THE HOMOLOGY VERSION OF THE CAUCHY INTEGRAL THEOREM

Mathematics 503 classroom notes—Fall 2005

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GRIDS. A ***grid*** is a finite set G of lines in the plane such that (i) every line $L \in G$ is either horizontal or vertical, (ii) G contains at least two vertical lines and two horizontal lines.

A grid is specified by giving an ordered pair $(\mathbf{a}, \mathbf{b})$ of finite sequences $\mathbf{a} = (a_0, \ldots, a_N)$, $\mathbf{b} = (b_0, \ldots, b_M)$, of real numbers, such that N, M are positive integers, $a_0 < a_1 < \ldots < a_{N-1} < a_N$, and $b_0 < b_1 < \ldots < b_{M-1} < b_M$. Given such a pair $(\mathbf{a}, \mathbf{b})$, the grid $G(\mathbf{a}, \mathbf{b})$ (or $G(a_0, \ldots, a_N; b_0, \ldots, b_M)$) consists of the vertical lines $\{(x, y) : x = a_j\}$, for $j = 0, \ldots, N$, and the horizontal lines $\{(x, y) : y = b_k\}$, for $k = 0, \ldots, M$. Hence ***the grid*** $G(a_0, \ldots, a_N; b_0, \ldots, b_M)$ ***has*** $N + M + 2$ ***lines***.

From now on we assume, until further notice, that a grid

$$G = G(\mathbf{a}, \mathbf{b}) = G(a_0, \ldots, a_N; b_0, \ldots, b_M)\,.$$

has been specified.

A ***point*** of G is a point of the form (a_j, b_k), for $j = 0, \ldots, N$, $k = 0, \ldots, M$. We write $Point(G)$ to denote the set of points of G. Hence ***the set*** $Point(G)$ ***has*** $(N + 1)(M + 1)$ ***(i.e.,*** $NM + N + M + 1$***) members***.

A ***segment*** of G is a straight-line segment from a point (a_j, b_k) of the grid to a point (a_{j+1}, b_k) or (a_j, b_{k+1}). (That is, a segment goes horizontally from left to right or vertically up.) We write $Segm(G)$ to denote the set of all segments of the grid G. Clearly, ***the set*** $Segm(G)$ ***has*** $N(M + 1) + M(N + 1)$ ***(i.e.,*** $2NM + N + M$***) members*** (because there are $N(M + 1)$ horizontal segments and $M(N + 1)$ vertical ones).

A ***rectangle*** of G is a set of the form $[a_{j-1}, a_j] \times [b_{k-1}, b_k]$, for $j \in \{1, \ldots, N\}$, $k \in \{1, \ldots, M\}$. We write $Rect(G)$ to denote the set of all rectangles of G. Clearly, ***the set*** $Rect(G)$ ***has*** NM ***members***.

We now define the ***chains*** associated to the grid G. There will be -1-***chains***, 0-***chains***, 1-***chains***, and 2-***chains***.

Remark. In general, one wants to consider ***chains with coefficients*** in some ring $\mathcal{R}$ such as $\mathbb{R}$ or $\mathbb{Z}$. For us, the only important ring of coefficients is $\mathbb{R}$, so we will only consider chains with coefficients in $\mathbb{R}$. $\diamondsuit$

- A (-1)-***chain*** of G is a member of $\mathbb{R}$.
- A 0-***chain*** of G is a formal linear combination

$$C = \sum_{\ell=1}^{L} c_\ell P_\ell \,,$$

  where the coefficients c_ℓ belong to $\mathbb{R}$ and the P_ℓ are points of G.

- A 1-***chain*** of G is a formal linear combination

$$C = \sum_{\ell=1}^{L} c_\ell S_\ell \,,$$

  where the coefficients c_ℓ belong to $\mathbb{R}$ and the S_ℓ are segments of G.

- A 2-***chain*** of G is a formal linear combination

$$C = \sum_{\ell=1}^{L} c_\ell R_\ell \,,$$

  where the coefficients c_ℓ belong to $\mathbb{R}$ and the R_ℓ are rectangles of G.

For $m = -1, 0, 1, 2$, we use $\mathcal{C}_m(G, \mathbb{R})$ to denote the space of all m-chains of G. Then

$\mathcal{C}_{-1}(G, \mathbb{R})$	are linear	1	
$\mathcal{C}_0(G, \mathbb{R})$	spaces	$NM + N + M + 1$	respectively.
$\mathcal{C}_1(G, \mathbb{R})$	over $\mathbb{R}$ of	$2NM + N + M$	
$\mathcal{C}_2(G, \mathbb{R})$	dimension	NM	

The boundary operator:

- If $C \in \mathcal{C}_{-1}(G, \mathbb{R})$, the ***boundary*** of C is zero. (You can think of 0 as a "-2-chain," if you wish.
- If $C = \sum_{\ell=1}^{L} c_\ell P_\ell \in \mathcal{C}_0(G, \mathbb{R})$, the ***boundary*** of C is the -1-chain ∂C given by

$$\partial C = \sum_{\ell=1}^{L} c_\ell \,.$$

- If S is a segment of G, going from A to B, the ***boundary*** of S is the 0-chain ∂S given by $\partial S = B - A$.

- If $C = \sum_{\ell=1}^{L} c_\ell S_\ell \in \mathcal{C}_1(G,\mathbb{R})$, the ***boundary*** of C is the 0-chain ∂C given by
$$\partial C = \sum_{\ell=1}^{L} c_\ell \partial S_\ell \,.$$
- If $R = [a_{j-1}, a_j] \times [b_{k-1}, b_k]$ is a rectangle of G, we use $S_1(R)$, $S_2(R)$, $S_3(R)$, $S_4(R)$, to denote, respectively, (ii) the segment from (a_{j-1}, b_{k-1}) to (a_j, b_{k-1}), (ii) the segment from (a_j, b_{k-1}) to (a_j, b_k), (iii) the segment from (a_{j-1}, b_k) to (a_j, b_k), and (iv) the segment from (a_{j-1}, b_{k-1}) to (a_{j-1}, b_k). The ***boundary*** of R is the 1-chain ∂R given by
$$\partial R = S_1(R) + S_2(R) - S_3(R) - S_4(R) \,.$$
- If $C = \sum_{\ell=1}^{L} c_\ell R_\ell \in \mathcal{C}_2(G,\mathbb{R})$, the ***boundary*** of C is the 1-chain ∂C given by
$$\partial C = \sum_{\ell=1}^{L} c_\ell \partial R_\ell \,.$$

Clearly, the four boundary maps defined above are linear, and we have the following diagram of linear spaces and linear maps:
$$\mathcal{C}_2(G,\mathbb{R}) \stackrel{\partial}{\longrightarrow} C_1(G,\mathbb{R}) \stackrel{\partial}{\longrightarrow} \mathcal{C}_0(G,\mathbb{R}) \stackrel{\partial}{\longrightarrow} \mathcal{C}_{-1}(G,\mathbb{R}) \stackrel{\partial}{\longrightarrow} \{0\} \,. \tag{1}$$

The boundary of a boundary ia always zero:

THEOREM. *Let C be an m-chain of G, where $m = 0$ or $m = 1$ or $m = 2$. Then $\partial\partial C = 0$.*

Proof: If $m = 0$, the conclusion is trivial.

If $m = 1$, it suffices to prove that $\partial\partial S = 0$ if $S \in \mathit{Segm}(G)$. But, if S is the segment from A to B, then $\partial S = B - A = 1 \cdot B + (-1) \cdot A$, so $\partial\partial S = 1 + (-1) = 0$.

If $m = 2$, it suffices to prove that $\partial\partial R = 0$ if $R \in \mathit{Rect}(G)$. Let $\partial S_1(R) = P_2 - P_1$, $\partial S_2(R) = P_4 - P_3$, $\partial S_3(R) = P_6 - P_5$, $\partial S_4(R) = P_8 - P_7$. Then $P_3 = P_2$, $P_6 = P_4$, $P_5 = P_8$, and $P_1 = P_7$. So
$$\begin{aligned}
\partial R &= \partial S_1(R) + \partial S_2(R) - \partial S_3(R) - \partial S_4(R) \\
&= P_2 - P_1 + P_4 - P_3 - (P_6 - P_5) - (P_8 - P_7) \\
&= P_2 - P_1 + P_4 - P_2 - (P_4 - P_8) - (P_8 - P_1) \\
&= P_2 - P_1 + P_4 - P_2 - P_4 + P_8 - P_8 + P_1 \\
&= 0 \,.
\end{aligned}$$

This completes the proof of the theorem. $\diamondsuit$

So the diagram of (1) has the additional property that

$$\boxed{\boxed{\boxed{\partial\partial = 0}}}.$$

A diagram with this property is called a ***chain complex***.

Cycles: If $m = -1, 0, 1, 2$, then an m-chain C is ***closed*** if $\partial C = 0$. A closed m-chain is called an m-***cycle***. We use $\mathcal{Z}_m(G, \mathbb{R})$ to denote the space of all m-cycles of G.

Boundaries: If $m = -1, 0, 1$, an m-***boundary*** is an m-chain which is the boundary of some $m+1$-chain. We use $\mathcal{B}_m(G, \mathbb{R})$ to denote the space of all m-boundaries of G.

Boundaries are cycles:

TRIVIAL THEOREM. *Every boundary is a cycle. That is,*
$\boxed{\boxed{\mathcal{B}_m(G, \mathbb{R}) \subseteq \mathcal{Z}_m(G, \mathbb{R})}}$ if $m = -1, 0, 1$.

Proof: If C is an m-boundary, then $C = \partial D$ for some $m+1$-chain D. Hence $\partial C = \partial\partial D = 0$. So C is a cycle. $\diamondsuit$

Cycles are boundaries:

NONTRIVIAL THEOREM. *Every cycle is a boundary. That is, (since we already know that boundaries are cycles),*

$$\boxed{\boxed{\mathcal{B}_m(G, \mathbb{R}) = \mathcal{Z}_m(G, \mathbb{R})}} \text{ if } m = -1, 0, 1.$$

Proof: Let us first consider the (very easy) case when $m = 0$. (The case when $m = -1$ is so trivial that I will not even bother to discuss it.) We know that $\mathcal{B}_0(G, \mathbb{R}) \subseteq \mathcal{Z}_0(G, \mathbb{R})$, and we have to prove that $\mathcal{Z}_0(G, \mathbb{R}) \subseteq \mathcal{B}_0(G, \mathbb{R})$. Let $C = \sum_{\ell=1}^{L} c_\ell P_\ell$ be a 0-cycle. Pick any point $Q \in Point(G)$. Then $C = \sum_{\ell=1}^{L} c_\ell (P_\ell - Q)$, because $\sum_{\ell=1}^{L} c_\ell = \partial C = 0$, since C is closed. Therefore, to prove that C is a boundary it suffices to show that each cycle $P_\ell - Q$ is a boundary. In other words, it suffices to show that if $P, Q \in Point(G)$ then $Q - P$ is a boundary. So let P, Q be points of G. Assume first that P and Q lie on the same horizonal line, and Q is to the right of P. Then there are segments $S_1, S_2, \ldots, S_n$ of G and points $P_1, P_2, \ldots, P_n$, $Q_1, Q_2, \ldots, Q_n$, such that $\partial S_j = Q_j - P_j$ for $j = 1, \ldots, n$, $Q_j = P_{j+1}$ for $j = 1, \ldots, n-1$, $P_1 = P$, and $Q_n = Q$. Let $D = S_1 + S_2 + \cdots +, S_n$. Then $\partial D = Q_1 - P_1 + Q_2 - P_2 + Q_3 - P_3 + \cdots + Q_n - P_n = Q - P$, so $Q - P$ is a boundary. If P and Q lie on the same horizonal line, and Q is to the left of P, then P is to the right of Q, so $P - Q$ is a boundary, and then $Q - P$ is a boundary as well. Hence we have shown that $Q - P$ is a boundary whenever P and Q lie on the same horizonal line. A similar argument shows that $Q - P$ is a boundary whenever P and Q lie on the same vertical line. Finally, if P, Q are any two points of G, there is a point A of G that lies on the same horizontal line as P and on the same vertical line as Q. Hence $A - P$ is a boundary, and $Q - A$ is a boundary. Since $Q - P = (Q - A) + (A - P)$ $Q - A$ is a boundary, and we are done.

Now consider the case when $m = 1$. Let us compute some dimensions. The linear space $\mathcal{Z}_0(G, \mathbb{R})$ has dimension $NM + N + M$, because $\mathcal{C}_0(G, \mathbb{R})$ has dimension $NM + N + M + 1$ and $\mathcal{Z}_0(G, \mathbb{R})$ is the kernel of a nontrivial linear map $\partial : \mathcal{C}_0(G, \mathbb{R}) \mapsto \mathbb{R}$. (This linear map is nontrivial because we can pick any point P of G and regard P as a 0-chain. Then $\partial P = 1$, so ∂ does not vanish identically on $\mathcal{C}_0(G, \mathbb{R})$.) We have already proved that $\mathcal{B}_0(G, \mathbb{R}) = \mathcal{Z}_0(G, \mathbb{R})$, so $\mathcal{B}_0(G, \mathbb{R})$ has dimension $NM+N+M$ as well. Now, $\mathcal{B}_0(G, \mathbb{R})$ is the image of the space $C_1(G, \mathbb{R})$ under the linear map $\partial : \mathcal{C}_1(G, \mathbb{R}) \mapsto \mathcal{C}_0(G, \mathbb{R})$. And $\mathcal{Z}_1(G, \mathbb{R})$ is the kernel of this map. Therefore

$$\dim \mathcal{Z}_1(G, \mathbb{R}) = \dim \mathcal{C}_1(G, \mathbb{R}) - \dim \mathcal{B}_0(G, \mathbb{R}) \,,$$

because, if V, W are finite-dimensional linear spaces, and $\mu : V \mapsto W$ is a linear map, then $\dim \ker \mu + \dim \operatorname{im} \mu = \dim V$.

We showed a few pages ago that $\dim \mathcal{C}_1(G, \mathbb{R}) = 2NM + N + M$, and we have just established that $\dim \mathcal{B}_0(G, \mathbb{R}) = NM + N + M$. Hence

$$\boxed{\boxed{\boxed{\dim \mathcal{Z}_1(G, \mathbb{R}) = NM \,.}}}$$

Since $\mathcal{B}_1(G, \mathbb{R}) \subseteq Z_1(G, \mathbb{R})$, to prove that $\mathcal{B}_1(G, \mathbb{R}) = \mathcal{Z}_1(G, \mathbb{R})$ it suffices to show that $\dim \mathcal{B}_1(G, \mathbb{R}) = NM$. We know that $Rect(G)$ has exactly NM members, and it is clear that $\mathcal{B}_1(G, \mathbb{R})$ is the linear span of the boundaries ∂R of these NM rectangles. So the conclusion that $\dim \mathcal{B}_1(G, \mathbb{R}) = NM$ will follow if we prove that the boundaries $\partial R_1, \partial R_2, \ldots, \partial R_{NM}$ of the NM rectangles $R_1, R_2, \ldots, R_{NM}$ of G are linearly independent. Suppose that

$$\sum_{s=1}^{NM} c_s \partial R_s = 0 \,.$$

We have to show that all the c_s vanish. Pick any $s_* \in \{1, \ldots, NM\}$. Then pick a point z_* in the interior of R_{s_*}. Let $C = \sum_{s=1}^{NM} c_s \partial R_s = 0$. Then the integral

$$\int_C \frac{dz}{z - z_*}$$

is equal to zero, because $C = 0$. On the other hand,

$$\int_C \frac{dz}{z - z_*} = \sum_{s=1}^{NM} c_s \int_{\partial R_s} \frac{dz}{z - z_*} = 2\pi\, i \sum_{s=1}^{NM} c_s W(\partial R_s, z_*) = 2\pi\, i\, c_{s_*}\,.$$

So $c_{s_*} = 0$. Since s_* is an arbitrary member of the index set $\{1, \ldots, NM\}$, we have establised that all the c_s vanish, so $R_1, R_2, \ldots, R_{NM}$ of G are linearly independent, as desired, and our proof is complete. ♢

REMARK. *If you think that it is inelegant to bring in complex integrals and winding numbers in the middle of what ought to be a purely combinatorial argument, then I fully agree with you. (Notice that Serge Lang also does it, and in an even uglier way.) It is not too hard to give a purely combinatorial proof that every* 1*-cycle is a boundary. Try to find one.*

We now forget about the fixed grid G, and try to do some of the things that we did for a grid in a grid-independent way, so that instead of "points of the grid" we will deal with arbitrary points, and instead of "segments of the grid" we will deal with arbitrary arcs.

SINGULAR CHAINS. Let Ω be open in $\mathbb{C}$. An ***arc*** in Ω is a continuous map $\gamma : [a, b] \mapsto \Omega$, defined on some compact interval $[a, b]$. The ***boundary*** of an arc γ is the formal difference $\partial\gamma = \gamma(b) - \gamma(a)$. (If this "formal difference" makes you uncomfortable, try the following completely rigorous definition: a 0-***chain*** in $\mathbb{C}$ is a function $\varphi : \mathbb{C} \mapsto \mathbb{R}$ such that $\varphi(z) = 0$ for all except a finite set of values of z. Clearly, the 0-chains form a linear space over $\mathbb{R}$. For any $z \in \mathbb{C}$, let $\alpha_z : \mathbb{C} \mapsto \mathbb{R}$ be given by $\alpha_z(w) = 0$ if $w \neq z$, $\alpha_z(z) = 1$. Then every 0-chain is a sum $\sum_{\ell=1}^{L} c_\ell \alpha_{z_\ell}$, where the c_ℓ are real numbers and the z_ℓ belong to $\mathbb{C}$. Now relabel α_z as z, so our 0-chains are sums $\sum_{\ell=1}^{L} c_\ell z_\ell$. This is what we mean by "formal linear combinations of points of $\mathbb{C}$." In particular, if z_1, z_2 are two points of $\mathbb{C}$—which need not be different—then the "formal difference" $z_2 - z_1$ is, really, the function $\alpha_{z_2} - \alpha_{z_1}$. Notice that in the special case when $z_1 = z_2$ the "formal difference" $z_2 - z_1$ is just the zero function, as it should be.)

An arc $\gamma : [a, b] \mapsto \mathbb{C}$ is a ***segment*** if

$$\gamma(t) = \frac{b - t}{b - a}\gamma(a) + \frac{t - a}{b - a}\gamma(b) \qquad \text{for} \quad a \leq t \leq b\,.$$

A segment $\gamma : [a, b] \mapsto \mathbb{C}$ is ***horizontal*** if $\gamma(a)$ and $\gamma(b)$ have the same y-coordinate. A segment $\gamma : [a, b] \mapsto \mathbb{C}$ is ***vertical*** if $\gamma(a)$ and $\gamma(b)$ have the same x-coordinate.

An ***HV arc*** is an arc $\gamma : [a, b] \mapsto \mathbb{C}$ which is a finite concatenation of horizontal and vertical segments. (That is, there exist $t_0, \ldots, t_n$ such that $a = t_0 < t_1 < \cdots < t_n = b$ having the property that for $j = 1, \ldots, n$ the arc $\gamma_j : [t_{j-1}, t_j] \mapsto \mathbb{C}$ obtained by restricting γ to the interval $[t_{j-1}, t_j]$ is a horizontal or a vertical segment.)

THEOREM H. *Every arc $\gamma : [a,b] \mapsto \Omega$ is homotopic in Ω, with fixed endpoints, to an HV arc.*

Proof. Pick $\delta \in \mathbb{R}$ such that $\delta > 0$ and $z \in \Omega$ whenever $|z - \gamma(t)| \le \delta$ for some $t \in [a,b]$. (The existence of δ follows from the fact that the set $\{\gamma(t) : a \le t \le b\}$ is compact.) Using the fact that γ is uniformly continuous, pick a positive number α such that $|\gamma(t) - \gamma(s)| \le \frac{\delta}{6}$ whenever $t, s \in [a,b]$ and $|t - s| \le \alpha$. Then pick a positive integer N such that $\frac{b-a}{N} < \alpha$. Let $\tau_k = a + \frac{k}{N}(b-a)$ for $k = 0, 1, \ldots, N$. For $k = 1, \ldots, N$, let σ_k be the midpoint of the interval $[\tau_{k-1}, \tau_k]$. Let z_k be the point of $\mathbb{C}$ that has the same y-coordinate as $\gamma(\tau_{k-1})$ and the same x-coordinate as $\gamma(\tau_k)$. Define a new path $\tilde{\gamma} : [a,b] \mapsto \mathbb{C}$, by letting

$$\tilde{\gamma}(t) = \begin{cases} \frac{\sigma_k - t}{\sigma_k - \tau_{k-1}}\gamma(\tau_{k-1}) + \frac{t-\tau_{k-1}}{\sigma_k - \tau_{k-1}} z_k & \text{for} \quad \tau_{k-1} \le t \le \sigma_k \\ \frac{\tau_k - t}{\tau_k - \sigma_k} z_k + \frac{t - \sigma_k}{\tau_k - \sigma_k}\gamma(\tau_k) & \text{for} \quad \sigma_k \le t \le \tau_k \end{cases}$$

for $k = 1, \ldots, n$. Then $\tilde{\gamma}$ is an HV arc in $\mathbb{C}$ having the same endpoints as γ.

We now show that $\tilde{\gamma}$ and γ are homotopic in Ω with fixed endpoints. (This implies, in particular, that $\tilde{\gamma}$ is in fact an arc in Ω.) For this purpose, we construct a homotopy $H : [a,b] \times [0,1] \mapsto \mathbb{C}$ between $\tilde{\gamma}$ and γ, and then show that H actually takes values in Ω. We define

$$H(t,s) = s\tilde{\gamma}(t) + (1-s)\gamma(t) \qquad \text{for} \qquad a \le t \le b\,,\ 0 \le s \le 1\,.$$

Then $H(t,0) = \gamma(t)$, $H(t,1) = \tilde{\gamma}(t)$, $H(a,s) = \gamma(a)$, and $H(b,s) = \gamma(b)$, so H is a homotopy from γ to $\tilde{\gamma}$ with fixed endpoints.

To conclude our proof, we have to show that $H(t,s) \in \Omega$ for all $(t,s) \in [a,b] \times [0,1]$. Pick t, s. Pick k such that $t \in [\tau_{k-1}, \tau_k]$. Assume first that $t \in [\tau_{k-1}, \sigma_k]$. Then $|\gamma(\tau_k) - \gamma(\tau_{k-1})| \le \frac{\delta}{6}$, and this easily implies $|z_k - \gamma(\tau_{k-1})| \le \frac{\delta}{6}$. Since $\tilde{\gamma}(t)$ is a convex combination of $\gamma(\tau_{k-1})$ and z_k, we have $|\tilde{\gamma}(t) - \gamma(\tau_{k-1})| \le \frac{\delta}{6}$. On the other hand, $|\gamma(t) - \gamma(\tau_{k-1})| \le \frac{\delta}{6}$, because $|t - \tau_{k-1})| \le \alpha$. Hence $|\tilde{\gamma}(t) - \gamma(t)| \le \delta$. Since $H(t,s)$ is a convex combination of $\tilde{\gamma}(t)$ and $\gamma(t)$, we have $|H(t,s) - \gamma(t)| \le \delta$. Hence $H(t,s) \in \Omega$, and our proof is complete. ♢

A ***singular* 1*-chain*** in Ω is a formal linear combination $\sum_{\ell=1}^{L} c_\ell \gamma_\ell$ of arcs in Ω. We already know what the "boundary" of an arc is. The ***boundary*** of a singular 1-chain $\sum_{\ell=1}^{L} c_\ell \gamma_\ell$ in Ω is the 0-chain $\sum_{\ell=1}^{L} c_\ell \partial\gamma_\ell$. A singular 1-chain is ***closed*** if its boundary vanishes. A closed singular 1-chain is a ***singular* 1*-cycle***.

An important class of examples of 1-cycles is provided by the formal linear combinations $\sum_{\ell=1}^{L} c_\ell \Lambda_\ell$ of loops. Recall that a ***loop*** in Ω is an arc $\Lambda : [a,b] \mapsto \Omega$ such that $\Lambda(b) = \Lambda(a)$. Hence ***if*** Λ ***is a loop in*** Ω ***then*** $\partial\Lambda = 0$. It follows that ***every formal linear combination*** $\sum_{\ell=1}^{L} c_\ell \Lambda_\ell$ ***of loops in*** Ω ***is a singular* 1*-cycle in*** Ω.

A ***path*** in Ω is a finite sequence $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of arcs $\gamma_j : [a_j, b_j] \mapsto \Omega$ such that $\gamma_j(b_j) = \gamma_{j+1}(a_{j+1})$ for $j = 1, \ldots, n-1$. To such a path we can associate the singular 1-chain $\kappa_\Gamma = \gamma_1 + \ldots + \gamma_n$. Then $\partial\kappa_\Gamma = \gamma_n(b_n) - \gamma_1(a_1)$. Not surprisingly, we say that the path Γ is ***closed*** if $\gamma_n(b_n) = \gamma_1(a_1)$, i.e., if $\partial\kappa_\Gamma = 0$. In other words, a path Γ is closed if and only if its corresponding singular 1-chain is closed.

If $f \in \mathrm{Hol}(\Omega)$, and $\gamma : [a, b] \mapsto \Omega$ is an arc in Ω, we define $\int_\gamma f$ (also written as $\int_\gamma f(z)\,dz$) in the usual way. (That is, we let $h : [a, b] \mapsto \mathbb{C}$ be a primitive of f along γ, and define $\int_\gamma f = h(b) - h(a)$.) If $C = \sum_{\ell=1}^{L} c_\ell \gamma_\ell$ is a singular 1-chain, we define

$$\int_\gamma f \stackrel{\text{def}}{=} \sum_{\ell=1}^{L} c_\ell \int_{\gamma_\ell} f \,.$$

In particular, if $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is a path in Ω, then $\int_\Gamma f$ is, by definition, the sum $\int_{\gamma_1} f + \int_{\gamma_2} f + \cdots + \int_{\gamma_n} f$, which is exactly the integral $\int_{\kappa_\Gamma} f$ of f along the singular 1-chain κ_Γ.

HOMOLOGY. We now discuss the crucial question of these notes, namely, to what extent two cycles Λ_1, Λ_2, in an open set Ω can be distinguished by computing complex integrals in Ω.

Suppose that C, D are two singular 1-cycles in Ω (or, more generally, two singular 1-chains in Ω). We say that C and D are *strongly homologous in* Ω if $\int_C g = \int_D g$ for every holomorphic function g on Ω. Then it is clear that C and D are strongly homologous in Ω if and only if the singular 1-chain $C - D$ is homologous to zero (i.e. to the zero 1-chain) in Ω.

A useless observation that is nevertheless worth knowing: *If a* 1*-chain* C *in* Ω *is strongly homologous to zero in* Ω*, then* C *is closed.*

Proof: Let $C = \sum_{\ell=1}^{n} c_\ell \gamma_\ell$ be a singular 1-chain in Ω. Let g be a holomorphic function on Ω. Then, for each ℓ, if $\gamma_\ell : [a, b] \mapsto \Omega$, and we let $Q_\ell = \gamma_\ell(b_\ell)$, $P_\ell = \gamma_\ell(a_\ell)$, we have

$$\int_{\gamma_\ell} g' = g(Q_\ell) - g(P_\ell) \,.$$

Hence

$$0 = \int_C g' = \sum_{\ell=1}^{n} c_\ell \Big(g(Q_\ell) - g(P_\ell) \Big) \,.$$

Let S be the set whose members are the Q_ℓ and the P_ℓ. (This set could have fewer than $2n$ members, because a point A may be equal to Q_ℓ for more than one ℓ or to P_ℓ for more than one ℓ, and in addition some Q_ℓ may also be equal to some $P_{\ell'}$.) Let $A_1, A_2, \ldots, A_m$ be the distinct points of S. Then

$$\partial C \quad = \quad \sum_{\ell=1}^{n} c_\ell (Q_\ell - P_\ell)$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{n} c_\ell Q_\ell - \sum_{\ell=1}^{n} c_\ell P_\ell \\
&= \sum_{j=1}^{m} \left(\sum_{1\le \ell\le n, Q_\ell = A_j} c_\ell Q_\ell - \sum_{1\le \ell\le n, P_\ell = A_j} c_\ell P_\ell \right) \\
&= \sum_{j=1}^{m} d_j A_j \,,
\end{aligned}
$$

where

$$d_j = \sum_{1\le \ell\le n, Q_\ell = A_j} c_\ell - \sum_{1\le \ell\le n, P_\ell = A_j} c_\ell \,.$$

Suppose there exists a j such that $d_j \neq 0$. Pick one and call it j_*. Let g be a polynomial such that $g(A_{j_*}) = 1$ but $g(A_j) = 0$ for $j \neq j_*$. Then

$$
\begin{aligned}
0 &= \sum_{\ell=1}^{n} c_\ell \Big(g(Q_\ell) - g(P_\ell) \Big) \\
&= \Big(\sum_{1\le \ell\le n, Q_\ell = A_{j_*}} c_\ell - \sum_{1\le \ell\le n, P_\ell = A_{j_*}} c_\ell \Big) g(A_{j_*}) \\
&= \Big(\sum_{1\le \ell\le n, Q_\ell = A_{j_*}} c_\ell - \sum_{1\le \ell\le n, P_\ell = A_{j_*}} c_\ell \Big) \\
&= d_{j_*} \,.
\end{aligned}
$$

So $d_{j_*} - 0$, and we have reached a contradiction. Hence all the d_j vanish, and $\partial C = 0$. ♢

THE HOMOLOGY VERSION OF THE CAUCHY INTEGRAL THEOREM. We have called two singular 1-cycles C_1, C_2 in Ω "strongly homologous in Ω" if the integrals along them have the same value for every function $f \in \mathrm{Hol}(\Omega)$. The "homology version of the Cauchy integral theorem" tells us that, for C_1 and C_2 to be strongly homologous in Ω it suffices to have the equality $\int_{C_1} f = \int_{C_2} f$ for all f in a rather small subset $\mathcal{W}_\Omega$ of $\mathrm{Hol}(\Omega)$. The set $\mathcal{W}_\Omega$ is that of all the functions

$$\Omega \ni z \mapsto \frac{1}{z - z_*} \,,$$

for all complex numbers z_* that are not in Ω. (For example: if $\Omega = \mathbb{C}$ then $\mathcal{W}_\Omega$ is empty, so any two singular 1-cycles in Ω are strongly homologous. If Ω is $\mathbb{C}$ with the origin removed, then $\mathcal{W}_\Omega$ has just one member, namely, the function $z \mapsto \frac{1}{z}$, so two singular 1-cycles in Ω are strongly homologous if and only if their winding numbers about 0 are equal. More generally, if $\Omega = \mathbb{C} - \{z_1, z_2, \ldots, z_n\}$, where $z_1, z_2, \ldots, z_n$ are distinct complex numbers, then $\mathcal{W}_\Omega$ exactly n members, namely, the functions $z \mapsto \frac{1}{z - z_j}$, for $j = 1, \ldots, n$, so two singular 1-cycles in Ω are strongly homologous if and only if their winding numbers about each z_j are equal.)

If C_1, C_2 are singular 1-chains in Ω, we call C_1 and C_2 are ***homologous in*** Ω if

$$\int_{C_1} \frac{dz}{z - z_*} = \int_{C_2} \frac{dz}{z - z_*} \qquad \text{for every } z_* \in \mathbb{C} - \Omega\,. \tag{2}$$

THE MAIN THEOREM. *Let Ω be open in $\mathbb{C}$. Let $f : \Omega \mapsto \mathbb{C}$ be holomorphic. Let C_1, C_2, be singular 1-chains in Ω. Then the following two conditions are equivalent:*

(1) C_1 and C_2 are strongly homologous in Ω.

(2) $\partial C_1 = \partial C_2$ and C_1 and C_2 are homologous in Ω.

Proof. Suppose that C_1 and C_2 are strongly homologous in Ω. Then the chain $C_1 - C_2$ is strongly homologous to zero in Ω, and we already know that if a 1-chain is strongly homologous to zero then it is closed. Hence

$$\partial(C_1 - C_2) = 0\,,$$

so $\partial C_1 = \partial C_2$. Furthermore, $\int_{C_1} f = \int_{C_2} f$ for all holomorphic $f : \Omega \mapsto \mathbb{C}$, so in particular (2) holds. Therefore C_1 and C_2 are homologous in Ω, and we have proved the implication (1)⇒(2).

Let us prove that (2)⇒(1). Asume that (2) holds. let $\Lambda = C_1 - C_2$. We want to prove that Λ is strongly homologous to zero in Ω. For this purpose, we fix $f \in \mathrm{Hol}(\Omega)$ and prove that $\int_\Lambda f = 0$.

Clearly, $\partial\Lambda = 0$, so Λ is a singular 1-cycle. Write

$$\Lambda = \sum_{\ell=1}^{L} c_\ell \gamma_\ell\,,$$

where each γ_ℓ is an arc in Ω, so γ_ℓ is a continuous map from an interval $[a_\ell, b_\ell]$ to Ω. Then $\partial\gamma_\ell = \gamma_\ell(b_\ell) - \gamma_\ell(a_\ell)$. So

$$0 = \partial\Lambda = \sum_{\ell=1}^{L} c_\ell \Big(\gamma_\ell(b_\ell) - \gamma_\ell(a_\ell)\Big)\,.$$

By Theorem H, each γ_ℓ is homotopic in Ω with fixed endpoints to an HV arc $\tilde{\gamma}_\ell : [a_\ell, b_\ell] \mapsto \Omega$. Let $\tilde{\Lambda}$ be the singular 1-chain $\sum_{\ell=1}^{L} c_\ell \tilde{\gamma}_\ell$. Then, for each ℓ, $\partial\tilde{\gamma}_\ell = \tilde{\gamma}_\ell(b_\ell) - \tilde{\gamma}_\ell(a_\ell) = \gamma_\ell(b_\ell) - \gamma_\ell(a_\ell) = \partial\gamma_\ell$, because $\tilde{\gamma}_\ell$ has the same endpoints as γ_ℓ. Therefore

$$\partial\tilde{\Lambda} = \sum_{\ell=1}^{L} c_\ell \partial\tilde{\gamma}_\ell = \sum_{\ell=1}^{L} c_\ell \partial\gamma_\ell = \partial\Lambda = 0\,.$$

Then, if g is any holomorphic function on Ω, the integrals $\int_{\gamma_\ell} g$ and $\int_{\tilde{\gamma}_\ell} g$ are equal, because γ_ℓ and $\tilde{\gamma}_\ell$ are homotopic in Ω with fixed endpoints. It follows that

$\int_{\tilde{\Lambda}} g = \int_{\Lambda} g$. Therefore Λ and $\tilde{\Lambda}$ are strongly homologous in Ω. In particular, $\tilde{\Lambda}$ is homologous to zero in Ω, since Λ is homologous to zero in Ω. Furthermore, $\int_{\Lambda} f = \int_{\tilde{\Lambda}} f$, so it will suffice to prove that $\int_{\tilde{\Lambda}} f = 0$.

Let us use # to denote concatenation of arcs. Then each $\tilde{\gamma}_\ell$ is a concatenation $\xi_{1,\ell}\#\xi_{2,\ell}\#\cdots\xi_{\nu_\ell,\ell}$ of horizontal and vertical segments. Let

$$\hat{\Lambda} = \sum_{\ell=1}^{L}\sum_{j=1}^{\nu_\ell} c_\ell \xi_{j,\ell}\,.$$

Then $\hat{\Lambda}$ is clearly closed (because the identity

$$\partial(\xi_{1,\ell}\#\xi_{2,\ell}\#\cdots\xi_{\nu_\ell,\ell}) = \partial\xi_{1,\ell} + \partial\xi_{2,\ell} + \cdots + \partial\xi_{\nu_\ell,\ell}$$

is trivially true) and strongly homologous to $\tilde{\Lambda}$, so $\hat{\Lambda}$ is homologous to zero in Ω, because $\tilde{\Lambda}$ is homologous to zero in Ω. Furthermore, $\int_{\hat{\Lambda}} f = \int_{\tilde{\Lambda}} f$, so our conclusion will follow if we prove that $\int_{\hat{\Lambda}} f = 0$.

For each j, ℓ, let $\eta_{j,\ell}$ be $\xi_{j,\ell}$ if $\xi_{j,\ell}$ is either horizontal running from left to right or vertical going up, and let $\eta_{j,\ell}$ be $\xi_{j,\ell}$ run in reverse if $\xi_{j,\ell}$ is either horizontal running from right to left or vertical going down. Also, let $d_{j,\ell}$ be c_ℓ if $\eta_{j,\ell} = \xi_{j,\ell}$, and $-c_\ell$ otherwise. Then $d_{j,\ell}\eta_{j,\ell}$ is strongly homologous to $c_\ell\xi_{j,\ell}$, so the singular 1-chain $\check{\Lambda} = \sum_{\ell=1}^{L}\sum_{j=1}^{\nu_\ell} d_{j,\ell}\eta_{j,\ell}$ is strongly homologous to $\hat{\Lambda}$. Then $\check{\Lambda}$ is homologous to zero in Ω, because $\hat{\Lambda}$ is homologous to zero in Ω. Furthermore, $\int_{\check{\Lambda}} f = \int_{\hat{\Lambda}} f$, so our conclusion will follow if we prove that $\int_{\check{\Lambda}} f = 0$.

Now let G be the grid consisting of all the horizontal lines and all the vertical lines that contain some endpoint of some segment $\eta_{j,\ell}$. Then every horizontal segment $\eta_{j,\ell}$ goes from a point of the grid to another point of the grid further to the right, so after subdividing the horizontal $\eta_{j,\ell}$, if necessary, we may assume that every horizontal segment $\eta_{j,\ell}$ is a segment of the grid. (It is clear that subdivding a segment σ produces a sum of segments which is strongly homologous to σ.) Similarly, we may assume that every vertical $\eta_{j,\ell}$ is a segment of the grid. So $\check{\Lambda}$ is in fact a 1-cycle of the grid.

Using our general results about grids, we know that $\check{\Lambda}$ is the boundary of a 2-chain $\mathbf{R} = \sum_{s=1}^{n}\mu_s R_s$, where the R_s are distinct rectangles of G, and the μ_s are nonzero real numbers.

CLAIM: all the R_s are entirely contained in Ω.

Let prove this claim. Suppose the claim wasn't true. Pick s_* such that R_{s_*} is not entirely contained in Ω. Pick a point $z_* \in R_{s_*}$ such that $z_* \notin \Omega$. Suppose first that z_* is an interior point of R_{s_*}. Then $W(\check{\Lambda}, z_*) = 0$, but on the other hand $W(\partial R_{s_*}, z_*) = 1$ and $W(\partial R_s, z_*) = 0$ whenever $s \neq s_*$, so

$$W(\check{\Lambda}, z_*) = W(\partial\mathbf{R}, z_*) = \sum_{s=1}^{n}\mu_s W(\partial R_s, z_*) = \mu_{s_*} \neq 0\,,$$

and we have reached a contradiction.

Now let us consider the other possible case, when z_* lies on the boundary of R_{s_*}. In that case, we can pick a positive radius δ such that no point of the open disc $\{z : |z - z_*| < \delta\}$ lies in any of the segments $\eta_{j,\ell}$. In particular, we may pick an interior point $\hat{z}$ of R_{s_*} such that $|\hat{z} - z_*| < \delta$. Since the integer-valued function $z \mapsto W(\hat{\Lambda}, z)$ is continuous on the connected set $\{z : |z - z_*| < \delta\}$, it must be constant. Hence $W(\check{\Lambda}, \hat{z}) = W(\check{\Lambda}, z_*) = 0$. As before, we have $W(\partial R_{s_*}, \hat{z}) = 1$ and $W(\partial R_s, \hat{z}) = 0$ whenever $s \neq s_*$, so

$$W(\check{\Lambda}, \hat{z}_*) = W(\partial \mathbf{R}, \hat{z}) = \sum_{s=1}^{n} \mu_s W(\partial R_s, \hat{z}) = \mu_{s_*} \neq 0\,,$$

and we have reached a contradiction in this case as well. (*REMARK: the point* $\hat{z}$ ***could*** *be in* Ω. *For example,* Ω *could be the plane minus one point, and* z_* *could be that point, so any other point would have to belong to* Ω.)

We have therefore proved our claim. Now, for any s, since f is holomorphic in Ω and R_s is contained in Ω, we have $\int_{\partial R_s} f = 0$. So

$$\int_{\check{\Lambda}} f = \int_{\partial \mathbf{R}} f = \int_{\sum_{s=1}^{n} \mu_s \partial R_s} f = \sum_{s=1}^{n} \mu_s \int_{\partial R_s} f = 0\,,$$

and our proof is complete. ♢