MATHEMATICS 300 — SPRING 2005

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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Contents

6 Homework assignment No. 2, due on Thursday February 3

Do not worry too much if you cannot do the "hard" problems, but please give them a try.

1. Book, §1.3 (pages 24, 25, 26), Problem 6, all the non-starred items.

2. Book, §1.3 (pages 24, 25, 26), Problem 8.

3. Book, §1.3 (pages 24, 25, 26), Problem 10, all the non-starred items.

4. Book, §1.4 (pages 35, 36, 37), Problem 4, items (a), (b), (e).

5. Book, §1.4 (pages 35, 36, 37), Problem 5, items (a) and (e).

6. Book, §1.4 (pages 35, 36, 37), Problem 7.

7. Book, §1.4 (pages 35, 36, 37), Problem 10, part (b).

8. (*Optional. This could be quite hard! Do not worry if you cannot do it, but at least try to* **think** *about it.*) Express the number $4,582,623$ as the sum of the squares of two natural numbers.

9. (*Optional. Hard.*) Suppose two integers a and b are sums of two squares of integers. Prove that the product ab must also be the sum of two squares of integers. (NOTE: here is an example of the kind of situation we are talking about. The number 13 is the sum of two squares, because $13 = 4 + 9$. The number 8 is the sum of two squares, because $8 = 4 + 4$. So, if our statement is true, it ought to be the case that the number 13×8 is the sum of two squares. And, indeed, it *is* true, because $13 \times 8 = 104$ and $104 = 100 + 4 = 10^2 + 2^2$.)

10. (*Optional. Hard.*) Suppose two integers a and b are sums of three squares of integers. Prove that the product ab must also be the sum of three squares of integers.

11. Here are five sentences about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than."

(1) Cows are taller than sheep. (That is, "every cow is taller than every sheep," or, if you prefer, "for every x and every y, if x is a cow and y is a sheep, then x is taller than y.")

(2) Giraffes are taller than cows.

(3) If something is taller than something else which is in turn taller than a third thing, then the first one is taller than the third one. (That is "for every x, every y and every z, if x is taller than y and y is taller than z, then x is taller than z.")

(4) Olivia is a giraffe.

(5) Dolly is a sheep.

Using sentences (1), (2), (3), (4), (5), *and* **no other fact** *about giraffes, cows, sheep, Olivia, Dolly, and the binary relation "taller than,"* prove that Olivia is taller than Dolly.

7 Some types of valid propositional inference step forms

Now that we know how to represent the propositional structure of statements by means of propositional forms, we can give examples of inferences that are valid because of their propositional form.

Let us first clarify what we mean by "inference step," "inference step form," "propositional inference step form," and what we mean for any of these things to be "valid."

An *inference step* consists of a list of statements—called the *premisses*—followed by a statement called the *conclusion*, separated in a clear way (for example, by means of a line) from the premisses.

EXAMPLE:

$$\text{(A)} \qquad \left\{ \begin{array}{l} \text{All Rutgers professors are very smart} \\ \text{Hector Sussmann is a Rutgers professor} \\ \hline \text{Hector Sussmann is very smart} \end{array} \right.$$

and

$$\text{(B)} \qquad \left\{ \begin{array}{l} \text{All Rutgers professors are very smart} \\ \text{Hector Sussmann is very smart} \\ \hline \text{Hector Sussmann is a Rutgers professor} \end{array} \right.$$

are both inference steps. Inference Step (A) is *valid*, because the conclusion follows from the premisses. Inference Step (B) is *not valid*, because the conclusion does not follow from the premisses.

An inference step is *valid* if the conclusion is a consequence of the premisses, that is, if the conclusion is true provided that the premisses are true. (Even more precisely: an inference step is valid if either (a) one of the premisses is false, or (b) the conclusion is true. For example, the inference step:

$$\left\{ \begin{array}{l} \text{The moon is made of green cheese} \\ \hline \text{Elvis is alive} \end{array} \right.$$

is valid, because the premiss is false.[1] Also,

[1] Notice that it does not matter whether Elvis is alive or not.

$$\begin{cases} \text{There is intelligent life in some planet other than the Earth} \\ \overline{} \\ \text{President Lincoln was murdered} \end{cases}$$

is valid, because the conclusion is true[2].)

Some inference steps are valid because of the *form* of the premisses and the conclusion. For example,

$$\text{(MP1)} \qquad \begin{cases} \text{If Hector Sussmann is a Rutgers professor then} \\ \qquad\qquad \text{Hector Sussmann is very smart} \\ \text{Hector Sussmann is a Rutgers professor} \\ \overline{} \\ \text{Hector Sussmann is very smart} \end{cases}$$

$$\text{(MP2)} \qquad \begin{cases} \text{If Elvis is a borogove then Elvis is mimsy} \\ \text{Elvis is a borogove} \\ \overline{} \\ \text{Elvis is mimsy} \end{cases}$$

$$\text{(MP3)} \qquad \begin{cases} \text{If you are a senior citizen then you are entitled to a discount} \\ \text{You are a senior citizen} \\ \overline{} \\ \text{You are entitled to a discount} \end{cases}$$

$$\text{(MP4)} \qquad \begin{cases} \text{If } \sqrt{2} \text{ is rational then } 2\sqrt{2} \text{ is rational} \\ \sqrt{2} \text{ is rational} \\ \overline{} \\ 2\sqrt{2} \text{ is rational} \end{cases}$$

$$\text{(MP5)} \qquad \begin{cases} \text{If } \sqrt{2} \text{ is irrational then } 2\sqrt{2} \text{ is irrational} \\ \sqrt{2} \text{ is irrational} \\ \overline{} \\ 2\sqrt{2} \text{ is irrational} \end{cases}$$

[2]Notice that it does not matter whether it is true or false that there is intelligent life in some planet other than the Earth.

These five inference steps are of the form

$$(\mathrm{MP}) \qquad \begin{cases} P \Rightarrow Q \\ P \\ \overline{} \\ Q \end{cases} .$$

(This means that each of the five steps (MP1,2,3,4,5) can be obtained from (MP) by substituting one sentence for each of the letter variables. For example, we get (MP4) by substituting "$\sqrt{2}$ is rational" for P and "$2\sqrt{2}$ is rational" for Q.)

The five steps (MP1,2,3,4,5) are valid *because they are of the form (MP).* Since (MP) is a propositional inference step form that renders every inference step having this form valid, we call (MP) a *valid* propositional inference step form.

To understand what this means, let us contrast (MP) with a propositional inference step form which ixs *not valid.* For example,

$$(\mathrm{MP}-\mathrm{awful}) \qquad \begin{cases} P \Rightarrow Q \\ Q \\ \overline{} \\ P \end{cases} .$$

is not valid. (*Proof that (MP-awful) is not valid*: if P is F and Q is T, then both premisses are true but the conclusion is F. *Another proof*: if we substitute "Kermit likes spinach" for P, and "Kermit eats spinach" for Q, then it is clear that the conclusion does not follow from the premisses, because it could very well be that Kermit would eat spinach if he liked it, and Kermit does indeed eat spinach, but Kermit does not like spinach, and the reason he is eating spinach is not that he likes it but that his doctor told him he has to eat spinach.)

In general,

DEFINITION. A *propositional inference step form* consists of a list of propositional forms—called the *premisses*—followed by a propositional form, called the *conclusion*, separated in a clear way (for example, by means of a line) from the premisses. ♢

Notice that, exactly as a "propositional form" is a totally different kind of animal from a "proposition," **a "propositional inference step form" is a totally different kind of animal from an "inference step."**

DEFINITION. A propositional inference step form S is *valid* if, for every assignment of truth values to the letter variables that makes all the premisses true, it follows that the conclusion is true. $\diamondsuit$

Propositional inference step forms can be used to represent the form of inference steps. As we have already explained before, when discussing the example of the five steps (MP1,2,3,4,5) and the form (MP),

DEFINITION. If Σ is an inference step and S is a propositional inference step form, we say that Σ *is of the form* S if Σ can be obtained from S by substituting one sentence for each of the letter variables that occur in S. $\diamondsuit$

7.1 A preview of the list of valid propositional inference step forms

We have already seen one example of a valid propositional inference step form. We now want to list several more examples, and want to do it in a systematic way. The pattern is going to be as follows.

(1) There are five propositional connectives, namely, $\sim$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

(2) For each of these five propositional connectives, we will give two valid propositional inference step forms, one being an "out" rule, that is, a rule that enables you to *use* sentences having this particular connective, and the other one being an "in" rule, that is, a rule that enables you to *get* sentences having this particular connective. (Think of the various steps of a proof as being like positions in a chess game. An "out" rule is a rule that tells you how to make a move so as to get out of a particular position. An "in" rule is a rule that tells you how to make moves so as to get to a particular position. For example, if you have proved something of the form $P \Rightarrow Q$, and you are stuck and do not know how to get out of this situation, Rule $\Rightarrow_{out}$ will help you, by telling you how, if you have $P \Rightarrow Q$, you can do something with it to get something else. If you want to prove something of the form $P \Rightarrow Q$, and you are stuck and do not know how to get there, Rule $\Rightarrow_{in}$ will help you, by telling you how you can prove $P \Rightarrow Q$.)

(3) In addition, we will have one special rule (the tautology rule), so we will end up with a total of **11 basic rules**, that is, **11 basic valid propositional inference step forms.**

(4) Furthemore, we will later introduce two more logical connectives (the quantifiers $\forall$ and $\exists$) which are *not* propositional. Associated to these

two quantifiers there will be four rules (two "out" rules and two "in" rules") so we will end up with a total of **15 logical rules**. And that will be it. **Only 15 logical rules, not more!!!** [3]

(5) Finally, I would like to point out that **all these rules are just plain common sense**. Look, for example, at (MP) above. Do you have to work hard in order to remember this? No, because what the rule says is completely obvious.

(6) And **the rules are very easy to remember**, thanks to this pattern that you have an "out" rule and an "in" rule for each connective, plus the tautology rule.

7.2 Modus Ponens (a.k.a. Rule $\Rightarrow_{out}$)

Let us start with the one valid propositional inference step form that we have already discussed, namely, (MP).

This inference form step is called **Modus Ponens**, and is one of the oldest logical rules[4]. If you don't want to have to remember a Latin name, you might as well call it "Rule $\Rightarrow_{out}$."

EXAMPLE. The inference step

$$(\#) \qquad \left\{ \begin{array}{l} \text{If } f \text{ is differentiable then } f \text{ is continuous} \\ f \text{ is differentiable} \\ \hline f \text{ is continuous} \end{array} \right.$$

is valid, because it is of the form (MP). (We would say that "(#) is valid by Modus Ponens", or "(#) is valid by Rule $\Rightarrow_{out}$.")

7.3 Proving an implication: Rule $\Rightarrow_{in}$

This rule says that, if you have a proof of Q starting from P, then you can go to $P \Rightarrow Q$.

[3] In addition, once we move beyond logic and start doing mathematics, there will have to be rules for all the symbols involved. For example, for each of the symbols $=, >, <$, we need at least one rule involving that symbol, because if we didn't have, for example, any rules involving $=$, we would never be able to prove anything involving $=$, not even such simple things as $1 = 1$. Think once again of the analogy with chess. The rules of chess must contain at least one rule for each type of piece. For example, if there were no rules at all involving the bishop, you would not know how to move a bishop.

[4] Such rules were called "syllogisms" in pre-19th century logic.

Normally, we would write this as

$$\begin{cases} \text{Proof of } Q \text{ assuming } P \\ \overline{} \\ P \Rightarrow Q \end{cases}$$

or

$$\begin{cases} \text{Assume } P \\ \vdots \\ Q \\ \overline{} \\ P \Rightarrow Q \end{cases}$$

7.4 A remark on introducing assumptions. ("Am I allowed to assume that pigs have wings?")

1. In a proof you can introduce any asumption you want, at any time. It does not matter if the assumption is true, of if you know whether it is true.
2. However, what you prove after you introduce one or more assumptions has only been proved under those assumptions. That is, if you prove conclusion C under assumption A, you cannot conclude that C is true, but only that A⇒C, that is, that in an imaginary world in which A is true C would have to be true as well.

Students often find Statement 1 in the above box shocking, and ask questions such as "how can I assume something if I do not know that it is true?"

The answer is that you can, and in fact you do precisely that most of the time, but for some mysterious reason when students write about mathematics they do not believe that the same thing that they are used to doing in their everyday life is O.K. and works the same way in mathematics.

In you everyday life, you often imagine some possibility in order to figure out what would happen if that was the case. For example, before the last presidential election you probably tried to imagine what would happen if Candidate X won, and what would happen if Candidate Y won. Then you compared the results and came to the conclusion that you liked better what would happen if one of those two candidates won, and decided for that reason to vote for that candidate.

To imagine what would happen if Candidate X won, you almost certainly started thinking along the following lines: "let me suppose that X had been elected president; then I guess he would appoint so-and-so as secretary of state, and this person would probably push our foreign policy in such-and-such direction, given what I know about his views." And you would have continued like that.

In other words, **you made the assumption that X won without knowing that it was true, and then you explored its consequences**.

You do the same kind of thing all the time. For example, to decide if you will go to Restaurant R for dinner, you would try to imagine what you would eat if you were there, and you might very well end up concluding that none of the dishes that you can imagine yourself ordering looks appealing, so in fact you will not go to Restaurant R. In other words: **you assumed that you had gone to Restaurant R and then concluded that you did not like the consequences of this assumption, so in the end you made the assumption false by not going to the restaurant.**

In mathematics we do the same thing. We introduce assumptions, and then explore their consequences. We do not care if the assumptions are true, because all we are trying to find out is "what else would have to be the case if these assumptions were true?"

Another way to see that Rule $\Rightarrow_{in}$ is O.K., is to think in terms of truth values. Suppose you can prove Q starting with P. Then you are guaranteed that if P is true then Q is true. And this in turn guarantees that $P \Rightarrow Q$ is true. (Reason: the *only* case when $P \Rightarrow Q$ is false is when P is true and Q is false.) In other words,

In order to prove that P⇒ Q, we might as well assume that P is true and then prove Q (thus establishing that P⇒Q is true) because in the other case, if P was false, the implication P⇒Q would automatically be true, and our work would be done.

Or, to put it slightly differently,

We don't "assume P" because it's true, and we don't need to know that P is true in order to assume P. We "assume P" because if P isn't true then we already know that P⇒Q is true.

BOX NO. 1: HYPOTHESES (A.K.A. ASSUMPTIONS).

Hypotheses, or **assumptions**, are statements that we **assume** to be true in a particular situation.
When we discuss proofs later on, we will see that that there are certain statements that you are free to bring in any time you want, because you know them to be true. (Such statements are called "axioms," or "postulates.") In additions, *definitions* function exactly like axioms, the only difference being that an axiom is something you *accept* as true (say, because you are convinced it is true but you do not know how to prove it) whereas a definition (say, for example, $2 = 1 + 1$) is something that you *declare*, or *decide* to be true. **Within a particular proof, hypotheses/assumptions function exactly like axioms and definitions: they are statements that you can bring in any time.**
There is, however, a big difference. Axioms and definitions are things that we take to be true forever, because we believe they are true. Hypotheses/assumptions are taken to be true not because we believe they are true, but **because we want to know the consequences of their being true.**
For example: suppose we to find out what the world would be like if pigs had wings. To do this, we first make the assumption "pigs have wings," by saying

(...) Assume that pigs have wings.

Then we would set out to explore the consequences of this assumption. That is, if you prefer, we would set out to study an **imaginary world** in which pigs have wings, and try to understand what that world would have to be like. We might then be able to "prove" all kind of "theorems," such as

THEOREM 1: *Pigs can fly.*

THEOREM 2: *If a pigsty does not have a roof, then the pigs will fly away.*

(Proving these theorems might not be easy. For example, it is not true that "every animal that has wings can fly," since, for example, ostriches, penguins and dodos do not fly. But you might notice that all the exceptions to the general rules seems to be birds, so perhaps you could make a convincing case that "every animal that has wings and is not a bird can fly." If you manage to do that, then you might go on to argue that pigs can fly, since pigs clearly are not birds.)
All these theorems would be proved **under the hypothesis** that pigs have wings. That is, we would never be saying that we have proved that pigs can fly. All we would be saying is that **IF** pigs had wings **THEN** pigs could fly. In other words, all the facts we would "prove" about pigs under the "pigs have wings" hypothesis would be valid **in the imaginary world of pigs with wings**, which, in principle, is not our real world.

BOX NO. 2: ENTERING AND LEAVING IMAGINARY WORLDS.

As we explained in Box 1, when you assume something you are setting out to explore an **imaginary world**, namely, a world in which that thing you are assuming is true. When can you leave that world and go back to the real world, taking with you the things you have discovered in your imaginary world? The answer is simple: if the thing you assumed is actually true in the real world, then everything you proved **under that assumption** is true, because in that case the imaginary world really is the real world.
Our rules of inference capture this very obvious idea as follows:

- You are always allowed to introduce anything you want as an asumption, any time you want.
- When you do that, whatever you have proved will have been proved **under your assumption**.
- If, assuming a statement P, you prove some other statement Q, then you will be allowed to go back to "the real world" and assert there that $P \Longrightarrow Q$.
- If, in addition, you can prove P, then the Modus Ponens rule allows you to go to Q.

In simpler terms: whatever you prove under the assumption P will be valid unrestrictedly (that is, without having to assume that you are under the assumption P) if you can also prove P.

EXAMPLE: If you prove that "pigs can fly" under the assumption that "pigs have wings," then the conclusion that "pigs have wings" will have been established in a world where pigs have wings. You can also go back from such an imaginary world to the real world by stating that "if pigs have wings then pigs can fly." This, however, is not enough to be able to conclude that in the real world pigs can fly, because in the real world you do not know that pigs have wings, and **if you do not know that pigs have wings then you cannot go from "if pigs have wings then pigs can fly" to "pigs can fly."**

7.5 Reasoning by cases (a.k.a. Rule $\vee_{out}$)

This is the valid propositonal inference step form

$$\left\{\begin{array}{l} P \vee Q \\ P \Rightarrow R \\ Q \Rightarrow R \\ \overline{} \\ R \end{array}\right.$$

Since you can prove $P \Rightarrow R$ by "assuming P and proving R," and you can prove $Q \Rightarrow R$ by "assuming Q and proving R," you could equally well organize the above as follows:

$$\left\{\begin{array}{l} P \vee Q \\ \text{Proof of } R \text{ assuming } P \\ \text{Proof of } R \text{ assuming } Q \\ \overline{} \\ R \end{array}\right.$$

or[5]

$$\left\{\begin{array}{l} P \vee Q \\ \text{Assume } P \\ \quad\vdots \\ R \\ \text{Now assume } Q \\ \quad\vdots \\ R \\ \overline{} \\ R \end{array}\right.$$

[5]You may ask *why are we proving R three times? When I write "R" the second time, what is the point? Haven't we proved R already?* The answer is **no**. When we proved R the first time, we proved it *under assumption P*. (That is, we really proved $P \Rightarrow R$, not R). Then we started all over again, assuming Q. (That's why we put the word "now," to indicate that we are not adding another assumption on top of P, but we are restarting our work, this time under assumption Q instead of P.) So this second time we prove $Q \Rightarrow R$, not R. And then, in the last step, having shown that we can get R under assumption P and also under assumption Q, we are finally entitled to assert that we have really proved R.

or, in a more verbose way,

$$\left\{\begin{array}{l} P \vee Q \\ \text{So there are two cases, namely, } P \text{ and } Q \\ \text{Consider first the case when, } P \text{ is true} \\ \quad\vdots \\ R \\ \text{Now consider the case when, } Q \text{ is true} \\ \quad\vdots \\ R \\ \overline{} \\ \text{So we have proved that } R \text{ holds in both cases} \\ \text{We therefore conclude that } R \text{ is true} \end{array}\right.$$

7.6 Rule $\vee_{in}$

This is one of the stupidest rules, but it is perfectly valid. It's in fact two rules:

$$(\vee_{in}.a) \quad \left\{\begin{array}{l} P \\ \hline P \vee Q \end{array}\right. \qquad \text{and} \qquad (\vee_{in}.b) \quad \left\{\begin{array}{l} Q \\ \hline P \vee Q \end{array}\right.$$

All this says is that, for example, if you are told that you need to have passed Math 300 or Math 311 in order to take Math 312, and you know that you have passed Math 300 then you satisfy the requirement.

As I told you, you may object that this is very stupid, and I will agree, but please don't tell me that this is very difficult!

7.7 More stupid rules (Rules $\wedge_{out}$ and $\wedge_{in}$)

Here they are. They are in fact two rules:

$$(\wedge_{in}) \qquad \left\{\begin{array}{l} P \\ Q \\ \hline P \wedge Q \end{array}\right.$$

$$(\wedge_{out}.a) \quad \left\{\begin{array}{l} P \wedge Q \\ \hline P \end{array}\right. \qquad \text{and} \qquad (\wedge_{out}.b) \quad \left\{\begin{array}{l} P \wedge Q \\ \hline Q \end{array}\right.$$

Again, *this may be very stupid, but it sure isn't difficult!!!*

7.8 The tautology proof rule

This is an inference step form in which the number of premisses is *zero*, that is, there are no inferences. It says that **any propositional inference step form with zero premisses whose conclusion is a tautology is valid.**

For example, the following propositional inference step form is valid

$$\begin{cases} \overline{} \\ (P \wedge Q) \Rightarrow ((P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R)) \end{cases}$$

is valid, because

$$(P \wedge Q) \Rightarrow ((P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R))$$

is a tautology[6]

EXAMPLE. Suppose you don't want to use the "proof by cases" rule $\vee_{out}$ (because you don't like it, or because I don't let you use it). Suppose you find yourself exactly in a situation where you have proved $P \wedge Q$, $P \Rightarrow R$, and $Q \Rightarrow R$, and you would like to conclude that R is true. Naturally, this would follow using Rule $\vee_{out}$, but if you don't want to use this ruel, here is another way: first bring in the tautology $(P \wedge Q) \Rightarrow ((P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R))$; since you already have $P \wedge Q$, you can use Modus Ponens to go to $(P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R)$; since you already have $P \Rightarrow R$, you can use Modus Ponens again to go to $(Q \Rightarrow R) \Rightarrow R$; finally, since you already have $Q \Rightarrow R$, you can use Modus Ponens a third time to go to R, and you are done.

7.9 Preview of some coming attractions

We already have seven rules. I still owe you four more rules, namely, the in and out rules for $\sim$ and $\Leftrightarrow$. (And I also owe you the four "nonpropositional" rules involving $\forall$ and $\exists$.)

You will get these rules in next week's handouts.

[6]How do you prove that $(P \wedge Q) \Rightarrow ((P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R))$ is a tautology? You can do it with truth tables, but if you want a quick way to see it, here is one: ask yourself *when would* $(P \wedge Q) \Rightarrow ((P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R))$ *be false?*; suppose it was false; then $P \wedge Q$ has to be T and $(P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R)$ has to be F; but for $(P \Rightarrow R) \Rightarrow ((Q \Rightarrow R) \Rightarrow R)$ to be F, $P \Rightarrow R$ must be T and $(Q \Rightarrow R) \Rightarrow R$ must be F; and then $Q \Rightarrow R$ is T and R is F; but then Q has to be F, because if it was T then $Q \Rightarrow R$ would be F, and we have already showed it is T; for exactly the same reason, P has to be F; but, since P and Q are F, $P \wedge Q$ has to be F as well, and this is impossible because we have already shown that $P \wedge Q$ is T.