MATHEMATICS 300 — FALL 2001

Introduction to Mathematical Reasoning
(640 - 300, Section 04)

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INSTRUCTOR'S NOTES, PART 2
(October 8, 2001)

This is the final version (except for typos, of which there probably are quite a few still left). **This supersedes the previous preliminary version that you got on October 3.**

Contents

10 Additional information on the course

10.1 The midterms (a reminder)

Midterm dates: Monday October 15 and **Wednesday November 28**.

10.2 The final exam

The **final exam** will be on **Monday December 17, 8 to 11 pm**.

10.3 Readings for the period from Sept. 5 to Oct. 10

- the book's note "To the Student,"
- the book's Chapter 1 (**all of it!**),
- the instructor's notes, Part 1,
- the book's Chapter 2 (**all of it!**),
- the instructor's notes, Part 2.

10.4 Homework assignments for the first half of October

10.4.1 Assignment no. 4, due on Wednesday, October 3.

A. Redo Problem E of Assignment No. 1. Make sure that you use the precise definition of "sum of a series." (You can find this definition in any Calculus textbook.)

B. Problems 10 and 11 in page 41 of the book.

10.4.2 Assignment no. 5, due on Wednesday, October 10

A. Section 7 of the book, Problems 1, 4, 7(c)(d), 8, 9.

B. Section 8 of the book, Problems 1(b)(c)(d)(e)(g), 2(b)(c)(d)(e)(f)(g), 3(d)(e)(f)(g), 4, 6, 8, 9.

C. Here are five sentences about rabbits, unicorns, elephants, Peter, Dumbo, and the binary relation "taller than."

(1) Unicorns are taller than elephants. (That is, "every unicorn is taller than every elephant," or, if you prefer, "for every x and every y, if x is a unicorn and y is an elephant, then x is taller than y.")

(2) Rabbits are taller than unicorns.

(3) If something is taller than something else which is in turn taller than a third thing, then the first one is taller than the third one. (That is "for every x, every y and every z, if x is taller than y and y is taller than z, then x is taller than z.")

(4) Peter is a rabbit.

(5) Dumbo is an elephant.

Using sentences (1), (2), (3), (4), (5), *and* **no other fact** *about rabbits, unicorns, elephants, Peter, Dumbo, and the binary relation "taller than,"* prove that Peter is taller than Dumbo.

11 Proofs: an introduction

A **proof** of a statement S is a *convincing argument* that establishes **beyond the shadow of a doubt** that S is true.

You should think of proving statements as a **game** that you are playing against an opponent who does not believe that the statement you are proving is true. You make statements, your opponent may find them convincing or not, and it is up to you to clear up all your opponent's doubts by arguing your case persuasively every time your opponent isn't convinced. (Maybe you should think that I, the instructor, am the "opponent." **You** have to convince **me**. If you convince me, then your proof has worked. Otherwise it has not.)

11.1 What is a convincing argument?

The trouble with the above remarks is that it is not very clear what will count as a "convincing argument." Suppose your opponent says, after you have given what looks to you like a very convincing proof: "I am not convinced; I don't see why what you just said proves anything." How would you deal with something like that?

Moreover, any argument you give will have to *start* somewhere. (You cannot prove anything starting from nothing.) What can you do if your opponent refuses to accept your starting point?

EXAMPLE: suppose you want to prove that,

- *The product of two odd integers is odd.*

  Equivalently—and more concisely—you want to show that

- *If x and y are odd integers then xy is odd.*

Or—even more concisely—you want to prove that

$$\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \left(((x \text{ is odd}) \wedge (y \text{ is odd})) \longrightarrow (xy \text{ is odd})\right). \tag{11.1.1}$$

To prove this, you can proceed as follows:
To prove (11.1.1), we pick arbitrary $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$, and try to prove that

$$((x \text{ is odd}) \wedge (y \text{ is odd})) \longrightarrow (xy \text{ is odd}). \tag{11.1.2}$$

To prove (11.1.2), we assume that

$$(x \text{ is odd}) \wedge (y \text{ is odd}) \tag{11.1.3}$$

and try to prove that

$$xy \text{ is odd}\,. \tag{11.1.4}$$

So let us assume (11.1.3). Then

$$x \text{ is odd} \tag{11.1.5}$$

and

$$y \text{ is odd}\,. \tag{11.1.6}$$

Since x is odd, the definition of "odd" tells us that

$$\exists z \in \mathbb{Z}, (x = 2z + 1)\,. \tag{11.1.7}$$

Since y is odd, the definition of "odd" tells us that

$$\exists w \in \mathbb{Z}, (y = 2w + 1)\,. \tag{11.1.8}$$

In view of (11.1.7), we can pick an integer—call it a—such that

$$x = 2a + 1\,. \tag{11.1.9}$$

In view of (11.1.8), we can pick an integer—call it b—such that

$$y = 2b + 1\,. \tag{11.1.10}$$

Then

$$xy = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1\,.$$

Therefore, if we let

$$c = 2ab + a + b\,,$$

we have

$$xy = 2c + 1\,. \tag{11.1.11}$$

Moreover,

$$c \in \mathbb{Z}\,, \tag{11.1.12}$$

because $c = 2ab + a + b$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}$.

In view of (11.1.11)and (11.1.12),

$$\exists u \in \mathbb{Z}, (xy = 2u + 1)\,. \tag{11.1.13}$$

It then follows from the definition of "odd" that

$$xy \text{ is odd}\,.$$

END OF PROOF

Is this a truly convincing argument?

The answer is "yes" if all the steps are convincing. But **if one step is not convincing, then the whole argument fails.** (This is why you should **not** expect to get partial credit for a wrong proof if most steps are O.K. but just a few of them are wrong. A proof is like a chain. And **a chain is no stronger than its weakest link.** If a 16-Ton weight is hanging right above your head, held by a chain, and the chain breaks because of one weak link, and the weight falls on your head and kills you, do you think the chain manufacturer should be entitled to "partial credit" because "most links were O.K.; only one broke"? If the driver of Lady Diana's car didn't drink every night, but only drank on the particular night of the accident, should he be entitled to partial credit because "after all, most of the time he didn't drink"?)

So, **are ALL the steps in the above proof truly convincing?**

In the proof, we have given justifications for each step. For example, at one point we went from "x is odd" to "$\exists z \in \mathbb{Z}, (x = 2z + 1)$," and we said that this was done because of the definition of "odd." Now, why is this step permissible?

Suppose I tell you that

> I don't see how the definition of "odd" allows you to do this.

How would you persuade me? Well, let us give it a try. You might say:

> because the definition of "odd" says the following: an integer x is *odd* provided that $\exists z \in \mathbb{Z}, (x = 2z + 1)$.

But I could then say

> Fine. We know that "x is odd," and also that "x is odd provided that $\exists z \in \mathbb{Z}, (x = 2z + 1)$." How does it follow from these two statements that "$\exists z \in \mathbb{Z}, (x = 2z + 1)$"?

You would then have to answer something like

> because there is a rule that says that if you have a statement A, and you also have the statement "A provided that B," then you can conclude that B.

(REMARK: The words "provided that," when used in a definition, mean "if and only if." The "rule" you are invoking really says that "if you have a statement A, and you also have the statement 'A if and only if B,' then you can conclude that B.")

Now I could say

> How do I know that this new rule of yours is valid?

What would you answer?

If you are very impatient, you could say

> BECAUSE I SAY SO, AND STOP ASKING STUPID QUESTIONS!!!!!!

But this would be sort of rude, so you might try to reason with me, and say something like:

> if you have a statement A, and you also have the statement "A if and only if B," then you can conclude that B, for the following reason: "A if and only if B" really means "if A then B and if B then A." So if you have "A if and only if B" then you have "if A then B," and them, since you have A, you can go to B. SEE?

Let us suppose I say

> No, I don't see!!! You are telling me that "If you have 'if A then B,' and you have A, then you can go to B." WHY?

You might try to answer

> Becase there is a rule called "Modus Ponens" that says precisely that.

Can you guess what my answer would be? Yes, you guessed right!!! My answer is

> Oh, another rule!!! You seem to make up a new rule every time I ask you a question!!! Why is this "Modus Ponens" rule of yours valid?

A posible answer of yours could be

> BECAUSE ANYBODY WHO IS NOT A COMPLETE IDIOT MUST REALIZE THAT IT IS TOTALLY OBVIOUS THAT IF $\mathbf{A}$ IS TRUE AND $\mathbf{A} \longrightarrow \mathbf{B}$ IS TRUE THEN B MUST BE TRUE.!!!!

Naturally, I would answer

> WHY?

and then you would give up in despair.

The point I am trying to make should be clear by now: no matter how you justify a step in a proof, your justification will be based on some rule or rules. And then I can always refuse to accept these rules and ask you to justify them. But your justification of the rules would have to be based on other rules. And I could claim, once again, that I am not convinced of the validity of these rules.

> IF I WANTED TO, I COULD GO ON LIKE THIS FOREVER. NO MATTER WHAT JUSTIFICATION YOU GIVE FOR A STEP OF YOUR PROOF, I COULD ALWAYS ASK: "WHY IS THAT VALID?" THEN YOU WOULD GIVE ME A REASON, BUT I COULD ASK "WHY?" AGAIN. AND SO ON AND ON AND ON.

In other words, if your opponent is truly stubborn and uncooperative, then there is *nothing* you can do. Maybe you will come up with some reason, but this reason will be based on some other facts that look obviously true to you. But your opponent can still refuse to accept these other facts, and this process can go on for ever.

11.2 Why do we need formal proofs?

To take care of the "problem of the opponent that doesn't play fair and refuses to be convinced," mathematicians have come up with the concept of a **formal proof**. We give very precise rules defining what a "formal proof" is, and if you prove something correctly, and strictly following the rules, then you are *guaranteed* that your proof will be accepted as valid.

If you do *not* obey the rules, then you are on uncertain ground: you might be able to convince an instructor or grader, but you cannot be sure, because any time you take a step one of us might ask "how do you justify that step?"

You should think of giving formal proofs as similar to playing a game such as chess: you are given very strict, precise rules, and you are only allowed to make moves following the rules. As long as you obey the rules, your opponent does not have the right to reject your moves. (If your opponent unilaterally decides that a move of yours is not acceptable, because "you are not allowed to move a bishop if in the previous move you moved a knight," then you would say that your opponent is not playing fair, because this "rule" was not in the list of rules that had been agreed upon before the game started.)

In Mathematics we play fair. We tell you what the rules are, and if you prove something correctly following the rules, then you can be sure that it will be accepted. You may think that having to follow the rules is a restriction to your freedom. But this is not so. You should think of it as **your own protection against unfair play.** If you correctly justify every step according to the rules, then we promise you that we will not change the rules in the middle of the game, kick the board, or come up with other surprises.

Remark 1. I said that you should think of giving formal proofs of theorems as a game similar to chess, in which you are given a small number of very strict rules telling you which moves are permitted, and each move has to be justifiable by explaining exactly which rule you are using and how you are using it. However, exactly as in the case of chess, if all you know is the rules, and you always apply the rules carefully and scrupulously, you may still lose all the time. If you want to **win** (that is, in the case of a proof, if you want to get to the desired conclusion) then you have to **think**. You have to plan a strategy, make a move now because it will be useful five or six steps later, etc.

The one big difference between chess and theorem-proving is that **chess is a zero-sum game**, but **theorem-proving is a cooperative game.** What this means is that in chess your opponent wants you to lose, whereas in theorem-proving you and your opponent are working together, trying to find out whether something is true, and to understand why.

That's why in chess you don't explain openly why you make a move. You do *not* say to your opponent things like: "I am making this move now so that then in the next move I can do this other thing, so that three moves later I can do that, and then I will checkmate you." *You don't want your opponent to be able to figure out your strategy.*

In theorem-proving you definitely **do** want your opponent to understand your strategy. Your opponent is going to be ruthless in criticizing you if you don't obey the rules, but as long as you obey them you will get full coopertaion. In fact, usually you don't have to go through the whole business of explaining every detail of every application of every rule. *As long as you persuade your opponent that something can be done using the rules, there is no need to go through all the steps and actually do it.* And, since your opponent wants to be persuaded, the more you do to explain why you are doing something, the better. That's why *in theorem-proving it's always nice to explain your strategy, so that your opponent can understand where you are headed.*

Because theorem-proving is a cooperative game, when writing proofs it is always nice to use the first person plural —the so-called "editorial we"— as we did in our first example of a proof ("we pick," "we are assuming," "we have shown"). This is a good way to convey the idea that the writer and the reader are *working together*, that **the writer is leading the reader gently along a path of discovery.** ♢

11.3 Proofs keep us honest

One of the main purposes of giving proofs is **to keep ourselves honest**. Let me explain. Suppose you think a certain statement "feels true" to you. You are just "convinced" is is true. You think it must be true. That is very nice, except that people who are very convinced of something, people who have a strong feeling that something must be true, are likely to make mistakes. We have seen at least two examples of that in this course:

1. Almost all the students "proved" in their homework that *a positive integer which is not prime is composite.* Yet, this statement is **false**, and the reason is very simple: the number 1 is a positive integer which is not prime, and yet 1 is not composite either. **How come nobody noticed (with one exception)? And how come almost everybody wrote a "proof" and was perfectly happy**

with it? How come nobody had doubts?

2. Almost all the students "proved" in their homework that *even if one were to change the order of the months of the year, it would still be necessarily true that every year must have a Friday the 13th.* Yet, this statement is **false**. I showed you in class how one can reorder the months so as to produce years that do not have a Friday the 13th. Once again, you should ask yourself **how come nobody noticed (with a few exceptions)? And how come almost everybody wrote a "proof" and was perfectly happy with it? How come nobody had doubts?**

These are just two examples. And, believe me, there are going to be quite a few more. In fact, there is a at least one which is already there in homework you have handed in but has not been returned to you yet.

Why does this happen? I do not know for sure, but here is my guess: this is happening to a very large extent because students are not careful and write just lots of words without making sure that they do indeed prove anything. For example, in the problem of the Friday the 13th, students wrote things like: *The order of the months does not matter because when you add numbers the order in which you add them does not matter.*

Now, to me these words are unitelligible. It is indeed true that "when you add numbers the order in which you add them does not matter." But, what does this have to do with there being a Friday the 13th? The student did not tell, and I do not see it.

What can you do about this problem? **Think carefully!!!**

12 On giraffes, cows, sheep, rabbits, unicorns, elephants, and honesty

Here is an example of **how carelessness when you write proofs can lead you to make big big mistakes, and how paying attention to the rules can keep you honest and prevent you from fooling yourself.**

Let us look at the problem on giraffes, cows, sheep, Olivia and Dolly. (Cf. Part 1 of the notes.) You were asked to prove that "Olivia is taller than Dolly." The correct answer is *it cannot be done; the conclusion that Olivia is taller than Dolly does* **not follow** *from the five hypotheses listed in the problem. If you really want to deduce from these hypotheses that Olivia is taller than Dolly you need one extra fact, which is true but is not one of the statements (1), (2), (3), (4), (5). Without this fact, it cannot be proved that Olivia is taller than Dolly.*

I will tell you very soon why this is true, and what the "extra fact" is. But before we get there, let us try to prove that Olivia is taller than Dolly. I will show you that, if you are really careful, then you will discover that your "proof" does not work, and you really cannot prove that Olivia is taller than Dolly. And, by working backwards, you will discover why this is so.

It turns out that *nobody* realized that the answer is that "it cannot be proved that Olivia is taller than Dolly." Why did this happen? Again, I do not know, but my guess is that it happen because people are just trying to write something, without being careful about whether what they write is properly justifiable.

Let us set out to prove that Olivia is taller than Dolly, and see what happens.

For example, one student wrote: let x bea cow. Since giraffes are taller than cows, Olivia is a giraffe, and x is a cow, it follows that Olivia is taller than x. Since cows are taller than sheep, x is a cow, and Dolly is a sheep, it follows that x is taller than Dolly. So by (3) Olivia is taller than Dolly.

What is wrong with the above argument? What is wrong is that the student introduces a cow and calls the cow x, but by the rules that we have discussed in class quite extensively, **you are not allowed to introduce an object of a certain kind and give it a name unless you have established that such an object exists.** In this case, if we want to introduce a cow and call her x, we need to know that cows exist, that is, that "$\exists x, x$ is a cow".

Of course it is true that "$\exists x, x$ is a cow". But this statement is not in the list of five assertions that you were allowed to use. And the question said very clearly that you are allowed to use these five statements and nothing else! So the fact that "$\exists x, x$ is a cow" cannot be used, even though it is true. And if you cannot use that fact there is no way on Earth that you can prove that Olivia is taller than Dolly.

To make this point clearer to you, I then gave you the same question, just changing "giraffes" to "rabbits," "cows" to "unicorns," and "sheep" to "elephants." **This has exactly the same logical structure as the giraffes-cows-sheep problem.** If you could prove that Olivia is taller than Dolly, then exactly the same argument would also prove that Peter is taller than Dumbo!

But Peter is **not** taller than Dumbo! (If you proved that Peter is taller than Dumbo then please ask yourself "am I really able to prove anything, even things that are obviously false?") Now it should be clearer that the "proof" does not work, because the intermediate step ("let x be a unicorn") is not true, since there are no unicorns!!!

13 Proofs

13.1 The structure of a proof

Suppose we specify a collection **H** of sentences called **hypotheses**. Suppose in addition that we have a collection **A** of sentences declared to be **axioms**, and a collection **D** of sentences declared to be **definitions**. Then a **proof under hypotheses H**—under the axioms **A** and the definitions **D**— is a sequence of **justified steps**, each of which is a sentence (called the "assertion" of—or in—that particular step) together with a valid justification.

A **proof of a sentence $\mathcal{A}$ under hypotheses H**—under the axioms **A** and the definitions **D**— is a proof under **A** and **D** for which the assertion in the last step is $\mathcal{A}$. A **valid justification** for a step is an explanation showing how the step's assertion is either:

- an axiom,
- a definition,
- a hypothesis,
- a consequence of previous steps by a rule of inference.

To complete this definition of "proof," we have to list the rules of inference. This will be done below.

13.2 Some examples of rules

Before we present the complete list, we show some examples of rules and of steps involving them.

13.2.1 Modus Ponens

One of our rules (called "Modus Ponens," or "$\longrightarrow$-out") will say that if you have two sentences of the form $\mathcal{A}$ and $\mathcal{A} \longrightarrow \mathcal{B}$, then you can go to $\mathcal{B}$.

So, for example, suppose that you are writing a proof, and you just finished Step 94. Suppose you had the following in Step 25:

(25) $3 > 0,$ (. . . . .)

and, later, in Step 47:

(47) $3 > 0 \rightarrow (\exists x, (x^2 = 3)).$ (. . . . .)

Then you are permitted to add a new step to you proof (say Step 95), by saying:

(95) $\exists x, (x^2 = 3).$ (Modus Ponens, from (25) and (47)).

13.2.2 The rule for using a conjunction

Another rule ("Rule $\wedge$-out," or the "rule for using a conjunction") will tell us that if you have a step with a sentence of the form $\mathcal{A} \wedge \mathcal{B}$, then you can go to $\mathcal{A}$ and to $\mathcal{B}$. For example, suppose that at some point of a proof you have

(238) $3 > 0 \wedge 3 < 4\,.$ (. . . . .)

Then, if you have just finished Step 454, you can write

(455) $3 > 0.$ (By Rule $\wedge$-out, from (238).)

or

(455) $3 < 4.$ (By Rule $\wedge$-out, from (238).)

13.2.3 The rule for proving a conjunction

Another rule ("Rule $\wedge$-in," or the "rule for proving a conjunction") will tell us that if you have two steps with sentences $\mathcal{A}$ and to $\mathcal{B}$, then you can go to $\mathcal{A} \wedge \mathcal{B}$. For example, suppose that in a proof you have

(238) $3 > 0.$ (. . . . .)

and

(703) $3 < 4.$ (. . . . .)

Then, if you have just finished Step 40356, you can write

(40357) $3 > 0 \wedge 3 < 4\,.$ (By Rule $\wedge$-in from (238) and (703).)

13.2.4 Universal specialization

Another rule ("Rule $\forall$-out," also called the "rule for using a universal sentence," or "universal specialization rule") will tell us that if you have a universal sentence—say "all elephants have a trunk and tusks"—and a constant—say "Dumbo"—then you can apply the general statement to the special case, and conclude that "Dumbo has a trunk and tusks."

For example, suppose that at some point of a proof you have

(977) $\forall n, (n \in \mathbb{N} \longrightarrow (n+1)^2 = n^2 + 2 \cdot n + 1)\,.$ (. . . . .)

Then, if you want to, you can have a later step

(1088) $3 \in \mathbb{N} \longrightarrow (3+1)^2 = 3^2 + 2 \cdot 3 + 10.$ (Specialization, plugging in 3 for n in (977).)

13.3 Proofs within proofs

There is one more thing you need to know, which is quite easy to understand but somehow confuses many people:

- *Whenever you want, you can start a "subproof," or "proof within a proof," in which you can make* **any assumption you want**. However, when you do that then all the things you prove are proved **under that assumption**.

For example, suppose we are reading a mystery novel, and I am trying to prove to you that it was Bob that murdered Bill. Suppose we got to a point where I have established that the murderer must have been Bob or John. Now I want to convince you that it's not John, so it must have been Bob.

Here is one thing I could say: "Suppose John murdered Bill. We know that Bill's body was found in his house, so the killing must have taken place in the house itself, or nearby, because even if the body had been moved it still couldn't have been moved very far. So John must have been in or near Bill's house at the time of the murder. But Bill was killed on September 8 between 9 and 10 p.m. EST. So John must have been in or near Bill's house on September 8 between 9 and 10 p.m. EST. On the other hand, we know that John gave a public lecture in San Diego on September 8, starting at 6 p.m. Pacific time, i.e. at 9 p.m. EST. So he must have traveled from New York to San Diego in one hour or less, which is impossible. So John did not murder Bill."

Notice that what we did here is construct, within our main proof that Bob is the murderer, a "subproof," or "proof within a proof," whose goal is to prove that John is not the murderer. We did this by *introducing a new hypothesis*, namely, by assuming that John is the murderer. In other words, *we created and set out to explore an imaginary world*, in which John did indeed kill Bill, in order to find out what are the consequences and ramifications of the new hypothesis. In this imaginary world, we proved a number of things, such as, for example, that "John traveled from New York to San Diego in one hour or less." But we proved these things *under the hypothesis that John killed Bill*. In order words, we did not really prove that "John traveled from New York to San Diego in one hour or less," but only that this must have been so *if John killed Bill*.

The ultimate consequence of our "proof within a proof" is that this world we have imagined, in which John killed Bill, is impossible, because in this world something impossible happened. So we get out of this "proof within a proof" the conclusion that *John did not kill Bill*. And *this conclusion no longer depends on the new hypothesis that "John killed Bill."* The final conclusion, that *John did not kill Bill*, is valid *without any extra hypotheses*, so it is part of our main proof.

In our mathematical formalism, we will be allowed to start a "proof within a proof" any time we want, by assuming anything we want. However, anything we prove will only be part of this "proof within a proof." There will be rules telling us how to get out of the "proof within a proof" and go back to the main proof. The most important one is the following:

RULE $\longrightarrow$-in. (The "conditional proof rule," also called "Rule $\longrightarrow$-in," or "rule for proving an implication.") *If assuming $\mathcal{A}$ you get to $\mathcal{B}$, then you can go, without assuming $\mathcal{A}$, to $\mathcal{A} \longrightarrow \mathcal{B}$.*

For example, suppose that, in some argument you are writing, you have established that

(3591) If Gore wins Florida then Gore will become president. (............)

and, later, that

(4507) If Gore becomes president then the next Attorney General will not be a conservative. (.....................)

That means that, if we use the letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ to represent the sentences "Gore wins Florida," "Gore becomes president," and "the next Attorney General will not be a conservative," then we already got

(3591) $\mathcal{A} \longrightarrow \mathcal{B}$. (.....................)
(4507) $\mathcal{B} \longrightarrow \mathcal{C}$. (.....................)

Suppose you just finished writing Step 6812, and now you want to prove $\mathcal{A} \longrightarrow \mathcal{C}$. Here is one thing you could do:

(6813)		Assume $\mathcal{A}$.	(Preparation to apply Rule $\longrightarrow$-in)
(6814)		$\mathcal{B}$.	(Modus Ponens from (3591) and (6813))
(6815)		$\mathcal{C}$.	(Modus Ponens from (4507) and (6814))
(6816)	$\mathcal{A} \longrightarrow \mathcal{C}$.		(Rule $\longrightarrow$-in, from (6813) and (6815))

Steps 6813, 6814, 6815 constitute a "proof within a proof," in which you introduce the new assumption $\mathcal{A}$, and under this assumption you prove first $\mathcal{B}$ and then $\mathcal{C}$. Then in Step 6816 you go back to the main proof, using Rule 4.P.

Notice that we have used extra indentation to indicate that we are writing a proof within a proof. The way we have organized our proof makes it clear that *at no point are we proving $\mathcal{B}$ or $\mathcal{C}$*. We are proving $\mathcal{B}$ and $\mathcal{C}$ *under the assumption* $\mathcal{A}$, from which we can conclude that $\mathcal{A} \longrightarrow \mathcal{C}$, as we do in Step 6816. For all we know, *$\mathcal{B}$ and $\mathcal{C}$ could perfectly well be false*, and nothing in what we wrote says otherwise. (Actually, $\mathcal{B}$ and $\mathcal{C}$ *are* false.) All our "proof within a proof" says is that *if you assume that $\mathcal{A}$ is true, then $\mathcal{C}$ must be true.*

The "proof within a proof" tells us nothing about what happens if $\mathcal{A}$ is false. And our final conclusion, that $\mathcal{A} \to \mathcal{C}$, is of course valid if $\mathcal{A}$ is false, because **an implication such as $\mathcal{A} \to \mathcal{C}$ is true when $\mathcal{A}$ is false.**

13.4 Introducing assumptions

Let us look at what we have just said from a slightly different angle.

In a proof you can introduce any assumption you want, at any time. It does not matter if the assumption is true, of if you know whether it is true.

Students often find this statement shocking, and ask questions such as "how can I assume something if I do not know that it is true?"

The answer is that you can, and in fact you do precisely that most of the time, but for some mysterious reason when students write about mathematics they do not believe that the same thing that they are used to doing in their everyday life is O.K. and works the same way in mathematics.

In you everyday life, you often imagine some possibility in order to figure out what would happen if that was the case. For example, before the last presidential election you probably tried to imagine what would happen if Candidate X won, and what would happen if Candidate Y won. Then you compared the results and came to the conclusion that you liked better what would happen if one of those two candidates won, and decided for that reason to vote for that candidate.

To imagine what would happen if Candidate X won, you almost certainly started thinking along the following lines: "let me suppose that X been elected president; then I guess he would appoint so-and-so as secretary of state, and this person would probably push our foreign policy in sushc-and-such direction, given what I know about his views." And you would have continued like that.

In other words, **you made the assumption that X won without knowing that it was true, and then you explored its consequences**.

You do the same kind of thing all the time. For example, to decide if you will go to Restaurant R for dinner, you would try to imagine what you would eat if you were there, and you might very well end up concluding that none of the dishes that you can imagine yourself ordering looks appealing, so in fact you will not go to Restaurant R. In other words: **you made the assumption that you had gone to Restaurant R and then concluded that you did not like its consequences, so in the end you rendered the assumption**

BOX NO. 1: HYPOTHESES (A.K.A. ASSUMPTIONS).

Hypotheses, or **assumptions**, are statements that we **assume** to be true in a particular situation.
When we discuss proofs later on, we will see that that there are certain statements that you are free to bring in any time you want, because you know them to be true. (Such statements are called "axioms," or "postulates.") In additions, *definitions* function exactly like axioms, the only difference being that an axiom is something you *accept* as true (say, because you are convinced it is true but you do not know how to prove it) whereas a definition (say, for example, $2 = 1 + 1$) is something that you *declare*, or *decide* to be true. **Within a particular proof, hypotheses/assumptions function exactly like axioms and definitions: they are statements that you can bring in any time.**
There is, however, a big difference. Axioms and definitions are things that we take to be true forever, because we believe they are true. Hypotheses/assumptions are taken to be true not because we believe they are true, but **because we want to know the consequences of their being true.**
For example: suppose we want to find out what the world would be like if pigs had wings. To do this, we first make the assumption "pigs have wings," by saying

(...) Assume that pigs have wings.

Then we would set out to explore the consequences of this assumption. That is, if you prefer, we would set out to study an **imaginary world** in which pigs have wings, and try to understand what that world would have to be like. We might then be able to "prove" all kind of "theorems," such as

THEOREM 1: *Pigs can fly.*

THEOREM 2: *If a pigsty does not have a roof, then the pigs will fly away.*

(Proving these theorems might not be easy. For example, it is not true that "every animal that has wings can fly," since, for example, ostriches, penguins and dodos do not fly. But you might notice that all the exceptions to the general rule seem to be birds, so perhaps you could make a convincing case that "every animal that has wings and is not a bird can fly." If you manage to do that, then you might go on to argue that pigs can fly, since pigs clearly are not birds.)
All these theorems would be proved **under the hypothesis** that pigs have wings. That is, we would never be saying that we have proved that pigs can fly. All we would be saying is that **IF** pigs had wings **THEN** pigs could fly. In other words, all the facts we would "prove" about pigs under the "pigs have wings" hypothesis would be valid **in the imaginary world of pigs with wings**, which, in principle, is not our real world.

false by not going to the restaurant.

In mathematics we do the same thing. We introduce assumptions, and then explore their consequences. We do not care if the assumptions are true, because all we are trying to find out is "what else would have to be the case if these assumptions were true?"

BOX NO. 2: ENTERING AND LEAVING IMAGINARY WORLDS.

As we explained in Box 1, when you assume something you are setting out to explore an **imaginary world**, namely, a world in which that thing you are assuming is true. When can you leave that world and go back to the real world, taking with you the things you have discovered in your imaginary world? The answer is simple: if the thing you assumed is actually true in the real world, then everything you proved **under that assumption** is true, because in that case the imaginary world really is the real world.
Our rules of inference capture this very obvious idea as follows:

- You are always allowed to introduce anything you want as an assumption, any time you want.
- When you do that, whatever you have proved will have been proved **under your assumption**.
- If, assuming a statement $\mathcal{A}$, you prove some other statement $\mathcal{B}$, then you will be allowed to go back to "the real world" and assert there that $\mathcal{A} \longrightarrow \mathcal{B}$.
- If, in addition, you can prove $\mathcal{A}$, then the Modus Ponens rule allows you to go to $\mathcal{B}$.

In simpler terms: whatever you prove under the assumption $\mathcal{A}$ will be valid unrestrictedly (that is, without having to assume that you are under the assumption $\mathcal{A}$) if you can also prove $\mathcal{A}$.

EXAMPLE: If you prove that "pigs can fly" under the assumption that "pigs have wings," then the conclusion that "pigs have wings" will have been established in a world where pigs have wings. You can also go back from such an imaginary world to the real world by stating that "if pigs have wings then pigs can fly." This, however, is not enough to be able to conclude that in the real world pigs can fly, because in the real world you do not know that pigs have wings, and **if you do not know that pigs have wings then you cannot go from "if pigs have wings then pigs can fly" to "pigs can fly."**

13.4.1 Proofs by contradiction

Let us start with an example.

AN EXAMPLE OF A PROOF BY CONTRADICTION:
QUESTION: *Prove that the number* 30 *cannot be expressed as the difference of the squares of two integers.*

PROOF: Suppose 30 can be expressed as the difference of the squares of two integers. (*Comment: this is a typical proof by contradiction. We want to prove Sthat something isn't true. To do so, we imagine it is true, that is, we explore what things would be like in an imaginary world where it is true. We then set out to show that this imaginary world is impossible, because in it something impossible would have to be true.*) Pick two integers n and m such that $n^2-m^2 = 30$. Then $(n-m)(n+m) = 30$. Now $n+m = (n-m)+2m$. Therefore $n-m$ and $n+m$ are either both odd or both even. If $n-m$ and $n+m$ are both odd then their product, i.e., the number 30, would have to be odd. If $n-m$ and $n+m$ are both even then their product, i.e., the number 30, would have to be divisible by 4. So 30 is either odd or divisible by 4. But this is false, because 30 is neither odd nor divisible by 4. So the assumption that 30 can be expressed as the difference of the squares of two integers leads to an impossible conclusion. Therefore 30 cannot be expressed as the difference of the squares of two integers. $\diamondsuit$

ANOTHER EXAMPLE OF A PROOF BY CONTRADICTION:

QUESTION: *Prove that the number* 2 *is not the square of a rational number.* (That is, "prove that $\sqrt{2}$ is irrational.")

PROOF: Suppose 2 is the square of a rational number. This means that 2 can be expressed as the quotient of the square of an integer divided by the square of a nonzero integer. Pick two integers n and m such that $2 = \frac{n^2}{m^2}$ and $m \neq 0$. Then, after cancelling all common factors > 1, we may assume that n and m have no common factors > 1. In particular, n and m are not both even, for if they were both even then they would have the factor 2 in common. Now, the equality $2 = \frac{n^2}{m^2}$ implies $2m^2 = n^2$. So n^2 is even. But then n is even, because if n was odd then n^2 would be odd. So $n = 2k$ for some integer k. But then $2m^2 = n^2 = (2k)^2 = 4k^2$. So $m^2 = 2k^2$. So m^2 is even. Then m is even, because if m was odd then m^2 would be odd. Therefore n and m are both even, and at the same time n and m are not both even. These two assertions together add up to an impossible conclusion. Therefore 2 is not the square of a rational number. $\diamondsuit$

BOX NO. 3: PROOFS BY CONTRADICTION.

One of the most common reasons for assuming something is to prove it cannot happen. In other words, you set out to explore the consequences of something in order to prove it is impossible. This is called "proof by contradiction." We have just seen an example: to prove by contradiction that 30 cannot be expressed as the difference of the squares of two integers, we assumed that 30 could be expressed as the difference of the squares of two integers, and ended up with something false, namely, that 30 is either odd or divisible by 4. That is, **we set out to explore an imaginary world in which 30 can be expressed as the difference of the squares of two integers, until we saw that in this world impossible things would happen, and then we concluded that such a world could not exist, so 30 cannot be expressed as the difference of the squares of two integers.**
This should answer once and for all one of the most frequently asked students' questions: **How can I assume something if I don't know it is is true?** Or, even worse, **how can I assume something if I know it is not true?**
The answer is very simple: you assume something that you don't know to be true because you want to find out what would happen if it was true. And often you assume it **precisely in order to prove it is not true.**

13.5 Introducing arbitrary objects

A very important way of starting subproofs by introducing assumptions is to introduce *arbitrary objects*.

This will be useful because there will be a rule ("Rule $\forall$-in," also called "universal generalization") telling us that, if you prove that something is true for an object a under the assumption that a is arbitrary, then you can conclude that what you proved about a is true *for all* objects.

Example 1. Suppose you want to prove that "if you take a natural number and add it to its square, then the result is an even number." (Symbolically, this is translated as

$$\forall n, (n \in \mathbb{N} \longrightarrow 2|(n^2+n)), \qquad \text{or} \qquad \forall n \in \mathbb{N}, 2|(n^2+n),$$

or, if it has been stated before that we are working in the world of natural numbers, we could just write $\forall n, (2|(n^2+n))$.)

Then you could say: "let us pick an **arbitrary** natural number and let us call it k. Then $k^2+k = k(k+1)$. So k^2+k is the product of two consecutive integers. Since one of these two integers must be even, it follows that k^2+k is even. So we have proved that $\forall n(2|n^2+n)$."

Notice that here we introduced a new letter k, which serves as the name of a number that is *fixed* throughout our analysis, until we get to the final step, in which we say that, since what we did was O.K. no matter what k was, then we have shown that $\forall n(2|n^2+n)$. In our "proof within a proof," the letter k is treated as a constant.

We could organize this as follows:

(***) Let $k \in \mathbb{N}$ be arbitrary. (Preparation to apply Rule $\forall$-in)

$\vdots$

(!!!) k^2+k is even. ($\cdots\cdots\cdots\cdots$)
($$$) $\forall n \in \mathbb{N}, (n^2+n$ is even). (Rule $\forall$-in, from (***) and (!!!))

or

(***) Let k be arbitrary. (Preparation to apply Rule $\forall$-in)
(&&&) Assume $k \in \mathbb{N}$. (Preparation to apply Rule $\longrightarrow$-in)

$\vdots$

(!!!) k^2+k is even. ($\cdots\cdots\cdots\cdots$)
($$$) $k \in \mathbb{N} \longrightarrow (k^2+k$ is even). (Rule $\longrightarrow$-in, from (&&&) and (!!!))
(%%%) $\forall n, (n \in \mathbb{N} \longrightarrow (n^2+n$ is even)). (Rule $\forall$-in, from (***) and ($$$))

When we start a subproof by introducing a letter by saying "let such and such be arbitrary," then this has automatically the effect of declaring that letter to be a constant for this subproof. But, **this declaration is "local," that is, valid within the subproof only. Once the subproof is finished, then the effect of this declaration is undone.**

13.6 The basic logical rules for proofs

Let us first review our conditions telling us when a step in a proof is valid.

(P.a) It is always permitted to bring in an *axiom* (or "postulate") as a new step.

(P.b) It is always permitted to bring in a *definition* as a new step.

(P.c) It is always permitted to bring in one of the *hypotheses* as a new step.

(P.d) It is always permitted to bring in as a new step something that follows from previous steps by applying a valid *rule of inference*.

(P.e) It is always permitted to start a proof within a proof by introducing any sentence you want as an assumption, and in particular to bring in a **new** letter as an arbitrary constant. However, whatever is proved after this is done, is only valid in the "proof within a proof", and one can only get out of it by applying one of the rules that allow you to get out of a subproof (that is, Rules 1.P, 1.U, 3.P, 6.P, 7.U).

And here, finally, are the basic rules of inference.

BOX NO. 4: THE BASIC LOGICAL RULES.

RULE T. You are allowed to bring in any statement which is an instance of a tautology. (This is called the "tautology proof rule.")

RULE $\neg$-in. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have started a subproof P' of a proof P with "Assume $\mathcal{A}$" and have obtained $\mathcal{B}$ in a step of P', and $\mathcal{B}$ is a contradiction, then you can get out of P' and go to $\neg\mathcal{A}$ in P. (This is the "rule for proving a negation," and is one of several rules of "proof by contradiction.")

RULE $\neg$-out. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have $\neg\mathcal{A}$ in a step of your proof, and then you start a subproof P' of your proof P with "Assume $\mathcal{B}$" and obtain $\mathcal{A}$ in a step of P', then you can get out of P' and go to $\neg\mathcal{B}$ in P. (This is another version of the "rule of proof by contradiction.")

RULE $\wedge$-in. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have $\mathcal{A}$ and $\mathcal{B}$ in previous steps, then you can go to $\mathcal{A} \wedge \mathcal{B}$. (This is called the "rule for proving a conjunction.")

RULE $\wedge$-out. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have $\mathcal{A} \wedge \mathcal{B}$ in a previous step, then you can go to $\mathcal{A}$ and you can also go to $\mathcal{B}$. (This is called the "rule for using a conjunction.")

RULE $\vee$-in. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have $\mathcal{A}$ in a previous step, then you can go to $\mathcal{A} \vee \mathcal{B}$.If you have $\mathcal{B}$ in a previous step, then you can go to $\mathcal{A} \vee \mathcal{B}$. (This is the "rule for proving a disjunction.")

RULE $\vee$-out. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, be sentences. If you have $\mathcal{A} \vee \mathcal{B}$, $\mathcal{A} \longrightarrow \mathcal{C}$, and $\mathcal{B} \longrightarrow \mathcal{C}$, in previous steps, then you can go to $\mathcal{C}$. (This is the "rule for using a disjunction," or "rule of proof by cases.")

RULE $\longrightarrow$-in. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have started a subproof P' of a proof P with "Assume $\mathcal{A}$" and have obtained $\mathcal{B}$ in a step of P', then you can get out of P' and go to $\mathcal{A} \longrightarrow \mathcal{B}$ in P'. (This is the "rule for proving an implication," or "conditional proof rule.")

RULE $\longrightarrow$-out. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have $\mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{B}$ in previous steps, then you can go to $\mathcal{B}$. (This is the "rule for using an implication," or "modus ponens.")

RULE $\longleftrightarrow$-in. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have $\mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{B} \longrightarrow \mathcal{A}$ in previous steps then you can go to $\mathcal{A} \longleftrightarrow \mathcal{B}$ in your proof. (This is called the "rule for proving a biconditional.")

RULE $\longleftrightarrow$-out. Let $\mathcal{A}$ and $\mathcal{B}$ be sentences. If you have $\mathcal{A} \longleftrightarrow \mathcal{B}$ in a previous step then you can go to $\mathcal{A} \longrightarrow \mathcal{B}$, and also to $\mathcal{B} \longrightarrow \mathcal{A}$, in your proof. (This is called the "rule for using a biconditional.")

RULE $\forall$-in. Suppose that

S1. x is a variable and $\mathcal{A}(x)$ is a formula where there is no quantifier involving x,

S2. a is a constant and $\mathcal{A}(a)$ is what you get from $\mathcal{A}(x)$ by substituting a for x in **all** the occurrences of x in $\mathcal{A}(x)$.

Suppose you are in a proof P (which may be itself be a subproof of some other proof) and you have started a subproof P' of P by saying "Let a be arbitrary," and that until that point a had never appeared in P. Suppose that in P' you got to $\mathcal{A}(a)$. Then you can get out of P' and go to $\forall x, \mathcal{A}(x)$ in P. (This the "rule for proving a universal statement," or "universal generalization rule.")

RULE $\forall$-out. Suppose S1, S2 are true. Then, if you have $\forall x, \mathcal{A}(x)$ in a previous step, you can go to $\mathcal{A}(a)$. (This is the "rule for using a universal statement," or "universal specialization rule.")

RULE $\exists$-in. Suppose S1, S2 are true. Suppose you got $\mathcal{A}(a)$ in a previous step. Then you can go to $\exists x, \mathcal{A}(x)$. (This is the "rule for proving an existential statement, or "existential generalization rule.")

RULE $\exists$-out. Suppose S1, S2 are true. Suppose you have started a subsubproof P'' within a proof P by saying "Let a be arbitrary," and "Assume $\mathcal{A}(a)$." Suppose that until that point a had never appeared in P. Suppose that in P'' you got to $\mathcal{B}$, where $\mathcal{B}$ is a sentence that does not contain the constant a. Suppose that in a previous step of P, before you started P'', you had $(\exists x, \mathcal{A}(x))$. Then you can get out of P'' and go to $\mathcal{B}$ in P. (This is the "rule for using an existential statement, or "existential specification rule.")

> That's it. These are **all** the rules of inference.
> There are exactly **fifteen** rules. Only fifteen!!!!

13.7 Some comments on the logical rules

There are fifteen rules. Maybe you think this is too much, and that remembering 15 rules requires a lot of effort on your part. I will show you now that this is not so, because **the rules of Box 4 are just plain common sense, so you should have no difficulty understanding them and remembering them**.

Actually, there is no need to have 15 rules. The number of rules can be cut down a lot, because several of the rules in Box 4 are really not needed. On the other hand, having these 15 rules makes it much easier to write proofs than if you had fewer rules. (In many cases, our rules enable you to do in one step what might require five or six steps without using the rule.)

Let us notice that the rules obey a nice simple pattern. In addition to the "tautology rule" (i.e., Rule 0), our rules are neatly organized into seven pairs consisting of two rules each. The seven pairs correspond to the **seven logical connectives**:

$$\neg \qquad \wedge \qquad \vee \qquad \longrightarrow \qquad \longleftrightarrow \qquad \forall \qquad \exists \quad .$$

Remember that **every mathematical sentence is of one and only one of the following eight types**:

- negation ("not", i.e, "$\neg$"),
- conjunction ("and", i.e, "$\wedge$"),
- disjunction ("or", i.e, "$\vee$"),
- implication ("implies", or "if ... then", i.e, "$\longrightarrow$"),
- biconditional ("if and only if", i.e, "$\longleftrightarrow$"),
- universal ("for all", i.e, "$\forall$"),
- existential ("for some", or "there exists ... such that", i.e, "$\exists$"),
- atomic.

Atomic sentences (for example, $2+2=4$) cannot be analyzed into smaller parts put together by logical connectives. This means that if you want to prove or use an atomic sentence you need to do mathematics, rather than pure logic. The main part of the course will be devoted to the mathematics, but at this point we are concentrating on the logical part. That is why we are presenting rules that tell us how to work with non-atomic sentences and use their logical structure to figure out how to use them and how to prove them.

As we indicated before, all the rules other than Rule T come in pairs of two rules each. There are seven such pairs, one for each of the seven logical connectives. For each connective, there are two rules. One rule (the "in" rule) tells you how to *get to* a sentence S (i.e. how to *prove* S) if the logical form of

S is that of the corresponding connective. The other rule (the "out" rule) tells you how to *get out* of a sentence S (i.e. how to *use* S) if the logical form of S is that of the corresponding connective.

Now let us look at the rules one at a time, and show that they are no more than plain common sense.

13.7.1 The tautology proof rule

Rule T just tells us that you are always allowed to bring in instances of tautologies. *Why is this reasonable?* Because **instances of tautologies are always true**, so it is perfectly reasonable to allow yourself to bring them in any time you want.

13.7.2 Proofs by contradiction

Rules $\neg$-in and $\neg$-out enable us to do **proofs by contradiction**. How does a proof by contradiction work?

Let us say you want to prove that a statement $\mathcal{S}$ is false, i.e., that you want to prove $\neg S$. To do that, **assume that $\mathcal{S}$ is true and try to get to something that you can be sure is false**.

This can be done in two ways. One way is to **get a to a contradiction**. This works, because a contradiction is false for sure. Rule $\neg$-in just says that if you get to a contradiction by assuming $\mathcal{S}$, then you can go to $\neg\mathcal{S}$. *Why is this reasonable?* Because if assuming $\mathcal{S}$ you derive a contradiction then $\mathcal{S}$ cannot be true, since if $\mathcal{S}$ was true it would follow that the contradiction is true.

Another way is to proceed as in Rule $\neg$-out. That is, you assume that $\mathcal{S}$ is true and try to prove, under that assumption, a statement $\mathcal{A}$ that has been proved to be false in a previous step. Then you can conclude $\neg\mathcal{S}$ again.

Remark 2. **Proofs by contradiction are used all the time.** When you have to prove something, you should always seriously consider the possibility of doing your proof by contradiction.

Here is the main reason. Suppose you know several things, say for example you know five statements $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$. Suppose you are asked to prove some other statement, say $\mathcal{E}$. If you try a "direct proof" (that is, a proof which is not by contradiction) then you have five statements you can use, namely $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, and from these statements you have to work your way to $\mathcal{S}$. Suppose instead that you do a proof by contradiction. You do this by saying "let me assume $\neg\mathcal{S}$ and see what happens." Now you have *six* things that you can use, namely, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{S}$. So you have more room to manoeuvre, and you can do more things.

A piece of advice: Most of your proofs should be by contradiction. When you have to prove something and you do not know what to do, the best strategy is to try saying "let me imagine that what I want to prove was false, and let me see what would have to happen." $\diamondsuit$

13.7.3 The rules for conjunction and disjunction

These rules are fairly trivial, so let me just go over them quickly.

Rule $\wedge$-in tells us how to prove a conjunction. It is a reasonable rule because it is obvious that if you know that two things are true then you can conclude that their conjunction is true.

Rule $\wedge$-out tells us how to make use of a conjunction: if you know that the conjunction of two things is true then you can conclude that each one of them is true.

Rule $\vee$-in gives us a way to prove a disjunction: to prove that the disjunction of two things is true, prove one of them.

Rule $\vee$-in gives us a way to prove a disjunction: to prove that the disjunction of two things is true, prove one of them.

Rule $\vee$-out is the "rule of proof by cases." It says that, if we want to conclude $\mathcal{C}$ and we know $\mathcal{A} \wedge \mathcal{B}$, then we consider the case when $\mathcal{A}$ is true, and then the case when $\mathcal{B}$ is true, and show that in each case $\mathcal{C}$ must be true.

EXAMPLE: suppose I tell you that "Karl was born is Oslo or in Stockholm. I don't know in which city, exactly, but I know it was one of those two. Is true that Karl is Scandinavian?" Then you could argue as follows:

(1) Karl was born in Oslo $\vee$ Karl was born in Stockholm. [Given]
(2) Assume Karl was born in Oslo. [Preparation to apply Rule $\longrightarrow$-in]
(3) Oslo is in Norway. [...]
(4) Norway is a Scandinavian country. [...]
(5) Karl was born in a Scandinavian country. [...]
(6) Karl is Scandinavian. [...]
(7) Karl was born in Oslo $\longrightarrow$ Karl is Scandinavian. [Rule$\longrightarrow$-in, from (2) and (6)]
(8) Assume Karl was born in Stockholm. [Preparation to apply Rule $\longrightarrow$-in]
(9) Stockholm is in Sweden. [...]
(10) Sweden is a Scandinavian country. [...]
(11) Karl was born in a Scandinavian country. [...]
(12) Karl is Scandinavian. [...]
(13) Karl was born in Stockholm $\longrightarrow$ Karl is Scandinavian. [Rule$\longrightarrow$-in, from (2) and (6)]
(14) Karl is Scandinavian. [Rule $\wedge$-out, from (1), (7) and (12)
END OF PROOF.

13.7.4 The rules for implication

Rule $\longrightarrow$-in gives us a way to prove an implication: to prove $\mathcal{A} \longrightarrow \mathcal{B}$, assume $\mathcal{A}$ and prove $\mathcal{B}$. (When you "assume $\mathcal{A}$" you enter a new level of "proof within a proof," as was explained before.)

Rule $\longrightarrow$-out is called "Modus Ponens," and is one of the most important of all the rules. It is an "out" rule. It gives us a way to put to use an implication

$\mathcal{A} \longrightarrow \mathcal{B}$. If you manage in addition to prove $\mathcal{A}$, then the rule tells you that you can go to $\mathcal{B}$.

13.7.5 The rules for the biconditional

Rule $\longleftrightarrow$-in tells us that you can prove a biconditional of the form $\mathcal{A} \longleftrightarrow \mathcal{B}$ by proving $\mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{B} \longrightarrow \mathcal{A}$.

Rule $\longleftrightarrow$-out is a trivial "out" rule. It tells us that if you have a biconditional $\mathcal{A} \longleftrightarrow \mathcal{B}$ then you can use it by concluding from it the implications $\mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{B} \longrightarrow \mathcal{A}$.

13.7.6 The rules for the universal quantifier

Rule $\forall$-in gives us a way to prove a universal statement: to prove that some statement S is true for all possible values of the variable x, pick an arbitrary object, call it a, and prove that S is true of a.

Rule $\forall$-out—the universal specialization rule—is an "out" rule. It is extremely easy to understand, and the name is easy to remember. The rule basically says: if you know that something is true for all things, then you can plug in any particular thing and you will get a true statement. (For example: if you know that "everybody is smart" you can conclude that "Alice is smart.") The name *specialization* comes from the fact that in going from $\forall x, \mathcal{A}(x)$ to $\mathcal{A}(a)$, you are going from a *general statement* (for example, "all people are smart") to a *special case* ("Alice is smart").

13.7.7 The rules for the existential quantifier

Rule $\exists$-in gives us a way to prove an existential statement: to prove that $\exists x, \mathcal{A}(x)$, exhibit an a such that $\mathcal{A}(a)$. (Such as a is called a **witness** for $\exists x, \mathcal{A}(x)$.)

Finally, Rule $\exists$-out is another "out" rule which just reflects plain common sense. All it says is that if you know that there exists an object of a certain kind, then you can introduce one such object and give it a name.

The most common way you would apply this rule is this. Say you are trying to prove $\mathcal{B}$ and you know that $\exists x, \mathcal{A}(x)$. Then you would say "since I know that there exists an x such that $\mathcal{A}(x)$, let me pick one and call it a." You could do that by starting a subproof with "Let a be arbitrary," and then a subsubproof with "Assume $\mathcal{A}(a)$." Suppose you then work your way to $\mathcal{B}$. Then you can get out of the subsubproof and land in the proof with $\mathcal{B}$.

It is very important that the new object a be given a name that has not been used before. For example, suppose "$\mathcal{A}(x)$" stands for "x killed Polonius," and we are told that $\exists x, \mathcal{A}(x)$, i.e., that somebody killed Polonius. Then you can introduce a name for this individual. You can call him/her a or, if you prefer, "the killer," in which case you would be able to say that "$\mathcal{A}(a)$" or that "$\mathcal{A}$(the killer)." But you cannot say "let's call this person Hamlet," or "let's call him Laertes," because Hamlet and Laertes are names of characters that have already appeared in the play. If you call the killer "Hamlet" or "Laertes" then you would be *prejudging*, and declaring that $\mathcal{A}$(Hamlet), i.e. that Hamlet killed Polonius, or that $\mathcal{A}$(Laertes), i.e. that Laertes killed Polonius. One of these happens to be true, and the other one is false, but in either case you cannot just conclude that it is true by merely choosing a name for the killer.)

Also, notice that the rule only allows you to conclude that $\mathcal{B}$ holds if you know that $\exists x, \mathcal{A}(x)$. If you do not know that such an x exists, then you cannot do anything with the rule, and in particular you cannot get to $\mathcal{B}$.

For a typical example of the kind of outrageous mistakes that one can make by failing to pay attention to the requirement that you must know that $\exists x, \mathcal{A}(x)$, look carefully at the Peano paradox, described in Box No. 5 below. **Make sure you understand and can explain what is wrong with that proof.**

BOX NO. 5: THE PEANO PARADOX.

Here is a "theorem" that is obviously false:

THEOREM: 1 *is the largest natural number.*

And here is a "proof":

PROOF: Let n be the largest natural number. Then n^2 is a natural number too. Since n is the largest natural number, $n^2 \leq n$. On the other hand, $n^2 \geq n$. (Proof by cases: either $n \geq 1$ or $n = 0$. If $n \geq 1$ then so $n^2 = n.n \geq n.1 = n$. So $n^2 \geq n$. If $n = 0$ then $n^2 = 0$, so $n^2 = n$ and then $n^2 \geq n$.) Therefore $n^2 = n$. The only solutions x of $x^2 = x$ are $x = 1$ and $x = 0$. So $n = 1$ or $n = 0$. But n is the largest natural number, so n cannot be 0 because $1 > 0$. Therefore n has to be equal to 1. **QED**

WHAT IS WRONG WITH THIS "PROOF"? The mistake occurs in the very first sentence, "let n be the largest natural number." *This step is not permissible, because we do not know that there exists a "largest natural number."* The author must have been thinking of appliyng Rule $\exists$-out here. But the rule only allows to introduce an object of a certai kind *if you know that one such an object exists.*

13.8 An example of an almost completely formal proof

The definition of the empty set says:

$$\forall x, (\neg x \in \emptyset)\,.$$

(That is, "nothing belongs to the empty set.")

The definition of *set inclusion* says that if X, Y are sets then

$$X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y))\,.$$

(That is, "X is a subset of Y if and only if every member of X belongs to Y.")

Using these two definitions plus our logical rules, let us give a completely formal proof of the following.

Theorem 1. *If A is any set then $\emptyset \subseteq A$.*

PROOF:

(1)	$\forall x, (\neg x \in \emptyset)$.	[Definition of $\emptyset$]
(2)	Let a be arbitrary.	[Preparation to apply Rule $\forall$-in]
(3)	Assume $a \in \emptyset$.	[Preparation to apply Rule $\longrightarrow$-in]
(4)	Assume $\neg a \in A$.	[Preparation to apply Rule $\neg$-out]
(5)	$\neg a \in \emptyset$.	[Rule $\forall$-out, plugging a for x in (1)]
(6)	$a \in \emptyset \wedge (\neg a \in \emptyset)$.	[Rule $\wedge$-in, from (3) and (5)]
(7)	$\neg(\neg(a \in A))$.	[Rule $\neg$-out, from (4) and(6), since $a \in \emptyset \wedge (\neg a \in \emptyset)$ is a contradiction]
(8)	$(\neg(\neg(a \in A))) \longrightarrow a \in A$.	[Rule T]
(9)	$a \in A$.	[Rule $\longrightarrow$-out, from (7) and (8)]
(10)	$a \in \emptyset \longrightarrow a \in A$.	[Rule $\longrightarrow$-in, from (3) and (9)]
(11)	$\forall x, (x \in \emptyset \longrightarrow x \in A)$.	[Rule $\forall$-in, from (2) and (10)]
(12)	$\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A))$.	[Definition of $\subseteq$]
(13)	$(\forall x, (x \in \emptyset \longrightarrow x \in A)) \longrightarrow \emptyset \subseteq A$.	[Rule $\longleftrightarrow$-out, from (12)]
(14)	$\emptyset \subseteq A$.	[Rule $\longrightarrow$-out, from (11) (13).]

END OF PROOF

13.9 An example of a completely formal proof

The previous example was "almost completely formal" but something was still missing: we had not formalized the definitions of "empty set" and "subset." Now let us write down these definitions in formal language, and give a completely formal proof of the statement of the previous section.

Furthermore, we have not been careful about specifying the ranges of our variables. Are there things other than sets, or is everything is a set? This is important because, for example, if we want to characterize the empty set $\emptyset$ by saying $\forall x, (\neg x \in \emptyset)$ (that is, "nothing belongs to $\emptyset$") then this does not tell us that $\emptyset$ is a set. If everything is a set, then this is no problem. But if we work in universe where there are sets as well as other things that are not sets, then if we want to guarantee that $\emptyset$ is a set we have to say it. This leads us to the following formula, which you can regard as the definition of the empty set, or as an axiom characterizing the empty set, as you prefer:

$$\emptyset \text{ is a set} \wedge \forall x, (\neg x \in \emptyset)\,. \tag{13.9.14}$$

(That is, "nothing belongs to the empty set.")

The definition of *set inclusion* says that if X, Y are sets then X is a subset of Y if and only if every member of X belongs to Y. In other words,

FORMAL DEFINITION OF "SUBSET" IN A UNIVERSE WHERE THERE ARE SETS AND OTHER THINGS:

$\forall X, \forall Y, ((X \text{ is a set} \wedge Y \text{ is a set}) \longrightarrow (X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y))))\,.$

Using these two definitions plus our logical rules, let us give a completely formal proof of the following.

Theorem 2. *If X is any set then $\emptyset \subseteq X$.*

PROOF:

Comment: we have to prove that "$\forall X, (X \text{ is a set} \longrightarrow \emptyset \subseteq X)$."

(1) $\forall X, \forall Y, ((X \text{ is a set} \wedge Y \text{ is a set}) \longrightarrow$
$(X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y))))$ [Definition of set inclusion]
(2) $\emptyset$ is a set $\wedge\, \forall x, (\neg x \in \emptyset)$. [Definition of $\emptyset$]
(3) $\emptyset$ is a set. [Rule $\wedge$-out, from (2)]
(4) $\forall x, (\neg x \in \emptyset)$. [Rule $\wedge$-out, from (2)]
(5) Let A be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we want to prove that "A is a set $\longrightarrow \emptyset \subseteq A$."

(6) Assume A is a set. [Preparation to apply Rule $\longrightarrow$-in]

Comment: we want to prove that "$\emptyset \subseteq A$."

(7) $\forall Y, ((\emptyset \text{ is a set} \wedge Y \text{ is a set}) \longrightarrow (\emptyset \subseteq Y \longleftrightarrow$
$(\forall x, (x \in \emptyset \longrightarrow x \in Y))))$ [Rule $\forall$-out, from (1), specializing X to $\emptyset$]
(8) $(\emptyset \text{ is a set} \wedge A \text{ is a set}) \longrightarrow (\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A)))$
[Rule $\forall$-out, from (7), specializing Y to A]
(9) $\emptyset$ is a set $\wedge$ A is a set [Rule $\wedge$-in, from (3) and (6)]
(10) $\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A))$ [Rule $\longrightarrow$-out, from (8)&(9)]
(11) $\forall x, (x \in \emptyset \longrightarrow x \in A) \longrightarrow \emptyset \subseteq A$ [Rule $\longleftrightarrow$-out, from (10)]

Comment: we now set out to prove that "$\forall x, (x \in \emptyset \longrightarrow x \in A)\,$."

(12) Let a be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we now set out to prove that "$a \in \emptyset \longrightarrow a \in A\,$."

(13) Assume $a \in \emptyset$. [Preparation to apply Rule $\longrightarrow$-out]

Comment: we now set out to prove that "$a \in A\,$." We do it by contradiction. That is, we assume $\neg(a \in A)$ and get something impossible.

(14) Assume $\neg(a \in A)$. [Preparation to apply Rule $\neg$-out]
(15) $\neg a \in \emptyset$. [Rule $\forall$-out, plugging a for x in (4)]
(16) $a \in \emptyset \wedge (\neg a \in \emptyset)$. [Rule $\wedge$-in, from (13) and (15)]
(17) $\neg(\neg(a \in A))$. [Rule $\neg$-in, from (14) and (16),
since $a \in \emptyset \wedge (\neg a \in \emptyset)$ is a contradiction]
(18) $(\neg(\neg(a \in A))) \longrightarrow a \in A$. [Rule T]
(19) $a \in A$. [Rule $\longrightarrow$-out, from (17) and (18)]

(20) $a \in \emptyset \longrightarrow a \in A$. [Rule $\longrightarrow$-in, from (13) and (19)]
(21) $\forall x, (x \in \emptyset \longrightarrow x \in A)$. [Rule $\forall$-in, from (12) and (21)]
(22) $\emptyset \subseteq A$. [Rule $\longrightarrow$-out, from (11) and (21)]
(23) A is a set $\longrightarrow \emptyset \subseteq A$. [Rule $\longrightarrow$-in, from (6) and (22)]
(24) $\forall X, (X$ is a set $\longrightarrow \emptyset \subseteq X)$. [Rule $\forall$-in, from (3) and (23)]
END OF PROOF

13.10 Another example of a completely formal proof

Now let us redo the above proof but *working in a universe where there are only sets and nothing but sets.* Since everything is a set, now there is no need to specify, every time we bring in an object, that it is a set.

Now our definition of the empty set (or our axiom characterizing the empty set, if you prefer), just says:

$$\forall x, (\neg x \in \emptyset)\,. \tag{13.10.15}$$

(That is, "nothing belongs to the empty set.")

The definition of *set inclusion* says that if X, Y are any objects (that is, any sets) then X is a subset of Y if and only if every member of X belongs to Y. In other words,

FORMAL DEFINITION OF "SUBSET" IN A UNIVERSE WHERE THERE ARE SETS AND NOTHING ELSE:

$$\forall X, \forall Y, (X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y)))\,.$$

Using these two definitions plus our logical rules, let us give a completely formal proof of the following.

Theorem 3. *If X is anything (that is, any set), then $\emptyset \subseteq X$.*

PROOF:

Comment: we have to prove that "$\forall X, (\emptyset \subseteq X)$."

(1) $\forall X, \forall Y, (X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y)))$ [Definition of set inclusion]
(2) $\forall x, (\neg x \in \emptyset)$. [Definition of $\emptyset$]
(3) Let A be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we want to prove that "$\emptyset \subseteq A$."

(4) $\forall Y, (\emptyset \subseteq Y \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in Y)))$
[Rule $\forall$-out, from (1), specializing X to $\emptyset$]
(5) $\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A))$
[Rule $\forall$-out, from (4), specializing Y to A]
(6) $\forall x, (x \in \emptyset \longrightarrow x \in A) \longrightarrow \emptyset \subseteq A$ [Rule $\longleftrightarrow$-out, from (10)]

Comment: we now set out to prove that "$\forall x, (x \in \emptyset \longrightarrow x \in A)$."

(7) Let a be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we now set out to prove that "$a \in \emptyset \longrightarrow a \in A$."

(8) Assume $a \in \emptyset$. [Preparation to apply Rule $\longrightarrow$-out]

Comment: we now set out to prove that "$a \in A$." We do it by contradiction. That is, we assume $\neg(a \in A)$ and get something impossible.

(9) Assume $\neg(a \in A)$. [Preparation to apply Rule $\neg$-out]
(10) $\neg a \in \emptyset$. [Rule $\forall$-out, plugging a for x in (2)]
(11) $a \in \emptyset \wedge (\neg a \in \emptyset)$. [Rule $\wedge$-in, from (8) and (10)]
(12) $\neg(\neg(a \in A))$. [Rule $\neg$-in, from (9) and (11), since $a \in \emptyset \wedge (\neg a \in \emptyset)$ is a contradiction]
(13) $(\neg(\neg(a \in A))) \longrightarrow a \in A$. [Rule T]
(14) $a \in A$. [Rule $\longrightarrow$-out, from (12) and (13)]
(15) $a \in \emptyset \longrightarrow a \in A$. [Rule $\longrightarrow$-in, from (8) and (14)]
(16) $\forall x, (x \in \emptyset \longrightarrow x \in A)$. [Rule $\forall$-in, from (7) and (15)]
(17) $\emptyset \subseteq A$. [Rule $\longrightarrow$-out, from (6) and (16)]
(18) $\forall X, \emptyset \subseteq X$. [Rule $\forall$-in, from (3) and (17)]
END OF PROOF

13.11 Remarks on proof levels

We have explained earlier that *every time we introduce an assumption (for example, when we say "let a be arbitrary," or "assume $a \in A$") we initiate a "subproof," or "proof within a proof," in which we place ourselves in an imaginary world where this assumption is true.*

For example, if I say "assume I was born in Oslo," then I am setting out to explore an imaginary world in which I was born in Oslo. Whatever I prove is proved *under the assumption that I was born in Oslo.* So, if I show that "I am Norwegian," it does not mean that I now know that I am Norwegian. All it means is that I know that *the fact that I am Norwegian has been proved under the assumption that I was born in Oslo.* We can then, if we want, get out from being "under the assumption that I was born in Oslo" and assert that *if I was born in Oslo then I am Norwegian.*

We indicate these different "levels of proof" by introducing extra indentation:

$\vdots$
(77) I am a mathematician. [...]
(78) Assume I was born in Oslo. [Preparation to apply Rule $\longrightarrow$-in]
(79) Oslo is in Norway [...]
(80) I was born in Norway. [...]
(81) I am Norwegian. [...]
(82) If I was born in Olso then I am Norwegian. [Rule $\longrightarrow$-in, from (78)&(80)]

13.12 A different way of indicating proof levels

If we have several proof levels, then it becomes cumbersome to use indentation. An alternative possibility is to use a number to indicate what proof level we are at. For example, the above lines about being a Norwegian could be written as follows:

$\vdots$

(77.0) I am a mathematician. [...]
(78.1) Assume I was born in Oslo. [Preparation to apply Rule $\longrightarrow$-in]
(79.1) Oslo is in Norway [...]
(80.1) I was born in Norway. [...]
(81.1) I am Norwegian. [...]
(82.0) If I was born in Olso then I am Norwegian.[Rule $\longrightarrow$-in, from (78)&(80)]

AN EXAMPLE: As an example of how to write proofs using numbers instead of indentation to indicate proof levels, here is the proof of §13.9, rewritten in the new way:

PROOF:

Comment: we have to prove that "$\forall X, (X \text{ is a set} \longrightarrow \emptyset \subseteq X)$."

(1.0) $\forall X, \forall Y, ((X \text{ is a set} \wedge Y \text{ is a set}) \longrightarrow$
$(X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y))))$ [Definition of set inclusion]
(2.0) $\emptyset$ is a set $\wedge \forall x, (\neg x \in \emptyset)$. [Definition of $\emptyset$]
(3.0) $\emptyset$ is a set. [Rule $\wedge$-out, from (2)]
(4.0) $\forall x, (\neg x \in \emptyset)$. [Rule $\wedge$-out, from (2)]
(5.1) Let A be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we want to prove that "A is a set $\longrightarrow \emptyset \subseteq A$."

(6.2) Assume A is a set. [Preparation to apply Rule $\longrightarrow$-in]

Comment: we want to prove that "$\emptyset \subseteq A$."

(7.2) $\forall Y, ((\emptyset \text{ is a set} \wedge Y \text{ is a set}) \longrightarrow (\emptyset \subseteq Y \longleftrightarrow$
$(\forall x, (x \in \emptyset \longrightarrow x \in Y))))$ [Rule $\forall$-out, from (1), specializing X to $\emptyset$]
(8.3) $(\emptyset \text{ is a set} \wedge A \text{ is a set}) \longrightarrow (\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A)))$
[Rule $\forall$-out, from (7), specializing Y to A]
(9.2) $\emptyset$ is a set $\wedge$ A is a set [Rule $\wedge$-in, from (3) and (6)]
(10.2) $\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A))$ [Rule $\longrightarrow$-out, from (8)&(9)]
(11.2) $\forall x, (x \in \emptyset \longrightarrow x \in A) \longrightarrow \emptyset \subseteq A$ [Rule $\longleftrightarrow$-out, from (10)]

Comment: we now set out to prove that "$\forall x, (x \in \emptyset \longrightarrow x \in A)$."

(12.3) Let a be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we now set out to prove that "$a \in \emptyset \longrightarrow a \in A$."

(13.4) Assume $a \in \emptyset$. [Preparation to apply Rule $\longrightarrow$-out]

Comment: we now set out to prove that "$a \in A$." We do it by contradiction. That is, we assume $\neg(a \in A)$ and get something impossible.

(14.5) Assume $\neg(a \in A)$.[Preparation to apply Rule $\neg$-out]
(15.5) $\neg a \in \emptyset$. [Rule $\forall$-out, plugging a for x in (4)]
(16.5) $a \in \emptyset \wedge (\neg a \in \emptyset)$. [Rule $\wedge$-in, from (13) and (15)]
(17.4) $\neg(\neg(a \in A))$. [Rule $\neg$-in, from (14) and (16), since $a \in \emptyset \wedge (\neg a \in \emptyset)$ is a contradiction]
(18.4) $(\neg(\neg(a \in A))) \longrightarrow a \in A$. [Rule T]
(19.4) $a \in A$. [Rule $\longrightarrow$-out, from (17) and (18)]
(20.3) $a \in \emptyset \longrightarrow a \in A$. [Rule $\longrightarrow$-in, from (13) and (19)]
(21.2) $\forall x, (x \in \emptyset \longrightarrow x \in A)$. [Rule $\forall$-in, from (12) and (21)]
(22.2) $\emptyset \subseteq A$. [Rule $\longrightarrow$-out, from (11) and (21)]
(23.1) A is a set $\longrightarrow \emptyset \subseteq A$. [Rule $\longrightarrow$-in, from (6) and (22)]
(24.0) $\forall X, (X$ is a set $\longrightarrow \emptyset \subseteq X)$. [Rule $\forall$-in, from (3) and (23)]
END OF PROOF

A SECOND EXAMPLE: As a second example of how to write proofs using numbers instead of indentation to indicate proof levels, here is the proof of §13.10, rewritten in the new way:

PROOF:

Comment: we have to prove that "$\forall X, (\emptyset \subseteq X)$."

(1.0) $\forall X, \forall Y, (X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y)))$
[Definition of set inclusion]
(2.0) $\forall x, (\neg x \in \emptyset)$. [Definition of $\emptyset$]
(3.1) Let A be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we want to prove that "$\emptyset \subseteq A$."

(4.1) $\forall Y, (\emptyset \subseteq Y \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in Y)))$
[Rule $\forall$-out, from (1), specializing X to $\emptyset$]
(5.1) $\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A))$
[Rule $\forall$-out, from (4), specializing Y to A]
(6,1) $\forall x, (x \in \emptyset \longrightarrow x \in A) \longrightarrow \emptyset \subseteq A$ [Rule $\longleftrightarrow$-out, from (10)]

Comment: we now set out to prove that "$\forall x, (x \in \emptyset \longrightarrow x \in A)$."

(7.2) Let a be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we now set out to prove that "$a \in \emptyset \longrightarrow a \in A$."

(8.3) Assume $a \in \emptyset$. [Preparation to apply Rule $\longrightarrow$-out]

Comment: we now set out to prove that "$a \in A$." We do it by contradiction. That is, we assume $\neg(a \in A)$ and get something impossible.

(9.4) Assume $\neg(a \in A)$.[Preparation to apply Rule $\neg$-out]
(10.4) $\neg a \in \emptyset$. [Rule $\forall$-out, plugging a for x in (2)]
(11.4) $a \in \emptyset \wedge (\neg a \in \emptyset)$. [Rule $\wedge$-in, from (8) and (10)]
(12.3) $\neg(\neg(a \in A))$. [Rule $\neg$-in, from (9) and (11), since $a \in \emptyset \wedge (\neg a \in \emptyset)$ is a contradiction]
(13.3) $(\neg(\neg(a \in A))) \longrightarrow a \in A$. [Rule T]

(14.3) $a \in A$. [Rule $\longrightarrow$-out, from (12) and (13)]
(15.2) $a \in \emptyset \longrightarrow a \in A$. [Rule $\longrightarrow$-in, from (8) and (14)]
(16.1) $\forall x, (x \in \emptyset \longrightarrow x \in A)$. [Rule $\forall$-in, from (7) and (15)]
(17.1) $\emptyset \subseteq A$. [Rule $\longrightarrow$-out, from (6) and (16)]
(18.0) $\forall X, \emptyset \subseteq X$. [Rule $\forall$-in, from (3) and (17)]
END OF PROOF

PROOF:

Comment: we have to prove that
"$\forall x, \forall A((A \textit{ is a set}) \longrightarrow (\{x\} \subseteq A \longleftrightarrow x \in A))$."

(1.0) $\forall X, \forall Y, ((X$ is a set $\wedge\, Y$ is a set$) \longrightarrow$
$(X \subseteq Y \longleftrightarrow (\forall x, (x \in X \longrightarrow x \in Y))))$ [Definition of set inclusion]
(2.0) $\forall x, (\{x\}$ is a set $\wedge\, \forall y, (y \in \{x\} \longleftrightarrow y = x \neg x \in \emptyset)$. [Definition of $\emptyset$]
(3.0) $\emptyset$ is a set. [Rule $\wedge$-out, from (2)]
(4.0) $\forall x, (\neg x \in \emptyset)$. [Rule $\wedge$-out, from (2)]
(5.1) Let A be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we want to prove that "A is a set $\longrightarrow \emptyset \subseteq A$."

(6.2) Assume A is a set. [Preparation to apply Rule $\longrightarrow$-in]

Comment: we want to prove that "$\emptyset \subseteq A$."

(7.2) $\forall Y, ((\emptyset$ is a set $\wedge\, Y$ is a set$) \longrightarrow (\emptyset \subseteq Y \longleftrightarrow$
$(\forall x, (x \in \emptyset \longrightarrow x \in Y))))$ [Rule $\forall$-out, from (1), specializing X to $\emptyset$]
(8.3) $(\emptyset$ is a set $\wedge\, A$ is a set$) \longrightarrow (\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A)))$
[Rule $\forall$-out, from (7), specializing Y to A]
(9.2) $\emptyset$ is a set $\wedge\, A$ is a set [Rule $\wedge$-in, from (3) and (6)]
(10.2) $\emptyset \subseteq A \longleftrightarrow (\forall x, (x \in \emptyset \longrightarrow x \in A))$ [Rule $\longrightarrow$-out, from (8)&(9)]
(11.2) $\forall x, (x \in \emptyset \longrightarrow x \in A) \longrightarrow \emptyset \subseteq A$ [Rule $\longleftrightarrow$-out, from (10)]

Comment: we now set out to prove that "$\forall x, (x \in \emptyset \longrightarrow x \in A)$."

(12.3) Let a be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: we now set out to prove that "$a \in \emptyset \longrightarrow a \in A$."

(13.4) Assume $a \in \emptyset$. [Preparation to apply Rule $\longrightarrow$-out]

Comment: we now set out to prove that "$a \in A$." We do it by contradiction. That is, we assume $\neg(a \in A)$ and get something impossible.

(14.5) Assume $\neg(a \in A)$.[Preparation to apply Rule $\neg$-out]
(15.5) $\neg a \in \emptyset$. [Rule $\forall$-out, plugging a for x in (4)]
(16.5) $a \in \emptyset \wedge (\neg a \in \emptyset)$. [Rule $\wedge$-in, from (13) and (15)]
(17.4) $\neg(\neg(a \in A))$. [Rule $\neg$-in, from (14) and (16), since $a \in \emptyset \wedge (\neg a \in \emptyset)$ is a contradiction]
(18.4) $(\neg(\neg(a \in A))) \longrightarrow a \in A$. [Rule T]
(19.4) $a \in A$. [Rule $\longrightarrow$-out, from (17) and (18)]
(20.3) $a \in \emptyset \longrightarrow a \in A$. [Rule $\longrightarrow$-in, from (13) and (19)]
(21.2) $\forall x, (x \in \emptyset \longrightarrow x \in A)$. [Rule $\forall$-in, from (12) and (21)]

(22.2) $\emptyset \subseteq A$. [Rule $\longrightarrow$-out, from (11) and (21)]
(23.1) A is a set $\longrightarrow \emptyset \subseteq A$. [Rule $\longrightarrow$-in, from (6) and (22)]
(24.0) $\forall X, (X$ is a set $\longrightarrow \emptyset \subseteq X)$. [Rule $\forall$-in, from (3) and (23)]
END OF PROOF

ANOTHER EXAMPLE: Given that "not all borogoves are mimsy," prove that "there exist borogoves that are not mimsy." (That is, given the hypothesis that "$\neg(\forall x, (x$ is a borogove $\longrightarrow x$ is mimsy$))$," we want to conclude that "$\exists x, (x$ is a borogove $\wedge\, x$ is not mimsy$)$.")

PROOF:

Comment: We will prove "$\exists x, (x$ is a borogove $\wedge\, x$ is not mimsy$)$" by contradiction. That is, we will assume that "$\neg\exists x, (x$ is a borogove $\wedge\, x$ is not mimsy$)$" and will get to something that we know is false. The most obvious false thing we could try to get is "$\forall x, (x$ is a borogove $\longrightarrow x$ is mimsy$)$." So the basic structure of our proof will be: assume "$\neg\exists x, (x$ is a borogove $\wedge\, x$ is not mimsy$)$" and prove "$\forall x, (x$ is a borogove $\longrightarrow x$ is mimsy$)$."

(1) $\neg(\forall x, (x$ is a borogove$\longrightarrow x$ is mimsy$))$. [Assumption]
(1a) Assume $\neg\exists x, (x$ is a borogove $\wedge$ $(\neg\, x$ is mimsy$))$. [Preparation to apply Rule $\neg$-out]

Comment: We set out to prove "$\forall x, (x$ is a borogove $\longrightarrow x$ is mimsy$)$."

(2) Let a be arbitrary. [Preparation to apply Rule $\forall$-in]

Comment: We try to prove "a is a borogove $\longrightarrow a$ is mimsy."

(3) Assume a is a borogove. [Preparation to apply Rule $\longrightarrow$-in]

Comment: We try to prove "a is mimsy." Since we cannot think of any other way to do it, we try a proof by contradiction: assume a is not mimsy and see what happens.

(4) Assume $\neg\, a$ is mimsy. [Preparation to apply Rule $\neg$-in]
(5) a is a borogove $\wedge\neg\, a$ is mimsy. [Rule $\wedge$-in, from (3)&(4)]
(6) $\exists x, (x$ is a borogove $\wedge\neg\, x$ is mimsy$)$. [Rule $\exists$-in, from (5)]
(7) $\neg\exists x, (x$ is a borogove $\wedge$ $(\neg\, x$ is mimsy$))$ $\wedge$ $\exists x, (x$ is a borogove $\wedge\neg\, x$ is mimsy$)$. [Rule $\wedge$-in, from (1a) and (6)]
(8) $\neg(\neg\, a$ is mimsy$)$. [Rule $\neg$-in, from (4) and (7), since the statement of (7) is a contradiction]
(9) $(\neg(\neg\, a$ is mimsy$))$ $\longrightarrow a$ is mimsy. [Rule T]
(10) a is mimsy. [Rule $\longrightarrow$-out, from (8) and (9)]
(11) a is a borogove$\longrightarrow a$ is mimsy. [Rule $\longrightarrow$-in, from (3) and (10)]
(12) $\forall x, (x$ is a borogove$\longrightarrow x$ is mimsy$)$. [Rule $\forall$-in, from (2) and (11)]

(13) $\neg(\neg, (\exists x, (x$ is a borogove $\wedge$ $(\neg\, x$ is mimsy$))))$.

[Rule $\neg$-out, from (1), (1a) and (12)

(14) $\neg(\neg, (\exists x, (x$ is a borogove $\wedge$ $(\neg\, x$ is mimsy$)))) \longrightarrow$.

$\exists x, (x$ is a borogove $\wedge$ $(\neg\, x$ is mimsy$))$. [Rule T]

(15) $\exists x, (x$ is a borogove $\wedge$ $(\neg\, x$ is mimsy$))$. [Rule $\longrightarrow$-out, from (13) and (14)]

END OF PROOF.