MATHEMATICS 300 — FALL 2002

Introduction to Mathematical Reasoning
(640 - 300, Section 01)

H. J. Sussmann

INSTRUCTOR'S NOTES, PART 1

(September 9, 2002)

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1 Information on the course

1.1 Course schedule

Our class meets on **Mondays and Thursdays, 3rd period (11:30 am to 12:50 pm)**, in ARC 205.

1.2 About the instructor

My *name* is **H.J. Sussmann.** My *office* is **Hill 538**.
My *Rutgers phone extension* is 5-5407.
My *e-mail address* is **sussmann@math.rutgers.edu**.

1.3 Web page

I have set up a *Web page* for our Math 300 section:

http://www.math.rutgers.edu/~sussmann/math300page.html

All the instructor's notes will be available there.

1.4 Office hours

My office hours will be:

- Monday and Thursday, 4th period (1:10 pm to 2:30 pm) in my office, Hill 538;
- any other time, by appointment, in my office, Hill 538.

You should let me know that you are coming to see me *even if you are planning to do so during my regular office hours.* REASON: I don't want to have to stay in my office waiting just in case someone might show up. If I know that you are coming then I will be there, but if you don't tell me that I should expect you then I will feel free to go away and do other things.

1.5 The Fall 2002 semester

The semester starts on *Tuesday, September 2* and ends on *Wednesday, Dec. 11.* The *Thanksgiving recess* goes from *Thursday, November 28* to *Sunday, December 1.* (In addition, there are the following *changes in designation of class days:* Tuesday November 26 has Thursday Classes; Wednesday November 27 has Friday Classes. The only effect of these changes on our Math 300 section is that we have a meeting on Tuesday November 26.)

1.6 Our Math 300 calendar:

We will have **28 meetings**, on the following dates:

- September: Th5, M9, Th12, M16, Th19, M23, Th26, M30 (8 meetings)
- October: Th3, M7, Th10, M14, Th17, M21, Th24, M28, Th31 (9 meetings)
- November: M4, Th7, M11, Th14, M18, Th21, M25, Tu26 (8 meetings)
- December: M2, Th5, M9 (3 meetings)

1.7 Homework, midterms, etc.

Homework and quizzes will count for about one third of your grade. There will be *two midterms*, which will count—together—for another third of your grade. The *final exam* will count for the remaining third.

Midterm dates: Monday October 14 and **Monday November 25**.

1.8 The textbook and the instructor's notes

We will be using:

- the book *Mathematics, a discrete introduction*, by **Edward R. Scheinerman** (Brooks/Cole Publishing Co., Pacific Grove, CA, 2000),
- the notes written by the instructor.

You should read and study carefully all the instructor's notes.

The material of these notes is an integral part of the course, as much as that of the book. *Furthermore, the notes contain all kinds of important information. For example, in this set of notes there are lots of things you need to know in order to do your homework.*

1.9 Readings for the first 6 days (Sept. 5, 9, 12, 16, 19, 23)

- the book's note "To the Student,"
- the book's Chapter 1 (**all of it!**),
- the instructor's notes, Part 1 (omitting §3.4, on quaternions).

1.10 Homework assignments for September

Before you start writing your homework, read carefully the rest of this handout, in particular §2 on "writing mathematics and submitting homework," and §4 on mathematical writing. Pay attention to §2.3, on "answering questions in this course."

1.10.1 Homework assignment no. 1, due on Monday September 16.

You can also hand in this assignment on Tuesday, September 17, by bringing it to the Hill Center and putting it in my mailbox in the Math Department.

A. Problems 1, 2, 3, 4, 5, 6 in page 5 of the book. (Note: in Problem 4 you are asked to define "less than" and a few other predicates. The question, as stated in the book, is incomplete, because the author is not telling you what you are allowed to use. For example, a student could define "less than" by saying "let x, y be integers; we say that x is *less than* y if y is greater than x," and then define "greater than" by saying "let x, y be integers; we say that x is *greater than* y if y is less than x." **This is not acceptable.** Why? **Because, together, these two definitions are circular.** In order to know what "less than" means you have to know what "greater than" means, and then, in order to know what "greater than" means you have to know what "less than" means. Obviously, the author does not want to allow that, but he is not telling us clearly what he is allowing. So let me tell you: you should write your definition using the concepts of natural number, integer, addition (that is, $+$), equality (that is, $=$), as well as the existential quantifier ("there exists x such that ..."), the universal quantifier ("for all x ..."), **and nothing else**[1].)

B. Is the number π even or odd? Justify your answer.

C. (a) Is "$\mathbb{N}$ is a set of natural numbers" a true statement? (b) Is "$\mathbb{N}$ is a set of natural numbers" a satisfactory answer to the question "What is $\mathbb{N}$?"?

D. (a) Is "elephants have four legs" a true statement? (b) Is "elephants have four legs" a satisfactory answer to the question "What is an elephant?"?

E. Write a definition of "even integer" without mentioning the concept of divisibility.

F. Write a definition of "odd integer" without using the number 2.

G. The definitions of "even," "divisible," and "odd" given in the book (Definitions 1.1, 1.2 and 1.4) also make sense in other number systems such as $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, etc. It suffices to replace the word "integer" throughout by "member of $\mathbb{Z}_2$" (or of $\mathbb{Z}_3$, or of $\mathbb{Z}_4$, etc.), or, equivalently, by "integer

[1] In the Spring 2001 term, a student wrote "let x, y be integers; we say that x is *less than* y if x is to the left of y." This is an example of what is **not** allowed, because you are not allowed to talk about "left" and "right."

modulo 2" (or "modulo 3," or "modulo 4," etc.), and to carry out all the operations such as addition in $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, etc.

For example, 32 is a member of $\mathbb{Z}_{37}$, and 32 is odd in $\mathbb{Z}_{37}$. *Proof*: in $\mathbb{Z}_{37}$ we have $2 \times 34 + 1 = 68 + 1 = 69 = 32$, so there exists an integer k modulo 37 (namely, $k = 34$), such that $32 = 2 \times k + 1$ in $\mathbb{Z}_{37}$, and this says that 32 is odd in $\mathbb{Z}_{37}$.

For what values of the positive integer n is it true that "in $\mathbb{Z}_n$, a number cannot be both even and odd"?

1.10.2 Homework assignment no. 2, due on Monday September 23

A. Express the number 2026 as the sum of two primes. Justify your answer. (In particular, make sure that you prove that the two numbers you found are prime.)

B. Section 2 of the book, Problems 1(b)(c), 2, 4, 6, 7.

C. Section 3 of the book, Problems 2, 5, 6, 10, 12, 13.

D. Here are nine answers given by students to the question "define 'composite number'".

1. Composite is when its divisible by two numbers.
2. A positive integer is *composite* if there exists an integer b such that $1 < b < n$ and $b|a$.
3. A positive integer a is *composite* if there exists an integer b such that $1 < b < n$ and $b|a$.
4. A positive integer a is *composite* if there exists an integer b and $1 < b < a$ and $a|b$.
5. A positive integer a is *composite* if there exists an integer b such that $1 < b < a$ and $a|b$.
6. A positive integer n is *composite* if there exist positive integers a and b such that $n = ab$.
7. A positive integer n is *composite* if there exist integers a and b such that $n = ab$ and $1 < a < n$.
8. A positive integer is *composite* if there exist integers a and b such that $n = ab$ and $1 < a < n$.
9. A positive integer n is *composite* if there exist integers a and b such that $n = ab$, $1 < a$, and $1 < b$.

Grade these answers on a scale from 0 to 10 and justify your grades. (Do not hesitate to be harsh and to make as many negative comments as may be appropriate.)

E. Of the nine answers given in Problem D to the question "define 'composite number'", one of them is such that, according to that purported definition, there do not exist any composite numbers at all. Determine which of the eight answers is the one having this undesirable feature, and prove that according to that definition there are no composite numbers.

F. Of the nine answers given in Problem D to the question "define 'composite number'", one of them is such that, according to that purported definition, every positive integer is composite. Determine which of the eight answers is the one having this undesirable feature, and prove that according to that definition every positive integer is composite.

G. Prove that every positive integer which is not prime is composite.

1.10.3 Homework assignment no. 3, due on Monday September 30

A. Section 4 of the book, Problems 3, 5, 6, 7, 9.

B. Section 5 of the book, Problems 1(a)(d)(e), 3, 4, 8, 11(b)(c)(e), 12(b), 18.

C. Section 6 of the book, Problems 2, 6, 7, 9(d)(f)(g), 12, 15.

1.10.4 Supplementary problems for the month of September.

Here are some supplementary problems. Don't worry if some of these problems seem too hard. They are meant to be challenging. Just do as much as you can. Hand in the solutions **not later than October 3**.

One of the main purposes of this set of questions is to get you to experience what it is like to think about problems that could be quite hard (even if they seem easy!), so that you will appreciate what it means to find a **truly good solution**. For you to see this, you have to think about the problems first, and if you cannot solve them **after you have tried to** and then you see the solution, you will understand much better what a **real proof** is.

A. Prove that every year must have a Friday the 13th.

B. In the planet of the Klingons, they have a calendar exactly like the one we use here on Earth (there are 12 months, January has 31 days, February has 28 or 29, March has 31, April has 30, etc.), except for one thing: the order of the months is different. We do not know what it is, but we know for sure that it is different from ours. (For example, it may be that June is the first month, September is the second one, etc.) Is it still necessarily true that every year must have a Friday the 13th? *Prove that it is true*, no matter what the order of the months is, *or prove that it need not be true* (that is, that there is a way to reorder the months so that not every year has a Friday the 13th).

C. Express the number 28 as the difference of the squares of two natural numbers.

D. Express the number 29 as the difference of the squares of two natural numbers.

E. Express the number 30 as the difference of the squares of two natural numbers.

F. (*Optional. This could be quite hard! Do not worry if you cannot do it, but at least try to* **think** *about it.*) Express the number $4,582,623$ as the sum of the squares of two natural numbers.

G. (*Optional. Hard.*) Suppose two integers a and b are sums of two squares of integers. Prove that the product ab must also be the sum of two squares of integers. (NOTE: here is an example of the kind of situation we are talking about. The number 13 is the sum of two squares, because $13 = 4 + 9$. The number 8 is the sum of two squares, because $8 = 4 + 4$. So, if our statement is true, it ought to be the case that the number 13×8 is the sum of two squares. And, indeed, it *is* true, because $13 \times 8 = 104$ and $104 = 100 + 4 = 10^2 + 2^2$.)

H. Here are five sentences about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than."

(1) Cows are taller than sheep. (That is, "every cow is taller than every sheep," or, if you prefer, "for every x and every y, if x is a cow and y is a sheep, then x is taller than y.")

(2) Giraffes are taller than cows.

(3) If something is taller than something else which is in turn taller than a third thing, then the first one is taller than the third one. (That is "for every x, every y and every z, if x is taller than y and y is taller than z, then x is taller than z.")

(4) Olivia is a giraffe.

(5) Dolly is a sheep.

Using sentences (1), (2), (3), (4), (5), *and* **no other fact** *about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than,"* prove that Olivia is taller than Dolly.

2 Writing mathematics & submitting homework

2.1 Write clearly in complete sentences

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements* with a *clear meaning* that can can be understood by anyone reading what you wrote. For example:

  - If you tell me that "she is very smart," but you haven't told me who "she" is, then I don't know what you are talking about, so you haven't made a statement with a clear meaning.
  - If you write "$x > 0$," but you haven't told me who "x" is, then I don't know what you are talking about, so you haven't made a statement.
  - If I ask you to state Pythagoras' theorem and your answer only says "$a^2 + b^2 = c^2$," then nobody will know what you are talking about, because you have not said what "a," "b," and "c" are supposed to be.[2]

- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in detail in class, and is also explained in §4.
- Provide proper connectives between equations as well as between ideas.
- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.
- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.* If you have to memorize a definition or a statement, then that is not a good sign, because it indicates lack of understanding.
- Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you've tried to read any complicated document.
- *Do not assume that the people reading your paper can read your mind. Do assume that they are intelligent, but also assume that they are busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.*

Effective written exposition will be worth at least 50% of your grade. Conversely, bad or unclear exposition may be penalized as much as 50% of the grade or even more.

[2]Here is a correct statement of Pyhtagoras' theorem: *Let c be the lenght of the hypothenuse of a right triangle, and let a, b be the lengths of the other two sides. Then $a^2 + b^2 = c^2$.*

- The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

2.2 Your written work

You should pay attention to presentation, especially for the homework:

- A nicely typed homework (e.g., using a word processor) is preferable to handwritten work. Handwritten work is acceptable too, but in that case:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**. If you are not willing to spend some of your time doing this; if what you hand in shows that you were in a hurry and that you did not make the effort to write things neatly and properly, then there is no reason for the instructor or the grader to spend any of our time reading what you wrote, and we will not do it.
  - You should use a pen. Never use a pencil.
  - Use any color other than red (for example, black, blue, or green), but DO NOT USE RED. (Reason: The use of red is reserved for the instructor's and grader's comments.)
- Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more than one sheet then the sheets are **stapled** and the pages are **numbered**.
- If you tear off the sheets from a spiral notebook, please make sure before you hand them in that there are none of those ugly hanging shreds of paper at the margins. Use scissors, or a cutter, if necessary.

If you hand in a homework assignment that has at least one of the following flaws:

- **it is written carelessly or in a hurry,**
- **it has lots of words crossed out,**
- **it has unreadable handwriting,**
- **it has pages that are unstapled or unnumbered or fail to show your name,**
- **it has shreds of paper at the margins,**
- **it is written using pencil rather than a pen,**
- **it is written in red,**

then the assignment will be marked "unacceptable" and returned unread.

2.3 Answering questions in this course

In this course, whenever you are given a problem where you are asked to do something, your answer should be either:

(a) doing what you were asked to do,

or

(b) showing—that is *proving*—that it cannot be done.

(See, for example, Problems 2, 3, 4, 5, 8, 10, 12 below, where the correct answer is "what you asked me to do cannot be done.")

Notice that, when the answer is that "it cannot be done," it is not enough for you to *say* that it cannot be done. *You have to tell me why.* In other words, you have to *prove* that it cannot be done.

This remark is very important, and will apply throughout the semester, not just during the first week. And it applies to all your work, to the homework, the quizzes, the midterm exams, and the final exam. So please read it and reread it until you are sure you got the point. ♢

2.4 Some examples of problems with correct answers

Here are some examples of problems with correct solutions:

PROBLEM 1: Express the number 26 as a sum of two odd natural numbers.

ANSWER: $26 = 3 + 23$. ♢

REMARK: *There are lots of other solutions, of course! For example, here are two solutions different from the one given above:* $26 = 7 + 19$, $26 = 13 + 13$.

PROBLEM 2: How many solutions does Problem 1 have?

ANSWER: This question cannot be answered in a unique way, because we have not told you exactly what is meant by a "solution of Problem 1," and, depending on how you define what is meant by a "solution," the answers to our question may be different.

Here are two equally legitimate ways to define "solution of Problem 1":

First way: A *solution of Problem 1* is an ordered pair (n, m) of odd natural numbers such that $n + m = 26$. (What this means is that a solution consists of specifying a first odd natural number n and a second odd natural number m such that $n + m = 26$. So "$n = 3$ and $m = 23$" is a solution, and "$n = 23$ and $m = 3$" is a *different* solution.)

According to this definition of "solution of Problem 1," there are exactly **13 solutions**, namely, the ordered pairs $(1, 25)$, $(3, 23)$, $(5, 21)$, $(7, 19)$, $(9, 17)$, $(11, 15)$, $(13, 13)$, $(15, 11)$, $(17, 9)$, $(19, 7)$, $(21, 5)$, $(23, 3)$, $(25, 1)$.

Second way: A *solution of Problem 1* is an unordered pair $\{n, m\}$ of odd natural numbers such that $n + m = 26$.

According to this definition of "solution of Problem 1," there are exactly **7 solutions**, namely, the unordered pairs $\{1, 25\}$, $\{3, 23\}$, $\{5, 21\}$, $\{7, 19\}$, $\{9, 17\}$, $\{11, 15\}$, $\{13, 13\}$). (What this means is that a solution consists of specifying two odd natural numbers whose sum is 26, without singling out one of them as the "first" one. So in this case "$n = 3$ and $m = 23$" is a solution, and "$n = 23$ and $m = 3$" is *the same* solution.) $\diamondsuit$

PROBLEM 3: Express the number 27 as the sum of two odd natural numbers.

SOLUTION: This is impossible. REASON: the sum of two odd numbers is always even. Since 27 is odd, it cannot be the sum of two odd numbers. $\diamondsuit$

PROBLEM 4: Prove that $2 + 2 = 5$.

SOLUTION: This cannot be done. REASON: The statement "$2 + 2 = 5$" is false, and false statements cannot be proved.

PROBLEM 5: Prove that for every positive integer x there is another positive integer y such that $0 < y < x$.

SOLUTION: This cannot be done. REASON: The statement that we are being asked to prove is false, and a false statement cannot be proved.

The reason that the statement is false is as follows: 1 is a positive integer, and there is no positive integer y such that $0 < y < 1$, because 1 is the smallest positive integer. So if we take x to be 1 then the conclusion is not true. So it is not true that "for *every* positive integer x there is another positive integer y such that $0 < y < x$." $\diamondsuit$

PROBLEM 6: Express the number 12 as the difference of the squares of two natural numbers.

SOLUTION: $12 = 16 - 4$, so $12 = 4^2 - 2^2$. $\diamondsuit$

PROBLEM 7: Express the number 20 as the sum of the squares of two natural numbers.

SOLUTION: $20 = 4 + 16 = 2^2 + 4^2$. $\diamondsuit$

PROBLEM 8: Express the number 4 as the sum of the squares of two positive integers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 4 as a sum of two positive integers:

$$4 = 1 + 3\,, \qquad 4 = 2 + 2\,.$$

In neither of these two equalities are the two terms on the right-hand side squares.

PROBLEM 9: Express the number 4 as the sum of the squares of two natural numbers.

SOLUTION: $4 = 0 + 4 = 0^2 + 2^2$. ♢

PROBLEM 10: Express the number 12 as the sum of the squares of two natural numbers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 12 as a sum of two natural numbers:

$12 = 0+12$, $12 = 1+11$, $12 = 2+10$, $12 = 3+9$, $12 = 4+8$, $12 = 5+7$, $12 = 6+6$.

In none of these expressions are the two terms on the right-hand side squares of natural numbers. ♢

PROBLEM 11: Express the number 12 as the sum of the squares of three natural numbers.

SOLUTION: $12 = 4 + 4 + 4$, so $12 = 2^2 + 2^2 + 2^2$. ♢

PROBLEM 12: Express the number 15 as the sum of the squares of three natural numbers.

SOLUTION: This cannot be done. REASON: If we are going to write 15 as the sum of three squares, then we should be able to write it as the sum of two natural numbers, one of which is a square, while the other is the sum of two squares.

So let us look at all the ways to write 15 as a sum of two natural numbers, see in which cases one of these numbers is square, and then see if the other number is the sum of two squares.

Here are all the ways to express 15 as a sum of two natural numbers:

$$\begin{array}{llll} 15 = 0 + 15, & 15 = 1 + 14, & 15 = 2 + 13, & 15 = 3 + 12, \\ 15 = 4 + 11, & 15 = 5 + 10, & 15 = 6 + 9, & 15 = 7 + 8\,. \end{array}$$

There are four expressions in the above list where one of the numbers is a square:

$$15 = 0 + 15, \quad 15 = 1 + 14, \quad 15 = 4 + 11, \quad 15 = 6 + 9\,.$$

In each case, we must ask whether the other number is a sum of two squares.

For "15=0+15," we have to see if 15 is the sum of two squares. The answer is "no," because we already have the list of all (eight) ways of writing 15 as a sum of two natural numbers, and in no case are both those numbers suqares.

For "15=1+14," we have to see if 14 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 14 are 0, 1, 4, and 9. So if are going to write 14 as a sum of two squares, the only possibilities are $14 = 0 + 14$, $14 = 1 + 13$, $14 = 4 + 10$, $14 = 9 + 6$. But in all these cases the other number—that is, 14, 13, 10, or 6—is not a square.)

For "15=4+11," we have to see if 11 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 11 are 0, 1, 4, and 9. So if are going to write 11 as a sum of two squares, the only possibilities are $11 = 0+11$, $11 = 1+10$, $11 = 4+7$, $14 = 9+5$. But in all these cases the other number (that is, 11, 10, 7, or 5) is not a square.)

Finally, for "15=6+9," we have to see if 6 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 6 are 0, 1, and 4. So if are going to write 6 as a sum of two squares, the only possibilities are $6 = 0+6$, $6 = 1+5$, $6 = 4+2$. But in all these cases the other number (that is, 6, 5, or 2) is not a square.) ♢

PROBLEM 13: Express the number 15 as the sum of the squares of four natural numbers.

SOLUTION: $15 = 1+1+4+9 = 1^2+1^2+2^2+3^2$. ♢

PROBLEM 14: Express the number 674 as the sum of the squares of four natural numbers.

SOLUTION: $674 = 144+400+121+9 = 12^2+20^2+11^2+3^2$. ♢

REMARK: You may be thinking *how on Earth did the author of these notes figure out how to solve problem 14?* That's a good question. It turns out that there is a technique for doing this, but it's not easy to explain. Maybe we will talk about it later in the course. But keep in mind that the question "how did I find a solution to problem 14?" is quite different from the question "is $674 = 12^2+20^2+11^2+3^2$ a correct solution?" This second question is easy to answer: you just compute $12^2+20^2+11^2+3^2$ and verify that what you get is 674. ♢

REMARK: You may have noticed that I gave you examples of natural numbers that could be expressed as the sum of two squares, and of natural numbers that could not. Then I gave you examples of natural numbers that could be expressed as the sum of three squares, and of natural numbers that could not. But for *four* squares what I have done is different: I gave you examples of natural numbers that could be expressed as the sum of four squares, but I did not give you examples of natural numbers that could not. Is there a reason for this? The answer is **yes**. The mathematician Lagrange (1736-1813) proved that **every natural number can be expressed as the sum of the squares of four natural numbers.** This is a hard theorem, and I will try to say a few words about it later in the course. ♢

3 A first look at the world of mathematical objects: number systems

"The time has come," the Walrus said,
"To talk of many things:
Of shoes—and ships—and sealing wax—
Of cabbages–and kings
And why the sea is boiling hot—
And whether pigs have wings."

BOX NO. 1: SUMMARY OF CHAPTER 3.

There are many different types of mathematical objects. Among them, there are **numbers** of various kinds, and also lots of other things that are not numbers.
Numbers belong to **number systems**. The most important number systems are:

- $\mathbb{N}$, the set of *natural numbers*,
- $\mathbb{Z}$, the set of *integers*,
- $\mathbb{Q}$, the set of *rational numbers*,
- $\mathbb{R}$, the set of *real numbers*,
- $\mathbb{C}$, the set of *complex numbers*,
- $\mathbb{K}$, the set of *quaternions*,
- $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, and, more generally $\mathbb{Z}_n$ (the set of *integers modulo* n) for every natural number n such that $n \geq 2$.

In addition to numbers, there are all kinds of other mathematical objects, such as *sets*, *lists*, *operations*, *functions*, *relations*, *arrays* of various kinds and, in particular, *matrices*, *linear spaces* (such as $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, etc.), *geometrical figures* such as *curves* of different kinds (for example, lines, parabolas, ellipses, hyperbolas, cycloids, exponential curves), *surfaces* (such as planes, spheres, cylinders, cones, ellipsoids, paraboloids, hyperboloids), and other figures (e.g., triangles, squares, rectangles, discs, balls), and many, many more things.
We will talk about those other things later. For the moment, we just concentrate on **number systems**.

Our first task is to take a look at the rich and varied collection of things that populate the mathematical world. Some of them should be familiar to you, others may be less so or not at all. Do not worry if you find in our list things that you have never heard of before: we will be coming back to the list later, and discussing all the items in much greater detail.

Mathematicians like to give names to things so that they can talk about complicated objects without repeating their long descriptions every time. So let

us start by agreeing to use ***WMO*** as our name for the world of mathematical objects.

3.1 Numbers: $\mathbb{Z}_{>0}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

The best known inhabitants of ***WMO*** are **numbers** and **number systems**. There are several different kinds of numbers, and they can be organized into *number systems*. A number can belong to different number systems, in the same way as, say, a person can belong to different associations. (For example, somebody could be a member, say, of the American Association of University Professors, the Rutgers Alumni Association, and the Sierra Club. Similarly, the number 3 belongs to lots of different number systems, such as, for example, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, ..., $\mathbb{Z}_{397}$, ...)

It is convenient to give the number systems **names**, and to introduce mathematical symbols to represent them. Here are some examples:

1. The set $\mathbb{N}$ of **natural numbers**. (Natural numbers are also called **non-negative integers**, or **whole numbers**.) The members of this set are the integers $0, 1, 2, 3 \ldots$. So, for example,

   0 is a member of $\mathbb{N}$ (or "0 belongs to $\mathbb{N}$"),

   1 is a member of $\mathbb{N}$ (or "1 belongs to $\mathbb{N}$"),

   2 is a member of $\mathbb{N}$ (or "2 belongs to $\mathbb{N}$"),

   .....

   385 is a member of $\mathbb{N}$ (or "385 belongs to $\mathbb{N}$"),

   .....

   $10,891,032$ is a member of $\mathbb{N}$ (or "$10,891,032$ belongs to $\mathbb{N}$"),

   .....

   -1 is not a member of $\mathbb{N}$ (or "-1 does not belong to $\mathbb{N}$"),

   3.5 is not a member of $\mathbb{N}$ (or "3.5 does not belong to $\mathbb{N}$"),

   and so on.

2. The set $\mathbb{Z}$ of all **integers**. The members of $\mathbb{Z}$ are the natural numbers as well as 0 and the negatives of natural numbers, i.e., the numbers -1, -2, -3, etc.

3. The set $\mathbb{Z}_{>0}$ of **positive integers** (or **nonzero natural numbers**). The members of this set are exactly the same as the members of $\mathbb{N}$, except only for the fact that 0 is not a member of $\mathbb{Z}_{>0}$ but is a member of $\mathbb{N}$.

BOX NO. 2: THE SYMBOL "$\in$" ("BELONGS TO").

We use the symbol $\in$ (read "belongs to," or "is a member of") to indicate *membership in a set*:

> If A is a set and x is an object, then "$x \in A$" means "x belongs to A," or "x is a member of A."

The symbol $\notin$ (read "does not belong to," or "is not a member of") indicates non-membership:

> If A is a set and x is an object, then "$x \notin A$" means "x does not belong to A," or "x is not a member of A."

For example the following statements are true:

$$\begin{array}{rrrr}
1 \in \mathbb{N}\,, & 2 \in \mathbb{N}\,, & 385 \in \mathbb{N}\,, & 10,891,032 \in \mathbb{N}\,, \\
0 \in \mathbb{N}\,, & -1 \notin \mathbb{N}\,, & 3.5 \notin \mathbb{N}\,, & \pi \notin \mathbb{N}\,, \\
\frac{8}{5} \notin \mathbb{N}\,, & \frac{10}{5} \in \mathbb{N}\,, & \frac{382{,}403{,}313}{9} \in \mathbb{N}\,, & \pi + 7 \notin \mathbb{N}\,, \\
3 + 7 \in \mathbb{N}\,, & 8 - 1 \in \mathbb{N}\,, & 1 - 8 \notin \mathbb{N}\,, & \pi - (2 - \pi) \notin \mathbb{N}\,, \\
1 \in \mathbb{Z}\,, & 2 \in \mathbb{Z}\,, & 385 \in \mathbb{Z}\,, & 10,891,032 \in \mathbb{Z}\,, \\
0 \in \mathbb{Z}\,, & -1 \in \mathbb{Z}\,, & 3.5 \notin \mathbb{Z}\,, & \pi \notin \mathbb{Z}\,, \\
\frac{8}{5} \notin \mathbb{Z}\,, & \frac{10}{5} \in \mathbb{Z}\,, & \frac{382{,}403{,}313}{9} \in \mathbb{Z}\,, & \pi + 7 \notin \mathbb{Z}\,, \\
3 + 7 \in \mathbb{Z}\,, & 8 - 1 \in \mathbb{Z}\,, & 1 - 8 \in \mathbb{Z}\,, & \pi - (2 - \pi) \notin \mathbb{Z}\,, \\
0 \notin \mathbb{Z}_{>0}\,, & 1 \in \mathbb{Z}_{>0}\,, & 4 \in \mathbb{Z}_{>0}\,, & -3 \notin \mathbb{Z}_{>0}\,.
\end{array}$$

4. The set $\mathbb{R}$ of all **real numbers**. Every natural number is a member of $\mathbb{R}$. Every integer is a member of $\mathbb{R}$. In addition, there are many other real numbers besides the integers.

   For example the following statements are true:

$$\begin{array}{rrrr}
1 \in \mathbb{R}\,, & 2 \in \mathbb{R}\,, & 385 \in \mathbb{R}\,, & 10,891,032 \in \mathbb{R}\,, \\
0 \in \mathbb{R}\,, & -1 \in \mathbb{R}\,, & 3.5 \in \mathbb{R}\,, & \pi \in \mathbb{R}\,, \\
\frac{8}{5} \in \mathbb{R}\,, & \frac{10}{5} \in \mathbb{R}\,, & \frac{382{,}403{,}313}{9} \in \mathbb{R}\,, & \pi + 7 \in \mathbb{R}\,, \\
3 + 7 \in \mathbb{R}\,, & 8 - 1 \in \mathbb{R}\,, & 1 - 8 \in \mathbb{R}\,, & \pi - (2 - \pi) \in \mathbb{R}\,.
\end{array}$$

   A real number (and, in particular, an integer) is "positive" if it is > 0. A number is "nonnegative" if it is ≥ 0.

   In particular, 0 **is not positive**, but 0 **is nonnegative.** 0 **is a nonnegative integer, but** 0 **is not a positive integer.**

5. The set $\mathbb{Q}$ of all **rational numbers**. (Rational numbers are often called "fractions"). By definition, a *rational number* is a real number that can

be expressed as the quotient (or "ratio[3]) of two integers. So, for example, $\frac{8}{3} \in \mathbb{Q}$, $6 \in \mathbb{Q}$, $\frac{9}{3} \in \mathbb{Q}$, $-6 \in \mathbb{Q}$, $-\frac{9}{3} \in \mathbb{Q}$.

Remark 1. Do not confuse the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ with the letters N, Z, R. They are different! (This is very convenient. If we agreed to use N as the name for the set of natural numbers, then the meaning of the letter N would be fixed for ever, and we would not be able to use N for other things.) ♢

Another way of saying that "$\mathbb{N}$ is the set consisting of 0, 1, 2, 3, 4, and so on," is to write

$$\mathbb{N} = \{\, 0, 1, 2, 3, 4, \ldots \}\,. \tag{3.1.1}$$

Simlarly,

$$\mathbb{Z} = \{\, 0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots \}\,.$$

3.2 The complex numbers

For a long time, people used the word "number" to mean "real number." But then the "complex numbers" were invented, and mathematicians decided, after long discussions on whether this was a good thing to do, that these "complex numbers" were objects worth admitting as *bona fide* numbers as well. So we can add a sixth example to our list of number systems:

6. The set $\mathbb{C}$ of all *complex numbers*. Here are some examples of these "numbers":

$$3\,, \qquad i\,, \qquad 3+i\,, \qquad 4-i\,, \qquad 35-67i\,, \qquad \frac{8}{5}\,, \qquad \frac{8}{5}-\pi i\,.$$

Complex numbers can be **added** and **multiplied**. To add two complex numbers, you just add separately the "i-parts" and the "non-i-parts." So, for example,

$$3+i+(4-2i) = 7-i\,,\;\; 3+(8+5i) = 11+5i\,,\;\; \frac{8}{5}+i+\Big(\frac{12}{5}+83i\Big) = 4+84i\,.$$

To multiply two complex numbers, you just multiply as you would real numbers, except that every time you run into $i.i$ you put -1 instead. For example:

$$\begin{aligned}(23+7i).(4+2i) &= 23.4+7.4.i+23.2.i+7.2.i.i\\ &= 92+28i+46i+14.i.i\\ &= 92-14+28i+46i\\ &= 78+72i\,.\end{aligned}$$

[3] "Ratio" is an old word for "quotient."

Remark 2. The invention, or discovery, of the complex numbers was a long process, driven by the effort to prove the "fundamental theorem of algebra." This theorem ought to say that an n-th order equation has n solutions, so for example the equation $x^2+1=0$ should have two solutions, the equation $x^7+3x^5+4x^2+5=0$ should have seven solutions, and so on. But the only way for $x^2+1=0$ to have two solutions is to stipulate that there is a number x such that $x^2=-1$, that is, that a "square root of -1" exists. This, however, leads to all kinds of difficulties, if you think that this square root—let us call it i—is an ordinary real number. For example, every real number is either positive or negative or zero. Now, i of course cannot be zero, so i is either positive or negative. Since the product of two positive numbers is positive, if i was positive then i^2 would have to be positive. Since the product of two negative numbers is positive, if i was negative then i^2 would have to be positive. So in either case, whether i is positive or negative, the number i^2 would have to be positive. But $i^2=-1$, so -1 would have to be positive, and this is not possible. The way out of this was to think of i as not being an ordinary number, really, but only a symbol to be manipulated according to certain rules. So, for example, following what **Descartes** wrote in 1637, one can "imagine" that the equation $x^2+1=0$ has two roots (i and $-i$) but these imagined roots are not "real" numbers. Earlier, in 1572, **Bombelli** had given precise rules for the manipulation of these imaginary numbers. **Argand** in 1814 had the idea of representing complex numbers as points in a two-dimensional plane. The term "complex number" was actually introduced by **Gauss** in 1831. The term "conjugate" had been introduced by **Cauchy** in 1821. $\diamondsuit$

BOX NO. 3: THE SYMBOL "$\subseteq$" ("IS A SUBSET OF").

We use the symbol $\subseteq$ (read "is a subset of," or "is contained in") to indicate *set inclusion*, and the symbol $\not\subseteq$ (read "is not a subset of") to indicate that the inclusion does not hold.
If A and B are sets, the expression "$A \subseteq B$ (read "A is a subset of B") means that *every member of A belongs to B*. For example: if A is the set of all U.S. Senators, and B is the set of all citizens of the U.S., then "$A \subseteq B$" and "$B \not\subseteq A$" are true statements.
The following statements about $\mathbb{Z}_{>0}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, are true:

$\mathbb{Z}_{>0} \subseteq \mathbb{Z}_{>0}$,	$\mathbb{Z}_{>0} \subseteq \mathbb{N}$,	$\mathbb{N} \not\subseteq \mathbb{Z}_{>0}$,	$\mathbb{N} \subseteq \mathbb{N}$,
$\mathbb{N} \subseteq \mathbb{Z}$,	$\mathbb{Z} \not\subseteq \mathbb{N}$,	$\mathbb{Z} \subseteq \mathbb{Z}$,	$\mathbb{Z}_{>0} \subseteq \mathbb{Z}$,
$\mathbb{Z} \subseteq \mathbb{Q}$,	$\mathbb{Q} \not\subseteq \mathbb{Z}$,	$\mathbb{Z} \subseteq \mathbb{R}$,	$\mathbb{N} \subseteq \mathbb{Q}$,
$\mathbb{Q} \not\subseteq \mathbb{N}$,	$\mathbb{Q} \not\subseteq \mathbb{Z}$,	$\mathbb{R} \not\subseteq \mathbb{Q}$,	$\mathbb{Q} \subseteq \mathbb{Q}$.

BOX NO. 4: NOT ALL REAL NUMBERS ARE RATIONAL.

In Box No. 3 we asserted that $\mathbb{R} \not\subseteq \mathbb{Q}$. How do we know that?
For "$\mathbb{R} \subseteq \mathbb{Q}$" to be true, it would have to be the case that every real number is rational. Yet, this is *not* the case, because there are real numbers that are not rational. A famous example is $\sqrt{2}$. We will prove later that

The number $\sqrt{2}$ is not rational.

Actually, many other real numbers are not rational, although in lots of interesting cases this is very hard to prove. For example, the numbers π and e are not rational.

3.3 The integers modulo n

Here are some more examples of important "number systems" of a rather different kind. These systems will be very important to us later.

7.a. The set $\mathbb{Z}_{12}$ (read this as "z-twelve") of all *integers modulo* 12. The members of $\mathbb{Z}_{12}$ are the following 12 integers:

$$0\,,\ 1\,,\ 2\,,\ 3\,,\ 4\,,\ 5\,,\ 6\,,\ 7\,,\ 8\,,\ 9\,,\ 10\,,\ 11\,.$$

Addition and multiplication are carried out in the usual way, except that any time you get 12 or something larger, you must "reduce modulo 12," that is, subtract 12 and keep subtracting 12 until you get a member of $\mathbb{Z}_{12}$. For example, the following equalities are true in $\mathbb{Z}_{12}$:

$$3+5=8\,,\quad 6+9=3\,,\quad 8+9=5\,,\quad 7+1=8\,,\quad 11+11=10\,,$$
$$4.2=8\,,\quad 6.9=6\,,\quad 8.7=8\,,\quad 7.11=5\,,\quad 7.7=1\ .$$

Remark 3. The numbers modulo 12 are very natural objects, and you are very much used to working with them. For example, suppose you start at 10 o'clock and work 8 hours. What time is it when you finish? The answer is "$10+8$ modulo 12," that is, 6 o'clock. If a movie theatre starts its show at 8 o'clock and shows a 2-hour movie 8 times, without intermission, what time is it at the end of the show? The answer is "$10+2.8$ modulo 12," that is, 2 o'clock.

Naturally, if you wanted to keep track of "am" vs. "pm" then you would want to use so-called "military time," that is, work in $\mathbb{Z}_{24}$, the numbers modulo 24. (In $\mathbb{Z}_{24}$, $16+13=5$, and $8.5=16$, and so on.) $\diamondsuit$

7.b. The set $\mathbb{Z}_7$ of all *integers modulo* 7.

The members of $\mathbb{Z}_7$ are the following 7 numbers:

$$0\,,\ 1\,,\ 2\,,\ 3\,,\ 4\,,\ 5\,,\ 6\,.$$

Addition and multiplication are carried out in the usual way, except that any time you get 7 or something larger, you must "reduce modulo 7," that is, subtract 7 and keep subtracting 7 until you get a member of $\mathbb{Z}_7$. For example, the following equalities are true in $\mathbb{Z}_7$:

$$\begin{array}{rrrrr} 3+2=5\,, & 6+5=11\,, & 4+3=0\,, & 2+6=1\,, & 4+2=6\,, \\ 4.2=1\,, & 6.5=2\,, & 3.4=5\,, & 3.3=2\,, & 3.5=1\ . \end{array}$$

Remark 4. Like the members of $\mathbb{Z}_{12}$, the numbers modulo 7 are also very natural objects that you are very much used to working with. For example, let us think of the days of the week as numbered, starting with Sunday as Day No. 0. In other words, think of

Sunday	as day No.	0
Monday	as day No.	1
Tuesday	as day No.	2
Wednesday	as day No.	3
Thursday	as day No.	4
Friday	as day No.	5
Saturday	as day No.	6

Suppose today is Thursday. What day of the week is it going to be 5 days from today? The answer is ***Tuesday***, of course. Why? Because Thursday is Day 4, so 5 days from today is Day $4+5$, that is, Day 2, modulo 7.

Suppose today, January 5, is a Wednesday. Whay day of the week is February 11 going to be? ANSWER: today is Day 3; to go from January 5 to February 11 you must add 36 days (that is, 31 days to go to February 5, and 6 more to go to February 11), so you'll end up in Day 3+36=39, which is equal to 4 modulo 7. So February 11 is going to be a Day 4, that is, ***a Thursday***.

Suppose today, January 5, is a Wednesday. Whay day of the week is July 5 going to be, assuming that we are NOT in a leap year? ANSWER: today is Day 3; to go from January 5 to July 5 we must add 31 days (to get to February 5), then 28 days (to get to March 5, using the fact that we are not in a leap year), then 31 days (to get to April 5), then 30 days (to get to May 5), then 31 days (to get to June 5), and, finally, 30 days (to get to July 5). So we end up in Day $3+31+28+30+31+30$ which, modulo 7, is the same as $3+3+0+2+3+2$, that is, 13, that is, 6 (modulo 7). So ***July 5 is going to be a Saturday***. $\diamondsuit$

7.c. Naturally, one could consider integers modulo any other number, not just 12 or 24. For example, $\mathbb{Z}_3$ is the set of **integers modulo** 3. So $\mathbb{Z}_3$ has exactly three members, namely,

$$0\,, \qquad 1\,, \qquad \text{and} \qquad 2\,.$$

Arithmetic modulo 3 is very simple. The addition table is as follows:

$$\begin{aligned} 0+0 &= 0\,, \\ 0+1 &= 1+0=1\,, \\ 0+2 &= 2+0=2\,, \\ 1+1 &= 2\,, \\ 1+2 &= 2+1=0\,, \\ 2+2 &= 1\,, \end{aligned}$$

and the multiplication table is

$$\begin{aligned} 0.0 &= 0.1=1.0=0.2=2.0=0\,, \\ 1.1 &= 2.2=1\,, \\ 1.2 &= 2.1=2\,. \end{aligned}$$

Remark 5. To say that an equality holds in $\mathbb{Z}_n$, we can say just that, or we can say that the equality holds (or is true) "modulo n."

For example, we can say that

$$5+8=2 \qquad \text{in} \quad \mathbb{Z}_{11}\,,$$

or that

$$5+8=2 \qquad \text{modulo} \quad 11\,.$$

Also, if it is clear from the context that we are working in $\mathbb{Z}_n$, then there is no need to say "in $\mathbb{Z}_n$" or "modulo n." (Example: after we said in Page 18 that "the following equalities are true in $\mathbb{Z}_{12}$," we then just went ahead and wrote things like $8.7 = 8$, without explaining again that this was meant to be "in $\mathbb{Z}_{12}$," or "modulo 12." There was no need, because we had already stated clearly that we were working in $\mathbb{Z}_{12}$. $\diamondsuit$

Remark 6. If you look at the way we defined the integers modulo n, you may think that the integers that are larger than n do not belong to $\mathbb{Z}_n$. (For example, the integer 8 does not belong to $\mathbb{Z}_7$, because $\mathbb{Z}_7$ consists of the integers 0, 1, 2, 3, 4, 5, and 6, and nothing else.) We can, however, assign a value to a symbolic expression such as 8 in $\mathbb{Z}_7$, by letting

$$8 = 1+1+1+1+1+1+1+1\,,$$

where the sum is understood to be in $\mathbb{Z}_7$. Then, in $\mathbb{Z}_7$, the expression "8" denotes the number 1, and we can write

$$8 = 1 \qquad \text{in} \qquad \mathbb{Z}_7\,.$$

Notice that

$$8 \neq 1 \qquad \text{in} \qquad \mathbb{Z}\,.$$

So the symbol 8, when we are working in $\mathbb{Z}_7$, is the name of a thing which is not the integer 8. (If it was the same, then "$8 \neq 1$" would also be true in $\mathbb{Z}_7$.)

One can also talk about *negatives* of integers modulo n. What do we mean by "the negative" of an integer modulo n? We just mean "the integer modulo n that added in $\mathbb{Z}_n$ to our given integer modulo n yields 0."

For example, what is the negative of 3 in $\mathbb{Z}_7$? Well, in $\mathbb{Z}_7$ the sum of 3 and 4 is 0. Therefore **the negative of** 3 **in** $\mathbb{Z}_7$ **is** 4. We can also write this using the symbol "$-$":

$$-3 = 4 \qquad \text{in} \qquad \mathbb{Z}_7\,,$$

or

$$-3 = 4 \qquad \text{modulo} \quad 7\,.$$

$\diamondsuit$

An important application of arithmetic modulo 3 is that it enables us to find a simple criterion to check if a given integer is divisible by 3. The key fact here is that $10 = 1$ modulo 3. This means that if we are working modulo 3 then a big number such as $4,687$ is just the same (in $\mathbb{Z}_3$) as $4+6+8+7$, i.e. as 25, which is equal to 1 in $\mathbb{Z}_3$. So $4,687$ is not divisible by 3.

BOX NO. 5: When is a natural number divisible by 3?

Take some natural number like $16,547$. Is it divisible by 3? Naturally, you can do the division and see what happens. But can you tell quickly and directly?
When we say that a number is "divisible by 3" what we mean is that if you start subtracting 3s from it until you cannot do that any more (that is, until you get to 0, 1 or 2), then you end up with 0. In other words, ***our number is equal to*** 0 ***modulo*** 3**.**
So in our example we need to find out whether $16,547$ is equal to 0, 1 or 2 modulo 3. Now:

$$16,547 = 1 \times 10,000 + 6 \times 1,000 + 5 \times 100 + 4 \times 10 + 7 \times 1\,.$$

Also, 10 is just equal to 1 modulo 3. So $100 = 10 \times 10 = 1 \times 1 = 1$ modulo 3. And $1,000 = 10 \times 10 \times 10 = 1 \times 1 \times 1 = 1$ modulo 3. And so on. Therefore, the following is true modulo 3:

$$16,547 = 1 + 6 + 5 + 4 + 7 = 23 = 2\,.$$

So $16,547$ ***is not divisible by*** 3**.**
The point of the above argument is that

$$\begin{aligned} 10,000 &= 1 \quad \text{modulo } 3\,,\\ 6,000 &= 6 \quad \text{modulo } 3\,,\\ 500 &= 5 \quad \text{modulo } 3\,,\\ 40 &= 4 \quad \text{modulo } 3\,,\\ 7 &= 7 \quad \text{modulo } 3\,, \end{aligned}$$

so

$$16,547 = 1 + 6 + 5 + 4 + 7 = 23 = 2 \qquad \text{modulo } 3.$$

The general criterion is this: **to find out if an integer is divisble by 3, just add the digits in its decimal expression. If the sum of the digits is divisible by 3, then the given number is divisible by 3. Otherwise it is not.**

QUESTION 1. Which of the following three natural numbers are divisible by 3?

$$\begin{aligned} &999,111,999,111\\ &543,043,664,987\\ &6,915,571,375,016,461,671,444,102,534 \end{aligned}$$

BOX NO. 6: When is a natural number divisible by 11?

The situation here is very similar to what we found when we looked at divisibility by 3, but with an important twist. Now 10 is not equal to 1 modulo 11, but to -1. So if we look at a natural number like $16,547$, the 7 should count as a 7 modulo 11, the 4 as a -4, the 5 as 5, the 6 as a -6, and the 1 as a 1.
(Reason: $16,547 = 1\times 10,000+6\times 1,000+5\times 100+4\times 10+7$. Modulo 11, we have $10 = -1$, $100 = 10^2 = 1$, $1,000 = -1$, $10,000 = 1$. So, $16,547 = 1-6+5-4+7$ modulo 11.) Therefore $16,547 = 3$ modulo 11, and then $16,547$ ***is not divisible by*** 11***.***

The general criterion is this: ***to find out if an integer is divisble by*** 11***, just add the digits in its decimal expression with alternating signs (plus, minus, plus, minus, and so on). If the alternating sum of the digits is divisible by*** 11***, then the given number is divisible by*** 11***. Otherwise it is not.***

QUESTION 2. Which of the three natural numbers of the previous question are divisible by 11?

QUESTION 3. Formulate and justify a criterion for testing if a number is divisible by 9.

QUESTION 4. Is the following true or false? If you take a natural number, reverse the order of its decimal figures, and subtract the smaller of the numbers you got from the larger, then what you get is divisible by 3. (Here are two examples: (1) start with 925, then get 529 by reversing the order of the figures, then subtract, and you get $925-529$, which is equal to $\boxed{396}$; (2) start with 386, then get 683 by reversing the order of the figures, then subtract, and you get $683-386$, which is equal to $\boxed{297}$. In both cases, we got numbers that are divisible by 3. The question for you is: is this going to to happen no matter which natural number we take?)

QUESTION 5. Is the following true or false? If you take a natural number, reverse the order of its decimal figures, and subtract the smaller of the numbers you got from the larger, then what you get is divisible by 11. (Here are two examples: (1) start with 925, then get 529 by reversing the order of the figures, then subtract, and you get $925-529$, which is equal to $\boxed{396}$; (2) start with 386, then get 683 by reversing the order of the figures, then subtract, and you get $683-386$, which is equal to $\boxed{297}$. In both cases, we got numbers that are divisible by 11. The question for you is: is this going to to happen no matter which natural number we take?)

QUESTION 6. Is the following true or false? If you take a natural number, reverse the order of its decimal figures, and add the two numbers, then what you get is divisible by 3. (Here are two examples: (1) start with 321, then get 123 by reversing the order of the figures, then add, and you get $321 + 123$, which is equal to $\boxed{444}$; (2) start with 801, then get 108 by reversing the order of the figures, then add, and you get $801 + 108$, which is equal to $\boxed{909}$. In both cases, we got numbers that are divisible by 3. The question for you is: is this going to to happen no matter which natural number we take?)

7.d. Even simpler is $\mathbb{Z}_2$, the set of **integers modulo** 2. Clearly, $\mathbb{Z}_2$ has exactly two members, namely,

$$0 \qquad \text{and} \qquad 1\,.$$

Arithmetic modulo 2 is so ridiculously simple that we can fit the complete addition and multiplication tables together in one line:

$$0+0=0\,,\ 0+1=1+0=1\,,\ 1+1=0\,,\ 0.0=0.1=1.0=0\,,\ 1.1=1\,.$$

QUESTION 7. Does the equation $x^2 + 1 = 0$ have a solution in $\mathbb{Z}_7$?

ANSWER. Let us write down explicitly the squares of all members of $\mathbb{Z}_7$:

$$0^2=0\,,\ 1^2=1\,,\ 2^2=4\,,\ 3^2=2\,,\ 4^2=2\,,\ 5^2=4\,,\ 6^2=1\,.$$

The only way for $x^2 + 1$ to be equal to zero would be for x^2 to be equal to 6. But in the above list we see clearly that there is no member of $\mathbb{Z}_7$ whose square is 6. So the answer is **NO**.

QUESTION 8. Does the equation $x^2 + 1 = 0$ have a solution in $\mathbb{Z}_{17}$?

ANSWER. In $\mathbb{Z}_{17}$, $4.4 = 16$, so $4.4 + 1 = 16 + 1 = 0$. So the answer is **YES**.

QUESTION 9. Does the equation $x^2 + 1 = 0$ have a solution in $\mathbb{Z}_{101}$?

ANSWER. In $\mathbb{Z}_{101}$, $10^2 + 1 = 101 = 0$, so the answer is **YES**.

QUESTION 10. Does the equation $x^2 + 1 = 0$ have a solution in $\mathbb{Z}_{103}$?

COMMENT. This is not easy. It turns out that the answer is "no." Of course, you could verify this by listing the squares of all 103 members of $\mathbb{Z}_{103}$ and checking that you never get 102 (which is -1 in $\mathbb{Z}_{103}$). But this would be a lot of work. There ought to be a smarter way. Maybe we can **prove** that the answer is "no" without having to do lengthy boring computations? Try to think about it. We will come back to this problem later.

QUESTION 11. In $\mathbb{Z}_{24}$, are there two nonzero numbers whose product is equal to zero?

ANSWER. In $\mathbb{Z}_{24}$, $8.3 = 0$. So the answer is **YES**.

QUESTION 12. In $\mathbb{Z}_4$, does the equations $a^2 + b^2 = 3$ have a solution?

ANSWER. Here are the values of $a^2 + b^2$ for all possible choices of $a, b \in \mathbb{Z}_4$,

$$\begin{aligned} 0^2 + 0^2 &= 0\,, \\ 0^2 + 1^2 = 1^2 + 0^2 &= 1\,, \\ 0^2 + 2^2 = 2^2 + 0^2 &= 1\,, \\ 1^2 + 1^2 &= 2\,, \\ 1^2 + 2^2 = 2^2 + 1^2 &= 2\,, \\ 2^2 + 2^2 = &= 0\,. \end{aligned}$$

We see that the right-hand side is never equal to 3. So the answer is **NO**.

QUESTION 13. In $\mathbb{Z}_{17}$, are there two nonzero numbers whose product is equal to zero?

ANSWER. If you multiply two numbers in $\mathbb{Z}_{17}$, the only way the product can be equal to zero in $\mathbb{Z}_{17}$ is if, when you reduce modulo 17, you get zero. That is, the product of these two numbers must be divisible by 17. But 17 is a prime number, so there is no way to multiply two nonzero numbers smaller than 17 and get a product which is divisible by 17. So the answer is **NO**.

Remark 7. Clearly, the class of number systems presented in this subsection consists of *infinitely many* "number systems," namely, the systems

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_5, \ldots, \quad \mathbb{Z}_{397}, \ldots, \quad \text{and so on}\,. \qquad \diamondsuit$$

3.4 The quaternions

In 1843 **W. R. Hamilton** (1805-1865) invented the **quaternions**, which we include as the eighth item in our list.

8. The set $\mathbb{K}$ of all **quaternions**. Quaternions are like complex numbers, but with four components rather than two. So for example $3 + 4i - 7j + 25k$ and $-1 + 2i + 9j + k$ are quaternions. To multiply quaternions, you just multiply as you would ordinary real or complex numbers, except that when you run into products such as $i.j$ or $k.i$ you rewrite them using the following "multiplication table":

$$\begin{aligned} i.j &= k\,, & j.i &= -k\,, \\ j.k &= i\,, & k.j &= -i\,, \\ k.i &= j\,, & i.k &= -j\,, \\ i.i = j.j &= k.k = -1\,. \end{aligned}$$

For example, suppose that we want to multiply

$$(3 + 4i - 7j + 25k).(-1 + 2i + 9j + k)\,.$$

Here is the calculation in full:

$$
\begin{aligned}
&(3+4i-7j+25k).(-1+2i+9j+k)\\
&\quad = 3.(-1)+3.2.i+3.9.j+3.k+4.(-1).i+4.2.i.i+4.9.i.j+4.i.k\\
&\qquad +(-7).(-1).j+(-7).2.j.i+(-7).9.j.j+(-7).j.k+25.(-1).k\\
&\qquad +25.2.k.i+25.9.k.j+25.k.k\\
&\quad = -3+6i+27j+3k-4i+8i.i+36i.j+4i.k+7j\\
&\qquad -14j.i-63j.j-7j.k-25k+50k.i+225k.j+25k.k\\
&\quad = -3+6i+27j+3k-4i-8+36k-4j+7j\\
&\qquad +14k+63-7i-25k+50j-225i-25\\
&\quad = (-3-8+63-25)+(6-4-7-225)i+(27-4+7+50)j\\
&\qquad +(3+36+14-25)k\\
&\quad = \quad 27-230i+80j+28k\,.
\end{aligned}
$$

BOX NO. 7: Hamilton discovers the quaternions.

Hamilton was knighted in 1835 and that year his second son, Archibald Henry, was born. But the next few years did not bring him much happiness. After the discovery of algebraic couples, he tried to extend the theory to triplets, and this became an obsession that plagued him for many years. The following autumn he went to Bristol for a meeting of the British Association, and Helen took the children with her to Bayly Farm for ten months. His cousin Arthur died, and not long after Helen returned from her mother's she went away again to England this time leaving the children behind after the birth of a daughter, Helen Eliza Amelia. At this point, William became depressed and started to have problems with alcohol so his sister came back to live at Dunsink.
Helen returned in 1842 when Hamilton was so preoccupied with the triplets that even his children were aware of it. Every morning they would inquire:

Well, Papa can you multiply triplets?

but he had to admit that he could still only add and subtract them.
On 16 October 1843 (a Monday) Hamilton was walking in along the Royal Canal with his wife to preside at a Council meeting of the Royal Irish Academy. Although his wife talked to him now and again Hamilton hardly heard, for the discovery of the quaternions, the first noncommutative algebra to be studied, was taking shape in his mind:

> *And here there dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples ... An electric circuit seemed to close, and a spark flashed forth.*

He could not resist the impulse to carve the formulae for the quaternions

$$i = j = k = ijk = -1$$

in the stone of Brougham Bridge as he and his wife passed it. Hamilton felt this discovery would revolutionise mathematical physics and he spent the rest of his life working on quaternions. He wrote:-

> *I still must assert that this discovery appears to me to be as important for the middle of the nineteenth century as the discovery of fluxions [the calculus] was for the close of the seventeenth.*

3.5 Reading formulas precisely

It is important that you acquire from the very beginning the habit of reading mathematical formulas aloud precisely and correctly. For example, the correct way to read the formula

$$3 \in \mathbb{N} \tag{3.5.2}$$

aloud is

$$3 \text{ belongs to } \mathbb{N}$$

or

$$3 \text{ is a member of } \mathbb{N}$$

or

$$3 \text{ is in } \mathbb{N}$$

or

$$3 \text{ is a natural number.}$$

But you **cannot** read (3.5.2) as, for example, "3 is contained in $\mathbb{N}$." "Is contained" means something else!

Here is another example: Formula (3.1.1) is read as

$$\mathbb{N} \text{ is the set consisting of 0, 1, 2, 3, 4, etc.}$$

It would be **wrong** (**very** wrong!) to read it as "$\mathbb{N}$ is 0, 1, 2, 3, 4, etc." Do you see why this is bad? The best way to understand it is to think of more familiar examples.

For example, you cannot say that "the U.S. Supreme Court is Chief Justice Rehnquist and Justice Scalia and Justice Thomas and Justice O'Connor and Justice Souter and Justice Kennedy and Justice Ginsburg and Justice Breyer and Justice Stevens." Why not? Because the U.S. Supreme Court is **one single thing, not nine things!** What you *can* say is:

> The U.S. Supreme Court is a court whose members are Chief Justice Rehnquist and Justices Scalia, Thomas, O'Connor, Souter, Kennedy, Ginsburg, Breyer, and Stevens.

Similarly, the set $\mathbb{N}$ of natural numbers is **one single object, not many objects**, so you can say "$\mathbb{N}$ is the set consisting of 0, 1, 2, 3, 4, etc." but you cannot say "$\mathbb{N}$ is 0, 1, 2, 3, 4, etc."

Also, you cannot say "$\mathbb{N}$ is the natural numbers," or "$\mathbb{N}$ is natural numbers." (Actually, "$\mathbb{N}$ is the natural numbers" is bad, but "$\mathbb{N}$ is natural numbers" is even worse. Why?) What you should say is "$\mathbb{N}$ is the set of natural numbers," or "$\mathbb{N}$ is the set of all natural numbers."[4]

4 Some remarks on mathematical writing

When we talk about mathematical objects, we often use *letters* (or other special symbols such as $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, 0, 1, 2, etc.). The way letters are used in mathematical discourse may often appear mysterious to the students. The first thing you need to do is to realize that it is not mysterious at all. It is, in fact, quite simple!

[4] Recommended reading: the short story "Pigs is pigs," by Ellis Parker Butler. You can find it in the Web site *www.bookvalley.com/cgibin/bv?b=162*. It's funny, and one of the things that make it funny is precisely the misuse of the verb "is" in "Pigs is pigs."

Letters are only used in mathematics in order to shorten certain statements that would otherwise be too long.

4.1 Some examples

Example 1. Suppose you want to define the word "cousin." You could say:

> A person is said to be a *cousin* of another person if one of the first person's parents is a sibling of one of the second person's parents. (4.1.1)

This is O.K., but having to talk about the "first person" and the "second person" and having to repeat the word "person" so many times makes it awkward. So what we can do is give names to those persons, for example x and y, and say the following:

> Let x and y be persons. Then x is said to be a *cousin* of y if one of x's parents is a sibling of one of y's parents. (4.1.2)

Example 2. Suppose you want to say

> If you take two real numbers, add them, and square the sum, then you get the same result as if you had squared each number separately, added the squares, and added to the result twice the product of the two numbers. (4.1.3)

This is awfully complicated, awkward, and ugly. You can make it much shorter by saying

> Let x, y be two real numbers. Then $(x+y)^2 = x^2 + y^2 + 2xy$. (4.1.4)

or

> For all real numbers x, y, $(x+y)^2 = x^2 + y^2 + 2xy$. (4.1.5)

We can say (4.1.5) even more concisely, by writing

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \Big((x+y)^2 = x^2 + y^2 + 2xy\Big). \tag{4.1.6}$$

Here the symbol $\forall$ is the **universal quantifier**. Statement (4.1.6) is read as follows:

> For all real numbers x and for all real numbers y, the square of x plus y is equal to the sum of the square of x, the square of y, and twice the product of x and y. (4.1.7)

Actually **you can even read (4.1.6) without any letter variables at all.** How? Well, that's precisely what (4.1.3) does! ♢

Example 3. Suppose you want to say

If you take the number $\frac{\sqrt{5}-1}{2}$, and add it to its square, you get 1. (4.1.8)

You could say this as follows:

$$\Big(\frac{\sqrt{5}-1}{2}\Big)^2 + \frac{\sqrt{5}-1}{2} = 1\,. \tag{4.1.9}$$

Or you could give a name to the number, say a, and write

Let $a = \frac{\sqrt{5}-1}{2}$. Then $a^2 + a = 1$. (4.1.10)

Example 4. Suppose you want to say

It is possible to find a real number whose cube, added to its square, is equal to the number itself. (4.1.11)

Again, this looks horribly complicated and ugly, and you can make it much shorter by saying

There exists a real number x such that $x^2 + x^3 = x\,.$ (4.1.12)

or

There is a real number x such that $x^2 + x^3 = x\,.$ (4.1.13)

or

$x^2 + x^3 = x$ for some real number x. (4.1.14)

We can say this thing even more concisely, by writing

$$\exists x \in \mathbb{R}, (x^2 + x^3 = x)\,. \tag{4.1.15}$$

Here the symbol $\exists$ is the **existential quantifier**. Statement (4.1.15) is read as follows:

There exists a real number x such that $x^2 + x^3 = x$. (4.1.16)

There exists a real number x such that the sum of the square of x and the cube of x cube equals x. (4.1.17)

or even, if you don't want to use any letter variables at all

There exists a real number such that the sum of the square of the number plus the cube of the number equals the number itself. (4.1.18)

or, if you prefer, you can read it as statement (4.1.11) above. ♢

QUANTIFIERS

The symbols $\exists$ and $\forall$ are the **quantifiers**.

" $\exists$ " is the **existential quantifier**.

" $\forall$ " is the **universal quantifier**.

A statement of the form "$\exists x$,blablabla-about-x" is an **existential statement**. We read it "There exists x such that blablabla-about-x."

A statement of the form "$\forall x$,blablabla-about-x" is a **universal statement**. We read it "For all x blablabla-about-x."

EXAMPLE: What kind of statement is "$\forall x, \exists y, y < x$"?
ANSWER: It is a universal statement, because it says that "for all , x something is true." The "something" that is true about x is an *existential statement*, namely, that "there exists a y such that $y < x$."
Let us be more precise. Let us give a name to our statement, say S. So from now until the end of this box the letter S stands for[a] the statement "$\forall x, \exists y, y < x$." Then what we just said is that S *is a universal statement* and, in fact, S is of the form "$\forall x, T$," where T *is an existential statement*, namely, "$\exists y, y < x$." So T is of the form "$\exists y, W$," where W is the statement "$y < x$."

QUESTION: Suppose we are talking about integers. What does "$\forall x, \exists y, y < x$" say in plain English? Is it true?
ANSWER: it says "for every integer there is a smaller integer." This is true, because given any integer x we can let $y = x - 1$, and then $y < x$.

QUESTION: Suppose we are talking about natural numbers. What does "$\forall x, \exists y, y < x$" say in plain English? Is it true?
ANSWER: it says "for every natural number there is a smaller natural number." This is false, because if we take x to be 0 then there is no natural number y such that $y < x$.

---

[a] Pay attention to what we have just done! Does it look familiar? If it does not, read §4.4 and then come back to this box. You will see that what we have done is this: we introduced a new letter, namely, S, and declared it to be a constant, by declaring its value to be the statement "$\forall x, \exists y, y < x$." This is exactly the same kind of thing we did in Example 3 when we introduced the letter a and declared it to be a constant with value $\frac{\sqrt{5}-1}{2}$. In Example 3 the letter a is used as a *numerical constant*, that is, a constant whose value is a number. Here we are using the letter S as a *statement constant*, that is, a constant whose value is a statement. In both cases, the letters are thing-symbols (that is, they stand for a thing, which is a number in one case and a statement in the other case) and are constants (that is, they stand for a fixed thing).

4.2 A very important remark on reading existential statements

An **existential statement** is a statement that asserts that there exists an object of a certain kind. For example:

Somebody killed Polonius (4.2.1)

is an existential statement, because it says that there is a person who killed Polonius. Using the symbolism of existential quantifiers, we could rewrite (4.2.1) as

$\exists x, x$ killed Polonius. (4.2.2)

The correct way to read the three-character expression "$\exists x,$" (including the comma) is "there exists an x such that". For example:

- you should read (4.2.2) as "there exists x such that x killed Polonius," or "there exists x who killed Polonius."
- a statement of the form "$\exists x \in \mathbb{R}$, something about x" is read as

  There exists a real number x such that something about x. (4.2.3)

  or

  There exists an x belonging to $\mathbb{R}$ such that something about x. (4.2.4)

It is **wrong** (**very** wrong!) to read (4.2.2) as "there exists x and x killed Polonius."

Similarly, you should **never** read "$\exists x \in \mathbb{R}$, something about x" as

There exists a real number x and something about x. (4.2.5)

If you read (4.2.2) as "there exists x and x killed Polonius," or "$\exists x \in \mathbb{R}$, something about x" as in (4.2.5) then you are completely changing the meaning of the statement, and you are not conveying what it actually means.

EXAMPLE: Suppose you want to explain what it means for two people to be "siblings or half-siblings" using an existential quantifier. You could say:

Let x and y be persons. We say that x *and* y *are siblings or half-siblings* if $x \neq y$ and $\exists z, z$ is a parent of x and a parent of y. (4.2.6)

How would you read this in plain English? You would certainly **not** say "there exists a z and z is a parent of x and a parent of y."!

What you would say (somewhat awkwardly, I know!) is "there exists a z who is parent of x and a parent of y," or (even more awkwardly, but still correctly) "there exists a z such that z is a parent of x and a parent of y."

It is important that you acquire as soon as possible the habit of using "such that" correctly. Please start right now! ♢

RECOMMENDED QUESTION FOR YOU TO THINK ABOUT: If I were to say "There exists a real number such that $x^2 + x^3 = x$," this would **not** be acceptable. Why? What is missing?

ANOTHER RECOMMENDED QUESTION FOR YOU TO THINK ABOUT: Is (4.1.15) true? Why?

4.3 More examples

Example 5. Suppose you want to say

> For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result (that is, 1 over the result). Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20. (4.3.1)

Once again, this is awful, so you can shorten the whole thing and just say

$$\text{Let } N = \textstyle\sum_{k=2}^{25} \frac{1}{k^2+5}\,. \text{ Then } N < 20. \tag{4.3.2}$$

Or, if you prefer, you could just say:

$$\textstyle\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,. \tag{4.3.3}$$

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: Is it true or false that $\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$? How about $\sum_{k=2}^{25} \frac{1}{k^2} < 20$? And $\sum_{k=2}^{25} 1 < 20$? And what would you say if I ask you whether it's true that $\sum_{n=2}^{25} \frac{1}{k^2+5} < 20$?

Example 6. Suppose you want to say

> If you take the function that to each real number assigns its cube, and integrate it from 0 to 4, then the result is 64. (4.3.4)

Again, this can be made much shorter by saying

$$\text{Let } N = \textstyle\int_0^4 x^3\,dx\,. \text{ Then } N = 64. \tag{4.3.5}$$

Or you could just say:

$$\textstyle\int_0^4 x^3\,dx = 64\,. \tag{4.3.6}$$

RECOMMENDED QUESTIONS FOR YOU TO THINK ABOUT: Is it true or false that $\int_0^4 x^3\,dx = 64$? Is it true or false that $\int_0^4 y^3\,dx = 64$?

Example 7. Suppose you want to say:

> The set of all real numbers whose square is larger than the cube of the number has at least two different members. (4.3.7)

We can make this shorter by saying

> Let S be the set of all real numbers x such that $x^2 > x^3$. Then S has at least two members. (4.3.8)

or

> Let $S = \{x \in \mathbb{R} : x^2 > x^3\}$. Then S has at least two members. (4.3.9)

Or you could just say:

$$\text{Let } S = \{x \in \mathbb{R} : x^2 > x^3\}. \text{ Then } \exists x \exists y, (x \in S, y \in S, \text{ and } x \neq y). \quad (4.3.10)$$

Example 8. Suppose you want to say that

> I am older than 20 and less than 70, and my brother is younger than I am, but his age is larger than half my age. (4.3.11)

Here too we can be more concise, and say

> Let x be my age and let y be my brother's age. Then $20 < x < 70$ and $\frac{x}{2} < y < x$. (4.3.12)

4.4 Constants and variables

In all the examples we have discussed so far, we have used **special symbols** (most commonly letters, such as x, y, a, k, N, n, S) as labels, so that we can avoid writing clumsy phrases such as "the number that we have been talking about." Sometimes we use other special symbols, such as Greek letters (α, β, ε, etc.), letters modified in a particular way (for example $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$), numerals (0, 1, 2, ...)

These special symbols stand for *objects*, or *things* of various kinds (numbers, human beings, sets), so we are going to call them *individual symbols*, or *thing-symbols*. (There are other kinds of mathematical symbols that do not stand for things. For example: the universal quantifier $\forall$, the existential quantifier $\exists$, the symbol + for the addition operation. We will talk about many of these symbols later.)

All the thing-symbols we have used are either

constants or **variables** (also called **dummy variables**).

A *constant* is a symbol that stands for a thing that stays fixed throughout an explanation or argument.

For instance, in Example 3, the letter a stands for the number $\frac{\sqrt{5}-1}{2}$. We made this clear by **introducing** the letter a by means of the **declaration**

$$\text{Let } a = \frac{\sqrt{5}-1}{2}.$$

That is, we **declared** the value of a to be $\frac{\sqrt{5}-1}{2}$, so from that moment on we know that a stands for "$\frac{\sqrt{5}-1}{2}$."

Variables (also called "dummy variables") are letters used as an *internal label* within a statement, but which do not have a value outside this statement.

This may look complicated, but it is in fact quite simple. Actually, it is something you are probably used to if you write computer programs in a language like Pascal. You often introduce a variable to keep track of some number with which you perform several operations. For example, you may want to keep track of a list of 20 numbers, called list[1], list[2], ..., up to list[20]. Suppose you want to add the 20 numbers, and end up with a number called "sum" as the result. You could write the following program:

begin

sum := 0;

For $i := 1$ to 20 do sum := sum + list[i];

end.

In this program, the letter i is just a label used to keep track of the variable index that keeps changing all the time. You could have written instead

begin

sum := 0;

For $k := 1$ to 20 do sum := sum + list[k];

end.

and you would have achieved the same thing. *Within the program*, the letter i (or k in our second version) has a value that keeps changing all the time. *Outside the program*, it does not make sense to ask "what is the value of i?"

Notice that what we are doing with this program is computing the sum

$$\text{Sum} = \sum_{i=1}^{20} \text{list}[i]\,,$$

which is exactly the same as saying

$$\text{Sum} = \sum_{k=1}^{20} \text{list}[k]\,,$$

or

$$\text{Sum} = \sum_{\alpha=1}^{20} \text{list}[\alpha]\,,$$

or

$$\text{Sum} = \sum_{\clubsuit=1}^{20} \text{list}[\clubsuit]\,.$$

The distinctive feature of a dummy variable is that you could have used any other symbol instead, and you would be saying exactly the same thing. For example, if you write

$$\text{For all real numbers } x,\ y,\ (x+y)^2 = x^2 + y^2 + 2xy\,, \tag{4.4.1}$$

or

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \left((x+y)^2 = x^2 + y^2 + 2xy\right), \tag{4.4.2}$$

then that has exactly the same meaning as you had written

$$\text{For all real numbers } a,\ b,\ (a+b)^2 = a^2 + b^2 + 2ab\,, \tag{4.4.3}$$

or

$$\forall a \in \mathbb{R}, \forall b \in \mathbb{R}, \left((a+b)^2 = a^2 + b^2 + 2ab\right), \tag{4.4.4}$$

or

$$\text{For all real numbers } \alpha,\ \beta,\ (\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta\,, \tag{4.4.5}$$

or

$$\forall \alpha \in \mathbb{R}, \forall \beta \in \mathbb{R}, \left((\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta\right), \tag{4.4.6}$$

or

$$\text{For all real numbers } m,\ n,\ (m+n)^2 = m^2 + n^2 + 2mn\,, \tag{4.4.7}$$

or

$$\forall m \in \mathbb{R}, \forall n \in \mathbb{R}, \left((m+n)^2 = m^2 + n^2 + 2mn\right), \tag{4.4.8}$$

or

$$\text{For all real numbers } \diamond,\ \#,\ (\diamond+\#)^2 = \diamond^2 + \#^2 + 2\diamond\#\,. \tag{4.4.9}$$

or

$$\forall \diamond \in \mathbb{R}, \forall \# \in \mathbb{R}, \left((\diamond+\#)^2 = \diamond^2 + \#^2 + 2\diamond\#\right). \tag{4.4.10}$$

Similarly, instead of $\exists x \in \mathbb{R}, \left(x^2 + x^3 = x\right),$

you could have said $\exists z \in \mathbb{R}, \left(z^2 + z^3 = z\right),$

or $\exists a \in \mathbb{R}, \left(a^2 + a^3 = a\right),$

or $\exists \omega \in \mathbb{R}, \left(\omega^2 + \omega^3 = \omega\right),$

or $\exists \aleph \in \mathbb{R}, \left(\aleph^2 + \aleph^3 = \aleph\right),$

or $\exists \diamond \in \mathbb{R}, \left(\diamond^2 + \diamond^3 = \diamond\right).$

Just one more example:

Instead of $\displaystyle\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$ you could have said $\displaystyle\sum_{n=2}^{25} \frac{1}{n^2+5} < 20\,,$

or $\displaystyle\sum_{\rho=2}^{25} \frac{1}{\rho^2+5} < 20\,,$ or $\displaystyle\sum_{GNU=2}^{25} \frac{1}{GNU^2+5} < 20\,.$

The "dummy variables" in the above examples (that is, i, k, α, $\clubsuit$, x, y, a, b, β, m, n, $\diamond$, $\#$, z, ω, $\aleph$, ρ, GNU) are really just **internal tags**, using to keep track of numbers **within a particular statement**. Outside this text, they variables do not have any meaning. For example, after you have written

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20\,,$$

it would not make sense for me to ask you "what is the value of k?" The letter k is just a tag to specify a *range of values*, as part of a set of instructions on how to compute a sum. Any other letter could be used instead, so in fact "k isn't really there." You can persuade yourself of this in another way, by remembering that

$$\sum_{k=2}^{25} \frac{1}{k^2+5} < 20$$

is just another way of saying

- For each integer between 2 and 25, take that integer, square it, add 5, and compute the multiplicative inverse of the result. Having done this for all the integers between 2 and 25 (including 2 and 25 themselves), add all the resulting numbers. Then the number you get is smaller than 20.

As you can see, there is no value of k there!

Similarly, when we say $\int_0^4 x^3\,dx = 64$, this is just a way of saying "if you take the function that to each real number assigns its cube, and integrate it from 0 to 4, you get 64." Once again, there is no value of x there! You could equally well have said $\int_0^4 y^3\,dy = 64$, or $\int_0^4 u^3\,du = 64$, or $\int_0^4 \aleph^3\,d\aleph = 64$.

And when we say $\exists x \in \mathbb{R}, x^2 = 2$, this is just a way of saying "there is a real number whose square is 2" (or, even better, "2 has a square root"). This statement does not talk about x.

This way of speaking and writing, using dummy variables, is very useful, especially when what we are trying to say is quite long. But **you have to be careful** when writing this way, and have to make sure that you are using the letters correctly.

A thing-symbol can only be used if either its value has been declared before or it is a dummy variable.

4.5 Examples of things you should NOT write

Suppose you write

$$x^2 + x = 90$$

just like that, without any words before or after explaining what x is, or making it a dummy variable. Then what you wrote is **meaningless**[5]. We have no way to tell if what you wrote is true or false, because we have no idea what you are talking about.

If you write something like this in your homework, then I will write in big red letters "**what is x?**"

For comparison, here are some perfectly meaningful things you could have said:

1. Let $x = 9$. Then $x^2 + x = 90$.
2. Let $x = 6$. Then $x^2 + x = 90$.
3. Let x be the number of planets of the solar system. Then $x^2 + x = 90$.
4. $x^2 + x = 90$ for every real number x.
5. $x^2 + x = 90$ for some real number x.

[5]It's more or less the same situation as if somebody walked into our classroom and said "she has arrived." If we already know who "she" is supposed to be then this makes sense. But if we don't then our reaction would probably be to ask "who are you talking about?" The word "she" is a *pronoun*. It stands for somebody, and a sentence using "she" is meaningless if we do not know which person "she" stands for. Similarly, the letter x in "$x^2 + x = 90$" functions like a pronoun. The sentence "$x^2 + x = 90$" is meaningless if you do not know who—or what— x stands for.

6. $\forall x \in \mathbb{R}, (x^2 + x = 90)$.

7. $\exists x \in \mathbb{R}, (x^2 + x = 90)$.

8. The equation $x^2 + x = 90$ has a solution $x \in \mathbb{R}$. (Or "the equation $x^2 + x = 90$ has a real solution.")

Some of these are true, while the others are false. (Precisely: 1,3,5,7,8 are true; 2,4,6 are false. Moreover: 6 is just another way of saying the same thing as 4, and 7 and 8 are just alternative ways of stating 5.)

Notice that in 1, 2 and 3 the letter x is a constant: its value is declared before the letter is used to make a statement. In 4, 5, 6, 7 and 8 the letter x is a dummy variable.

Again for comparison, let us now write some things that do **not** have a completely precise meaning:

A. The equation $x^2 = 2$ has a solution.

B. The equation $x^2 + 1 = 0$ has a solution.

C. The equation $3x = 5$ has a solution.

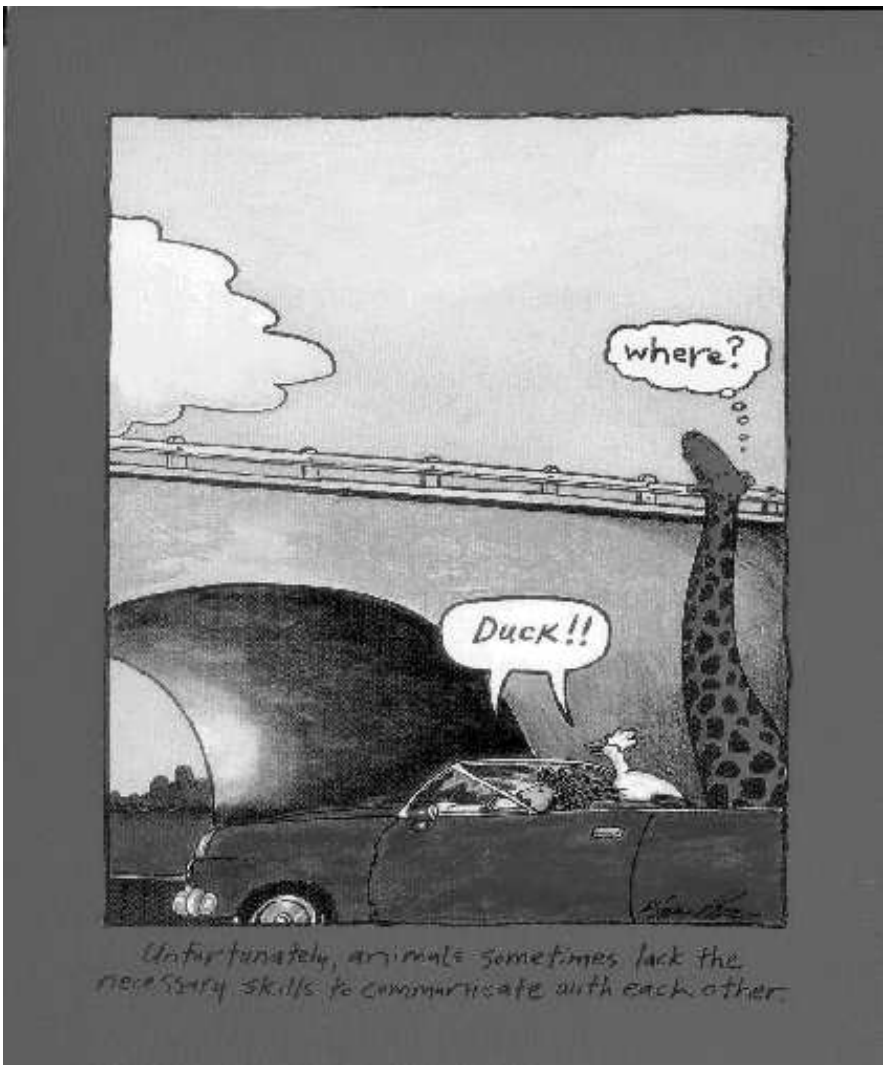
Why are these things not good enough? Because they do not specify the **range** of values within which the solution x is to be sought. For example, A is true if we look for solutions $x \in \mathbb{R}$, but false if we look for solutions $x \in \mathbb{Q}$. B is true if we look for solutions $x \in \mathbf{C}$ (the set of complex numbers) but false if we look for solutions $x \in \mathbb{R}$. C is true if we look for solutions $x \in \mathbb{Q}$ (or $x \in \mathbb{R}$) but false if we look for solutions $x \in \mathbb{Z}$.

On the other hand, the following are perfectly meaningful statements:

A1. The equation $x^2 = 2$ has a solution $x \in \mathbb{R}$. (That is: $\exists x \in \mathbb{R}, (x^2 = 2)$.)
A2. The equation $x^2 = 2$ has a solution $x \in \mathbb{Q}$. (That is: $\exists x \in \mathbb{Q}, (x^2 = 2)$.)
B1. The equation $x^2+1 = 0$ has a solution $x \in \mathbb{Z}$. (That is: $\exists x \in \mathbb{Z}, (x^2 + 1 = 0)$.)
B2. The equation $x^2+1 = 0$ has a solution $x \in \mathbb{R}$. (That is: $\exists x \in \mathbb{R}, (x^2 + 1 = 0)$.)
B3. The equation $x^2+1 = 0$ has a solution $x \in \mathbf{C}$. (That is: $\exists x \in \mathbf{C}, (x^2 + 1 = 0)$.)
C1. The equation $3x = 5$ has a solution $x \in \mathbb{Z}$. (That is: $\exists x \in \mathbb{Z}, (3x = 5)$.)
C2. The equation $3x = 5$ has a solution $x \in \mathbb{Q}$. (That is: $\exists x \in \mathbb{Q}, (3x = 5)$.)
C3. The equation $3x = 5$ has a solution $x \in \mathbb{R}$. (That is: $\exists x \in \mathbb{R}, (3x = 5)$.)
C4. The equation $3x = 5$ has a solution $x \in \mathbf{C}$. (That is: $\exists x \in \mathbf{C}, (3x = 5)$.)

(Actually, A1, B3, C2, C3 and C4 are true, whereas A2, B1, B2 and C1 are false. Of course, in each case you could say the same thing in lots of different ways. For example, instead of C2 you could say "the equation $3x = 5$ has a rational solution," or "the equation $3x = 5$ can be solved in the set of rational numbers," or "there is a rational number that satisfies the equation $3x = 5$," or "$3x = 5$ is solvable in the set of rational numbers.")

4.6 Ambiguous sentences



A sentence whose meaning is imprecise because it can be interpreted in more than one way is said to be *ambiguous*. For example, the sentences

> Flying planes can be dangerous
> College students are turning to vegetables
> The University president promised to stop drinking on campus
> I shot an elephant in my pajamas[6]

[6] "Last night I shot an elephant in my pajamas. How the elephant got into my pajamas I have no idea." (Groucho Marx)

Mary had a little lamb[7]
George W. Bush and Hillary R. Clinton are married

are ambiguous. Why? Think of other examples of ambiguous sentences.

Similarly, A, B, C above are ambiguous. Another example of ambiguous sentence is a sentence where a letter does not have a declared value and is not a dummy variable either. For example, if I tell you to "read chapter x," then that's ambiguous—unless I have told you before what "x" stands for, or given you clues so you can find out, or perhaps I have said "read Chapter x for all x" (that is "read all the chapters") or "read Chapter x for some x" (that is "read one chapter")— because you don't know what x is supposed to be.

Avoid ambigous sentences!

4.7 Words are important

Formulas without words are usually meaningless and should be avoided like the plague.

It's the **words** that glue the formulas together and give a structure and a meaning to mathematical writing. If you just write a bunch of formulas then you probably do not know what you are saying, and if you don't know then you should not expect that the instructor or the grader will know.

Look, for example, at Statements 1, 2, 4 and 5 above. In all cases, the "formula" $x^2 + x = 90$ is the same, but the statements are quite different and have totally different meanings. That's because the words are different!

[7] Mary had a little lamb,
Its fleece was white as snow,
Everywhere that Mary went,
The lamb was sure to go.
But also:
Mary had a little lamb,
A loaf of bread, a pot of jam,
An ice cream soda topped with fizz,
And oh! - How sick our Mary is!