# MATHEMATICS 300 — SPRING 2005

*Introduction to Mathematical Reasoning*

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# INSTRUCTOR'S NOTES

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## Contents



## 10 The fourteen basic rules of inference.

*REMARK: In a previous handout, I talked about "fifteen rules." I have changed my mind and decided that it's going to be only fourteen, because I am having only one rule involving the negation connective* $\sim$.

Here are the basic rules of inference. You will see that there are exactly fourteen of them. We could get away with fewer rules (actually, it is easy to bring the number down to seven, and with some more work one can bring it down to five), but these fourteen rules are very easy to remember, so I really like them this way.

**RULE 1.** (*The tautology rule.*) You are allowed to bring in any statement which is an instance of a tautology.

**RULE 2.** (Rule $\Longrightarrow_{out}$, also called *Modus Ponens.*) If you have $P$ and $P \Longrightarrow Q$ then you can go to $Q$.

**RULE 3.** (Rule $\Longrightarrow_{in}$.) If you have started a proof within a proof by assuming $P$, and have proved $Q$, then you can get out of the proof within a proof and go back to the main proof with $P \Longrightarrow Q$.

**RULE 4.** (Rule $\wedge_{out}$.) (a) If you have $P \wedge Q$ then you can go to $P$; (b) if you have $P \wedge Q$ then you can go to $Q$.

**RULE 5.** (Rule $\wedge_{in}$.) If you have $P$ and $Q$ then you can go to $P \wedge Q$.

**RULE 6.** (Rule $\vee_{out}$, a.k.a, "proof by cases".) If you have $P \vee Q$ and $P \Longrightarrow R$ and $Q \Longrightarrow R$ then you can go to $R$.

**RULE 7.** (Rule $\vee_{in}$.) (a) If you have $P$ then you can go to $P \vee Q$; (b) If you have $Q$ then you can go to $P \vee Q$.

**RULE 8.** (The "proof by contradiction" rule.) (a) If assuming $\sim P$ you get a contradiction, then you can go to $P$; (b) If assuming $P$ you get a contradiction, then you can go to $\sim P$

REMARK. The most common way to use this rule is by assuming that $P$, the thing you are rying to prove, is not true—i.e., by assuming $\sim P$—and proving some other proposition $Q$ that is already known to be false, because $\sim Q$ has been proved before. Then one can use Rule $\wedge_{in}$ to get $Q \wedge (\sim Q)$, which is a contradiction, and apply Rule 8 to get $P$.

**RULE 9.** (Rule $\Longleftrightarrow_{out}$.) If you have $P \Longleftrightarrow Q$ then (a) if you have $P$ then you can go to $Q$; (b) if you have $Q$ then you can go to $P$.

**RULE 10.** (Rule $\Longleftrightarrow_{in}$.) If you prove $P \Longrightarrow Q$ and $Q \Longrightarrow P$ then you can go to $P \Longleftrightarrow Q$.

**RULE 11.** (Rule $\forall_{out}$, a.k.a. the "specialization rule.") Suppose $p(x)$ is a formula where there is no quantifier involving $x$, $t$ is a constant term, and $p(x \to t)$ is what you get from $p(x)$ by substituting $t$ for $x$ in **all** the occurrences of $x$ in $p(x)$. Then, if you have $\forall x, p(x)$, you can go to $p(x \to t)$.

REMARK: a ***term*** is an expression such as $x$, or $a$, or $x + a$, or $(x + a) + 3$, or $(3x + (a + b)) \cdot (u + 25 - \pi)$, or $A \cup B$, which stands for a thing (object, entity) which may or may not involve letter variables. A ***constant term*** is one that does not contain any variables. For example, if the letters $a$, $b$ have been declared to be constants before (e.g. by saying "let $a$, $b$ be arbitrary integers", or "let $a = 3$"), then $a + b$ is a constant term. Rule $\forall_{out}$ says that, for example, if you have $\forall x, (x \in \mathbb{Z} \Longrightarrow x \cdot x \geq 0)$, and you have a constant term such as $-3$, or $3 - 7$, or $-3 + a$ (if $a$ has been declared to be a constant) then you can apply the statement $\forall x, (x \in \mathbb{Z} \Longrightarrow x \cdot x \geq 0)$ to the special case $x = -3$ (or $x = 3 - 7$, or $x = -3 + a$) and conclude that $-3 \in \mathbb{Z} \Longrightarrow (-3) \cdot (-3) \geq 0$, $3 - 7 \in \mathbb{Z} \Longrightarrow (3 - 7) \cdot (3 - 7) \geq 0$, and $-3 + a \in \mathbb{Z} \Longrightarrow (-3 + a) \cdot (-3 + a) \geq 0$.

**RULE 12.** (Rule $\forall_{in}$.) Suppose $p(x)$ is a formula where there is no quantifier involving $x$, $a$ is a letter, and $p(x \to a)$ is what you get from $p(x)$ by substituting $a$ for $x$ in **all** the occurrences of $x$ in $p(x)$. ***Suppose the letter $a$ has not appeared before***. Then you can start a proof within a proof by saying "Let $a$ be arbitrary." If in this proof within a proof you get to $p(x \to a)$, then you can go to $\forall x, p(x)$ in your main proof.

REMARK: Naturally, instead of "$a$" you could use "$b$," or "$z$," or "$\alpha$," or "$\beta$," or "$\aleph$," or "$\diamond$," or any symbol you want. What is important is that what you do should apply to a *completely arbitrary* object in our universe of discourse. Otherwise, you will not be proving that $p(x)$ is true for *all* $x$. For example —if our universe of discourse is $\mathbf{R}$— it would not be O.K. to prove that $\forall x, \exists y, y^2 = x$ (i.e., that every real number has a real square root) by saying "Let 9 be arbitrary. Then $\exists y, y^2 = 9$ is true, so we have proved that $\forall x, \exists y, y^2 = x$." What is wrong here? What is wrong is that 9 is not an *arbitrary* real number. 9 is 9, and just because something is true for the number 9 it doesn't mean that it is true for **all** real numbers.

**RULE 13.** (Rule $\exists_{out}$.) Suppose $p(x)$ is a formula where there is no quantifier involving $x$, $a$ is a letter, and $p(x \to a)$ is what you get from $p(x)$ by substituting $a$ for $x$ in **all** the occurrences of $x$ in $p(x)$. ***Suppose that the letter $a$ has NOT appeared before. Suppose you have proved $\exists x, p(x)$***. Then you can start a proof within a proof by introducing a new object, calling it $a$, and stipulating that $p(x \to a)$. This effectively declares $a$ to be a constant, locally, within the "proof within a proof." If you ever get to something that does *not* contain $a$, then you can use it outside your proof within a proof, in the main proof.

REMARK: **It is important that the new object be given a name that has not been used before.** For example, suppose $p(x)$ stands for "$x$ killed Polonius," and our universe of discourse is the set of all people. Suppose you are told that $\exists x, p(x)$, i.e. that somebody killed Polonius. Then you can introduce a name for this individual. You can call him/her $a$ or, if you prefer, "the killer," in which case you would be able to say that $p(x \to a)$, i.e., that $a$ killed Polonius. But you cannot say "let's call this person Hamlet," or "let's call him Laertes," because Hamlet and Laertes are names of characters that have already appeared in the play. If you call the killer "Hamlet" or "Laertes" then you would be *prejudging*, and declaring that $p(x \to \text{Hamlet})$, i.e. that Hamlet killed Polonius, or that $p(x \to \text{Laertes})$, i.e. that Laertes killed Polonius. One of these happens to be true, and the other one is false, but in either case you cannot just conclude that it is true by merely choosing a name for the killer.)

**RULE 14.** (Rule $\exists_{in}$, a.k.a. the witness rule.) Suppose $p(x)$ is a formula where there is no quantifier involving $x$, $t$ is a constant term, and $p(x \to t)$ is what you get from $p(x)$ by substituting $t$ for $x$ in **all** the occurrences of $x$ in $p(x)$. Then from $p(x \to t)$ you can go to $\exists x, p(x)$.

REMARK: Here is an example. Suppose we are working in $\mathbb{Z}$, and you want to prove that $\exists x, x^2 + 3 \cdot x = 10$. You would first show that $2^2 + 3 \cdot 2 = 10$. Now, if $p(x)$ is the formula "$x^2 + 3 \cdot x = 10$", then $p(x \to 2)$ is the formula "$2^2 + 3 \cdot 2 = 10$". So we have proved $p(x \to 2)$, and Rule $\exists_{in}$ allows us to go to $\exists x, x^2 + 3 \cdot x = 10$.

AN EXAMPLE. Here is an example that combines several of our rules. Suppose you want to prove that the sum of two odd integers is even, working in

$\mathbb{Z}$. That is, you want to prove that

$$\forall x, \forall y, \Big(\big((\exists k, x = 2k+1) \wedge (\exists k, y = 2k+1)\big) \Longrightarrow (\exists k, x+y = 2k)\Big).$$

Here is how you could do it.

| | | |
|---|---|---|
| 1. | Let $a$ be arbitrary. | [V.d] |
| 2. | Let $b$ be arbitrary. | [V.d] |
| 3. | Assume $(\exists k, a = 2k+1) \wedge (\exists k, b = 2k+1)$. | [V.d] |
| 4. | $\exists k, a = 2k+1$. | [Rule $\wedge_{out}$] |
| 5. | $\exists k, b = 2k+1$. | [Rule $\wedge_{out}$] |
| 6. | Pick a $k$ such that $a = 2k+1$ and call it $k_1$. | [Rule $\exists_{out}$] |
| 7. | Pick a $k$ such that $b = 2k+1$ and call it $k_2$. | [Rule $\exists_{out}$] |
| 8. | $a = 2k_1 + 1$. | [From 6] |
| 9. | $b = 2k_2 + 1$. | [From 7] |
| 10. | $a + b = (2k_1+1) + (2k_2+1) = 2(k_1+k_2+1)$. | |
| 11. | $\exists k, a+b = 2k$. | [Rule $\exists_{in}$] |
| 12. | $\exists k, a+b = 2k$. | [Rule $\exists_{in}$] |
| 13. | $\exists k, a+b = 2k$. | [Rule $\exists_{in}$] |
| 14. | $\big((\exists k, a = 2k+1) \wedge (\exists k, b = 2k+1)\big) \Longrightarrow \exists k, a+b = 2k$. | [R. $\Longrightarrow_{in}$] |
| 15. | $\forall y, \big((\exists k, a = 2k+1) \wedge (\exists k, y = 2k+1)\big) \Longrightarrow \exists k, a+y = 2k$. | [R. $\forall_{in}$] |
| 16. | $\forall x, \forall y, \big((\exists k, x = 2k+1) \wedge (\exists k, y = 2k+1)\big) \Longrightarrow \exists k, x+y = 2k$. | [R. $\forall_{in}$] |

# 11 Homework assignment No. 4, due on Thursday February 17

**1.** Book, §1.7 (pages 63-64), Problem 1, items (a), (b), (c).

**2.** Book, §1.7 (pages 63-64), Problem 3.

**3.** Book, §1.7 (pages 63-64), Problem 5.

**4.** Book, §1.7 (pages 63-64), Problem 6 (a).

**5.** Book, §1.7 (pages 63-64), Problem 7 (c).

**6.** Book, §1.7 (pages 63-64), Problem 9.

**7.** Book, §1.7 (pages 63-64), Problem 11, items (b), (d), (e), (f).