MATHEMATICS 300 — FALL 2006

Introduction to Mathematical Reasoning

H. J. Sussmann

INSTRUCTOR'S NOTES

Pages 1 to 52

(September 7, 2006)

Contents

1 Information on the course

1.1 Course schedule

Our class (Introduction to Mathematical Reasoning, Mathematics 300, section 12) meets on **Tuesdays and Thursdays, 8th period (7:40pm to 9:00pm)** in Room B5, Hardenbergh Hall (College Avenue Campus).

1.2 About the instructor

My *name* is **H.J. Sussmann.** My *office* is **Hill 538**.
My *Rutgers phone extension* is 5-5407.
My *e-mail address* is **sussmann@math.rutgers.edu**.

1.3 Web page

I have set up a *Web page* for our Math 300 section:

http://www.math.rutgers.edu/˜sussmann/math300page-Fall06.html

All the instructor's notes will be available there.

1.4 Office hours

My office is **Hill 538**. My office hours will be:

- **Tuesday and Thursday, 3:30pm to 5:00pm**, in my office,
- any other time (possibly including weekends), by appointment, in my office.

1.5 Lectures and exams

We will have **26 lectures**, on

- September 5, 7, 12, 14, 19, 21, 26, 28,
- October 3, 10, 12, 17, 19, 24, 26, 31,
- November 2, 7, 9, 14, 16, 28, 30,
- December 5, 10, 12,

and **two midterm exams**, on **Thursday, October 5** and **Tuesday November 21**.

The **Final exam date** will be announced as soon as it becomes available.

1.6 Your final grade

Homework and quizzes will count for about 20% of your grade. (Homework problems labeled "optional" will not be counted towards the 20% of your grade, but you will get bonus points for doing them correctly, and you will lose nothing for not doing them or for doing them incorrectly, so you have nothing to lose by submitting solutions.)

The *two midterms*, which will count—together—for about 40%. The *final exam* will count for the remaining 40%.

In addition, I will give you another option: if your grade based on the final exam alone is higher than the one computed with the formula of the previous paragraph, then you will get the higher grade.

1.7 The textbook and the instructor's notes

We will be using:

- the book *A Transition to Advanced Mathematics* (sixth edition), by Douglas Smith, Maurice Eggen, and Richard St. Andre;
- the notes written by the instructor.

The material of the instructor's notes is an integral part of the course, as much as that of the book. *Furthermore, the notes contain all kinds of important information. For example, in this set of notes there are lots of things you need to know in order to do your homework.*

1.8 Always bring the book to class!

In the lectures, we are going to spend a lot of time looking at the book and analyzing definitions, arguments and proofs given there. So

Please always bring the book to class! You are going to need it.

1.9 Readings for the first 4 days (September 5, 7, 12 1nd 14)

- the book's "Preface to the student,"
- the book's Chapter 1 (**all of it!**),
- the instructor's notes, Pages 1 to 52 (omitting §3.3 and §3.4).

1.10 Homework assignment no. 1, due on Thursday September 14

Before you start writing your homework, read carefully the rest of this handout, in particular §1.11 on writing mathematics and submitting homework." Pay special attention to §1.12, on "answering questions in this course."

1. Book, Exercises 1.1. (pages 8-9-10-11): Problems 1 (non-starred items), and 10 (non-starred items),

2. (i) Prove that there exist integers x, y such that $x^2-y^2 = 28$, (ii) Prove that there exist integers x, y such that $x^2 - y^2 = 29$, (iii) Prove that there exist integers x, y such that $x^2 - y^2 = 30$.

3. Prove that every prime number greater than 2 is odd. (*NOTE: The definitions of "odd" and "prime number" are given in the book, page xii. A* **natural number** *is an integer n such that $n \geq 1$.*)

4. (Optional) Prove that every year must have a Friday the 13th.

5. (Optional) In the planet of the Klingons, they have a calendar exactly like the one we use here on Earth (there are 12 months, January has 31 days, February has 28 or 29, March has 31, April has 30, etc.), except for one thing: the order of the months is different. We do not know what it is, but we know for sure that it is different from ours. (For example, it may be that June is the first month, September is the second one, etc.) Is it still necessarily true that every year must have a Friday the 13th? *Prove that it is true*, no matter what the order of the months is, *or prove that it need not be true* (that is, that there is a way to reorder the months so that not every year has a Friday the 13th).

1.11 Some remarks about mathematical writing

1.11.1 Write clearly in complete sentences

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements* with a *clear meaning* that can can be understood by anyone reading what you wrote. For example:
  - If you tell me that "she is very smart," but you haven't told me who "she" is, then I don't know who you are talking about, so you haven't made a statement with a clear meaning.
  - If you write "$x > 0$," but you haven't told me who "x" is, then I don't know what you are talking about, so you haven't made a statement.
  - If I ask you to state Pythagoras' theorem and your answer only says "$a^2 + b^2 = c^2$," then nobody will know what you are talking about[1], because you have not said what "a," "b," and "c" are supposed to be.[2]
- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are used correctly, that is that either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in detail in class, and will also be explained later in these notes.
- Provide proper connectives between equations as well as between ideas.

[1] Of course, your teacher will know what you are trying to say, and anybody who already knows the statement of Pythagoras' theorem will know. But when you are asked to state a theorem or a definition **you should write it as if you were talking to somebody who does not know yet what the theorem or the definition say.**

[2] Here is a correct statement of Pyhtagoras' theorem: *Let c be the lenght of the hypothenuse of a right triangle, and let a, b be the lengths of the other two sides. Then* $a^2 + b^2 = c^2$.

- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed. (Here are two very entertaining books about punctuation that I recommend to you: (1) *Eats, Shoots and Leaves; the Zero Tolerance Approach to Punctuation*, by Lynne Truss, (2) *Eats, Shoots and Leaves; why Commas Really Do Make a Difference!*, by Lynne Truss and Bonnie Timmons.)

- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.* If you have to memorize a definition or a statement, then that is not a good sign, because it indicates lack of understanding.

- Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you've tried to read any complicated document.

- *Do not assume that the people reading your paper can read your mind. Do assume that they are intelligent, but also assume that they are busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.*

Effective written exposition will be worth at least 50% of your grade. Conversely, bad or unclear exposition may be penalized as much as 50% of the grade or even more.

- The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

1.11.2 Your written work

You should pay attention to presentation, especially for the homework:

- A nicely typed homework (e.g., using a word processor) is preferable to handwritten work. Handwritten work is acceptable too, but in that case:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**. If you are not willing to spend some of your time doing this; if what you hand in shows that you were in a hurry and that you did not make the effort to write things neatly and properly, then there is no reason for the instructor or the grader to spend any of our time reading what you wrote, and we will not do it.
  - Use a pen. Never use a pencil.
  - Use any color other than red (for example, black, blue, or green), but DO NOT USE RED. (Reason: The use of red is reserved for the instructor's and grader's comments.)
  - If you tear off the sheets from a spiral notebook, please make sure before you hand them in that there are none of those ugly hanging shreds of paper at the margins. Use scissors, or a cutter, if necessary.
- Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more than one sheet then the sheets are **stapled** and **numbered**.

If you hand in a homework assignment that has one of the following flaws:

- **it is written carelessly or in a hurry,**
- **it has lots of words crossed out,**
- **it has unreadable handwriting,**
- **it has unstapled sheets,**
- **it has unnumbered sheets,**
- **it has sheets that fail to show your name,**
- **it has shreds of paper at the margins,**
- **it is written using pencil rather than a pen,**
- **it is written in red,**

then you will lose points. If it has two or more of those flaws, then the assignment will be marked "unacceptable" and returned unread and, from Assignment No. 3 on, you will not get a chance to redo it and hand it in again.

1.12 Answering questions in this course

In this course, whenever you are given a problem where you are asked to do something, your answer should be either:

(a) doing what you were asked to do,

or

(b) showing—that is *proving*—that it cannot be done.

(See, for example, Problems 2, 4, 6, 7 below, where the correct answer is "what you asked me to do cannot be done.")

Notice that, when the answer is that "it cannot be done," it is not enough for you to *say* that it cannot be done. *You have to tell me why.* In other words, you have to *prove* that it cannot be done.

This remark is very important, and will apply throughout the semester, not just during the first week. And it applies to all your work, to the homework, the quizzes, the midterm exams, and the final exam. So please read it until you are sure you got the point. $\diamondsuit$

1.13 Some examples of problems with correct answers

Here are some examples of problems with correct solutions:

PROBLEM 1: Express the number 26 as a sum of two odd natural numbers.

ANSWER: $26 = 3 + 23$. $\diamondsuit$

REMARK: *There are lots of other solutions, of course! For example, here are two solutions different from the one given above:* $26 = 7 + 19$, $26 = 13 + 13$.

PROBLEM 2: Express the number 27 as the sum of two odd natural numbers.

SOLUTION: This is impossible. REASON: the sum of two odd numbers is always even. Since 27 is odd, it cannot be the sum of two odd numbers. $\diamondsuit$

PROBLEM 3: Prove that the number 26 can be expressed as the sum of the squares of two integers. That is, prove that there exist integers m, n such that $26 = m^2 + n^2$.

ANSWER: $26 = 5^1 + 1^2$. So, if we take $m = 5$, $n = 1$, then $26 = m^2 + n^2$. $\diamondsuit$

REMARK: What we have used here is the standard technique for proving that an object of a certain kind exists, namely, **exhibiting one**. We wanted to show that a pair m, n of integers having a certain property, namely, $m^2 + n^2 = 26$, exists, so we produced one such pair.

PROBLEM 4: Prove that the number 22 can be expressed as the sum of the squares of two integers. That is, prove that there exist integers m, n such that $22 = m^2 + n^2$.

SOLUTION: This cannot be proved because it is not true. REASON: Suppose it was possible to express 22 as the sum of the squares of two integers. Pick two integers m, n such that $m^2 + n^2 = 22$. Then we may assume that $m \geq 0$ and $n \geq 0$, because if m or n was < 0 then we could replace it by its negative and the equality $m^2 + n^2 = 22$ would still hold. Now, m cannot be > 4, because if $m > 4$ then $m \geq 5$, so $m^2 \geq 25$, and then $m^2 + n^2$ cannot be equal to 22, since $n^2 \geq 0$. So the only possible values of m are 0, 1, 2, 3, and 4. If $m = 0$, then $m^2 + n^2 = n^2$, so $n^2 = 22$, which is not possible because 22 is not the square of an integer. If $m = 1$, then $22 = m^2 + n^2 = 1 + n^2$, so $n^2 = 21$, which is not possible because 21 is not the square of an integer. If $m = 2$, then $22 = m^2 + n^2 = 4 + n^2$, so $n^2 = 18$, which is not possible because 18 is not the square of an integer. If $m = 3$, then $22 = m^2 + n^2 = 9 + n^2$, so $n^2 = 13$, which is not possible because 13 is not the square of an integer. If $m = 4$, then $22 = m^2 + n^2 = 16 + n^2$, so $n^2 = 6$, which is not possible because 6 is not the square of an integer. So all five cases $m = 0, 1, 2, 3, 4$ have been excluded. Since we have shown that these are all the possible values of m, it follows that m, n cannot exist. $\diamondsuit$

PROBLEM 5: Prove that the number 22 can be expressed as the sum of the squares of three integers. That is, prove that there exist integers m, n, q such that $22 = m^2 + n^2 + q^2$.

SOLUTION: $22 = 3^2 + 3^2 + 2^2$. So we can take $m = 3$, $n = 3$, $q = 2$. $\diamondsuit$

PROBLEM 6: Prove that $2 + 2 = 5$.

SOLUTION: This cannot be done. REASON: The statement "$2 + 2 = 5$" is false, and false statements cannot be proved. $\diamondsuit$

2 First remarks on reasoning and proofs

This course is about reasoning and proofs, so perhaps it is not a bad idea to start by discussing the meaning of the words **reasoning** and **proof**. A proof is a special type of argument, so we will have to talk about **arguments**. An argument (and in particular a proof) has **hypotheses** and a **conclusion**, so we will have to analyze those two concepts as well. (Usually, the number of hypotheses is zero—that is, no hypoteses at all—or one or two, but there could be more.)

Furthermore, arguments consist of **steps**, and each step involves an **assertion** and a **justifications**, so we will have to talk about explain how steps, assertions and justifications. Arguments can be **convincing** or **unconvincing**, so we will describe what makes an argument convincing. A proof which is convincing in a very strong sense will be said to be **valid**, so we will explain what makes a proof valid. It will turn out that a proof is valid when all the steps are **valid proof steps**, so we will need to study what a valid proof step is. This will bring us to the notion of **logical form** of a sentence and of a proof step, so we will have to introduce the very important notion of logical form. The study of all these concepts, which we have to understand in order to understand what a valid proof is, is called **logic**, so you may think of this chapter as an "Introduction to Logic".

2.1 Reasoning and arguments

> It is one of those cases where the art of the reasoner should be used rather for the sifting of details than for the acquiring of fresh evidence. ... The difficulty is to detach the framework of fact—of absolute undeniable fact—from the embellishments of theorists and reporters. Then, having established ourselves upon this sound basis, it is our duty to see what inferences may be drawn.
>
> (Words spoken by Sherlock Holmes in A. Conan Doyle's *Silver Blaze*)

What, exactly, is "reasoning"? The dictionary[3] says that reasoning is "forming or trying to reach conclusions by connected thought."

As a starting point, this is a fairly adequate explanation. When we *reason*,

[3] *Concise Oxford Dictionary*, 8th Ed., Copyright 1991, Oxford Univ. Press. Henceforth, this will be referred to as COD8.

we *express thoughts* (by speaking or writing), and we do it *in a connected way*, one step after the other, explaining what the connection is.

We reason by producing *arguments*. An **argument** is a "reasoning process" (as COD8 says). Or "an episode of thinking." (S. Guttenplan, *The Languages of Logic*, Blackwell, 1998, p. 1). I like "reasoning process" better, because it conveys the idea of *movement*: in an argument (especially a "sequential argument", which is the kind that will concern us): we *move* from thought to thought; an argument is a *series of thoughts*; each thought *follows* from the previous ones or from some external reason, as will be explained below; and we end up with the *conclusion*. Notice that the word "follows" means both "comes after" and "is a consequence of", and "conclusion" means both "what you have at the end" and "the thought that is intended to be the consequence of all the others".

I will, however, add some extra ingredients to these explanations of the meaning of "reasoning" and "argument." To understand these, let me start by giving a few examples of arguments:

Example 1. Here is our first example of an argument. You should imagine that Mr. and Mrs. Nice are talking to each other; Mr. Nice has just suggested that they might invite Mr. and Mrs. Awful for dinner tomorrow, and then Mrs. Nice reasons as follows:

- Suppose we invite Mr. and Mrs. Awful to come to dinner tomorrow.

Step 1. We know that if we invite the Awfuls for dinner, they will come.

Step 2. It follows that tomorrow the Awfuls will come to dinner.

Step 3. We also know that every time they come to dinner the Awfuls smoke a lot, without asking us for permission to smoke.

Step 4. Therefore, tomorrow the Awfuls will smoke a lot, without asking us for permission to smoke.

Step 5. And we know that when someone smokes a lot in hour house, without asking us for permission to smoke, we get very upset.

Step 6. So tomorrow we will get very upset.

End of Example 1.

In the argument of Example 1 there is a **hypothesis**, that is, a statement that we imagine to be true "for the sake of the argument". This means that ***we set out to explore an imaginary world in which the hypothesis is true, in order to find out what other things of interest to us are***

true there. The hypothesis of the argument of Example 1 is the statement that "we invite Mr. and Mrs. Awful to come to dinner tomorrow.)"

We then move step by step. Each of our steps consists of an **assertion** as well as a **justification**.

The assertion of Step 1 is that "if we invite the Awfuls for dinner, they will come". The assertion of Step 2 is that "tomorrow the Awfuls will come for dinner". The assertion of Step 3 is that "every time they come for dinner the Awfuls smoke a lot, without asking us for permission to smoke". The assertion of Step 4 is that "tomorrow the Awfuls will smoke a lot, without asking us for permission to smoke". The assertion of Step 5 is that "when someone smokes a lot in hour house, without asking us for permission to smoke, we get very upset". The assertion of Step 6 is that "tomorrow we will get very upset".

Mrs. Nice's argument ends with Step 6, and the assertion of this last step is called the **conclusion** of the argument.

Now let me list the justifications of the six steps. The justification of Step 1 consists of the words "we know that", which state—or reaffirm—the fact that we[4] both agree on this. For Step 2, the justification is in the words "it follows that", which tell us that the assertion is a consequence of those of the previous steps, that is, the hypothesis and Step 1. The justification of Step 3 consists of the words "we also know". That of Step 4 is the word[5] "Therefore". That of Step 5 is the phrase "we know that". Finally, the justification of Step 6 is the word "So".

Notice that the steps are of two kinds, depending on the justification:

- An ***accepted fact step*** (or ***known fact step***, or ***given fact step***), in which Mrs. Nice brings in a fact that both she and Mr. Nice know to be true or accept as true.

- An ***inference step***, in which a new fact is brought in because it is a consequence of the assertions of the previous steps and the hypotheses.

The key point is that, as we move step by step towards the conclusion, ***the assertion of every step ought to be accepted as true by both***

[4] That is, Mr. and Mrs. Nice.

[5] This is, of course, only an *incomplete justification*, because all it is saying is that the assertion of Step 4 follows from those of previous steps, but it does not tell us *from which steps it follows*, and *how*. But one would imagine that Mr. Nice is smart enough to realize that what Mrs. Nice has in mind is that the assertion of Step 4 follows from those of Steps 2 and 3.

Mr. and Mrs. Nice. Why? Because in every new step the assertion is either (a) something that both Mr. amd Mrs. Nice agree to accept as true, or (b) something that follows from the previous steps, so that if the assertions of the previous steps are true then that of the new step must be true as well. (The assertion of Step 1 is true—or, at least, is accepted as such by Mr. and Mrs. Nice—because they already know that it is true, and do not need to be persuaded. The assertion of Step 2 is true because ***it is a consequence of*** (that is, ***follows from***) the assertion of Step 1 and the hypothesis[6]

The assertion of Step 3 is true—for Mr. and Mrs. Nice—because they already know that it is true. The assertion of Step 4 is true because it is a consequence of those of Step 2 and Step 3. The assertion of Step 5 is true—for Mr. and Mrs. Nice—because they both agree that it is true. The assertion of Step 6 is true because it follows from those of Step 4 and Step 5.

Since the assertions of all the steps are true, it follows that in particular the conclusion is true.

So ***what the argument is supposed to do is persuade Mr. Nice that the conclusion must be true in an imaginary world in which the hypothesis is true***. As long as Mr. Nice grants or accepts that all the accepted facts that occur in the given fact steps are true, and that all the inference steps are valid (in the sense that the assertion of each of these inference steps really does follow from those of previous steps and the assumptions), he has to accept the conclusion.

Is the argument truly convincing? That is, should Mr. Nice accept it? Well, he could raise objections, by pointing out, for example, that one of Mrs. Nice's "known facts" isn't really, true, so he does not have to accepot it. (He might, for example, remind his wife that once, six months ago, the Awfuls came to dinner and did not smoke.) Or he might argue that one of the inference steps is invalid. (For example, he might say that Step 4 does not really follow from Steps 2 and 3 because, just because in the past the Awfuls have always smoked when they came to dinner, it does not follow that they will do it again tomorrow, because they may have changed and quit smoking.) Any argument by Mr. Nice giving reasons why Mrs. Nice's argument might not be convincing is a **refutation** of Mrs. Nice's argument. If Mr. Nice provides a refutation, then Mrs. Nice may accept it,

[6]Keep in mind that, "for the sake of the argument", both Mr. and Mrs. Nice have agreed to accepot the hypothesis as true, that is, to put themselves in an imaginary world in which the hypothesis is true.

and withdraw her argument, or refute the refutation, by giving an argument of her own showing that the refutation is not convincing. And this process of arguments and refutations may go on and on, and never stop.

2.1.1 Some examples of good and bad arguments

I am now going to give you a few examples of arguments, of which some are convincing and others are not.

Example 2. Here is an example of an argument with no hypotheses. The conclusion is that "The Earth is not round". This argument is of course unconvincing, as it should be, because we all know that the Earth *is* round[7].

Step 1. If the Earth was round, then many things would fall off it.

Step 2. Things do not fall off the Earth.

Step 3. Therefore, the Earth is not round.

In real life, the argument would probably be presented in a more colloquial way, as follows:

> You will surely agree with me that if the Earth was round then many things would fall off it. And I am sure you also agree that things do not fall off the Earth. Therefore, you must agree that the Earth is not round.

(The words "You will surely agree with me that" and "I am sure you also agree that" are the *justifications* of Steps 1 and 2, intended to persuade you that the sentences on these steps are "accepted facts", or "known facts", so that you must agree with them. The word "Therefore" is the justification of Step 3, telling you that this step follows from the previous ones.) *End of Example 2.*

Example 3. Here is another argument, also without hypotheses. The conclusion is that "The Earth is not flat".

[7] Well, not exactly round, but almost!

Step 1. If the Earth was flat, then when you see a ship sailing into the horizon, then all the parts of the ship would fade away gradually at the same rate until they disappear.

Step 2. But what one actually sees is that the bottom disappears first, then the middle, and finally the top.

Step 3. Therefore, the Earth is not flat.

Here again, the argument would probably be presented to you in a more colloquial way, as follows:

> You will surely agree with me that if the Earth was flat then when you see a ship sailing into the horizon all the parts of the ship would fade away gradually at the same rate until they disappear. But you must have noticed that what one actually sees is that the bottom disappears first, then the middle, and finally the top. Therefore, the Earth is not flat.

Is this argument convincing? Personally, I find it quite convincing, but you have to be careful! Some people would say that "the argument is convincing because the Earth is round, not flat". In other words, they would say that the argument is convincing *because the conclusion is true*. This is a mistake that you should avoid. **Just because the conclusion is true is does not follow that the argument is convincing**. To understand this, look at my next example.

End of Example 3.

Example 4. Here is an argument which is obviously unconvincing, even though the conclusion is true.

Step 1. The ghost of my late grandmother just whispered to me that the earth is round.

Step 2. Therefore, the Earth is round.

End of Example 4.

Example 5. Here is an argument, adapted and simplified from A. Conan Doyle's short story *Silver Blaze*. Sherlock Holmes, the famous detective, is speaking, and says

1. The only people who could have stolen the horse were Mr. Straker, the stable-boy, and Fitzroy Simpson—a perfect stranger.
2. The stable boy was sound asleep, because he had eaten curried mutton heavily laced with powdered opium, which put him to sleep.
3. Therefore the stable boy was not the one who stole the horse.
4. The dog always barks when a stranger approaches the stables.
5. The dog did not bark that night.
6. Therefore Mr. Simpson did not steal the horse.
7. Therefore it was Mr. Straker that stole the horse.

End of Example 5.

Example 6. The following argument is adapted and simplified from an episode of the TV series *Columbo.* Inspector Columbo is speaking, and says

1. You and Congressman Jones both said that you were together in your office on the night of June 16 to 17, from 10pm to 2am
2. You both said that you arrived to your parking lot at 10pm and Congressman Jones arrived at the same time, and then you both parked there and walked uo to your office.
3. You both said that you two were in your office together all the time from 10:00 to 2:00 and that at 2:00 you left together, you drove off in your car and Congressman Jones drove off in his car.
4. However, on June 17 at 6am, when the police went to the building where you work, they found that one parking place in the parking lot was dry, but all the other places were wet.
5. According to the weather bureau, the only time it rained in the area on June 16 or 17 was on June 16 from 11pm to midnight.
6. If both your car and Congressman Jones' car had been there parked during the time you both said they were, there would have been two dry parking places rather than one.

7. Therefore, there was only one car parked in your office's parking lot between 11pm and midnight.

8. Therefore, you and Congressman Jones are lying.

End of Example 6.

2.1.2 An argument part of a dialogue between an advocate and a skeptic

In each of the arguments we have looked at so far, somebody is trying to make a case for something, to convince someone else of the truth of some assertion, to get the other person to accept something. Let us call the first person, the one who is trying to do the convincing, the **advocate**. And let us refer to the other person, the one whom the advocate is trying to persuade, as the **skeptic**[8]. The assertion of which the advocate is trying to persuade the skeptic is the **conclusion** of the argument. (In our six examples, the conclusions are: (a) in Example 1, the statement that "tomorrow we will get very upset"; (b) in Example 2, the statement that "the Earth is not round"; (c) in Example 3, the statement that "the Earth is not flat"; (d) in Example 4, the statement that "the Earth is round"; (e) in Example 5, the statement that "it was Mr. Straker that stole the horse"; (f) in Example 6, the statement that "you and Congressman Jones are lying".) And the advocate is trying to persuade the skeptic that the conclusion is true **if the hypothesis are true**, that is—if you prefer—in an imaginary world in which the hypotheses are true[9].

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[8]The job of the skeptic is to take take a position contrary to that of the advocate, for the sake of the argument (not necessarily believing that the advocate's position is wrong). Another name commonly used for this is "devil's advocate". This name comes from the fact that "formerly, during the canonization process of the Roman Catholic Church, the Promoter of the Faith (Latin *Promotor Fidei*), or Devil's Advocate (Latin *advocatus diaboli*), was a canon lawyer appointed by the Church to argue against the canonization of the candidate. It was his job to take a skeptical view of the candidate's character, to look for holes in the evidence, and to argue that any miracles attributed to the candidate were fraudulent, etc." I stress that there is nothing bad, let alone "diabolic", about playing devil's advocate. The work of the devil's advocate "can be used to test the quality of the original argument and identify weaknesses in its structure", which is likely to be help the advocate improve and strengthen the original argument. (The quotations are from http://en.wikipedia.org/w/index.php?title=Devil%27s_advocate&oldid=68572051)

[9]If there are no hypotheses, then the advocate is just trying to persuade the skeptic that the conclusion is true. That is, when there are no hypotheses, the "imaginary world in

The way the advocate's case is made is as follows:

- To begin with, advocate and skeptic must have a common starting ground, that is, they must agree on some things[10]. So they must have a collection of statements that both sides accept to be true. We will call this the **Stock of Accepted Facts**, or SAF.
- In a particular argument, there is no need for the advocate to list *all* the accepted facts. It is enough if the advocate lists those accepted facts that will be needed in the argument[11]. Furthermore, it is not necessary to list at the very beginning all the accepted facts to be used. They may be presented as they are needed. (For example, in the argument of Example 5, the accepted facts are Steps 1, 2, 4, and 5. Steps 4 and 5 are brought in when they are needed, in order to exclude Mr. Simpson as a possible thief.)
- The advocate then proceeds step by step. Each step is of one of two kinds:
  1. An "accepted fact step", or "AF-step",
  2. An "inference step", or "I-step",
- Each step contains an *assertion*, which is either
  - A sentence drawn from the SAF, if the step is an AF-step,

  or

which the hypotheses are true" is just the real world, because you do not need to imagine anything.

[10] It is impossible to have an intelligent discussion with someone with whom you do not agree on anything whatsoever.

[11] Naturally, the skeptic may refuse to accept some of these facts, and in that case the argument cannot proceed. But, if the advocate and the skeptic have agreed to accept certain statements, then the skeptic cannot back off and refuse to accept one of those statements. For example, in Mathematics there will be certain statements that will be declared to be "axioms", or "postulates", which means that both sides agree to accept them. If, for example, there is an axiom that says that $1 > 0$ (that is, "one is larger than zero") then when the advocate brings in the statement "$1 > 0$" the skeptic cannot refuse to accept it, because this refusal would be playing against the rules both sides have agreed to obey.

  - A new sentence, alleged to follow from the assertions of the previous steps and the hypotheses, so that anyone who agrees with the assertions of all the sentences preceding a particular step is also obliged to agree with the assertion of the new step, if the step is an I-step.

- In addition, each step contains a *justification*, which explains whether the step is an AF-step or an I-step, and tells us, if the step is an I-step, how it follows from previous steps and the hypotheses.

(For example, (1) in Example 2, the AF-steps are Steps 1 and 2, and Step 3 is an I-step; (2) in Example 3, the AF-steps are Steps 1 and 2, and Step 3 is an I-step; (3) in Example 5, the AF-steps are Steps 1, 2, 4 and 5, and the I-steps are Steps 3, 6 and 7; (4) in Example 6, the AF-steps are Steps 1, 2, 3, 4, 5, and 6, and the I-steps are Steps 7 and 8.)

Let us summarize what we have learned from these examples and the discussion following them. *First of all*, reasoning can always be thought to be part of a conversation, in which somebody—the "advocate"—is trying to convince somebody else—the "skeptic"—of something, called the *conclusion*.

Even if you are "just thinking", without talking to anybody else, you should think of this process as a conversation with yourself. Actually, it is better to see your reasoning process as involving a debate between two distinct parts of your self, namely, one playing the role of advocate and the other one playing the role of skeptic. This means, in particular, that

> In reasoning, **you should always be your own critic**, that is, you should always act as a skeptic about your own argument, by questioning every step.
>
> In Math 300, when you write a proof, you should see yourself as presenting your case to an audience of skeptics, consisting of the other students, the instructor and the grader. And you should include yourself among the skeptics, by imagining that there is a part of you—your "critical self"—which is always looking for reasons to doubt what the other self is doing, by asking "why is this true?", or "how do we know that this is true?". In other words, you should ask yourself "what could be wrong with my argument" before anyone else does.

Second, when we reason we first seek to elucidate a common ground between the "for" side and the "against" side, by, in the words of Sherlock Holmes quoted above, "detaching the framework of fact, of absolutely undeniable fact" (that is, making sure that all sides agree on calling certain statements "undeniable"), or, in other words, agreeing on a SAF (Stock of Accepted Facts).

Third, once we have settled on a SAF, we move from these facts by *inferring* (i.e., *deducing*) consequences from them. If all our deductions are correct, then all the new facts arrived at in this way should be something that all sides agree on. And, in particular, the skeptic should agree that the conclusion is true.

In other words,

Reasoning is a conversation between two (real or imaginary) sides, in which one of the sides tries to persuade the other one that something is true, by "forming or trying to reach conclusions by connected thought.", using statements drawn from a supply of facts on whose truth both sides agree (the "Stock of Accepted Facts", or SAF), and then progressing step by step by inferring (i.e., deducing) new statements.

Reasoning proceeds by producing *arguments*.

An **argument**, from a given SAF, is a list of *steps*, each of which consists of a sentence together with a justification. The justification can be of two kinds: either an explanation of why the sentence is part of the SAF, or an explanation of why it is a consequence of the previous steps.

2.1.3 Arguments must be sequential

In our explanation of the meaning of "argument", we have insisted that every I-step should be a consequence of *previous* steps. This would disqualify the following.

Example 7. Consider the "argument":

You have to eat your soup, because I say so.

Strictly speaking, this *not* a argument in the sense explained above, because it consists of two sentences, namely,

- You have to eat your soup,
- I say so.

These sentences are connected by means of the conjunction "because", but this conjunction actually indicates that *the first sentence is supposed to be a consequence of the second one*, whereas *in a true argument, the second sentence could be a consequence of the first one, but not the other way around.* *End of Example 7.*

Example 8. Even though the "argument" of Example 7 is not a true argument in the sense of our definition, it is possible to express the same ideas in the form of a true argument, by writing

> *Step 1.* I say that you have to eat your soup.
>
> *Step 2.* Therefore you have to eat your soup.

This is clearly an argument in our sense. *End of Example 8.*

Often, one wants to accept "arguments" such as that of Example 7 as true arguments. This could be done by changing a little bit the definition of "argument", and saying something like this: "an **argument**, from a given SAF, is a list of *steps*, each of which consists of a sentence together with a justification. The justification can be of two kinds: either an explanation of why the sentence is part of the SAF, or an explanation of why it is a consequence of the other steps". Notice that this is exactly the same as the definition we gave in the box of Page 21, except only that we are now no longer requiring that each step which is not drawn from the SAF be a consequence of *previous* steps, but only that it be a consequence of *other* steps, as in our Example 7, where Step 1 follows from Step 2.

If we had followed this route, then the "arguments" in the strict sense of the box of Page 21 would be called **sequential arguments**, to distinguish them from the more general "arguments" in which a step can be a consequence of other steps that come *after*.

We will *not*, however, follow this route, because if we were to allow "non-sequential" arguments then we would get into a lot of trouble.

2.1.4 Circular arguments are completely invalid

The reason that allowing non-sequential arguments is bad, is that if you allow them then you open the door to ***circular arguments***, and circular arguments are completely invalid.

To see why, consider the following example.

Example 9.

Step 1. The Republic of Ruritania is a democracy.

Step 2. In a democracy, the government is legitimate.

Step 3. Therefore the government of the Republic of Ruritania is legitimate.

Step 4. A legitimate government is one that respects the will of the people.

Step 5. Therefore the government of the Republic of Ruritania respects the will of the people.

Step 6. If the government of a country respects the will of the people then the contry is a democracy.

Step 7. Therefore the Republic of Ruritania is a democracy.

In this “argument”, the steps can be justified as follows: Steps 2, 4, and 6 are AF-steps; Steps 1, 3, 5 and 7 areI-steps; Step 1 follows from Steps 5 and 6; Step 3 follows from Steps 1 and 2; Step 5 follows from Steps 3 and 4, and Step 7 follows from Steps 5 and 6.

It is clear, however, that this should *not* qualify as an argument, because in order to know that Step 1 is true you need to know, among other things, that Step 5 is true, and in order to know *that* you need to know that Step 3 is true, and in order to know *that* you need to know that Step 1 is true. So, ultimately, in order to know that that Step 1 is true you need to know that it is true. That is, *you are using a statement to establish that very* *same statement*[12]. This is called **circularity**, and a purported argument that exhibits circularity is a **circular argument**.

Circular arguments have absolutely no value, and should always be avoided.

You probably think that this observation is silly, because it is clear that circular arguments are bad, and you would never be so dumb as to write one. Having taught Math 300 many times, I can assure you that students have a tendency to use circular arguments all the time, so I insist on warning you against them.

The most frequently used kind of circular argument is one where the student uses the conclusion, at the beginning or in the middle of a proof. You may think that you will never do that, but many students will, and you are not careful you will do it too. So allow me to say it again, with my apologies to you if you think you do not need to be reminded of this:

Any argument that makes use of the conclusion is circular, and therefore has absolutely no value, and should be avoided.

SOMETHING FOR YOU TO THINK ABOUT. Try to come up with a definition of "argument" that allows for "arguments" that are not sequential

[12] I might as well written: "*Step 1*, Ruritania is a democracy; *Step 2*, Ruritania is a democracy", and "justified" this by saying "Step 1 follows from Step 2, and Step 2 follows from Step 1", which shows more blatantly how ridiculous it is to argue this way, as if I was saying "I am the King of Norway because I am the King of Norway". or "I gave you an F grade because I gave you and F grade". You would clearly not accept that kind of argument, would you? Here, I only wrote it in seven steps rather than two in order to hide a little bit the fact that this "argument" is bad.

(such as the one of Example 7), but does not allow circularity. *Do not be discouraged if you cannot figure this out. It is hard!*

2.1.5 When is an argument convincing?

So far, we have given some examples of arguments, but have not said much about whether they are convincing, that is, whether you should accept them if they are presented to you. (We just said one thing, namely, that circular arguments are noit convincing. But there are lots of other reasons that can make an argument unconvincing, and that's what we want to talk about now.)

For example, are the arguments of Examples 2 and 3 convincing? That is, do you have to accept them it because you have been convinced? And if they are not convincing, then what is wrong with them?

You might answer that (i) the argument of Example 2 is "wrong", because the conclusion is false, that is, because the Earth *is* round, and (ii) the argument of Example 3 is "right", because the conclusion is true, that is, because the Earth is *not* flat.

This is, however, the wrong way to go about deciding if an argument is right or wrong, and why it is wrong when it is. In Example 2, suppose that you are the skeptic, and you say to the advocate "your argument is wrong because the Earth is round, but your conclusion says that it is not". Then the advocate might answer back: "you say that the conclusion is false, but this does not tell me what is wrong with my argument. If every AF-step is indeed a fact that you and I accept, and every I-step is a consequence

That is, you might think of *refuting* the argument by saying that "it is wrong because the conclusion is false". *This is not a good way to refute an argument.*

Observing that the conclusion is false is not a satisfactory way to refute an argument. Of course, if the conclusion is false then the argument must be wrong, but just pointing out that the conclusion is false does not tell us *what* is wrong with the argument.

This is a very delicate but very important point, so let me say more about it.

Imagining yourself on trial, accused of some crime that you did not commit. You know that you are innocent, but the prosecutor does not know, and the jurors do not know. Now imagine that the prosecutor presents a long and detailed argument with which he aims to "prove" that you did commit the crime. You know that the prosecutor's conclusion is false, so you know that there must be a flaw in the prosecutor's argument. But *you would not try to refute the prosecutor argument bu just saying that the conclusion is wrong, because if you do that then the jurors will probably not believe you!* What you must do is persuade the jury that the prosecutor's argument is wrong. (For example, you may argue that the DNA evidence produced by the prosecutor is not valid, because the police could have tampered with the evidence.)

Now let us go back to our situation of an argument that the skeptic wants to refute by pointing out that the conclusion is false. How would the skeptic argue that the conclusion is wrong? The skeptic would have to give an argument for that. And then, if the advocate says "the skeptic is wrong because I have given a very good argument showing that the conclusion is true", and the skeptic says "the advocate is wrong because I have given a very good argument showing that the conclusion is false", we find ourselves in an impasse. There is a *mystery* here, a situation of tension that neeeds to be resolved. One of them must be wrong, and the thing to do is for the skeptic to study the advocate's argument, find a flaw in it, and persuade the advocate, or for the advocate to study the skeptic's argument, find a flaw in it, and persuade the skeptic.

Remark 1. In my own personal experience as a mathematician, I have found myself many times in this situation. Typically, what happens is this: somebody submits a paper to a journal, claiming to have proved a theorem. The journal then sends me the paper and asks me to act as referee. I read the paper, and become convinced that the theorem is false. I then try to prove that the theorem is false. If I succeed, then maybe I will also take the extra time to look for the mistake in the proof, or maybe not. If I find the mistake, then I write a report proving that the theorem is false and showing where the proof goes wrong. If I do not find the mistake (perhaps because I did not bother to look for it), then I will just say in my report that the theorem is false and give a proof, leaving it up to the author to figure why his or her proof is wrong. Usually, the author finds the mistake and agrees with me. But sometimes the author cannot find it, and becomes convinced that his/her proof is right. Then the author will try to find a mistake in *my* proof, and may succeed, and write a rebuttal to my report, explaining why my proof is wrong. And in most cases, when this happens, I will acknowledge my mistake. On some rare occasions, this report-and-rebuttal game may go on for several more rounds. Or we may get stuck, with author and referee each claiming to be right and insisting that the other one is wrong. In this case, the journal's editor may read all the reports and rebuttals and decide that the referee—or the author—is right, or may ask a third person to do it. Usually, when the situation is resolved, *never happens that somebody is left with a proof of a false conclusion and does not where the proof went wrong.* That's because no mathematician would be able to tolerate such a situation. If I think that I have proved something, but I realize that what I proved is false, then I will be unable to sleep until I figure out where my proof is wrong. ***You should do the same: if you think that you have proved something, but then you realize that what you have proved is actually false, then you should feel a need to figure out where your proof is wrong, and should be unable to sleep until you find out***. *End of Remark 1.*

If you want to refute an argument, you have to find one step that is not convincing. This means that you have to do one of the following two things: either

- establish that one of the steps claimed to be part of the SAF really isn't part of the SAF,

or

- establish that one of the steps that are claimed to follow from previous steps and the hypotheses does not really follow from them.

Notice that I did *not* say "point out"; I said "establish". In other words, it is not enough to *say* that a step should not really be part of the SAF, or does not follow from the steps from which it is asserted to follow. You have to explain *why*. (We will see how to do that in many examples.)

Example 10. Here is a refutation of the argument of Example 2:

Refutation of the argument of Example 2. The argument assumes that the statement

S: If the Earth was round, then many things would fall off it.

is part of the SAF, that is, that we all agree that it is true[13] But, why should S be true? It might be the case that, say, there exists some force (let me make up a name for it, for example "gravitational attraction"), that causes things to move towards the center of the Earth, and that what we call "falling" isn't really "falling down" but "moving towards the center of the Earth". If this was true, then the Earth could still be round, but things would not fall off its surface, so S would be false[14]. *End of the refutation of the argument of Example 2. End of Example 10*[15].

So the argument of Example 2 is not convincing, because we have refuted it. How about the argument of Example 3? Is that convincing? Again, you may think that the answer is "yes", because you know that the Earth is not flat. But, once again, this is not a good answer, because it might be that the argument is wrong anyhow.

Example 11. Here is an attempt to refute the argument of Example 3.

Attempted refutation of the argument of Example 3. Step 1 is not entirely convincing because, for example, there might exist some strange optical effect that causes images of ships to vanish in precisely the way described in that step. *End of attempted refutation of the argument of Example 3. End of Example 11.*

[13]Notice that at this point I have done something that we will be doing again and again in this course: I have introduced a *letter variable*, namely, the letter S; and I have done it in the only way that will be allowed in the course, *by declaring a value*. I have declared that "S" stands for the statement "If the Earth was round, then many things would fall off it".

[14]Naturally, when I wrote this refutation I got a lot of help from knowing that the imaginary "gravitational force" is in fact real. But this is irrelevant. What matters is whether the argument can be refuted or not.

[15]Since the example has ended, the validity of our declaration of a value for the letter S ends as well. So the letter S becomes a *free variable*, by which we mean that we are free to give it another value whenever we want to.

Remark 2. Personally, I do not find this refutation very convincing, because the author gives no idea of what that "optical effect" might be, and no good reason for us to believe that it maight exist.

This illustrates a general problem with arguments in everyday discourse: usually, arguments are not clear-cut; an argument may sound convincing to some people and not to others; and in those cases it is not easy to resolve the disagreement, and people end up getting angry at one another and fighting. In Mathematics, we will propose a notion of "argument" that does not have this problem. This is what we will call "mathematical proof". We will get there eventually, but it is going to require some work. *End of Remark 2.*

Question 1. Write a refutation of Sherlock Holmes' argument in Example 5. *End of Question 1.*

Question 2. Write a refutation of Inspector Columbo's argument in Example 6. *End of Question 2.*

So far, we have looked at refutations of arguments when the conclusion is false. Is it possible to refute an argument when the conclusion is true? And how does one do it?

The answer is very simple: *to refute an argument, you do exactly what was explained in the box of Page 27.* That box does not say anything about the conclusion being true or false. So *what you have to do is exactly the same thing whether the conclusion is true or false.*

Does this mean that "in order to refute an argument it does not make any difference whether the the conclusion is true or false"? Not quite. *What you have to do* is the same in both cases. But *whether you can do it* depends very much on which case you are in[16]:

[16] Notice that in the box I have done again something that I had already done once (see Footnote 13), and this time I am doing it twice. I am introducing two *letter variables*, namely, **A** and **C**; and I have done it in the only way that will be allowed in the course, *by declaring a value.* I have declared that "**A**" stands for "an arbitrary argument", and "**C**" stands for "the conclusion of **A**." (The meaning of "arbitrary" will be explained later, see XXX.) At this point, **A** and **C**, which were "free variables", become "constants", that is, have a fixed value. The declarations are valid until the box ends. After we are finished with the box, "**A**" and "**C**" become free variables again, so we can assign other values to them whenever we want to.

Let A be an argument and let C be its conclusion. Then

- **If C false, then something must be wrong with A, so you have to be able to refute A, by doing what is explained in the box of Page 27.**
- **If C is true, then A could be right, in which case it is not possible to refute it, or A could be wrong, in which case you have to be able to refute A, by doing what is explained in the box of Page 27.**

Question 3. Would it have been possible, in the box of Page 29, to declare the value for **C** *before* rather than *after* declaring the value for **A**. If not, explain why not. *End of Question 3.*

The second part of the box of Page 29 makes a very important point, that students often find surprising, so let me say the same thing again, with different words:

Just because the conclusion of an argument is true, it does not follow that the argument itself is correct. An argument can be wrong even if it ends up with a true conclusion.

As I mentioned a few lines ago, this is a point that the students often find hard to accept. I have to confess that I do not quite understand why. To me, it is evident that I can say anything I want, no matter how stupid, and then at the end put the words "Therefore the Earth is round". This would give an "argument" whose "conclusion" is perfectly true, while the argument itself is complete nonsense. I have already given you an example of this, in Example 4. And here is another example of a wrong argument with a true conclusion:

Example 12.

1. $2 + 2 = 4$.
2. Therefore the Earth is round.

Silly, isn't it. Of course the Earth is round! But that's not because $2 + 2 = 4$! What does "$2 + 2 = 4$" have to with the Earth being round? *End of Example 12.*

2.1.6 The "Same Kind of Argument" principle.

Example 13. Here is a slightly more subtle example of a wrong argument used to establish a true conclusion.

Step 1. In a U.S. presidential election, the candidate who gets the largest number of votes wins.

Step 2. In the 1976 presidential election, Jimmy Carter got 40,830,763 votes and Gerald R. Ford got 39,147,973 votes.

Step 3. So Jimmy Carter got the largest number of votes.

Step 4. Therefore Jimmy Carter won.

It is indeed true that Jimmy Carter won the 1976 presidential election. But we want to know whether this particular argument is convincing. And the answer is that it is not. Here is a refutation.

Refutation. The sentence of Step 1 is presumably, part of the Stock of Agreed Upon Facts. But this sentence is *not* true. It is not true that "In a U.S. presidential election, the candidate who gets the largest number of votes wins". The true rule determining the winner is as follows: *the candidate who gets a majority of votes in the Electoral College wins.* This candidate need not be the one who got the largest number of votes. It is possible for a candidate to get a majority of the popular vote and yet fail to get a majority of the Electoral College votes and to win the election. For example, in 2000, Al Gore got a majority of the popular vote, but George W. Bush got a majority of the Electoral College votes, so Bush won.

Here is another way to formulate our refutation: if our "1976 election" argument was convincing, then the following argument would also be convincing:

Step 1. In a U.S. presidential election, the candidate who gets the largest number of votes wins.

Step 2. In the 2000 presidential election, Albert A. Gore got 50,999,897 votes and George W. Bush got 50,456,002 votes.

Step 3. So Gore got the largest number of votes.

Step 4. Therefore Gore won.

Yet, Al Gore did *not* win. So the "2000 election" is not convincing, and then the "1976 election" argument is not convincing either, because it is "exactly the same kind of argument". *End of Example 13.*

Here is an important lesson to be learned from Example 13.

THE "SAME KIND OF ARGUMENT" PRINCIPLE

One way to refute an argument is to show that, using "exactly the same kind of argument", one can establish a conclusion that is definitely false.

Remark 3. The question of what is meant, precisely, by "exactly the same kind of argument", is very delicate. But for *mathematical proofs* we will make this perfectly clear later.

Let us illustrate the use of the "same kind of argument" principle with some examples.

Example 14. Here is an argument, purporting to establish the conclusion that “Every year must have a Friday the 13th”.

Step 1. The year has 12 months.

Step 2. So there are 12 days that are Day 13 of a month.

Step 3. There are 7 possible days of the week.

Step 4. Each of the twelve “Day 13” must fall on one of the seven days of the week.

Step 5. Therefore, when we look at the twelve Day 13 and which day of the week they fall on, all seven days of the week must occur, and in particular one of them must be a Friday.

Is this convincing?

In order to answer this question, it might help if we know whether the conclusion is true, but you probably do not know that. If you could show that the conclusion is false, then you would know that *something* must be wrong with this argument, but you still would not know exactly *what* is wrong. If, on the other hand, you could show in some other way that the conclusion is true, then you would know that this particular argument *could* be right, but you would not know that it *is* right.

Let me try to persuade you that this particular argument is wrong. I will do it by using the “same kind of argument” principle: I will show you that with this same kind of argument you could also prove something else which is clearly false.

Imagine another civilization for which the year has 364 days, divided into 13 months, each month having 28 days. (Or, if you prefer, you can imagine that the year has 336 days, and is divided into 12 months, each one having 28 days.) Also, imagine that the 13th of the first month is a Tuesday. Then the 13th of every month will also be a Tuesday, and there will not be a Friday the 13th. On the other hand, the argument used above would equally well apply to this case, so we would be able to prove that every year has a Friday the 13th even in a situation where this is manifestly false.

Now we are still left with the original question: is is true that every year has a Friday the 13th? I am asking you to answer this question in one of the optional homework problems. To do this problem, you must either find a correct argument showing that the conclusion is true, or a correct argument showing that it is false. And, in order to do this, it will be useful to you to learn some lessons from our wrong argument. What, exactly, can we learn? Clearly, in our argument we have not used all the facts about the months that matter. We only used the fact that there are 12 months, but we did not use the fact that the lengths of the months are 30, 31 or 28 days (or 29 days for a leap year), and that this happens in a special way. (For example, there are four 30-day months, seven 31-day months, and one 28- or 29-day month.) Our refutation shows that *these facts should matter*. So our argument should use this extra information about the months. Will this be enough? Does it suffice to know that there are four 30-day months, seven 31-day months, and one 28- or 29-day month? Or maybe the order in which these months occur matters? (That is, maybe it matters that the order is 31-28-31-30-31-30-31-31-30-31-30-31, rather than, say 30-31-30-30-31-31-31-30-31-28-31-31.) This equestion is the subject of another optional homework problem, on the planet of the Klingons. *End of Example 14*

Example 15. Here is an argument, purporting to establish the conclusion that "There do not exist integers m, n such that $30 = m^2 - n^2$".

Step 1. The number 30 is divisible by 6.

Step 2. A number which is divisible by 6 cannot be expressed as the difference of the squares of two integers.

Step 3. Therefore 30 cannot be expressed as the difference of the squares of two integers.

Is this convincing? Obviously not! If this argument was valid, we could also use it to prove that "There do not exist integers m, n such that $60 = m^2 - n^2$", since 60 is also divisible by 6. But

$$60 = 256 - 196 = 16^2 - 14^2 \,,$$

so there *do* exist integers m, n such that $60 = m^2 - n^2$, as we can see by taking $m = 16$ and $n = 14$. *End of Example 15*

2.2 Some examples of mathematical proofs

A **proof** is an argument which is so convincing that every reasonable person has to accept it. I acknowledge that this is not a very easy criterior to apply, but in mathematics I am going to make it perfectly clear what constitutes a proof.

First, let us look at some examples. In all these examples,

- There will be *Steps* consisting of an *assertion* and a *justification*.
- The justification will be written at the end of the step, after the assertion, preceded by the string "[J:" and followed by "]".
- We will be using the following rules as justifications for our steps:

R1 (*The rule for introducing assumptions.*) We are always allowed to "assume" (or "suppose") anything we want, because "assuming" something means "imagining that it is true", and we are always free to imagine anything we want. On the other hand, when we assume something, we are starting a "proof within a proof", in which we are no longer in the real world, but in an imaginary world in which the assumption is true. Then everything we prove from then on will be in this imaginary world, not in the real world. As you will see later, there are rules for going back from an imaginary world to the real world, that is, for ending a "proof within a proof" and returning to the main proof.

R2 (*The rule for introducing existing things.*) We are always allowed to introduce an object, give it a name (usually, but not always, a letter), and stipulate that that object has certain properties, provided that (a) the name is a letter or string that has not been used before, and (b) it has been established in a previous step that an object with those properties exists. For example, if we have proved that "there are cows", that is, that "there exists x such that x is a cow", then we can say "pick a cow and call her Suzy".

R3 (*The rule for bringing in a known fact.*) We are always allowed to have a new step in which a known fact is asserted. Known facts include: (a) the hypotheses, if any, (b) definitions, (c) anything that has been agreed to be allowed as a known fact. (Typically, you will either be allowed to use as a known fact anything that has been proved before, or you will be told that only some facts can be used, in which case it will be made clear to you which known facts are allowed.)

R4 (*The special case rule.*) If you have established in a previous step that some assertion A is true "for arbitrary[17] x of a certain kind", and you have a particular object p of that kind, then you can conclude that A is true of p. For example, if you have established that "every Rutgers professor is a great teacher", then you can conclude that "H.J. Sussmann is a great teacher".

R5 (*The and-get rule.*) If you have established two statements P and Q in previous steps, then you can go to "P and Q."

R5 (*The proof-by-contradiction rule.*) If you have assumed a statement P and proved a contradiction under that assumption (that is, if you have shown that in an imaginary world in which P happens something impossible must also happen) then you can go back to the real world and conclude there that $\sim P$. (NOTE: If S is a statement, then "$\sim S$" stands for the statement "not S", that is, "it is not true that S", or "it is not the case that S". In particular, "$\sim c^2 + 1 = 0$" should be read as "c-squared plus one is not equal to zero".) What, exactly, is a "contradiction"? We will say a lot about contradictions later, but at this point all you need to know is that any statement of the form "P and $\sim P$" is a contradiction. (There are many other statements that are contradictions, but you don't need to know that right now.)

2.2.1 Proof that the equation $x^2 + 1 = 0$ does not have a real solution

THEOREM. *There does not exist a real number x such that $x^2 + 1 = 0$.*

PROOF. We are going to prove our result by contradiction.

Step 1. Suppose a real number x such that $x^2 + 1 = 0$ exists. [J: Introducing assumption]

Step 2. Pick a real number x such that $x^2 + 1 = 0$ and call it c, so c is a real number and $c^2 + 1 = 0$. [J: From S. 1, introducing existing things]

Step 3. On the other hand, we know that if x is an arbitrary real number then $x^2 \geq 0$. [J: Known fact]

Step 4. So $c^2 \geq 0$. [J: From S. 3, by Special Case Rule]

[17] The meaning of the word "arbitrary" will be discussed at great length later. At this point, all you need to know is that "for arbitary x of a certain kind" means "for all possible choices of an object of that kind". For example, "for an arbitrary Rutgers professor x, it is true that x is a great teacher" means exactly the same as "every Rutgers professor is a great teacher".

Step 5. Furthermore, $1 > 0$. [J: Known fact]
Step 6. Also, if x and y are arbitrary real numbers such that $x \geq 0$ and $y > 0$ then $x + y > 0$. [J: Known fact]
Step 7. So $c^2 + 1 > 0$. [J: From S. 4, 5 and 6, by Special Case Rule, plugging in c^2 for x and *1* for y]
Step 8. But, if x is an arbitrary real number such that $x > 0$, then $\sim x = 0$. [J: Known fact]
Step 9. So $\sim c^2 + 1 = 0$. [J: From S. 7 and 8, by Special Case Rule, plugging in c^2+1 for x]
Step 10. Therefore $c^2 + 1 = 0$ and $\sim c^2 + 1 = 0$. [J: From S. 2 and 9, using the "and-get" rule: from P and Q you can go to "P and Q"]
Step 11. But "$c^2 + 1 = 0$ and $\sim c^2 + 1 = 0$" is a contradiction. [J: It's of the form "P and $\sim P$"]
Step 12. So a real number x such that $x^2 = -1$ does not exist. [J: Proof-by-contradiction Rule, from S. 1 and S. 11]

QUESTION. You probably know that there are numbers (called "complex numbers") such that in the universe of those numbers the equation $x^2+1 = 0$ does have a solution. (Actually, it has two solutions, namely, i and $-i$.) It follows that the proof we just gave cannot work in thw world of complex numbers. *Which step or steps of the proof go wrong if we deal with complex numbers rather than real numbers?*

This section is incomplete. I am going to add several more examples. You will get the completed version on Tuesday September 12.

3 A first look at the mathematical zoo: number systems

SUMMARY OF THIS CHAPTER

There are many different types of mathematical objects. Among them, there are **numbers** of various kinds, and also lots of other things that are not numbers.

Numbers belong to **number systems**. The most important number systems are:

- $\mathbb{N}$, the set of *natural numbers*,
- $\mathbb{Z}$, the set of *integers*,
- $\mathbb{Q}$, the set of *rational numbers*,
- $\mathbb{R}$, the set of *real numbers*,
- $\mathbb{C}$, the set of *complex numbers*,
- $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, and, more generally $\mathbb{Z}_n$ (the set of *integers modulo* n) for every natural number n such that $n \geq 2$.

In addition to numbers, there are all kinds of other mathematical objects, such as *sets*, *lists*, *operations*, *functions*, *relations*, *arrays* of various kinds and, in particular, *matrices*, *linear spaces* (such as $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, etc.), *geometrical figures* such as *curves* of different kinds (for example, lines, parabolas, ellipses, hyperbolas, cycloids, exponential curves), *surfaces* (such as planes, spheres, cylinders, cones, ellipsoids, paraboloids, hyperboloids), and other figures (e.g., triangles, squares, rectangles, discs, balls), and many, many more things.

We will talk about those other things later. For the moment, we just concentrate on **number systems**.

Our first task is to take a look at the rich and varied collection of things that populate the mathematical world. Some of them should be familiar to you, others manybe less so or not at all. Do not worry if you find in our list things that you have never heard of before: we will be coming back to the list later, and discussing all the items in much greater detail.

3.1 Numbers: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Q}$ and $\mathbb{R}$.

The best known mathematical objects are **numbers** and **number systems**. There are several different kinds of numbers, and they can be organized into *number systems*. A number can belong to different number systems, in the same way as, say, a person can belong to different associations. (For example, somebody could be a member, say, of the American Association of University Professors, the Rutgers Alumni Association, and the Sierra Club. Similarly, the number 3 belongs to lots of different number systems, such as, for example, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, ..., $\mathbb{Z}_{397}$, ...)

It is convenient to give the number systems **names**, and to introduce mathematical symbols to represent them. Here are some examples:

1. The set $\mathbb{N}$ of **natural numbers**, also known as **positive integers**, or **whole numbers**. The members of this set are the integers $1, 2, 3 \ldots$. So for example 1 is a member of $\mathbb{N}$, 2 is a member of $\mathbb{N}$, 385 is a member of $\mathbb{N}$, $10, 891, 032$ is a member of $\mathbb{N}$, but 0 is not a member of $\mathbb{N}$, -1 is not a member of $\mathbb{N}$, 3.5 is not a member of $\mathbb{N}$, and so on.

2. The set $\mathbb{Z}$ of all **integers**. The members of $\mathbb{Z}$ are the natural numbers as well as 0 and the negatives of natural numbers, i.e., the numbers -1, -2, -3, etc.

3. The set $\mathbb{Z}_+$ of **nonnegative integers** The members of this set are exactly the same as the members of $\mathbb{N}$, except only for the fact that 0 is not a member of $\mathbb{N}$ but is a member of $\mathbb{Z}_+$.

4. The set $\mathbb{R}$ of all **real numbers**. Every natural number is a member of $\mathbb{R}$. Every integer is a member of $\mathbb{R}$. In addition, there are many other real numbers besides the integers.

   For example the following statements are true:

$$\begin{array}{rrrr}
1 \in \mathbb{R}\,, & 2 \in \mathbb{R}\,, & 385 \in \mathbb{R}\,, & 10,891,032 \in \mathbb{R}\,, \\
0 \in \mathbb{R}\,, & -1 \in \mathbb{R}\,, & 3.5 \in \mathbb{R}\,, & \pi \in \mathbb{R}\,, \\
\frac{8}{5} \in \mathbb{R}\,, & \frac{10}{5} \in \mathbb{R}\,, & \frac{382,403,313}{9} \in \mathbb{R}\,, & \pi + 7 \in \mathbb{R}\,, \\
3 + 7 \in \mathbb{R}\,, & 8 - 1 \in \mathbb{R}\,, & 1 - 8 \in \mathbb{R}\,, & \pi - (2 - \pi) \in \mathbb{R}\,.
\end{array}$$

THE SYMBOL "$\in$" ("BELONGS TO")

We use the symbol $\in$ (read "belongs to", or "belonging to") to indicate *membership in a set*, and the symbol $\notin$ (read "does not belong to", or "not belonging to") to indicate non-membership.
For example the following statements are true:

$$
\begin{array}{rrrr}
1 \in \mathbb{N}\,, & 2 \in \mathbb{N}\,, & 385 \in \mathbb{N}\,, & 10,891,032 \in \mathbb{N}\,, \\
0 \notin \mathbb{N}\,, & -1 \notin \mathbb{N}\,, & 3.5 \notin \mathbb{N}\,, & \pi \notin \mathbb{N}\,, \\
\frac{8}{5} \notin \mathbb{N}\,, & \frac{10}{5} \in \mathbb{N}\,, & \frac{382,403,313}{9} \in \mathbb{N}\,, & \pi + 7 \notin \mathbb{N}\,, \\
3+7 \in \mathbb{N}\,, & 8-1 \in \mathbb{N}\,, & 1-8 \notin \mathbb{N}\,, & \pi - (2-\pi) \notin \mathbb{N}\,, \\
1 \in \mathbb{Z}\,, & 2 \in \mathbb{Z}\,, & 385 \in \mathbb{Z}\,, & 10,891,032 \in \mathbb{Z}\,, \\
0 \in \mathbb{Z}\,, & -1 \in \mathbb{Z}\,, & 3.5 \notin \mathbb{Z}\,, & \pi \notin \mathbb{Z}\,, \\
\frac{8}{5} \notin \mathbb{Z}\,, & \frac{10}{5} \in \mathbb{Z}\,, & \frac{382,403,313}{9} \in \mathbb{Z}\,, & \pi + 7 \notin \mathbb{Z}\,, \\
3+7 \in \mathbb{Z}\,, & 8-1 \in \mathbb{Z}\,, & 1-8 \in \mathbb{Z}\,, & \pi - (2-\pi) \in \mathbb{Z}\,, \\
0 \in \mathbb{Z}_+\,, & 1 \in \mathbb{Z}_+\,, & 4 \in \mathbb{Z}_+\,, & -3 \notin \mathbb{Z}_+\,.
\end{array}
$$

5. The set $\mathbb{Q}$ of all **rational numbers** (often called "fractions") By definition, a *rational number* is a real number that can be expressed as the quotient (or "ratio[18]) of two integers. So, for example, $\frac{8}{3} \in \mathbb{Q}$, $6 \in \mathbb{Q}$, $\frac{9}{3} \in \mathbb{Q}$, $-6 \in \mathbb{Q}$, $-\frac{9}{3} \in \mathbb{Q}$.

[18] "Ratio" is an old word for "quotient."

THE SYMBOL "$\subseteq$" ("IS A SUBSET OF")

We use the symbol $\subseteq$ (read "is a subset of," or "is contained in") to indicate *set inclusion*, and the symbol $\not\subseteq$ (read "is not a subset of") to indicate that the inclusion does not hold.
If A and B are sets, the expression "$A \subseteq B$ (read "A is a subset of B") means that *every member of A belongs to B*. For example: if A is the set of all U.S. Senators, and B is the set of all citizens of the U.S., then "$A \subseteq B$" and "$B \not\subseteq A$" are true statements.
The following statements about $\mathbb{Z}_{>0}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, are true:

$\mathbb{Z}_+ \subseteq \mathbb{Z}_+$,	$\mathbb{N} \subseteq \mathbb{Z}_+$,	$\mathbb{Z}_+ \not\subseteq \mathbb{N}$,	$\mathbb{N} \subseteq \mathbb{N}$,
$\mathbb{N} \subseteq \mathbb{Z}$,	$\mathbb{Z} \not\subseteq \mathbb{N}$,	$\mathbb{Z} \subseteq \mathbb{Z}$,	$\mathbb{Z}_+ \subseteq \mathbb{Z}$,
$\mathbb{Z} \subseteq \mathbb{Q}$,	$\mathbb{Q} \not\subseteq \mathbb{Z}$,	$\mathbb{Z} \subseteq \mathbb{R}$,	$\mathbb{N} \subseteq \mathbb{Q}$,
$\mathbb{Q} \not\subseteq \mathbb{N}$,	$\mathbb{Q} \not\subseteq \mathbb{Z}$,	$\mathbb{R} \not\subseteq \mathbb{Q}$,	$\mathbb{Q} \subseteq \mathbb{Q}$.

NOT ALL REAL NUMBERS ARE RATIONAL

In the previous box we asserted that $\mathbb{R} \not\subseteq \mathbb{Q}$. How do we know that?
For "$\mathbb{R} \subseteq \mathbb{Q}$" to be true, it would have to be the case that every real number is rational. Yet, this is *not* the case, because there are real numbers that are not rational. A famous example is $\sqrt{2}$. We will prove later that

$$\boxed{\text{The number } \sqrt{2} \text{ is not rational.}}$$

Actually, many other real numbers are not rational, although in lots of interesting cases this is very hard to prove. For example, the numbers π and e are not rational.

Remark 4. Do not confuse the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ with the letters N, Z, R. They are different! (This is very convenient. If we agreed to use N as the name for the set of natural numbers, then the meaning of the letter N would be fixed for ever, and we would not be able to use N for other things.) $\diamondsuit$

3.2 Reading formulas with $\in$ and $\subseteq$

It is important that you acquire from the very beginning the habit of reading mathematical formulas aloud precisely and correctly. For example, the correct way to read the formula "$3 \in \mathbb{N}$" aloud is

3 belongs to $\mathbb{N}$

or

3 is a member of $\mathbb{N}$

or

3 is in $\mathbb{N}$.

But you **cannot** read "$3 \in \mathbb{N}$" as, for example, "3 is contained in $\mathbb{N}$." "Is contained" means something else!

Here is another example: the formula

$$\mathbb{N} = \{1, 2, 3. \ldots\}\,, \tag{3.2.1}$$

is read as

$\mathbb{N}$ is the set consisting of 1, 2, 3, 4, etc.

It would be **wrong** (**very** wrong!) to read it as "$\mathbb{N}$ is 1, 2, 3, 4, etc." Do you see why this is bad? The best way to understand it is to think of more familiar examples.

For example, you cannot say that "the U.S. Supreme Court is Chief Justice Roberts and Justice Scalia and Justice Thomas and Justice Alito and Justice Kennedy and Justice Souter and Justice Ginsburg and Justice Breyer and Justice Stevens." Why not? Because the U.S. Supreme Court is **one single thing, not nine things!** What you *can* say is:

> The U.S. Supreme Court is a tribunal whose members are Chief Justice Roberts and Justices Scalia, Thomas, Alito, Kennedy, Souter, Ginsburg, Breyer, and Stevens.

Similarly, the set $\mathbb{N}$ of natural numbers is **one single object, not many objects**, so you can say "$\mathbb{N}$ is the set consisting of 1, 2, 3, 4, etc." but you cannot say "$\mathbb{N}$ is 1, 2, 3, 4, etc."

Also, you cannot say "$\mathbb{N}$ is the natural numbers," or "$\mathbb{N}$ is natural numbers." (Actually, "$\mathbb{N}$ is the natural numbers" is bad, but "$\mathbb{N}$ is natural numbers" is even worse. Why?) What you should say is "$\mathbb{N}$ is the set of natural numbers," or "$\mathbb{N}$ is the set of all natural numbers."[19]

3.3 The complex numbers

For a long time, people used the word "number" to mean "real number." But then the "complex numbers" were invented, and mathematicians decided, after long discussions on whether this was a good thing to do, that these "complex numbers" were objects worth admitting as *bona fide* numbers as well. So we can add a sixth example to our list of number systems:

6. The set $\mathbb{C}$ of all *complex numbers*. Here are some examples of these "numbers":

$$3\,, \qquad i\,, \qquad 3+i\,, \qquad 4-i\,, \qquad 35-67i\,, \qquad \frac{8}{5}\,, \qquad \frac{8}{5}-\pi i\,.$$

Complex numbers can be **added** and **multiplied**. To add two complex numbers, you just add separately the "i-parts" and the "non-i-parts." So, for example,

$$3+i+(4-2i)=7-i\,,\ \ 3+(8+5i)=11+5i\,,\ \ \frac{8}{5}+i+\left(\frac{12}{5}+83i\right)=4+84i\,.$$

To multiply two complex numbers, you just multiply as you would real numbers, except that every time you run into $i.i$ you put -1 instead. For example:

$$\begin{aligned}(23+7i).(4+2i) &= 23.4+7.4.i+23.2.i+7.2.i.i\\ &= 92+28i+46i+14.i.i\\ &= 92-14+28i+46i\\ &= 78+72i\,.\end{aligned}$$

[19]Recommended reading: the short story "Pigs is pigs," by Ellis Parker Butler. You can find it in the Web site *www.bookvalley.com/cgibin/bv?b=162*. It's funny, and one of the things that make it funny is precisely the misuse of the verb "is" in "Pigs is pigs."

Remark 5. The invention, or discovery, of the complex numbers was a long process, driven by the effort to prove the "fundamental theorem of algebra." This theorem ought to say that an n-th order equation has n solutions, so for example the equation $x^2 + 1 = 0$ should have two solutions, the equation $x^7 + 3x^5 + 4x^2 + 5 = 0$ should have seven solutions, and so on. But the only way for $x^2 + 1 = 0$ to have two solutions is to stipulate that there is a number x such that $x^2 = -1$, that is, that a "square root of -1" exists. This, however, leads to all kinds of difficulties, if you think that this square root—let us call it i—is an ordinary real number. For example, every real number is either positive or negative or zero. Now, i of course cannot be zero, so i is either positive or negative. Since the product of two positive numbers is positive, if i was positive then i^2 would have to be positive. Since the product of two negative numbers is positive, if i was negative then i^2 would have to be positive. So in either case, whether i is positive or negative, the number i^2 would have to be positive. But $i^2 = -1$, so -1 would have to be positive, and this is not possible. The way out of this was to think of i as not being an ordinary number, really, but only a symbol to be manipulated according to certain rules. So, for example, following what **Descartes** wrote in 1637, one can "imagine" that the equation $x^2 + 1 = 0$ has two roots—i and $-i$—but these imagined roots are not "real" numbers. Earlier, in 1572, **Bombelli** had given precise rules for the manipulation of these imaginary numbers. **Argand** in 1814 had the idea of representing complex numbers as points in a two-dimensional plane. The term "complex number" was actually introduced by **Gauss** in 1831. The term "conjugate" had been introduced by **Cauchy** in 1821. ♢

3.4 The integers modulo n

Here are some more examples of important "number systems" of a rather different kind. These systems will be very important to us later.

7.a. The set $\mathbb{Z}_{12}$ of all *integers modulo* 12. The members of $\mathbb{Z}_{12}$ are the following 12 integers:

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11\,.$$

Addition and multiplication are carried out in the usual way, except that any time you get 12 or something larger, you must "reduce modulo 12," that is, subtract 12 and keep subtracting 12 until you get a member of $\mathbb{Z}_{12}$. For example, the following equalities are true in $\mathbb{Z}_{12}$:

$$\begin{array}{ccccc} 3+5=8\,, & 6+9=3\,, & 8+9=5\,, & 7+1=8\,, & 11+11=10\,, \\ 4.2=8\,, & 6.9=6\,, & 8.7=8\,, & 7.11=5\,, & 7.7=1 \quad . \end{array}$$

Remark 6. The numbers modulo 12 are very natural objects, and you are very much used to working with them. For example, suppose you start at 10 o'clock and work 8 hours. What time is it when you finish? The answer is "$10+8$ modulo 12," that is, 6 o'clock. If a movie theatre starts its show at 8 o'clock and shows a 2-hour movie 8 times, without intermission, what time is it at the end of the show? The answer is "$10+2.8$ modulo 12," that is, 2 o'clock.

Naturally, if you wanted to keep track of "am" vs. "pm" then you would want to use so-called "military time," that is, work in $\mathbb{Z}_{24}$, the numbers modulo 24. (In $\mathbb{Z}_{24}$, $16+13=5$, and $8.5=16$, and so on.)$\diamondsuit$

8.b. The set $\mathbb{Z}_7$ of all *integers modulo* 7. The members of $\mathbb{Z}_7$ are the following 7 numbers:

$$0\,,\ 1\,,\ 2\,,\ 3\,,\ 4\,,\ 5\,,\ 6\,.$$

Addition and multiplication are carried out in the usual way, except that any time you get 7 or something larger, you must "reduce modulo 7," that is, subtract 7 and keep subtracting 7 until you get a member of $\mathbb{Z}_7$. For example, the following equalities are true in $\mathbb{Z}_7$:

$$\begin{array}{ccccc} 3+2=5\,, & 6+5=11\,, & 4+3=0\,, & 2+6=1\,, & 4+2=6\,, \\ 4.2=1\,, & 6.5=2\,, & 3.4=5\,, & 3.3=2\,, & 3.5=1 \quad . \end{array}$$

Remark 7. Like the members of $\mathbb{Z}_{12}$, the numbers modulo 7 are also very natural objects that you are very much used to working with. For example, let us think of the days of the week as numbered, starting with Sunday as Day No. 0. In other words, think of

Sunday	as day No.	0
Monday	as day No.	1
Tuesday	as day No.	2
Wednesday	as day No.	3
Thursday	as day No.	4
Friday	as day No.	5
Saturday	as day No.	6

Suppose today is Thursday. What day of the week is it going to be 5 days from today? The answer is ***Tuesday***, of course. Why? Because Thursday is Day 4, so 5 days from today is Day $4 + 5$, that is, Day 2, modulo 7.

Suppose today, January 5, is a Wednesday. Whay day of the week is February 11 going to be? ANSWER: today is Day 3; to go from January 5 to February 11 you must add 36 days (that is, 31 days to go to February 5, and 6 more to go to February 11), so you'll end up in Day 3+36=39, which is equal to 4 modulo 7. So February 11 is going to be a Day 4, that is, ***a Thursday***.

Suppose today, January 5, is a Wednesday. Whay day of the week is July 5 going to be, assuming that we are NOT in a leap year? ANSWER: today is Day 3; to go from January 5 to July 5 we must add 31 days (to get to February 5), then 28 days (to get to March 5, using the fact that we are not in a leap year), then 31 days (to get to April 5), then 30 days (to get to May 5), then 31 days (to get to June 5), and, finally, 30 days (to get to July 5). So we end up in Day 3+31+28+30+31+30 which, modulo 7, is the same as $3 + 3 + 0 + 2 + 3 + 2$, that is, 13, that is, 6 (modulo 7). So ***July 5 is going to be a Saturday***. ♢

8.c. Naturally, one could consider integers modulo any other number, not just 12 or 24. For example, $\mathbb{Z}_3$ is the set of **integers modulo** 3. So $\mathbb{Z}_3$ has exactly three members, namely,

$$0\,, \qquad 1\,, \qquad \text{and} \qquad 2\,.$$

Arithmetic modulo 3 is very simple. The addition table is as follows:

$$0 + 0 \;=\; 0\,,$$

$$\begin{aligned} 0+1 &= 1+0=1\,, \\ 0+2 &= 2+0=2\,, \\ 1+1 &= 2\,, \\ 1+2 &= 2+1=0\,, \\ 2+2 &= 1\,, \end{aligned}$$

and the multiplication table is

$$\begin{aligned} 0.0 &= 0.1=1.0=0.2=2.0=0\,, \\ 1.1 &= 2.2=1\,, \\ 1.2 &= 2.1=2\,. \end{aligned}$$

Remark 8. To say that an equality holds in $\mathbb{Z}_n$, we can say just that, or we can say that the equality holds (or is true) "modulo n."

For example, we can say that

$$5+8=2 \qquad \text{in} \quad \mathbb{Z}_{11}\,,$$

or that

$$5+8=2 \qquad \text{modulo} \quad 11\,.$$

Also, it is clear from the context that we are working in $\mathbb{Z}_n$, then there is no need to say "in $\mathbb{Z}_n$" or "modulo n." (Example: after we said in Page 45 that "the following equalities are true in $\mathbb{Z}_{12}$," we then just went ahead and wrote things like $8.7 = 8$, without explaining again that this was meant to be "in $\mathbb{Z}_{12}$," or "modulo 12." There was no need, because we had already stated clearly that we were working in $\mathbb{Z}_{12}$. ♢

Remark 9. If you look at the way we defined the integers modulo n, you may think that the integers that are larger than n do not belong to $\mathbb{Z}_n$. (For example, the integer 8 does not belong to $\mathbb{Z}_7$, because $\mathbb{Z}_7$ consists of the integers 0, 1, 2, 3, 4, 5, and 6, and nothing else.) We can, however, assign a value to a symbolic expression such as 8 in $\mathbb{Z}_7$, by letting

$$8=1+1+1+1+1+1+1+1\,,$$

where the sum is understood to be in $\mathbb{Z}_7$. Then, in $\mathbb{Z}_7$, the expression "8" denotes the number 1, and we can write

$$8 = 1 \qquad \text{in} \qquad \mathbb{Z}_7 \,.$$

Notice that

$$8 \neq 1 \qquad \text{in} \qquad \mathbb{Z} \,.$$

So the symbol 8, when we are working in $\mathbb{Z}_7$, is the name of a thing which is not the integer 8. (If it was the same, then "$8 \neq 1$" would also be true in $\mathbb{Z}_7$.)

One can also talk about *negatives* of integers modulo n. What do we mean by "the negative" of an integer modulo n? We just mean "the integer modulo n that added in $\mathbb{Z}_n$ to our given integer modulo n yields 0."

For example, what is the negative of 3 in $\mathbb{Z}_7$? Well, in $\mathbb{Z}_7$ the sum of 3 and 4 is 0. Therefore **the negative of** 3 **in** $\mathbb{Z}_7$ **is** 4. We can also write this using the symbol "$-$":

$$-3 = 4 \qquad \text{in} \qquad \mathbb{Z}_7 \,,$$

or

$$-3 = 4 \qquad \text{modulo} \quad 7 \,.$$

$\diamondsuit$

An important application of arithmetic modulo 3 is that it enables us to find a simple criterion to check if a given integer is divisible by 3. The key fact here is that $10 = 1$ modulo 3. This means that if we working modulo 3 then a big number such as $4,687$ is just the same (in $\mathbb{Z}_3$) as $4 + 6 + 8 + 7$, i.e. as 25, which is equal to 1 in zz_3. So $4,687$ is not divisible by 3.

BOX NO. 1: When is a natural number divisible by 3?

Take some natural number like $16,547$. Is it divisible by 3? Naturally, you can do the division and see what happens. But can you tell quickly and directly?
When we say that a number is "divisible by 3" what we mean is that if you start subtracting 3s from it until you cannot do that any more (that is, until you get to 0, 1 or 2), then you end up with 0. In other words, ***our number is equal to* 0 *modulo* 3.**
So in our example we need to find out whether $16,547$ is equal to 0, 1 or 2 modulo 3. Now:

$$16,547 = 1 \times 10,000 + 6 \times 1,000 + 5 \times 100 + 4 \times 10 + 7 \times 1\,.$$

Also, 10 is just equal to 1 modulo 3. So $100 = 10 \times 10 = 1 \times 1 = 1$ modulo 3. And $1,000 = 10 \times 10 \times 10 = 1 \times 1 \times 1 = 1$ modulo 3. And so on. Therefore, the following is true modulo 3:

$$16,547 = 1 + 6 + 5 + 4 + 7 = 23 = 2\,.$$

So $16,547$ ***is not divisible by* 3.**
The point of the above argument is that

$$\begin{aligned} 10,000 &= 1 \quad \text{modulo } 3\,, \\ 6,000 &= 6 \quad \text{modulo } 3\,, \\ 500 &= 5 \quad \text{modulo } 3\,, \\ 40 &= 4 \quad \text{modulo } 3\,, \\ 7 &= 7 \quad \text{modulo } 3\,, \end{aligned}$$

so

$$16,547 = 1 + 6 + 5 + 4 + 7 = 23 = 2 \qquad \text{modulo } 3.$$

The general criterion is this: **to find out if an integer is divisble by 3, just add the digits in its decimal expression. If the sum of the digits is divisible by 3, then the given number is divisible by 3. Otherwise it is not.**

BOX NO. 2: When is a natural number divisible by 11?

The situation here is very similar to what we found when we looked at divisibility by 3, but with an important twist. Now 10 is not equal to 1 modulo 11, but to -1. So if we look at a natural number like $16,547$, the 7 should count as a 7 modulo 11, the 4 as a -4, the 5 as 5, the 6 as a -6, and the 1 as a 1.
(Reason: $16,547 = 1\times 10,000 - 6\times 1,000 + 5\times 100 - 4\times 10 + 7$. Modulo 11, we have $10 = -1$, $100 = 10^2 = 1$, $1,000 = -1$, $10,000 = 1$. So, $16,547 = 1 - 6 + 5 - 4 + 7$ modulo 11.) Therefore $16,547 = 3$ modulo 11, and then $16,547$ ***is not divisible by*** 3**.**

The general criterion is this: ***to find out if an integer is divisble by*** 11***, just add the digits in its decimal expression with alternating signs (plus, minus, plus, minus, and so on). If the alternating sum of the digits is divisible by*** 11***, then the given number is divisible by*** 11***. Otherwise it is not.***

QUESTION 1. Which of the following three natural numbers are divisible by 3?

$$\begin{array}{r} 999,111,999,111 \\ 543,043,664,987 \\ 6,915,571,375,016,461,671,444,102,534 \end{array}$$

QUESTION 2. Which of the three natural numbers of the previous question are divisible by 11?

QUESTION 3. Formulate and justify a criterion for testing if a number is divisible by 9.

QUESTION 4. Is the following true or false? If you take a natural number, reverse the order of its decimal figures, and subtract the smaller of the numbers you got from the larger, then what you get is divisible by 3. (Here are two examples: (1) start with 925, then get

529 by reversing the order of the figures, then subtract, and you get $925 - 529$, which is equal to $\boxed{396}$; (2) start with 386, then get 683 by reversing the order of the figures, then subtract, and you get $683 - 386$, which is equal to $\boxed{297}$. In both cases, we got numbers that are divisble by 3. The question for you is: is this going to to happen no matter which natural number we take?)

QUESTION 5. Is the following true or false? If you take a natural number, reverse the order of its decimal figures, and subtract the smaller of the numbers you got from the larger, then what you get is divisible by 11. (Here are two examples: (1) start with 925, then get 529 by reversing the order of the figures, then subtract, and you get $925 - 529$, which is equal to $\boxed{396}$; (2) start with 386, then get 683 by reversing the order of the figures, then subtract, and you get $683 - 386$, which is equal to $\boxed{297}$. In both cases, we got numbers that are divisble by 11. The question for you is: is this going to to happen no matter which natural number we take?)

QUESTION 6. Is the following true or false? If you take a natural number, reverse the order of its decimal figures, and add the two numbers, then what you get is divisible by 3. (Here are two examples: (1) start with 321, then get 123 by reversing the order of the figures, then add, and you get $321 + 123$, which is equal to $\boxed{444}$; (2) start with 801, then get 108 by reversing the order of the figures, then add, and you get $801 + 108$, which is equal to $\boxed{909}$. In both cases, we got numbers that are divisble by 3. The question for you is: is this going to to happen no matter which natural number we take?)

8.d. Even simpler is $\mathbb{Z}_2$, the set of **integers modulo** 2. Clearly, $\mathbb{Z}_2$ has exactly two members, namely,

$$0 \qquad\qquad \text{and} \qquad\qquad 1\,.$$

Arithmetic modulo 2 is so ridiculously simple that we can fit the complete addition and multiplication tables together in one line:

$$0+0=0\,,\ 0+1=1+0=1\,,\ 1+1=0\,,\ 0.0=0.1=1.0=0\,,\ 1.1=1\,.$$

QUESTION 7. Does the equation $x^2 + 1 = 0$ have a solution in $\mathbb{Z}_7$?

ANSWER. Let us write down explicitly the squares of all members of $\mathbb{Z}_7$:

$$0^2 = 0\,,\ \ 1^1 = 1\,,\ \ 2^2 = 4\,,\ \ 3^2 = 2\,,\ \ 4^2 = 2\,,\ \ 5^2 = 4\,,\ \ 6^2 = 1\,.$$

The only way for $x^2 + 1$ to be equal to zero would be for x^2 to be equal to 6. But in the above list we see clearly that there is no member of $\mathbb{Z}_7$ whose square is 6. So the answer is **NO**.

QUESTION 8. Does the equation $x^2 + 1 = 0$ have a solution in $\mathbb{Z}_{17}$?

ANSWER. In $\mathbb{Z}_{17}$, $4.4 = 16$, so $4.4 + 1 = 16 + 1 = 0$. So the answer is **YES**.

QUESTION 9. In $\mathbb{Z}_{24}$, are there two nonzero numbers whose product is equal to zero?

ANSWER. In $\mathbb{Z}_{24}$, $8.3 = 0$. So the answer is **YES**.

QUESTION 10. In $\mathbb{Z}_4$, does the equation $a^2 + b^2 = 3$ have a solution?

ANSWER. Here are the values of a^2+b^2 for all possible choices of $a, b \in \mathbb{Z}_4$,

$$\begin{aligned} 0^2 + 0^2 &= 0\,,\\ 0^2 + 1^2 = 1^2 + 0^2 &= 1\,,\\ 0^2 + 2^2 = 2^2 + 0^2 &= 1\,,\\ 1^2 + 1^2 &= 2\,,\\ 1^2 + 2^2 = 2^2 + 1^2 &= 2\,,\\ 2^2 + 2^2 = &= 0\,. \end{aligned}$$

We see that the right-hand side is never equal to 3. So the answer is **NO**.

QUESTION 11. In $\mathbb{Z}_{17}$, are there two nonzero numbers whose product is equal to zero?

ANSWER. If you multiply two numbers in $\mathbb{Z}_{17}$, the only way the product can be equal to zero in $\mathbb{Z}_{17}$ is if, when you reduce modulo 17, you get zero. That is, the product of these two numbers must be divisible by 17. But 17 is a prime number, so there is no way to multiply two nonzero numbers smaller than 17 and get a product which is divisible by 17. So the answer is **NO**.

Remark 10. Clearly, the class of number systems presented in this subsection consists of *infinitely many* "number systems," namely, the systems

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_5, \ldots, \quad \mathbb{Z}_{397}\,, \ldots\,, \quad \text{and so on}\,.$$

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