition there are no composite numbers.

F. Of the eight answers given in Problem D to the question "define 'composite number' ", one of them is such that, according to that purported definition, every positive integer is composite. Determine which of the eight answers is the one having this undesirable feature, and prove that according to that definition every positive integer is composite.

G. Prove that every positive integer which is not prime is composite.

5.3 Assignment no. 3, due on Wednesday, September 26

A. Section 5 of the book, Problems 1(a)(d)(e), 3, 4, 8, 11(b)(c)(e), 12(b), 18.

B. Section 6 of the book, Problems 2, 6, 7, 9(d)(f)(g), 12, 15.

5.4 Supplementary problems for the month of September.

Here are some supplementary problems. Don't worry if some of these problems seem too hard. They are meant to be challenging. Just do as much as you can. Hand in the solutions **not later than October 1**.

One of the main purposes of this set of questions is to get you to experience what it is like to think about problems that could be quite hard (even if they seem easy!), so that you will appreciate what it means to find a **truly good solution**. For you to see this, you have to think about the problems first, and if you cannot solve them **after you have tried to** and then you see the solution, you will understand much better what a **real proof** is.

A. Prove that every year must have a Friday the 13th.

B. In the planet of the Klingons, they have a calendar exactly like the one we use here on Earth (there are 12 months, January has 31 days, February has 28 or 29, March has 31, April has 30, etc.), except for one thing: the order of the months is different. We do not know what it is, but we know for sure that it is different from ours. (For example, it may be that June is the first month, September is the second one, etc.) Is it still necessarily true that every year must have a Friday the 13th? *Prove that it is true*, no matter what the order of the months is, *or prove that it need not be true* (that is, that there is a way to reorder the months so that not every year has a Friday the 13th).

C. Express the number 28 as the difference of the squares of two natural numbers.

D. Express the number 29 as the difference of the squares of two natural numbers.

E. Express the number 30 as the difference of the squares of two natural numbers.

F. (*Optional. This could be quite hard! Do not worry if you cannot do it, but at least try to* **think** *about it.*) Express the number $4,582,623$ as the sum of the squares of two natural numbers.

G. (*Optional. Hard.*) Suppose two integers a and b are sums of two squares of integers. Prove that the product ab must also be the sum of two squares of integers. (NOTE: here is an example of the kind of situation we are talking about. The number 13 is the sum of two squares, because $13 = 4 + 9$. The number 8 is the sum of two squares, because $8 = 4 + 4$. So, if our statement is true, it ought to be the case that the number 13×8 is the sum of two squares. And, indeed, it *is* true, because $13 \times 8 = 104$ and $104 = 100 + 4 = 10^2 + 2^2$.)

H. Here are five sentences about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than."

(1) Cows are taller than sheep. (That is, "every cow is taller than every sheep," or, if you prefer, "for every x and every y, if x is a cow and y is a sheep, then x is taller than y.")
(2) Giraffes are taller than cows.
(3) If something is taller than something else which is in turn taller than a third thing, then the first one is taller than the third one. (That is "for every x, every y and every z, if x is taller than y and y is taller than z, then x is taller than z.")
(4) Olivia is a giraffe.
(5) Dolly is a sheep.

Using sentences (1), (2), (3), (4), (5), *and* **no other fact** *about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than,"* prove that Olivia is taller than Dolly.

6 Writing mathematics & submitting homework

6.1 Write clearly in complete sentences

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements* with a *clear meaning* that can can be understood by anyone reading what you wrote. For example:
  - If you tell me that "she is very smart," but you haven't told me who "she" is, then I don't know what you are talking about, so you haven't made a statement with a clear meaning.
  - If you write "$x > 0$," but you haven't told me who "x" is, then I don't know what you are talking about, so you haven't made a statement.
  - If I ask you to state Pythagoras' theorem and your answer only says "$a^2 + b^2 = c^2$," then nobody will know what you are talking about, because you have not said what "a," "b," and "c" are supposed to be.[3]
- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in detail in class, and is also explained in §8.

[3]Here is a correct statement of Pyhtagoras' theorem: *Let c be the lenght of the hypothenuse of a right triangle, and let a, b be the lengths of the other two sides. Then $a^2 + b^2 = c^2$.*

- Provide proper connectives between equations as well as between ideas.
- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.
- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.* If you have to memorize a definition or a statement, then that is not a good sign, because it indicates lack of understanding.
- Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you've tried to read any complicated document.
- *Do not assume that the people reading your paper can read your mind. Do assume that they are intelligent, but also assume that they are busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.*

Effective written exposition will be worth at least 50% of your grade. Conversely, bad or unclear exposition may be penalized as much as 50% of the grade or even more.

- The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

6.2 Your written work

You should pay attention to presentation, especially for the homework:

- A nicely typed homework (e.g., using a word processor) is preferable to handwritten work. Handwritten work is acceptable too, but in that case:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**. If you are not willing to spend some of your time doing this; if what you hand in shows that you were in a hurry and that you did not make the effort to write things neatly and properly, then there is no reason for the instructor or the grader to spend any of our time reading what you wrote, and we will not do it.

  - You should use a pen. Never use a pencil.
  - Use any color other than red (for example, black, blue, or green), but DO NOT USE RED. (Reason: The use of red is reserved for the instructor's and grader's comments.)

- Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more than one sheet then the sheets are **stapled** and the pages are **numbered**.
- If you tear off the sheets from a spiral notebook, please make sure before you hand them in that there are none of those ugly hanging shreds of paper at the margins. Use scissors, or a cutter, if necessary.

If you hand in a homework assignment that has at least one of the following flaws:

- **it is written carelessly or in a hurry,**
- **it has lots of words crossed out,**
- **it has unreadable handwriting,**
- **it has pages that are unstapled or unnumbered or fail to show your name,**
- **it has shreds of paper at the margins**

(in other words, anything that looks ugly, or anything that you would be ashamed to hand in, say, if you were writing a letter to apply for a job), then

- **this is a form of disrespect towards your instructor and your grader, and that's exactly how you should expect that it will be treated.**

Homework assignments that fail to meet the above requirements will be marked "unacceptable" and returned unread.

7 Some important remarks

7.1 Integers, positive integers, natural numbers

The *integers* are the numbers $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$. The *natural numbers* (also called "nonnegative integers") are the numbers $0, 1, 2, 3, \ldots$. (Sometimes the natural numbers are also called "whole numbers.") In particular, **every natural number is an integer,** but of course not every integer is a natural number, (For example, -1 is an integer but is not a natural number.) Also, **the number 0 is a natural number.**

Later on, after we have started to work with set notation, we will use $\mathbb{N}$ as a name for the set of all natural numbers, so

$$\mathbb{N} = \{ 0, 1, 2, 3, 4, \ldots \} . \tag{7.1.1}$$

Therefore the *members* of $\mathbb{N}$ are the natural numbers. For example, 3 is a member of $\mathbb{N}$ (or, equivalently, "3 belongs to $\mathbb{N}$"). We indicate this by writing

$$3 \in \mathbb{N} . \tag{7.1.2}$$

Similarly, $5 \in \mathbb{N}$, $22 \in \mathbb{N}$, $60 \in \mathbb{N}$, $0 \in \mathbb{N}$. But $-1 \notin \mathbb{N}$ (that is, "-1 does not belong to $\mathbb{N}$."

We will use $\mathbb{Z}$ to denote the set of *all integers.* That is,

$$\mathbb{Z} = \{ 0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots \} .$$

Also, we will write $\mathbb{R}$ to denote the set of *real numbers.*

A real number (and, in particular, an integer) is "positive" if it is > 0. A number is "nonnegative" if it is ≥ 0.

In particular, 0 **is not positive**, but 0 **is nonnegative. 0 is a nonnegative integer, but 0 is not a positive integer.**

The **positive integers** are $1, 2, 3, 4, \ldots$.

Remark 1. Do not confuse the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ with the letters N, Z, R. They are different! (This is very convenient. If we agreed to use N as the name for the set of natural numbers, then the meaning of the letter N would be fixed for ever, and we would not be able to use N for other things.) ♢

7.2 Reading formulas precisely

It is important that you acquire from the very beginning the habit of reading mathematical formulas aloud precisely and correctly. For example, the correct way to read Formula (7.1.2) aloud is

$$3 \text{ belongs to } \mathbb{N}$$

or

3 is a member of $\mathbb{N}$

or

3 is in $\mathbb{N}$.

But you **cannot** read (7.1.2) as, for example, "3 is contained in $\mathbb{N}$." "Is contained" means something else!

Here is another example: Formula (7.1.1) is read as

$\mathbb{N}$ is the set consisting of 0, 1, 2, 3, 4, etc.

It would be **wrong** (**very** wrong!) to read it as "$\mathbb{N}$ is 0, 1, 2, 3, 4, etc." Do you see why this is bad? The best way to understand it is to think of more familiar examples.

For example, you cannot say that "the U.S. Supreme Court is Chief Justice Rehnquist and Justice Scalia and Justice Thomas and Justice O'Connor and Justice Souter and Justice Kennedy and Justice Ginsburg and Justice Breyer and Justice Stevens." Why not? Because the U.S. Supreme Court is **one single thing, not nine things!** What you *can* say is:

> The U.S. Supreme Court is a court whose members are Chief Justice Rehnquist and Justices Scalia, Thomas, O'Connor, Souter, Kennedy, Ginsburg, Breyer, and Stevens.

Similarly, the set $\mathbb{N}$ of natural numbers is **one single object, not many objects**, so you can say "$\mathbb{N}$ is the set consisting of 0, 1, 2, 3, 4, etc." but you cannot say "$\mathbb{N}$ is 0, 1, 2, 3, 4, etc."

Also, you cannot say "$\mathbb{N}$ is the natural numbers," or "$\mathbb{N}$ is natural numbers." (Actually, "$\mathbb{N}$ is the natural numbers" is bad, but "$\mathbb{N}$ is natural numbers" is even worse. Why?) What you should say is "$\mathbb{N}$ is the set of natural numbers," or "$\mathbb{N}$ is the set of all natural numbers."[4]

7.3 Answering questions in this course

In this course, whenever you are given a problem where you are asked to do something, your answer should be either:

(a) doing what you were asked to do,

or

(b) showing—that is *proving*—that it cannot be done.

[4]Recommended reading: the short story "Pigs is pigs," by Ellis Parker Butler. You can find it in the Web site *www.bookvalley.com/cgibin/bv?b=162*. It's funny, and one of the things that make it funny is precisely the misuse of the verb "is" in "Pigs is pigs."

(See, for example, Problems 2, 3, 4, 5, 8, 10, 12 below, where the correct answer is "what you asked me to do cannot be done.")

Notice that, when the answer is that "it cannot be done," it is not enough for you to *say* that it cannot be done. *You have to tell me why.* In other words, you have to *prove* that it cannot be done.

This remark is very important, and will apply throughout the semester, not just during the first week. And it applies to all your work, to the homework, the quizzes, the midterm exams, and the final exam. So please read it and reread it until you are sure you got the point. $\diamondsuit$

7.4 Some examples of problems with correct answers

Here are some examples of problems with correct solutions:

PROBLEM 1: Express the number 26 as a sum of two odd natural numbers.

ANSWER: $26 = 3 + 23$. $\diamondsuit$

REMARK: *There are lots of other solutions, of course! For example, here are two solutions different from the one given above:* $26 = 7 + 19$*,* $26 = 13 + 13$.

PROBLEM 2: How many solutions does Problem 1 have?

ANSWER: This question cannot be answered in a unique way, because we have not told you exactly what is meant by a "solution of Problem 1," and, depending on how you define what is meant by a "solution," the answers to our question may be different.

Here are two equally legitimate ways to define "solution of Problem 1":

First way: A *solution of Problem 1* is an ordered pair (n, m) of odd natural numbers such that $n + m = 26$. (What this means is that a solution consists of specifying a first odd natural number n and a second odd natural number m such that $n + m = 26$. So "$n = 3$ and $m = 23$" is a solution, and "$n = 23$ and $m = 3$" is a *different* solution.)

According to this definition of "solution of Problem 1," there are exactly **13 solutions**, namely, the ordered pairs $(1, 25)$, $(3, 23)$, $(5, 21)$, $(7, 19)$, $(9, 17)$, $(11, 15)$, $(13, 13)$, $(15, 11)$, $(17, 9)$, $(19, 7)$, $(21, 5)$, $(23, 3)$, $(25, 1)$.

Second way: A *solution of Problem 1* is an unordered pair $\{n, m\}$ of odd natural numbers such that $n + m = 26$.

According to this definition of "solution of Problem 1," there are exactly **7 solutions**, namely, the unordered pairs $\{1, 25\}$, $\{3, 23\}$, $\{5, 21\}$, $\{7, 19\}$, $\{9, 17\}$, $\{11, 15\}$, $\{13, 13\}$). (What this means is that a solution consists of specifying two odd natural numbers whose sum is 26, without singling out one of them as the "first" one. So in this case "$n = 3$ and $m = 23$" is a solution, and "$n = 23$ and $m = 3$" is *the same* solution.) $\diamondsuit$

PROBLEM 3: Express the number 27 as the sum of two odd natural numbers.

SOLUTION: This is impossible. REASON: the sum of two odd numbers is always even. Since 27 is odd, it cannot be the sum of two odd numbers. $\diamondsuit$

PROBLEM 4: Prove that $2+2=5$.

SOLUTION: This cannot be done. REASON: The statement "$2+2=5$" is false, and false statements cannot be proved.

PROBLEM 5: Prove that for every positive integer x there is another positive integer y such that $0 < y < x$.

SOLUTION: This cannot be done. REASON: The statement that we are being asked to prove is false, and a false statement cannot be proved.

The reason that the statement is false is as follows: 1 is a positive integer, and there is no positive integer y such that $0 < y < 1$, because 1 is the smallest positive integer. So if we take x to be 1 then the conclusion is not true. So it is not true that "for *every* positive integer x there is another positive integer y such that $0 < y < x$." $\diamondsuit$

PROBLEM 6: Express the number 12 as the difference of the squares of two natural numbers.

SOLUTION: $12 = 16 - 4$, so $12 = 4^2 - 2^2$. $\diamondsuit$

PROBLEM 7: Express the number 20 as the sum of the squares of two natural numbers.

SOLUTION: $20 = 4 + 16 = 2^2 + 4^2$. $\diamondsuit$

PROBLEM 8: Express the number 4 as the sum of the squares of two positive integers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 4 as a sum of two positive integers:

$$4 = 1 + 3\,, \qquad 4 = 2 + 2\,.$$

In neither of these two equalities are the two terms on the right-hand side squares.

PROBLEM 9: Express the number 4 as the sum of the squares of two natural numbers.

SOLUTION: $4 = 0 + 4 = 0^2 + 2^2$. $\diamondsuit$

PROBLEM 10: Express the number 12 as the sum of the squares of two natural numbers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 12 as a sum of two natural numbers:

$12 = 0{+}12$, $12 = 1{+}11$, $12 = 2{+}10$, $12 = 3{+}9$, $12 = 4{+}8$, $12 = 5{+}7$, $12 = 6{+}6$.

In none of these expressions are the two terms on the right-hand side squares of natural numbers. $\diamondsuit$

PROBLEM 11: Express the number 12 as the sum of the squares of three natural numbers.

SOLUTION: $12 = 4 + 4 + 4$, so $12 = 2^2 + 2^2 + 2^2$. $\diamondsuit$

PROBLEM 12: Express the number 15 as the sum of the squares of three natural numbers.

SOLUTION: This cannot be done. REASON: If we are going to write 15 as the sum of three squares, then we should be able to write it as the sum of two natural numbers, one of which is a square, while the other is the sum of two squares.

So let us look at all the ways to write 15 as a sum of two natural numbers, see in which cases one of these numbers is square, and then see if the other number is the sum of two squares.

Here are all the ways to express 15 as a sum of two natural numbers:

$$\begin{array}{llll} 15 = 0 + 15, & 15 = 1 + 14, & 15 = 2 + 13, & 15 = 3 + 12, \\ 15 = 4 + 11, & 15 = 5 + 10, & 15 = 6 + 9, & 15 = 7 + 8\,. \end{array}$$

There are four expressions in the above list where one of the numbers is a square:

$$15 = 0 + 15, \quad 15 = 1 + 14, \quad 15 = 4 + 11, 15 = 6 + 9\,.$$

In each case, we must ask whether the other number is a sum of two squares.

For “15=0+15,” we have to see if 15 is the sum of two squares. The answer is “no,” because we already have the list of all (eight) ways of writing 15 as a sum of two natural numbers, and in no case are both those numbers suqares.

For “15=1+14,” we have to see if 14 is the sum of two squares. The answer is “no.” (Reason: the only squares not greater than 14 are 0, 1, 4, and 9. So if are going to write 14 as a sum of two squares, the only possibilities are $14 = 0 + 14$, $14 = 1 + 13$, $14 = 4 + 10$, $14 = 9 + 6$. But in all these cases the other number (that is, 14, 13, 10, or 6) is not a square.)

For “15=4+11,” we have to see if 11 is the sum of two squares. The answer is “no.” (Reason: the only squares not greater than 11 are 0, 1, 4, and 9. So if are going to write 11 as a sum of two squares, the only possibilities are $11 = 0 + 11$, $11 = 1 + 10$, $11 = 4 + 7$, $14 = 9 + 5$. But in all these cases the other number (that is, 11, 10, 7, or 5) is not a square.)

Finally, for “15=6+9,” we have to see if 6 is the sum of two squares. The answer is “no.” (Reason: the only squares not greater than 6 are 0, 1, and 4. So if are going to write 6 as a sum of two squares, the only possibilities are $6 = 0 + 6$, $6 = 1 + 5$, $6 = 4 + 2$. But in all these cases the other number (that is, 6, 5, or 2) is not a square.) $\diamondsuit$

PROBLEM 13: Express the number 15 as the sum of the squares of four natural numbers.

SOLUTION: $15 = 1 + 1 + 4 + 9 = 1^2 + 1^2 + 2^2 + 3^2$. ♢

PROBLEM 14: Express the number 674 as the sum of the squares of four natural numbers.

SOLUTION: $674 = 144 + 400 + 121 + 9 = 12^2 + 20^2 + 11^2 + 3^2$. ♢

REMARK: You may be thinking *how on Earth did the author of these notes figure out how to solve problem 14?* That's a good question. It turns out that there is a technique for doing this, but it's not easy to explain. Maybe we will talk about it later in the course. But keep in mind that the question "how did I find a solution to problem 14?" is quite different from the question "is $674 = 12^2 + 20^2 + 11^2 + 3^2$ a correct solution?" This second question is easy to answer: you just compute $12^2 + 20^2 + 11^2 + 3^2$ and verify that what you get is 674. ♢

REMARK: You may have noticed that I gave you examples of natural numbers that could be expressed as the sum of two squares, and of natural numbers that could not. Then I gave you examples of natural numbers that could be expressed as the sum of three squares, and of natural numbers that could not. But for *four* squares what I have done is different: I gave you examples of natural numbers that could be expressed as the sum of three squares, but I did not give you examples of natural numbers that could not. Is there a reason for this? The answer is **yes**. The mathematician Lagrange (1736-1813) proved that **every natural number can be expressed as the sum of the squares of four natural numbers.** This is a hard theorem, and I will try to say a few words about it later in the course. ♢

8 Using letters in mathematical writing

When we talk about mathematical objects, we often use *letters* (or other special symbols such as $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, 0, 1, 2, etc.). The way letters are used in mathematical discourse may often appear mysterious to the students. The first thing you need to do is to realize that it is not mysterious at all. It is, in fact, quite simple!

Letters are only used in mathematics in order to shorten certain statements that would otherwise be too long.

8.1 Some examples

Example 1. Suppose you want to define the word "cousin." You could say:

> A person is said to be a *cousin* of another person if one of the first person's parents is a sibling of one of the second person's parents. (8.1.1)

This is O.K., but having to talk about the "first person" and the "second person" and having to repeat the word "person" so many times makes it awkward. So what we can do is give names to those persons, for example x and y, and say the following:

> Let x and y be persons. Then x is said to be a *cousin* of y if one of x's parents is a sibling of one of y's parents. (8.1.2)

Example 2. Suppose you want to say

> If you take two real numbers, add them, and square the sum, then you get the same result as if you had squared each number separately, added the squares, and added to the result twice the product of the two numbers. (8.1.3)

This is awfully complicated, awkward, and ugly. You can make it much shorter by saying

> Let x, y be two real numbers. Then $(x+y)^2 = x^2 + y^2 + 2xy$. (8.1.4)

or

> For all real numbers x, y, $(x+y)^2 = x^2 + y^2 + 2xy$. (8.1.5)

We can say (8.1.5) even more concisely, by writing

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \Big((x+y)^2 = x^2 + y^2 + 2xy\Big) . \tag{8.1.6}$$

Here the symbol $\forall$ is the **universal quantifier**. Statement (8.1.6) is read as follows:

> For all real numbers x and for all real numbers y, the square of x plus y is equal to the sum of the square of x, the square of y, and twice the product of x and y. (8.1.7)

Actually **you can even read (8.1.6) without any letter variables at all.** How? Well, that's precisely what (8.1.3) does! ♢

Example 3. Suppose you want to say

If you take the number $\frac{\sqrt{5}-1}{2}$, and add it to its square, you get 1. (8.1.8)

You could say this as follows:

$$\left(\frac{\sqrt{5}-1}{2}\right)^2 + \frac{\sqrt{5}-1}{2} = 1\,. \tag{8.1.9}$$

Or you could give a name to the number, say a, and write

Let $a = \frac{\sqrt{5}-1}{2}$. Then $a^2 + a = 1$. (8.1.10)

Example 4. Suppose you want to say

It is possible to find a real number whose cube, added to its square, is equal to the number itself. (8.1.11)

Again, this looks horribly complicated and ugly, and you can make it much shorter by saying

There exists a real number x such that $x^2 + x^3 = x\,.$ (8.1.12)

or

There is a real number x such that $x^2 + x^3 = x\,.$ (8.1.13)

or

$x^2 + x^3 = x$ for some real number x. (8.1.14)

We can say this thing even more concisely, by writing

$$\exists x \in \mathbb{R}, (x^2 + x^3 = x)\,. \tag{8.1.15}$$

Here the symbol $\exists$ is the **existential quantifier**. Statement (8.1.15) is read as follows:

There exists a real number x such that $x^2 + x^3 = x$. (8.1.16)

There exists a real number x such that the sum of the square of x and the cube of x cube equals x. (8.1.17)

or even, if you don't want to use any letter variables at all

There exists a real number such that the sum of the square of the number plus the cube of the number equals the number itself. (8.1.18)

or, if you prefer, you can read it as statement (8.1.11) above. $\diamondsuit$

QUANTIFIERS

The symbols $\exists$ and $\forall$ are the **quantifiers**.

" $\exists$ " is the **existential quantifier**.

" $\forall$ " is the **universal quantifier**.

A statement of the form "$\exists x$,blablabla-about-x" is an **existential statement**. We read it "There exists x such that blablabla-about-x."

A statement of the form "$\forall x$,blablabla-about-x" is a **universal statement**. We read it "For all x blablabla-about-x."

EXAMPLE: What kind of statement is "$\forall x, \exists y, y < x$"?
ANSWER: It is a universal statement, because it says that "for all , x something is true." The "something" that is true about x is an *existential statement*, namely, that "there exists a y such that $y < x$."
Let us be more precise. Let us give a name to our statement, say S. So from now until the end of this box the letter S stands for[a] the statement "$\forall x, \exists y, y < x$." Then what we just said is that S *is a universal statement* and, in fact, S is of the form "$\forall x, T$," where T *is an existential statement*, namely, "$\exists y, y < x$." So T is of the form "$\exists y, W$," where W is the statement "$y < x$."

QUESTION: Suppose we are talking about integers. What does "$\forall x, \exists y, y < x$" say in plain English? Is it true?
ANSWER: it says "for every integer there is a smaller integer." This is true, because given any integer x we can let $y = x - 1$, and then $y < x$.

QUESTION: Suppose we are talking about natural numbers. What does "$\forall x, \exists y, y < x$" say in plain English? Is it true?
ANSWER: it says "for every natural number there is a smaller natural number." This is false, because if we take x to be 0 then there is no natural number y such that $y < x$.

[a]Pay attention to what we have just done! Does it look familiar? If it does not, read §8.4 and then come back to this box. You will see that what we have done is this: we introduced a new letter, namely, S, and declared it to be a constant, by declaring its value to be the statement "$\forall x, \exists y, y < x$." This is exactly the same kind of thing we did in Example 3 when we introduced the letter a and declared it to be a constant with value $\frac{\sqrt{5}-1}{2}$. In Example 3 the letter a is used as a *numerical constant*, that is, a constant whose value is a number. Here we are using the letter S as a *statement constant*, that is, a constant whose value is a statement. In both cases, the letters are thing-symbols (that is, they stand for a thing, which is a number in one case and a statement in the other case) and are constants (that is, they stand for a fixed thing).

8.2 A very important remark on reading existential statements

An **existential statement** is a statement that asserts that there exists an object of a certain kind. For example:

$$\text{Somebody killed Polonius} \tag{8.2.1}$$

is an existential statement, because it says that there is a person who killed Polonius. Using the symbolism of existential quantifiers, we could rewrite (8.2.1) as

$$\exists x, x \text{ killed Polonius.} \tag{8.2.2}$$

The correct way to read the three-character expression "$\exists x$," (including the comma) is "there exists an x such that". For example:

- you should read (8.2.2) as "there exists x such that x killed Polonius," or "there exists x who killed Polonius."
- a statement of the form "$\exists x \in \mathbb{R}$, something about x" is read as

  $$\text{There exists a real number } x \text{ such that something about } x. \tag{8.2.3}$$

  or

  $$\text{There exists an } x \text{ belonging to } \mathbb{R} \text{ such that something about } x. \tag{8.2.4}$$

It is **wrong** (**very** wrong!) to read (8.2.2) as "there exists x and x killed Polonius."

Similarly, you should **never** read "$\exists x \in \mathbb{R}$, something about x" as

$$\text{There exists a real number } x \text{ and something about } x. \tag{8.2.5}$$

If you read (8.2.2) as "there exists x and x killed Polonius," or "$\exists x \in \mathbb{R}$, something about x" as in (8.2.5) then you are completely changing the meaning of the statement, and you are not conveying what it actually means.

EXAMPLE: Suppose you want to explain what it means for two people to be "siblings or half-siblings" using an existential quantifier. You could say:

Let x and y be persons. We say that x *and* y *are siblings or half-siblings* if $x \neq y$ and $\exists z, z$ is a parent of x and a parent of y. (8.2.6)

How would you read this in plain English? You would certainly **not** say "there exists a z and z is a parent of x and a parent of y."!

What you would say (somewhat awkwardly, I know!) is "there exists a z who is parent of x and a parent of y," or (even more awkwardly, but still correctly) "there exists a z such that z is a parent of x and a parent of y."

It is important that you acquire as soon as possible the habit of using "such that" correctly. Please start right now! $\diamondsuit$

RECOMMENDED QUESTION FOR YOU TO THINK ABOUT: If I were to say "There exists a real number such that $x^2 + x^3 = x$," this would **not** be acceptable. Why? What is missing?

ANOTHER RECOMMENDED QUESTION FOR YOU TO THINK ABOUT: Is (8.1.15) true? Why?

8.3 More examples

Example 5. Suppose you want to say

> For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result (that is, 1 over the result). Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20. (8.3.1)

Once again, this is awful, so you can shorten the whole thing and just say

$$\text{Let } N = \sum_{k=2}^{25} \frac{1}{k^2+5}\,. \text{ Then } N < 20. \tag{8.3.2}$$

Or, if you prefer, you could just say:

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,. \tag{8.3.3}$$

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: Is it true or false that $\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$? How about $\sum_{k=2}^{25} \frac{1}{k^2} < 20$? And $\sum_{k=2}^{25} 1 < 20$? And what would you say if I ask you whether it's true that $\sum_{n=2}^{25} \frac{1}{k^2+5} < 20$?

Example 6. Suppose you want to say

> If you take the function that to each real number assigns its cube, and integrate it from 0 to 4, then the result is 64. (8.3.4)

Again, this can be made much shorter by saying

$$\text{Let } N = \int_0^4 x^3\,dx\,. \text{ Then } N = 64. \tag{8.3.5}$$

Or you could just say:

$$\int_0^4 x^3\,dx = 64\,. \tag{8.3.6}$$

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: Is it true or false that $\int_0^4 x^3\,dx = 64$? Is it true or false that $\int_0^4 y^3\,dx = 64$?

Example 7. Suppose you want to say:

The set of all real numbers whose square is larger than the cube of the number has at least two different members. (8.3.7)

We can make this shorter by saying

Let S be the set of all real numbers x such that $x^2 > x^3$. Then S has at least two members. (8.3.8)

or

Let $S = \{x \in \mathbb{R} : x^2 > x^3\}$. Then S has at least two members. (8.3.9)

Or you could just say:

$$\text{Let } S = \{x \in \mathbb{R} : x^2 > x^3\}\text{. Then } \exists x \exists y, (x \in S, y \in S, \text{ and } x \neq y). \tag{8.3.10}$$

Example 8. Suppose you want to say that

I am older than 20 and less than 70, and my brother is younger than I am, but his age is larger than half my age. (8.3.11)

Here too we can be more concise, and say

Let x be my age and let y be my brother's age. Then $20 < x < 70$ and $\frac{x}{2} < y < x$. (8.3.12)

8.4 Constants and variables

In all the examples we have discussed so far, we have used **special symbols** (most commonly letters, such as x, y, a, k, N, n, S) as labels, so that we can avoid writing clumsy phrases such as "the number that we have been talking about." Sometimes we use other special symbols, such as Greek letters (α, β, ε, etc.), letters modified in a particular way (for example $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$), numerals (0, 1, 2 , . . .)

These special symbols stand for *objects*, or *things* of various kinds (numbers, human beings, sets), so we are going to call them *individual symbols*, or *thing-symbols*. (There are other kinds of mathematical symbols that do not stand for things. For example: the universal quantifier $\forall$, the existential quantifier $\exists$, the symbol $+$ for the addition operation. We will talk about many of these symbols later.)

All the thing-symbols we have used are either

constants	or	**variables**	(also called **dummy variables**).

A *constant* is a symbol that stands for a thing that stays fixed throughout an explanation or argument.

For instance, in Example 3, the letter a stands for the number $\frac{\sqrt{5}-1}{2}$. We made this clear by **introducing** the letter a by means of the **declaration**

$$\text{Let } a = \tfrac{\sqrt{5}-1}{2}.$$

That is, we **declared** the value of a to be $\frac{\sqrt{5}-1}{2}$, so from that moment on we know that a stands for "$\frac{\sqrt{5}-1}{2}$."

Variables (also called "dummy variables") are letters used as an *internal label* within a statement, but which do not have a value outside this statement.

This may look complicated, but it is in fact quite simple. Actually, it is something you are probably used to if you write computer programs in a language like Pascal. You often introduce a variable to keep track of some number with which you perform several operations. For example, you may want to keep track of a list of 20 numbers, called list[1], list[2], ..., up to list[20]. Suppose you want to add the 20 numbers, and end up with a number called "sum" as the result. You could write the following program:

```
begin
    sum := 0;
    For i := 1 to 20 do sum := sum + list[i];
end.
```

In this program, the letter i is just a label used to keep track of the variable index that keeps changing all the time. You could have written instead

```
begin
    sum := 0;
    For k := 1 to 20 do sum := sum + list[k];
end.
```

and you would have achieved the same thing. *Within the program*, the letter i (or k in our second version) has a value that keeps changing all the time. *Outside the program*, it does not make sense to ask "what is the value of i?"

Notice that what we are doing with this program is computing the sum

$$\text{Sum} = \sum_{i=1}^{20} \text{list}[i]\,,$$

which is exactly the same as saying

$$\text{Sum} = \sum_{k=1}^{20} \text{list}[k]\,,$$

or

$$\mathrm{Sum} = \sum_{\alpha=1}^{20} \mathrm{list}[\alpha]\,,$$

or

$$\mathrm{Sum} = \sum_{\clubsuit=1}^{20} \mathrm{list}[\clubsuit]\,.$$

The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing. For example, if you write

$$\text{For all real numbers } x,\ y,\ (x+y)^2 = x^2+y^2+2xy\,, \tag{8.4.1}$$

or

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, ((x+y)^2 = x^2+y^2+2xy)\,, \tag{8.4.2}$$

then that has exactly the same meaning as you had written

$$\text{For all real numbers } a,\ b,\ (a+b)^2 = a^2+b^2+2ab\,, \tag{8.4.3}$$

or

$$\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, ((a+b)^2 = a^2+b^2+2ab)\,, \tag{8.4.4}$$

or

$$\text{For all real numbers } \alpha,\ \beta,\ (\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta\,, \tag{8.4.5}$$

or

$$\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, ((\alpha+\beta)^2 = \alpha^2+\beta^2+2\alpha\beta)\,, \tag{8.4.6}$$

or

$$\text{For all real numbers } m,\ n,\ (m+n)^2 = m^2+n^2+2mn\,, \tag{8.4.7}$$

or

$$\forall m \in \mathbb{R}, \forall n \in \mathbb{R}, ((m+n)^2 = m^2+n^2+2mn)\,, \tag{8.4.8}$$

or

$$\text{For all real numbers } \diamond,\ \#,\ (\diamond+\#)^2 = \diamond^2+\#^2+2\diamond\#\,. \tag{8.4.9}$$

or

$$\forall \diamond \in \mathbb{R}, \forall \# \in \mathbb{R}, ((\diamond+\#)^2 = \diamond^2+\#^2+2\diamond\#)\,. \tag{8.4.10}$$

Similarly, instead of $\exists x \in \mathbb{R}, \left(x^2+x^3 = x\right)$,

you could have said $\exists z \in \mathbb{R}, \left(z^2+z^3 = z\right)$,

or $\exists a \in \mathbb{R}, \left(a^2+a^3 = a\right)$,

or $\exists \omega \in \mathbb{R}, \left(\omega^2+\omega^3 = \omega\right)$,

or $\exists \aleph \in \mathbb{R}, \left(\aleph^2+\aleph^3 = \aleph\right)$,

or $\exists \diamond \in \mathbb{R}, \left(\diamond^2+\diamond^3 = \diamond\right)$.

Just one more example:

Instead of $\displaystyle\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$ you could have said $\displaystyle\sum_{n=2}^{25} \frac{1}{n^2+5} < 20\,,$

or $\displaystyle\sum_{\rho=2}^{25} \frac{1}{\rho^2+5} < 20\,,$ or $\displaystyle\sum_{GNU=2}^{25} \frac{1}{GNU^2+5} < 20\,.$

The "dummy variables" in the above examples (that is, i, k, α, $\clubsuit$, x, y, a, b, β, m, n, $\diamond$, $\#$, z, ω, $\aleph$, ρ, GNU) are really just **internal tags**, using to keep track of numbers **within a particular statement**. Outside this text, they variables do not have any meaning. For example, after you have written

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$$

it would not make sense for me to ask you "what is the value of k?" The letter k is just a tag to specify a *range of values*, as part of a set of instructions on how to compute a sum. Any other letter could be used instead, so in fact "k isn't really there." You can persuade yourself of this in another way, by remembering that

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$$

is just another way of saying

- For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20.

As you can see, there is no value of k there!

Similarly, when we say $\int_0^4 x^3\,dx = 64$, this is just a way of saying "if you take the function that to each real number assigns its cube, and integrate it from 0 to 4, you get 64." Once again, there is no value of x there! You could equally well have said $\int_0^4 y^3\,dy = 64$, or $\int_0^4 u^3\,du = 64$, or $\int_0^4 \aleph^3\,d\aleph = 64$.

And when we say $\exists x \in \mathbb{R}, x^2 = 2$, this is just a way of saying "there is a real number whose square is 2" (or, even better, "2 has a square root"). This statement does not talk about x.

This way of speaking and writing, using dummy variables, is very useful, especially when what we are trying to say is quite long. But **you have to be careful** when writing this way, and have to make sure that you are using the letters correctly.

A thing-symbol can only be used if either its value has been declared before or it is a dummy variable.

8.5 Examples of things you should NOT write

Suppose you write

$$x^2 + x = 90$$

just like that, without any words before or after explaining what x is, or making it a dummy variable. Then what you wrote is **meaningless**[5]. We have no way to tell if what you wrote is true or false, because we have no idea what you are talking about.

If you write something like this in your homework, then I will write in big red letters "**what is x?**"

For comparison, here are some perfectly meaningful things you could have said:

1. Let $x = 9$. Then $x^2 + x = 90$.

2. Let $x = 6$. Then $x^2 + x = 90$.

3. Let x be the number of planets of the solar system. Then $x^2 + x = 90$.

4. $x^2 + x = 90$ for every real number x.

5. $x^2 + x = 90$ for some real number x.

[5]It's more or less the same situation as if somebody walked into our classroom and said "she has arrived." If we already know who "she" is supposed to be then this makes sense. But if we don't then our reaction would probably be to ask "who are you talking about?" The word "she" is a *pronoun*. It stands for somebody, and a sentence using "she" is meaningless if we do not know which person "she" stands for. Similarly, the letter x in "$x^2 + x = 90$" functions like a pronoun. The sentence "$x^2 + x = 90$" is meaningless if you do not know who—or what— x stands for.

6. $\forall x \in \mathbb{R}, (x^2 + x = 90)$.

7. $\exists x \in \mathbb{R}, (x^2 + x = 90)$.

8. The equation $x^2 + x = 90$ has a solution $x \in \mathbb{R}$. (Or "the equation $x^2 + x = 90$ has a real solution.")

Some of these are true, while the others are false. (Precisely: 1,3,5,7,8 are true; 2,4,6 are false. Moreover: 6 is just another way of saying the same thing as 4, and 7 and 8 are just alternative ways of stating 5.)

Notice that in 1, 2 and 3 the letter x is a constant: its value is declared before the letter is used to make a statement. In 4, 5, 6, 7 and 8 the letter x is a dummy variable.

Again for comparison, let us now write some things that do **not** have a completely precise meaning:

A. The equation $x^2 = 2$ has a solution.

B. The equation $x^2 + 1 = 0$ has a solution.

C. The equation $3x = 5$ has a solution.

Why are these things not good enough? Because they do not specify the **range** of values within which the solution x is to be sought. For example, A is true if we look for solutions $x \in \mathbb{R}$, but false if we look for solutions $x \in \mathbb{Q}$. B is true if we look for solutions $x \in \mathbf{C}$ (the set of complex numbers) but false if we look for solutions $x \in \mathbb{R}$. C is true if we look for solutions $x \in \mathbb{Q}$ (or $x \in \mathbb{R}$) but false if we look for solutions $x \in \mathbb{Z}$.

On the other hand, the following are perfectly meaningful statements:

A1. The equation $x^2 = 2$ has a solution $x \in \mathbb{R}$. (That is: $\exists x \in \mathbb{R}, (x^2 = 2)$.)
A2. The equation $x^2 = 2$ has a solution $x \in \mathbb{Q}$. (That is: $\exists x \in \mathbb{Q}, (x^2 = 2)$.)
B1. The equation $x^2+1 = 0$ has a solution $x \in \mathbb{Z}$. (That is: $\exists x \in \mathbb{Z}, (x^2 + 1 = 0)$.)
B2. The equation $x^2+1 = 0$ has a solution $x \in \mathbb{R}$. (That is: $\exists x \in \mathbb{R}, (x^2 + 1 = 0)$.)
B3. The equation $x^2+1 = 0$ has a solution $x \in \mathbf{C}$. (That is: $\exists x \in \mathbf{C}, (x^2 + 1 = 0)$.)
C1. The equation $3x = 5$ has a solution $x \in \mathbb{Z}$. (That is: $\exists x \in \mathbb{Z}, (3x = 5)$.)
C2. The equation $3x = 5$ has a solution $x \in \mathbb{Q}$. (That is: $\exists x \in \mathbb{Q}, (3x = 5)$.)
C3. The equation $3x = 5$ has a solution $x \in \mathbb{R}$. (That is: $\exists x \in \mathbb{R}, (3x = 5)$.)
C4. The equation $3x = 5$ has a solution $x \in \mathbf{C}$. (That is: $\exists x \in \mathbf{C}, (3x = 5)$.)

(Actually, A1, B3, C2, C3 and C4 are true, whereas A2, B1, B2 and C1 are false. Of course, in each case you could say the same thing in lots of different ways. For example, instead of C2 you could say "the equation $3x = 5$ has a rational solution," or "the equation $3x = 5$ can be solved in the set of rational numbers," or "there is a rational number that satisfies the equation $3x = 5$," or "$3x = 5$ is solvable in the set of rational numbers.")

8.6 Ambiguous sentences

A sentence whose meaning is imprecise because it can be interpreted in more than one way is said to be *ambiguous.* For example, the sentences

Flying planes can be dangerous

College students are turning to vegetables

The University president promised to stop drinking on campus

I shot an elephant in my pajamas[6]

Mary had a little lamb[7]

George W. Bush and Hillary R. Clinton are married

are ambiguous. Why? Think of other examples of ambiguous sentences.

Similarly, A, B, C above are ambiguous. Another example of ambiguous sentence is a sentence where a letter does not have a declared value and is not a dummy variable either. For example, if I tell you to "read chapter x," then that's ambiguous—unless I have told you before what "x" stands for, or given you clues so you can find out, or perhaps I have said "read Chapter x for all x" (that is "read all the chapters") or "read Chapter x for some x" (that is "read one chapter")— because you don't know what x is supposed to be.

Avoid ambigous sentences!

8.7 Words are important

Formulas without words are usually meaningless and should be avoided like the plague.

[6] "Last night I shot an elephant in my pajamas. How the elephant got into my pajamas I have no idea." (Groucho Marx)

[7] Mary had a little lamb, \\ A loaf of bread, a pot of jam, \\ An ice cream soda topped with fizz, \\ And oh! - How sick our Mary is!

It's the **words** that glue the formulas together and give a structure and a meaning to mathematical writing. If you just write a bunch of formulas then you probably do not know what you are saying, and if you don't know then you should not expect that the instructor or the grader will know.

Look, for example, at Statements 1, 2, 4 and 5 above. In all cases, the "formula" $x^2 + x = 90$ is the same, but the statements are quite different and have totally different meanings. That's because the words are different!

8.8 More on constants

8.8.1 Global (universal) constants

The simplest example of the use of letters in mathematics is provided by the *global constants*, or *universal constants*, as we now explain.

Example 9. The ratio of the length of a circle to the length of its diameter, whose value is approximately 3.141592, is a very important number. Since mathematicians don't want to have to say "the ratio of the length of a circle to the length of its diameter" every time wish to refer to this number, they have agreed to use the Greek letter π ("pi") as a name (or, if you prefer, as an abbreviation) for it. $\diamondsuit$

Example 10. Another important number is the basis of natural logarithms, whose value is approximately 2.718. This is a number that shows up all the time in calculus, so we give it a name and call it e. $\diamondsuit$

The symbols π and e are two examples of *global constants*, that is, symbols that mathematicians agreed once and for all to use for a fixed mathematical object. In this case, the constants are *numerical*, because they stand for numbers (real numbers, to be precise).

Here are some other examples of global numerical constants: 0 ("zero"), 1 ("one"), 2 ("two"), 3 ("three").

Actually, mathematicians want to be able to write a name for *every* natural number. For this purpose, it not possible to choose a different single symbol like 0 or 1 (or perhaps letters) as a name for every number, because there are infinitely many numbers. So we need to use *strings of symbols* to name numbers. (Such strings are called "numerals.") Moreover, to decide which string corresponds to which number, it would be silly to do it on a one-by-one basis, because then we would have to memorize an enormous "dictionary" translating numbers into strings of symbols and viceversa[8].

It is much better to have a systematic approach, enabling us to build the names of complicated numbers from some basic symbols according to simple *rules*. Since ancient times, many such approaches have been devised. (For example, the Roman numeral system use the letters I, V, X, L, C, and M. In the Roman system, the numerals III, IX, and $MMCCCLXIX$ represent the

[8] It's actually much worse! The "dictionary" would have to be infinitely long, because there are infinitely many natural numbers.

numbers that we would call today 3, 9, and $2,369$.) The one in use today is the system of so-called *decimal Arabic numerals*, that is, the numerals that you are undoubtedly familiar with: 0, 1, 2, 3, ..., 235, ..., $34,607$, etc.

8.8.2 Local (temporary) constants

Often, a number shows up repeatedly in a particular computation or argument, so we want to introduce a letter symbol to serve as a name for that number *for the duration of this particular computation or argument only.* Such letter symbols are called *local constants*.

We can introduce a local constant by *declaring its value.* For example, we can say:

$$\boxed{\text{Let } a = 35. \text{ Then } a \text{ is divisible by } 5. \qquad (8.8.1)}$$

Here, the words "let $a = 35$" effectively declare the symbol a to be a constant, representing the number 35. So the statements in the above box are just a roundabout way to say "35 is divisible by 5."

Once this particular use of the letter a is over, you could, if you want to, use the same letter to represent some other number. But in that case it would have to be very clear that the old declaration that $a = 35$ no longer applies. You could do this, for example, by saying something like

Let $a = 35$. Then a is divisible by 5.

$\vdots$

Now let $a = 36$. Then a is not divisible by 5.

The word "now" serves the purpose of telling the reader that "we are starting all over again, and the old declared value of a no longer applies."

Example 11. Your bank uses a computer program to prepare your monthly statements. Within that computer program, various numbers related to your bank account are represented by symbols or strings of symbols. Those representations are *local*. This means, quite simply, that those symbols represent what they do *only within the particular computation that the bank's computer is carrying out to prepare your statement.* Say, for example, that the bank's program uses the string "DEP" to represent the sum total of your deposits. Then "DEP" will stand for a particular number that the program will compute, but once that computation is over then the string "DEP" will no longer have a value, and the computer will be free to start another computation, for another customer, and use "DEP" again, this time to refer to the sum total of the new customer's deposits. Outside the bank's computations for your bank statement, the symbol "DEP" has no value. It only has a value *internally*, within the computation. $\diamondsuit$

Example 12. Local constants are used, for example, in contracts, or in trials. If you are buying a house, then the contract will probably contain a clause at the beginning declaring the words "the buyer" to stand for you for that particular contract. This means that the letter string "the buyer" is functioning as a local constant, whose value is you for this contract. Once your attorney goes to the signing for the next house sale, where the buyer is a different person, then he will present this person with a different contract, in which the string "the buyer" has a totally different value. So the value of the string "the buyer" is fixed only within a specific contract, and changes when yo go another contract.

A similar situation occurs in trials. At the beginning of a trial, the parties will agree that the words "the defendant" stand for a specific person. But this declaration of value is only applicable within a particular trial. Once we move on to another case, the string "the defendant" with be redeclared to have a different value. ♢

8.8.3 Constants with a declared initially unknown value

The declared value of a constant could be *unknown* to us when we start writing, even if it is fixed in some way. It may happen that the value is eventually fully determined, or that later on we find out more about it without fully determining it, or that at the end of our discussion we know as little about the value as we knew at the beginning.

Example 13. We could write

$$\boxed{\begin{array}{l}\text{Let } a \text{ be the number such that } 3a+5=17.\\ \text{Then } 3a=17-5=12. \text{ Therefore } a=4.\end{array}} \tag{8.8.2}$$

In this case, when we declare a to be a constant, we are not explicitly stating what its value is. We are just declaring that a must satisfy a certain condition, namely, the equation $3a+5=17$. Then we go on and solve the equation, so we end up knowing that $a=4$, but our knowledge of the value of a is a *consequence* of our original declaration together with some reasoning. ♢

Example 14. We could write

$$\boxed{\begin{array}{l}\text{Let}\\ \qquad a=\begin{cases}1 & \text{if Goldbach's conjecture is true},\\ 0 & \text{if Goldbach's conjecture is false}.\end{cases}\\ \text{Then } a^2=a, \text{ because } 0^2=0,\ 1^2=1, \text{ and } a \text{ is either 0 or 1.}\end{array}} \tag{8.8.3}$$

Example 15. Can we start a paragraph by writing

$$\boxed{\text{Let } a \text{ be the number such that } 3a^2+5=17.}\ ?$$

NO!!!!!!!!!!!!!! Why not? Because in what we have written there is a word (just one word!) that makes the whole thing unacceptable. That word is the word "the." Why is that word bad? Because when you talk about "the number such that ..." you are implying that there is only one such number[9], but in our situation there are two of them, because both 2 and -2 are solutions of the equation $3a^2 + 5 = 17$. ♢

Example 16. Can we start a paragraph by writing

$$\boxed{\text{Let } a \text{ be a number such that } 3a^2 + 5 = 17.} \ ?$$

YES!!!!!!!!!!!!!! Now there is no problem. We are bringing in the letter a and declaring it to be a constant with a value that is not completely known to us. We are stating that the value of a is a number such that, if we square it, multiply by 3, and add 5 to the result, we get 17. (This means that, in fact, a has two possible values, namely, 2 and -2. So this is a similar situation to the one of Example 14, where we could be sure that a is 0 or 1, but we could not tell which one.) ♢

8.8.4 Arbitrary constants

Another way in which a constant can be fixed but unknown is by declaring it to be an *arbitrary* object, and stipulating certain conditions that it must satisfy. (See Box 1 for a detailed explanation of how this works.)

For example, suppose you want to argue that the sum of two even integers is even. (See the proof in the book, page 16.) Then you could write the following:

1. Let x and y be arbitrary even integers.
2. Since x is even, we know from the definition of "even number" (book, page 1) that x is divisible by 2.
3. Since y is even, we know from the definition of "even number" (book, page 1) that y is divisible by 2.
4. Since x is divisible by 2, we know from the definition of "divisible" (book, page 2) that there is an integer c such that $2c = x$.
5. Since y is divisible by 2, we know from the definition of "divisible" (book, page 2) that there is an integer d such that $2d = y$.
6. Since $2c = x$ and $2d = y$, it follows that $2c + 2d = x + y$.
7. Since $2c + 2d = 2(c + d)$, it follows that $2(c + d) = x + y$.
8. Therefore there exists an integer e (namely, $c + d$)) such that $2e = x + y$.
9. Since $2e = x + y$ and e is an integer, it follows from the definition of "divisible" that $x + y$ is divisible by 2.
10. Since $x+y$ is divisible by 2, it follows from the definition of "even number" (book, page 1) that $x + y$ is even.
    END OF PROOF.

[9] You would say that "Trenton is the capital of New Jersey," but you would not say that "Hardenbergh Hall is the Rutgers University building." That's because New Jersey has only one capital, but Rutgers University has lots of buildings. You *could* say that "Hardenbergh Hall is a Rutgers University building." That would be O.K. Notice how much difference a little word ("a" versus "the") can make!.

BOX NO. 1: ARBITRARY OBJECTS.

An *arbitrary object* is a fixed object about which we know nothing. Arbitrary objects can be subjected to certain conditions; for example, an "arbitrary integer" is an arbitrary object that cannot be anything whatsoever (it cannot be, for instance, the State of New Jersey, or a cow, or the number 3/5; it has to be an integer), but other than that it is an integer we know nothing whatsoever about it.
The way you should think about "arbitrary objects" is as follows: imagine that there is somebody (a friend, or a computer, or an alien from the planet of the Klingons) whose job is, every time you write "let a be an arbitrary object," to pick one such object, write down what it is on a piece of paper, and then puts the paper in an envelope and seal the envelope.
We will call such a person (or computer, or alien) the **CAT** ("creator of arbitrary things"). So, for example, if you say "let a be arbitrary; suppose a is an integer" then the **CAT** will pick an integer—say 23—and write "$a = 23$" on the paper that will go inside the envelope. (Actually, the **CAT** is also allowed pick other objects such as Mount Everest or the king of Norway, which are not integers, but, as we shall soon see, a reasonable **CAT** would never do that.)
The crucial facts are:

1. the **CAT** is allowed to pick any object whatsoever and assign it to be the value of your constant;
2. only the **CAT** knows which object he has picked;
3. the **CAT** wants to prove you wrong;
4. the **CAT** is allowed to wait up to the last moment before disclosure time before choosing the values of your arbitrary objects;
5. at the end of your argument, the **CAT** will disclose the chosen values of all the arbitrary constants that you had introduced, and *if these values satisfy all the conditions that were imposed on them but fail to satisfy at least one of the conclusions that you drew about them in your argument the* **CAT** *wins and you lose.* (If the values do not satisfy all the conditions, the **CAT** loses. If the values satisfy all the conclusions, the **CAT** loses.)

(Rule 5 is temporary, and will later be replaced by the more precise Rule 5#.)

The above proof uses five letters, namely, x, y, c, d, and e. Notice that **each letter is properly introduced by declaring its value**:

- x and y are declared to be "arbitrary even integers";
- c is declared to be a "an integer such that $2c = x$";
- d is declared to be a "an integer such that $2d = y$";
- e is declared to be $c + d$.

In principle, the introduction of an arbitrary object must take the form:

Let a be an arbitrary object of such and such kind.

For example, one could write one of the following things:

- Let a be an arbitrary integer.
- Let x be an arbitrary even integer.
- Let p be an arbitrary prime number.
- Let x be an arbitrary real number.
- Let ε be an arbitrary positive real number.

The word "arbitrary" is often is omitted, and one writes things like

- Let a be an integer.
- Let x be an even integer.
- Let p be a prime number.
- Let x be a real number.
- Let ε be a positive real number.

It is strongly recommended that you acquire the habit, whenever you see a statement like "Let x be an even integer." to see it as really another way of saying "Let x be an arbitrary even integer."

This way of thinking will turn out to make things much easier for you.

Example 17. Suppose you say:

(#) Let a, b, c be a arbitrary integers such that $ab = ac$. Then $b = c$.

Is this correct? The answer is "yes" if the **CAT** cannot beat you, and "no" if the **CAT** can beat you. Let's see if the **CAT** can beat you.

The **CAT**, has to choose a, b and c to be integers, and has to pick them so that $ab = ac$. (Otherwise, if one of the objects a, b, c is not an integer—for example, if a is Mount Everest—then the condition that a, b, c are integers is not met, and the **CAT** loses. If a, b, c are integers but $ac = bc$ is not true, then once again you are not satisfying all the conditions, so the **CAT** loses.) The **CAT** could, for example, pick $a = 3$, $b = 4$, $c = 4$, but then "$b = c$" would be true, and the **CAT** would lose again.

However, remember that the **CAT** can wait until you have said all you were going to say. So the **CAT** will wait, and after hearing you say "then $b = c$," it will be clear how to win. The **CAT** will pick, for example, $a = 0$, $b = 3$, $c = 5$. *With these choices all the conditions you imposed (a, b, c are integers, and $ab = ac$) are satisfied, but your conclusion (that is, $b = c$) is not, so you lose and the* **CAT** *wins.*

Our final conclusion is that (#) is **not** correct. ♢

9 Definitions: why they matter and how you should write them.

We have already seen examples of why definitions are important. In page 12 we asked the question "how many solutions does Problem 1 of page 12 have?" We reached the conclusion that we could not answer the question because we did not know exactly what was meant by a "solution of Problem 1 of page 12."

If we are asked "is the number 23 small?", we have the same difficulty. I have no idea how to answer this question, because I do not know what "small" means. (Do you? If you think you do, please tell me!) I know what "smaller than" means, but that's a totally different thing!

On the other hand, if we are asked "is the number 18 even?", then we can answer, because we have a precise definition of "even." (See the book, page 1.) To prove that 18 is even we apply the definition. It says that an integer is *even* if it is divisible by 2. So to prove that 18 is even we have to prove that 18 is divisible by 2. How do we prove that? We have to apply the definition of "divisible." That definition appears in page 2 of the book. Furthermore, the definition starts with the words "let a and b be integers," which is exactly what we need, because we are precisely in this situation, of having two integers, namely, 18 and 2. The rest of the definition says that "we say that a is *divisible* by b provided there is an integer c so that $bc = a$."

Now, as far as the book's definition is concerned, a and b are arbitrary integers. (By now, you know what this means!) But we have two specific ones, namely, 18 and 2. So we can plug them in and conclude that 18 is divisible by 2 provided there is an integer c so that $2c = 18$." So now all we need is to prove that "there is an integer c so that $2c = 18$." (Notice that here c is a dummy variable, occurring in an existential statement.) That is, we have to prove that $\exists c \in \mathbb{Z}, 2c = 18$.

To prove that there exists an object such that something happens, you show one.
For example: to prove that there exists a town in New Jersey with a population larger that 10,000, you say "for instance, Piscataway has more than 10,000 people." (Naturally, you could also have used Trenton, or Princeton, or New Brunswick.)

So, to prove that $\exists c \in \mathbb{Z}, 2c = 18$ you must exhibit an integer c such that $2c = 18$. The obvious choice (in fact the only choice) is 9. Since $2.9 = 18$, it is true that $\exists c, 2c = 18$. So 18 is divisible by 2, and we are done.

Some key facts you should retain from this are

1. We wanted to prove that something was true of a number (namely, that the number was even).
2. To prove that, we used the definition of "even," which talks precisely about one number, and we plugged in the number we wanted, namely, 18.
3. This brought in another concept, "divisible," which is about two numbers rather than just one.

4. But when we applied the definition of "even" we went in with the number 18, and the definition of "even" gave us the other number (namely, 2), so we got exactly the two numbers that we needed to plug into the definition of "divisible" in the roles of a and b.

What you see from this is that

> In a definition there are "arguments." These arguments connect the definition with the rest of the world. For example, the definition of "even" has **one** argument. It enables you to plug in **one** object and get a true-false conclusion. The definition of "divisible" has **two** arguments. You plug in **two** things to get a true-false conclusion.

When you write a definition you should first of all figure out how many arguments will be involved, and then you should make sure that the definition begins by introducing these arguments.

Example 18. Let us figure out on our own, step by step, how to define "divisible." First of all, "divisible" is a word that talks about two numbers. (We do not talk about one number being "divisible." We talk about one number being "divisible" by another number.) So if we set out to write a definition of "divisible" we have to start by introducing our two arguments: "Let a and b be numbers." Moreover, the "numbers" involved, cannot just be any numbers. They have to be *integers*. So we really have to start with "Let a and b be integers." Now that we know who we are talking about (that is, we have properly introduced our two arguments) we are ready to say what we want to say. We want to explain what it means for a to be divisible by b. So we write "We say that a is *divisible* by b if (or 'provided that') XXXXXXXXXXX." Now what is missing is XXXXXXXXXXX. We already have the *structure* of our definition:

Let a and b be integers. We say that a is *divisible* by b if XXXXXXXXXXX.

and we have to fill in the contents. (So far, all that mattered is that "divisible" is something about two integers. Now we have to say what that "something" is.) In our case, the thing to put in the XXXXXXXXX slot is "$\exists c \in \mathbb{Z}, bc = a$," and we end up with

Let a and b be integers. We say that a is *divisible* by b if $\exists c \in \mathbb{Z}, bc = a$,

which is what the book has in page 2.

Remark 2. When you write a definition, you are defining a particular word or phrase. That word or phrase is called the *definiendum.* (This just means "the thing being defined.") **The definiendum should always be highlighted.** In the book, the author does this by using Italics. (Look, for example, at the five definitions in pages 1, 2, 3, 4.) When you write your homework or your exams, or when I write on the blackboard, it's hard to do Italics, so we use underlining instead. ♢

Example 19. Let us figure out on our own, step by step, how to define "grandson" for the benefit of someone who knows what "father," "mother" and "male" mean. First of all, "grandson" is a word that talks about two people. (We do not talk about one person being "a grandson." We talk about one person being another person's "grandson.") So if we set out to write a definition of "grandson" we have to start by introducing our two arguments: "Let a and b be persons." Now that we know who we are talking about (that is, we have introduced our two arguments) we are ready to say what we want to say. We want to explain what it means for a to be a grandson of b. So we write "We say that a is a *grandson* of b if (or 'provided that') XXXXXXXXX." Now what is missing is XXXXXXXXX. We already have the *structure* of our definition:

Let a and b be persons. We say that a is a *grandson* of b if XXXXXXXX.

and we have to fill in the contents. (So far, all that mattered is that "grandson" is something about two persons. Now we must say what that "something" is.)

In our case, the thing to put in the XXXXXX slot seems to be "b is a parent of someone who is a parent of a," that is,

"$\exists c, (c$ is a person and c is a parent of a and b is a parent of c)."

But a moment's thought shows that this is not enough. We are trying to define *grandson*, not *grandchild*. So we have to make sure that a is male.

The final result is

Let a and b be persons. We say that a is a *grandson* of b if a is male and $\exists c, (c$ is a person and c is a parent of a and b is a parent of c).

Remark 3. In the more high-brow language that we will start using later on, "even," "odd," "prime" are **one-variable predicates**, (also called **one-argument predicates**, or **one-place predicates**, or **unary predicates**), while "divisible," "larger than," "smaller than," "grandson" are examples of **two-variable predicates**, (also called **two-argument predicates**, or **two-place predicates**, or **binary predicates**). ♢

Remark 4. As you probably have guessed by now, there are three-variable predicates, four-variable predicates, and so on. For example, "common divisor" is a three-variable predicate. You can find the definition of "common divisor" in page 260 of the book. You will see that the definition talks about *three* integers, called a, b and d. ♢