MATHEMATICS 300 — SPRING 2005

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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Contents

10 The definition of "limit"

We want to define what "limit" means. Before we do that, we have to make sure we understand what objects the definition of limit will be talking about.

We talk about the limit of a ***function*** as x approaches a number a, having a certain ***value*** L. So the definition of limit will involve ***three ingredients***:

1. a function f,
2. the number $a \in \mathbb{R}$ where we want to find the limit,
3. the number $L \in \mathbb{R}$ which is the purported value of the limit.

10.1 Functions

Let us start by making it clear what a "function" is. A ***function from reals to reals*** is a rule f that enables us to associate to every real number x in a unique way a number $f(x)$ (called the **value of** f **at** x). We will write

$$f : \mathbb{R} \to \mathbb{R}$$

to indicate that f is a function from reals to reals.

10.1.1 How do we introduce a function?

The way one introduces a function f is by specifying the rule that gives us the value of $f(x)$ for each x. For example, we may introduce a function by saying

> Let f be the function from reals to reals such that $f(x) = x^2 + 1$ for every $x \in \mathbb{R}$.

or

> Let f be the function from reals to reals such that
>
> $$f(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases}$$

or

> Let f be the function from reals to reals such that, for $x \in \mathbb{R}$, $f(x)$ is the largest integer n such that $n \leq x$. (That is, "let $f : \mathbb{R} \to \mathbb{R}$ be such that
>
> $$\begin{aligned} &(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(f(x) = y \\ &\quad \Leftrightarrow ((y \in \mathbb{Z} \wedge y \leq x) \wedge (\forall z \in \mathbb{R})((z \in \mathbb{Z} \wedge z \leq x) \Rightarrow z \leq y))) \end{aligned}$$
>
> ")

10.2 Limits

DEFINITION. Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let a, L be real numbers. We say that ***the limit of*** $f(x)$ ***as*** $x \to a$ ***is*** L, and write $\lim_{x \to a} f(x) = L$, if

$$\begin{aligned} &(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow \\ &\quad (\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x - a| < \delta \to |f(x) - L| < \varepsilon))) \, . \end{aligned}$$

10.3 Examples

EXAMPLE 1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x$ for all $x \in \mathbb{R}$. Prove that $\lim_{x \to 0} f(x) = 0$.

Proof. We apply the definition of limit, with $a = 0$, $L = 0$, and f the function such that $f(x) = x$ for all $x \in \mathbb{R}$. We want to prove that

$$\begin{aligned} &(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow \\ &\quad (\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x| < \varepsilon))) \, . \end{aligned}$$

To prove this, let $\varepsilon \in \mathbb{R}$ be arbitrary. Assume $\varepsilon > 0$. Let $\bar{\delta} = \varepsilon$. Then $\bar{\delta} > 0$, because $\varepsilon > 0$. Let $x \in \mathbb{R}$ be arbitrary. Assume that $0 < |x| < \bar{\delta}$. Then in particular $|x| < \bar{\delta}$, so $|x| < \varepsilon$, because $\bar{\delta} = \varepsilon$.

We have proved that $|x| < \varepsilon$ under the assumption that $0 < |x| < \bar{\delta}$. So we can conclude that $0 < |x| < \bar{\delta} \Rightarrow |x| < \varepsilon$ by Rule $\Rightarrow_{in}$.

We have shown that $0 < |x| < \bar{\delta} \Rightarrow |x| < \varepsilon$ for an arbitrary $x \in \mathbb{R}$. Hence we can conclude that $(\forall x \in \mathbb{R})(0 < |x| < \bar{\delta} \Rightarrow |x| < \varepsilon)$ by Rule $\forall_{in}$. We also have found that $\bar{\delta} > 0$, so Rule $\wedge_{in}$ tells us that

$$\bar{\delta} > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \bar{\delta} \Rightarrow |x| < \varepsilon)))$$

Therefore, using $\bar{\delta}$ as a witness, we can use Rule $\exists_{in}$ to conclude that

$$(\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x| < \varepsilon))$$

This was proved under the assumption that $\varepsilon > 0$. So Rule $\Rightarrow_{in}$ allows us to conclude that

$$\varepsilon > 0 \Rightarrow \Big((\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x| < \varepsilon))\Big).$$

Finally, this last statement was derived for an arbitrary $\varepsilon \in \mathbb{R}$. Therefore

$$(\forall \varepsilon \in \mathbb{R})\left(\varepsilon > 0 \Rightarrow \Big((\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x| < \varepsilon))\Big)\right),$$

which is what we wanted to prove. $\diamondsuit$

EXAMPLE 2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x$ whenever $x \in \mathbb{R}$ and $x \neq 0$, and $f(0) = 1$. Prove that $\lim_{x\to 0} f(x) = 0$.

Proof. We apply the definition of limit, with $a = 0$, $L = 0$, and f the function such that $f(x) = x$ if $x \neq 0$, $f(0) = 1$. We want to prove that

$$\begin{aligned}&(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow \\ &\quad (\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon))).\end{aligned}$$

To prove this, let $\varepsilon \in \mathbb{R}$ be arbitrary. Assume $\varepsilon > 0$. Let $\bar{\delta} = \varepsilon$. Then $\bar{\delta} > 0$, because $\varepsilon > 0$. Let $x \in \mathbb{R}$ be arbitrary. Assume that $0 < |x| < \bar{\delta}$. Then in particular $|x| < \bar{\delta}$, so $|x| < \varepsilon$, because $\bar{\delta} = \varepsilon$. Furthermore, $x \neq 0$,because $0 < |x|$. So $f(x) = x$. Hence the fact that $|x| < \varepsilon$ implies that $|f(x)| < \varepsilon$.

We have proved that $|f(x)| < \varepsilon$ under the assumption that $0 < |x| < \bar{\delta}$. So we can conclude that $0 < |x| < \bar{\delta} \Rightarrow |f(x)| < \varepsilon$ by Rule $\Rightarrow_{in}$.

We have shown that $0 < |x| < \bar{\delta} \Rightarrow |f(x)| < \varepsilon$ for an arbitrary $x \in \mathbb{R}$. Hence we can conclude that $(\forall x \in \mathbb{R})(0 < |x| < \bar{\delta} \Rightarrow |f(x)| < \varepsilon)$ by Rule $\forall_{in}$. We also have found that $\bar{\delta} > 0$, so Rule $\wedge_{in}$ tells us that

$$\bar{\delta} > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \bar{\delta} \Rightarrow |f(x)| < \varepsilon)))$$

Therefore, using $\bar{\delta}$ as a witness, we can use Rule $\exists_{in}$ to conclude that

$$(\exists\delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon))$$

This was proved under the assumption that $\varepsilon > 0$. So Rule $\Rightarrow_{in}$ allows us to conclude that

$$\varepsilon > 0 \Rightarrow \Big((\exists\delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon))\Big)\,.$$

Finally, this last statement was derived for an arbitrary $\varepsilon \in \mathbb{R}$. Therefore

$$(\forall\varepsilon \in \mathbb{R})\Bigg(\varepsilon > 0 \\ \Rightarrow \Big((\exists\delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon))\Big)\Bigg)\,,$$

which is what we wanted to prove. ♢

EXAMPLE 3. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2$ whenever $x \in \mathbb{R}$. Prove that $\lim_{x\to 0} f(x) = 0$.

Proof. We apply the definition of limit, with $a = 0$, $L = 0$, and f the function such that $f(x) = x^2$ for $x \in \mathbb{R}$. So we have to prove that

$$(\forall\varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow \\ (\exists\delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x^2| < \varepsilon)))\,.$$

To prove this, let $\varepsilon \in \mathbb{R}$ be arbitrary. Assume $\varepsilon > 0$. Choose $\bar{\delta} = \sqrt{\varepsilon}$. Then $\bar{\delta} > 0$. Let $x \in \mathbb{R}$ be arbitrary. Assume that $0 < |x| < \bar{\delta}$. Then in particular $|x| < \bar{\delta}$, so $x^2 < \bar{\delta}^2 = \varepsilon$, and then $|x^2| < \varepsilon$.

We have proved that $|x^2| < \varepsilon$ under the assumption that $0 < |x| < \bar{\delta}$. So we can conclude that $0 < |x| < \bar{\delta} \Rightarrow |x^2| < \varepsilon$ by Rule $\Rightarrow_{in}$.

We have shown that $0 < |x| < \bar{\delta} \Rightarrow |x^2| < \varepsilon$ for an arbitrary $x \in \mathbb{R}$. Hence we can conclude that $(\forall x \in \mathbb{R})(0 < |x| < \bar{\delta} \Rightarrow |x^2| < \varepsilon)$ by Rule $\forall_{in}$. We also have found that $\bar{\delta} > 0$, so Rule $\wedge_{in}$ tells us that

$$\bar{\delta} > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \bar{\delta} \Rightarrow |x^2| < \varepsilon)))$$

Therefore, using $\bar{\delta}$ as a witness, we can use Rule $\exists_{in}$ to conclude that

$$(\exists\delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x^2| < \varepsilon))$$

This was proved under the assumption that $\varepsilon > 0$. So Rule $\Rightarrow_{in}$ allows us to conclude that

$$\varepsilon > 0 \Rightarrow \Big((\exists\delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x^2| < \varepsilon))\Big)\,.$$

Finally, this last statement was derived for an arbitrary $\varepsilon \in \mathbb{R}$. Therefore

$$(\forall \varepsilon \in \mathbb{R})\Bigg(\varepsilon > 0 \\ \Rightarrow \Big((\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x| < \delta \Rightarrow |x^2| < \varepsilon))\Big)\Bigg),$$

which is what we wanted to prove. ♢

EXAMPLE 4. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^2 + 1$ whenever $x \in \mathbb{R}$. Prove that $\lim_{x\to 2} f(x) = 5$.

Proof. We apply the definition of limit, with $a = 1$, $L = 5$, and f the function such that $f(x) = x^2 + 1$ for $x \in \mathbb{R}$. So we have to prove that

$$(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow \\ (\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x-2| < \delta \Rightarrow |(x^2+1) - 5| < \varepsilon))).$$

Since $(x^2 + 1) - 5 = x^2 - 4$, the statement we have to prove is equivalent to:

$$(\forall \varepsilon \in \mathbb{R})(\varepsilon > 0 \Rightarrow \\ (\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon))).$$

To prove this, let $\varepsilon \in \mathbb{R}$ be arbitrary. Assume $\varepsilon > 0$. Choose $\bar{\delta} = \min(1, \frac{\varepsilon}{6})$. Then $\bar{\delta} > 0$. Let $x \in \mathbb{R}$ be arbitrary. Assume that $0 < |x-2| < \bar{\delta}$. Then in particular $|x-2| < \bar{\delta}$, so $|x^2 - 4| = |x-2|\,|x+2| \le \bar{\delta}|x+2|$. Furthermore, $|x+2| = |x-2+4| \le |x-2| + 4 \le \bar{\delta} + 4 \le 5$, since $\bar{\delta} \le 1$. Hence $|x^2 - 4| \le 5\bar{\delta}$ so $|x^2 - 4| < \varepsilon$, because $\bar{\delta} \le \frac{\varepsilon}{6}$, so that $5\bar{\delta} < \varepsilon$.

We have proved that $|x^2 - 4| < \varepsilon$ under the assumption that $0 < |x-2| < \bar{\delta}$. So we can conclude that $0 < |x-2| < \bar{\delta} \Rightarrow |x^2 - 4| < \varepsilon$ by Rule $\Rightarrow_{in}$.

We have shown that $0 < |x-2| < \bar{\delta} \Rightarrow |x^2 - 4| < \varepsilon$ for an arbitrary $x \in \mathbb{R}$. Hence we can conclude that $(\forall x \in \mathbb{R})(0 < |x-2| < \bar{\delta} \Rightarrow |x^2-4| < \varepsilon)$ by Rule $\forall_{in}$. We also have found that $\bar{\delta} > 0$, so Rule $\wedge_{in}$ tells us that

$$\bar{\delta} > 0 \wedge (\forall x \in \mathbb{R})(0 < |x-2| < \bar{\delta} \Rightarrow |x^2 - 4| < \varepsilon)))$$

Therefore, using $\bar{\delta}$ as a witness, we can use Rule $\exists_{in}$ to conclude that

$$(\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon))$$

This was proved under the assumption that $\varepsilon > 0$. So Rule $\Rightarrow_{in}$ allows us to conclude that

$$\varepsilon > 0 \Rightarrow \Big((\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon))\Big).$$

Finally, this last statement was derived for an arbitrary $\varepsilon \in \mathbb{R}$. Therefore

$$(\forall \varepsilon \in \mathbb{R})\Bigg(\varepsilon > 0 \\ \Rightarrow \Big((\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon))\Big)\Bigg),$$

that is,

$$(\forall \varepsilon \in \mathbb{R})\Bigg(\varepsilon > 0 \\ \Rightarrow \Big((\exists \delta \in \mathbb{R})(\delta > 0 \wedge (\forall x \in \mathbb{R})(0 < |x-2| < \delta \Rightarrow |f(x) - 5| < \varepsilon))\Big)\Bigg),$$

which is what we wanted to prove. ♢

11 Homework assignment No. 5, due on Thursday Feb. 24

1. Book, §1.6 (pages 50-53), Problem 3.

2. Book, §1.6 (pages 50-53), Problem 4.

3. Book, §1.6 (pages 50-53), Problem 5, parts (a), (b), (c), (d), (e), (f), (g).

4. Book, §1.6 (pages 50-53), Problem 6.

5. Book, §1.6 (pages 50-53), Problem 7, parts (b), (e), (f).

6. Prove, using the definition of limit that $\lim_{x \to 3}(x^3 + 3x) = 33$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + 1$ when $x < 1$, and $f(x) = 2x$ when $x \geq 1$. Prove, using the definition of limit, that $\lim_{x \to 1} f(x) = 2$. (A piece of advice: although you don't need this for the proof, it would be a good idea for you to graph the function f.)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + 1$ when $x < 1$, and $f(x) = 3x$ when $x \geq 1$. Prove, using the definition of limit, that $\lim_{x \to 1} f(x)$ does not exist (that is, that $\sim (\exists L \in \mathbb{R})(\lim_{x \to 1} f(x) = L$. (A piece of advice: here too, even though you don't need if for the proof, it would be a good idea for you to graph the function f.)