MATHEMATICS 300 — SPRING 2005

Introduction to Mathematical Reasoning

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INSTRUCTOR'S NOTES

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5 Propositional Calculus

5.1 The basic symbols

To specify precisely what a **propositional calculus** is, we have to begin by listing its **basic symbols**.

THE BASIC SYMBOLS OF A PROPOSITIONAL CALCULUS

A propositional calculus uses **three** kinds of basic symbols:

1. the *propositional variables* (usually letters such as P, Q, R, S),

2. the *propositional connectives* $\vee$ (disjunction), $\wedge$ (conjunction), $\Longrightarrow$ (conditional, or implication), $\Longrightarrow$ (biconditional, or equivalence), $\sim$ (negation),

3. the *parentheses* (and) .

REMARK: In the above list, we have cheated a little bit, by being somewhat vague about the list of propositional variables. (The word "etc." is a strong symptom of vagueness!!! What could "etc." possibly mean?) Exactly what are the propositional variables? Just A, B, C, D? Or just P, Q, R, S? That is, do we have only four propositional variables? Suppose that for a particular proof or definition we need formulas with 6 variables. Can we make new variables as we go along?

Here is another way to see the problem. Suppose you wanted to write a computer program to check whether a particular string of symbols is a formula of your predicate calculus. (For example, $(P \wedge Q) \vee R$ is a formula, but $(P \wedge Q)(\vee R$ is not. Then you would have to tell your program which symbols count as variables. If you put in your program the stipulation that the variables are P, Q, R and S, then the program wil recognize $(P \wedge Q) \vee R$ as a formula, but it will not accept $((P \wedge Q) \vee (wedgeS)) \vee T$ because you have not told it that T is also a variable. The problem is, you do not know in advance how many variables you are going to need. You may want in some cases to use 4, 5, 6, …, 457, or any number of variables, so we really need an *infinite* collection of propositional variables. But you cannot list infinitely many variables by telling the computer that the variables are "P, Q, R, S, etc." The computer does not understand what "etc." means.

The way around this is to use a *fixed, finite* list of symbols to specify infinitely many variables. Actually, two symbols suffice. We pick just one letter (for example, the P) and a "marker" symbol (for example, the vertical bar $|$), and then let the propositional variables be P, $P|$, $P||$, $P|||$, $P||||$, $P||||$, and so on. Now we have a propositional calculus with *exactly* 9 *symbols*, namely,

$$\boxed{P \qquad | \qquad \vee \qquad \wedge \qquad \Rightarrow \qquad \Leftrightarrow \qquad \sim \qquad (\qquad)}\ .$$

In this calculus, instead of writing something like "$(A \wedge B) \Rightarrow (C \vee D)$", we would write "$(P \wedge P|) \Rightarrow (P|| \vee P|||)$". Thus, the formulas we end up writing get a little bit longer, but we have achieved something that in this course we are always striving for, namely, **complete precision**. ♢

5.2 Propositional forms (a.k.a. well-formed formulas)

Using the basic symbols of our propositional calculus, we want to write **propositional forms** (a.k.a. **sentence forms**, or **statement forms**, or **well-formed formulas**), that is, expressions such as "$(A \wedge B) \Rightarrow A$", where we can plug in a statement for each of the variables and get a "compound statement." (For example, if we plug in "I am rich" for A and "I am smart" for B, we get the statement "if I am rich and I am smart then I am rich," which is incredibly stupid but undoubtedly true.)

Not every string of symbols will be "acceptable" to us. For example, "AB" should not be acceptable, since we would not know how to plug in statements for A and B and get something meaningful. (If you think you can plug in "I am rich" and "I am smart" for A and B, think again! What you get if you do that is "I am rich I am smart," which is certainly not a correct English sentence.)

We are going to call those strings that we would like to accept as being really "forms of compound statements" **well formed**.

REMARK: Every time you use some basic symbols to create a language, there is going to be a distinction between strings that are acceptable and strings that are not. For example, in English the string "the cow is white" (consisting of 16 symbols; we *are* counting the blank space as a separate symbol) is acceptable, but the string "eo wt hics htiwe" is not. (And what do you think about "the white is cow", or "the is cow white"? I leave these ones up to you, but you should draw from these examples the conclusion that in English the notion of an "acceptable string" is not 100% precise.)

In the so-called "formal languages" there is a precise, totally unambiguous notion of acceptability of a string. The acceptable strings are called "well formed."

For example, suppose we use the symbols

0	1	2	3	4	5	6	7	8	9

(called "digits") to name nonnegative integers. Then the strings "0", "1", "3", "70", "63", "20954", "137828", are acceptable, but "01" is not. The precise rule is as follows: a string of digits is **well formed** if it is either the "0" string or a string whose leftmost symbol is not 0.

The well-formed formulas of the propositional calculus are also called **propositional forms.** The well-formed formulas are those strings of symbols that can be obtained by applying certain **rules**:

PRECISE DEFITION OF "PROPOSITIONAL FORM"

Propositional forms are well-formed formulas of a given propositional calculus. The rules determining which strings are well formed are as follows:

RULE PF1: One-letter strings consisting of a propositional variable are well-formed formulas.

RULE PF2: If A is a well-formed formula consisting of a single letter then $\sim A$ is a well-formed formula.

RULE PF3: If A is a well-formed formula consisting of more than a single letter then $\sim (A)$ is a well-formed formula.

RULE PF4: If A and B are well-formed formulas consisting of a single letter then $A \wedge B$, $A \vee B$, $A \Longrightarrow B$ and $A \Longrightarrow B$ are well-formed formulas.

RULE PF5: If A and B are well-formed formulas such that A consists of a single letter but B does not, then $A \wedge (B)$, $A \vee (B)$, $A \Longrightarrow (B)$ and $A \Longrightarrow (B)$ are well-formed formulas.

RULE PF6: If A and B are well-formed formulas such that B consists of a single letter but A does not, then $(A) \wedge B$, $(A) \vee B$, $(A) \Longrightarrow B$ and $(A) \Longrightarrow B$ are well-formed formulas.

RULE PF7: If A and B are well-formed formulas neither one of which consists of a single letter, then $(A) \wedge (B)$, $(A) \vee (B)$, $(A) \Longrightarrow (B)$ and $(A) \Longrightarrow (B)$ are well-formed formulas.

RULE PF8: Only those strings obtained by applying Rules PF1, PF2, PF3, PF4, PF5, PF6, and PF7 are well-formed formulas.

Here are some examples of propositional forms (assuming that the letters P, Q, R are propsitional variables):

$$
\begin{aligned}
&P \\
&Q \\
&\sim P \\
&(\sim P) \wedge Q \\
&\sim ((\sim P) \wedge Q) \\
&(P \vee R) \Longrightarrow (\sim ((\sim P) \wedge Q))
\end{aligned}
$$

EXAMPLE: the string

$$((P \vee Q) \wedge (\sim Q)) \Rightarrow (P \vee Q)$$

is well formed, because one can arrive at it by applying the rules as follows:

STEP 1: P and Q are well formed, by Rule 1.

STEP 2: Applying Rule 2 with Q in the role of $\mathcal{A}$, we see that $\sim Q$ is well formed.

STEP 3: Applying Rule 4 with P in the role of $\mathcal{A}$ and Q in the role of $\mathcal{B}$, we see that $P \vee Q$ is well formed.

STEP 4: Applying Rule 7 with $P \vee Q$ in the role of $\mathcal{A}$ and $\sim Q$ in the role of $\mathcal{B}$, we see that $(P \vee Q) \wedge (\sim Q)$ is well formed.

STEP 5: Applying Rule 7 with $(P \vee Q) \wedge (\sim Q)$ in the role of $\mathcal{A}$ and $P \vee Q$ in the role of $\mathcal{B}$, we see that $((P \vee Q) \wedge (\sim Q)) \Rightarrow (P \vee Q)$ is well formed.$\diamondsuit$

And now let us look at a more complicated example of a propositional form:

AN EXAMPLE OF THE USE OF RULES PF1 TO PF9

Let us look at the string

$$(\sim Q) \Longrightarrow (\sim (\sim (P \wedge (((\sim (\sim Q)) \vee R) \Longrightarrow (R \Longrightarrow (P \wedge Q)))))) \;.$$

Is this a well-formed formula? The answer is YES. To prove this, we exhibit a "derivation" of our string from the rules (using "WFF" to stand for "well-formed formula");

STEP 1: P is a WFF by Rule PF1.
STEP 2: Q is a WFF by Rule PF1.
STEP 3: $\sim Q$ is a WFF by Rule PF2.
STEP 4: $\sim (\sim Q)$ is a WFF by Rule PF3.
STEP 5: $P \wedge Q$ is a WFF by Rule PF4.
STEP 6: R is a WFF by Rule PF1.
STEP 7: $R \Longrightarrow (P \wedge Q)$ is a WFF by Rule PF5.
STEP 8: $(\sim (\sim Q)) \vee R$ is a WFF by Rule PF6.
STEP 9: $((\sim (\sim Q)) \vee R) \Longrightarrow (R \Longrightarrow (P \wedge Q))$ is a WFF by Rule PF6.
STEP 10: $P \wedge (((\sim (\sim Q)) \vee R) \Longrightarrow (R \Longrightarrow (P \wedge Q)))$ is a WFF by Rule PF5.
STEP 11: $\sim (P \wedge (((\sim (\sim Q)) \vee R) \Longrightarrow (R \Longrightarrow (P \wedge Q))))$ is a WFF by Rule PF3.
STEP 12: $\sim (\sim (P \wedge (((\sim (\sim Q)) \vee R) \Longrightarrow (R \Longrightarrow (P \wedge Q)))))$ is a WFF by Rule PF3.
STEP 13: $(\sim Q) \Longrightarrow (\sim (\sim (P \wedge (((\sim (\sim Q)) \vee R) \Longrightarrow (R \Longrightarrow (P \wedge Q))))))$ is a WFF

by Rule PF7.

5.3 How do we know that a string is not well formed?

In general, knowing that a string is well formed, when it is, is relatively easy, because one can do it by just applying the rules to obtain the string. (We have just done one example of that.)

But showing that a string S is *not* well formed is much harder, because to show that you would have to examine all possible ways to obtain well-formed strings by applying the rules, and see that you never get S. But there are infinitely many ways to apply the rules toet well-formed strings, so you cannot examine them all. (If you do not see the problem, ask yourself: how can I write a computer program that will produce a "yes" asnwer if you input a well-formed string such as $P \wedge Q$ and will give a "no" asnwer if you input a string such as $P\wedge)Q$, which is not well-formed? It turns out that this *can* be done, but it is not easy!)

However, in many cases how can show rigorously that a string isnot well-formed by resorting to special tricks.

EXAMPLE: I claim that the string $((P \wedge Q) \Rightarrow Q$ is ***not*** well formed. Let us try to understand *why*. First of all, you can try to produce $((P \wedge Q) \Rightarrow Q$ using the rules, and no matter how hard you try you will not be able to. But, just because you have not been able to do something it does not follow that it cannot be done. So, how do we know that $((P \wedge Q) \Rightarrow Q$ **cannot** be obtained using the rules? We need an *argument.* We need *reasoning.* We need a **proof.**

Here is a proof. First, we prove a small theorem.

THEOREM 1. *In every well-formed formula the number of left parentheses is equal to the number of right parentheses.*

PROOF. Call a string "balanced" if it has the same number of left and right parentheses. Any string obtained using Rules 1, 2 or 4 is balanced, because it has zero left parentheses and zero right parentheses. Any application of Rule 3 or 5 or 6 will add an extra left parenthesis and an extra right parenthesis, so when you apply any of these rules to a balanced string you get a balanced string. In any application of Rule 7, the number of left parentheses in the result is that of $\mathcal{A}$, plus that of $\mathcal{B}$, plus two, and the same is true for the number of right parentheses. So when you apply Rule 7 to two balanced strings you get a balanced string. So the application of Rules 1 through 7 can never create unbalanced strings. END OF PROOF.

Remark. Theorem 1 gives an example of a **necessary condition** for a string to be well formed. (We say that A is a *necessary condition* for B if A happens whenever B does, that is, if B implies A. For example: "he took the final exam" is a necessary condition for "he passed the course," because if he passed the course then he must have taken the final exam. But is of course not a "sufficient condition." Just taking the final exam is not enough to pass the course.) The theorem says that if a string is well formed then it has to have the same number of left and right parentheses. This condition is not sufficient: there are many strings—such as, for example, the string "$P)($"—that have the same number of left and right parentheses but are not well formed. How can we tell if a string is well formed? Even more precisely, can one give a **computer algorithm** to check? The answer is "yes," but this is a hard thing to do, and if you are interested you should look it up in a mathematical logic book. $\diamondsuit$

PROBLEM 3. Is $((P \vee Q) \wedge \sim Q) \Longrightarrow (P \vee Q)$ well formed? Justify your answer.

PROBLEM 4. Is $((P \vee Q) \wedge (\sim Q))(\Longrightarrow (P \vee Q))$ well formed? Justify your answer.

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QUESTION 1: Is the string $P)($ well-formed?

ANSWER: NO. This may look almost obvious to you but, once again, it is a good exercise to try to understand *how we know that this string is not well-formed.*

Theorem 1 will not help us, because this string has one left parenthesis and one right parenthesis, so the necessary condition of Theorem 1 is satisfied. So we need another theorem, giving us another necessary condition for well-formedness.

THEOREM 2. *In every well-formed formula, if the formula contains parentheses, then the first parenthesis—if you read the formula from left to right—must be a left parenthesis.*

PROOF. Call a string "bad" if it contains parentheses and the first parenthesis from left to right is a right parenthesis. Call string "good" if it is not bad. A string obtained using Rule 1 is good, because it has no parentheses at all. An application of Rules PF2 or PF4 will not produce any new parentheses, so if the strings you applied these rules to were good then the new string will also be good. An application of Rule PF3 will yield a string whose first symbol is $\sim$ and whose second symbol is (, so this string is clearly good. An application of Rule PF5 will yield a string whose first symbol is a letter, whose second symbol is one of the logical connectives, and whose third symbol is (, so this string is clearly good. An application of Rule PF6 or PF7 will yield a string whose first symbol is (, so this string is clearly good. So the application of Rules PF1 to PF7 can never yield bad strings. END OF PROOF.

Remark. Theorem 2 gives another example of a **necessary condition** for a string to be well formed: it says that if a string is well formed and contains parentheses then it first parenthesis—goin from left to right—has to be a left parenthesis. This condition is not sufficient, even if you use it together with that of Theorem 1: there are many strings that satisfy the conditions of Theorems 1 and 2 but are not well formed. ♢

QUESTION 2: Is the string $P()$ well-formed?

ANSWER: NO. Again, you may think this is obvious, but we want to have a real proof.

Theorem 1 will not help us, because this string has one left parenthesis and one right parenthesis, so the necessary condition of Theorem 1 is satisfied. Theorem 2 will not help us either, because the first parenthesis of our string is a left parenthesis.

So we need another theorem, giving us another necessary condition for well-formedness.

THEOREM 2. *A well-formed formula cannot contain a left parenthesis immediately followed by a right parenthesis.*

A WRONG PROOF. Call a string "bad" if it contains a left parentheses immediately followed by right parenthesis. Call string "good" if it is not bad. A string obtained using Rule 1 is good, because it has no parentheses at all. An application of Rules PF2 or PF4 will not produce any new parentheses, so if the strings you applied these rules to were good then the new string will also be good. An application of Rule PF3 will yield a string whose first symbol is $\sim$ and whose second symbol is (, so this string is clearly good. An application of Rule PF5 will yield a string whose first symbol is a letter, whose second symbol is one of the logical connectives, and whose third symbol is (, so this string is clearly good. An application of Rule PF6 or PF7 will yield a string whose first symbol is (, so this string is clearly good. So the application of Rules PF1 to PF7 can never yield bad strings. END OF PROOF.

Remark. Theorem 2 gives another example of a **necessary condition** for a string to be well formed: it says that if a string is well formed and contains parentheses then it first parenthesis—goin from left to right—has to be a right parenthesis. This condition is not sufficient, even if you use it together with that of Theorem 1: there are many strings that satisfy the conditions of Theorems 1 and 2 but are not well formed. ♢

PROBLEM 4. Is $(((P \vee Q) \wedge \sim Q) \Longrightarrow (P \vee Q))$ well formed? Justify your answer.

PROBLEM 5. Is $(((P \vee Q) \wedge (\sim Q))(\Longrightarrow (P \vee Q)))$ well formed? Justify your answer.

When we work with a propositional calculus, we usually do not want to have to write too many parentheses. So we add the

PARENTHESIS DELETION OPTION: If there is only one way to put the parentheses back in to produce a well-formed formula, then you are allowed to omit them.

For example, you can write $P \wedge Q$ rather than $(P \wedge Q)$, or $\sim P$ rather than $(\sim P)$.

Remark. The book, in the bottom half of page 6, gives you extra leeway to omit parentheses. For example, the book gives you **extra rules** so that, if you write $\sim P \wedge Q$, it is understood that this stands for $((\sim P) \wedge Q))$. (In principle, it could also have stood for $(\sim (P \wedge Q))$. The rules given in the book exclude this second possibility, because "$\sim$ applies to the smallest proposition following it.")

Notice that **if you want to make the formulas simpler by allowing deletion of lots of parentheses, as the book does, then the rules become more complicated.** In my own personal view, it is better to keep the rules simple, even at the cost of writing formulas with more parentheses. **You are free to do it my way, or the book's way, as you prefer, as long as you do it correctly.** $\diamondsuit$

5.4 What is the propositional calculus good for?

So far, we have just given some rules for writing certain weird strings of symbols and calling them "propositional forms." What do we want to with all that?

The answer is:

> **We use propositional forms to represent forms of propositions or statements.**

For example, the statement

- (S1) I like coffee, and if you offer me some coffe I'll drink it

could be represented by the propositional form $P \wedge (Q \Rightarrow R)$, where the propositional variables P, Q, R stand for "I like coffee," "you offer me some coffe," and "I'll drink it."

Notice that (S1) could also be represented in the form P, where the propositional variable P stands for the whole sentence "I like coffee, and if you offer me some coffe I'll drink it." Or it could be represented in the form $P \Rightarrow Q$, where P stands for "I like coffee," and Q stands for "if you offer me some coffe I'll drink it."

So **the representation of a statement by means of a propositional form is not unique.** There may be more than one way to represent the statement, depending on how deeply you want to go into the details its structure. In our example, "P" is a **coarser** propositional-form representation of (S1) than "$P \wedge Q$," which is in turn coarser that "$P \wedge (Q \Rightarrow R)$" (or, the other way around, ("$P \wedge (Q \Rightarrow R)$" is **finer** than "$P \wedge Q$," and "$P \wedge Q$" is **finer** than "$P$").

5.5 Tautologies and contradictions

As explained in the book, a propositional form has a **truth table**. The truth table tells you how to assing a "truth value" (True or False) to a propositional form if you assign truth values to the propositional variables.

DEFINITION: A **tautology** is a propositional form that has the value True for all possible ways of assigning truth values to its propositional variables.

DEFINITION:A **contradiction** is a propositional form that has the value False for all possible ways of assigning truth values to its propositional variables.

Often, we also apply the words "tautology" and "contradiction" to statements in ordinary English. In that case, what we mean by a "tautology" is "a statement that has a propositional form which is a tautology." Also, what we mean by a "contradiction" is "a statement that has a propositional form which is a contradiction."

For example: the statement

- (S2) Hector Sussmann is the greatest teacher in the whole world

is true (of course!), but it is of the form P, which is not a tautology, and it has no other, finer propositional form. So (S2) is not a tautology.

On the other hand, the statement

- (S3) Either Hector Sussmann is the greatest teacher in the whole world or he is not.

has the propositional form P, which is not a tautology, but is also has the form $(P \vee (\sim P))$, which is a tautology. So (S3) **is** a tautology.

Remark. The example of (S3) shows that, for something to be a tautology, it suffices that **one** of its propositional forms be a tautology. In the case of statement (S3), we can find at least three different propositional form for it, namely, P, $P \vee Q$, and $P \vee (\sim P)$, which are progressively finer, and only the third one is a tautology. This, however, suffices to make (S3) a tautology. $\diamondsuit$

Let us look at another example of a formal representation. Consider the statement (in which we assume that f is a fixed function described earlier, and x_0 is some given number)

- (S4) If f has a relative minimum at x_0 and f is differentiable at x_0 then $f'(x_0) = 0$.

This is of the form P (taking P to stand for the whole statement), or of the form $P \Longrightarrow Q$ (taking P to stand for "f has a relative minimum at x_0 and f is differentiable at x_0" and Q to stand for "$f'(x_0) = 0$"), or of the form $(P \wedge Q) \Longrightarrow R$ (taking P to stand for "f has a relative minimum at x_0," Q to stand for "f is differentiable at x_0," and R to stand for "$f'(x_0) = 0$"). Once again, notice that the logical form is not unique.