MATHEMATICS 300 — SPRING 2005

Introduction to Mathematical Reasoning

H. J. Sussmann

INSTRUCTOR'S NOTES

January 20, 2005

Contents

1 Information on the course

1.1 Course schedule

Our class meets on **Tuesdays and Thursdays, 5th period (2:50pm to 4:10pm)**, in Beck Hall 011.

1.2 About the instructor

My *name* is **H.J. Sussmann.** My *office* is **Hill 538**.
My *Rutgers phone extension* is 5-5407.
My *e-mail address* is **sussmann@math.rutgers.edu**.

1.3 Web page

I have set up a *Web page* for our Math 300 section:

http://www.math.rutgers.edu/˜sussmann/math300page.html

All the instructor's notes will be available there.

1.4 Office hours

My office hours will be:

- **Tuesday and Thursday, 4th period** (12:40 pm to 2:20 pm) in my office, **Hill 538**; (*NOTE: this is essentially the 4th period, but I am ending the office hours 10 minutes earlier, at 2:20 instead of 2:30, so as to have more time to get to Beck Hall at 2:50 for the class; also, I am starting 30 minutes earlier, at 12:40 instead of 1:10, to have more time to disucss your questions*);
- any other time, by appointment, in my office, Hill 538.

1.5 Homework, midterms, final exam.

Homework and quizzes will count for about one third of your grade. There will be *two midterms*, which will count—together—for another third of your grade. The *final exam* will count for the remaining third.

Midterm dates: Tuesday, March 1 and **Tuesday, April 19**.

Final exam date: Friday, May 6, 12:00-3:00 pm.

1.6 The textbook and the instructor's notes

We will be using:

- the book *A Transition to Advanced Mathematics* (fifth edition), by Douglas Smith, Maurice Eggen,and Richard St. Andre,
- the notes written by the instructor.

> *You should read and study carefully all the instructor's notes.*
>
> **The material of these notes is an integral part of the course, as much as that of the book.** *Furthermore, the notes contain all kinds of important information. For example, in this set of notes there are lots of things you need to know in order to do your homework.*

1.7 Readings for the first 4 days (Jan. 18, 20, 25, 27)

- the book's Chapter 1 (**all of it!**),
- the instructor's notes of January 20 and January 25.

2 Writing mathematics & submitting homework

2.1 Write clearly in complete meaningful sentences

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements* with a *clear meaning* that can can be understood by anyone reading what you wrote. For example:
  - If you tell me that "she is very smart," but you haven't told me who "she" is, then I don't know who you are talking about, so you haven't made a statement with a clear meaning.
  - If you write "$x > 0$," but you haven't told me who "x" is, then I don't know what you are talking about, so you haven't made a statement.
  - If I ask you to state Pythagoras' theorem and your answer only says "$a^2 + b^2 = c^2$," then you are not specifying what you are talking about, because you have not said what "a," "b," and "c" are supposed to be.[1]

[1] Here is a correct statement of Pyhtagoras' theorem: *Let T be a right triangle, let a, b be the lengths of the legs of T, and let c be the lenght of the hypothenuse. Then $a^2+b^2 = c^2$.*

- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in great detail in class, and will also be explained later in these notes.

  NOTE: Letter[2] variables function in mathematical writing very much like pronouns (he, she, they) in ordinary English. For example, the pronouns "he" and "she" stand for a person, but exactly which person this is we don't know unless it is somehow announced in advance or clear from the context. Furthermore, the person that the pronoun stands for can **vary** from paragraph to paragraph or even within a single paragraph. (For example, "Senator Collins said she would vote in favor, but Senator Boxer said she would vote against.") So pronouns are **variables**, in the sense that they stand for a thing or person that can **vary**. Letter variables work the same way: we can say things like "if $x = 5$ then $x^2 + x + 3$ is equal to 33, but if $x = 6$ then $x^2 + x + 3 = 45$." Notice that each time we make an assertion about x (for example, "$x^2 + x + 3$ is equal to 33," or "$x^2 + x + 3 = 45$") a value for x is **declared** before the assertion is made. A declaration can be overruled by declaring a different value for x. (In our example, x is declared to be 5, so "x" stands for "5" until we make a new declaration by letting x equal 6.) A particular declaration can have a very long **scope** (i.e., it can be valid for a long text following it). And, as long as a particular declaration holds, the letter x stands for a name of a particular thing, which is **fixed**, i.e., **constant**, so x is a **constant**. In other words, when we use a variable, and declare a particular value for it, then we "treat the variable as a constant." Some symbols such as 0, 1, 2, are assigned a fixed value once and for all for ever, and this can happen even to letter variables. (For example,π, e.) We will have a lot more to say about variables and constants later on.

[2] By "letter" I mean any of the usual letters of the English alphabet, lower case or capital. (NOTE: Capitals and lower case letters are **different** symbols. So if in a text we use both x and X as variables, then they are not the same thing. In addition, mathematicians often use as variables Greek letters and other special symbols such as $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$. In principle, you could use any symbol (for example, $\diamondsuit$, $\Box$, $\clubsuit$) as a variable.)

- Provide proper connectives between equations as well as between ideas.

  EXAMPLE: If you write

  $$\begin{aligned} x^2 &= 36\,, \\ x &= 6\,, \end{aligned}$$

  just like that, then you are not telling me what connection there is between these two statements. Presumably, what you have in mind is that the second equation follows from the first one, so you should indicate this by writing something like

  $$\begin{aligned} x^2 &= 36\,, \\ \text{therefore} \qquad x &= 6\,, \end{aligned}$$

  or

  (*) since $x^2 = 36$, it follows that $x = 6$.

  NOTE: Is (*) correct? That is, *does "$x = 6$" follow from "$x^2 = 36$?"* It depends on what else you know. If all you know is that x is a real number and $x^2 = 36$, then (*) is **not** a valid deduction, because x could also be equal to -6. If, on the other hand, you also know that x is nonnegative, then (*) is a valid deduction. This means that *in (*) there is a* **hidden premiss**, *namely, that* $x \geq 0$. Now, if you want to have a full justification of this step, you should bring this "hidden premiss" into the open, by writing, for example,

  (**) since $x^2 = 36$ and $x \geq 0$, it follows that $x = 6$,

  or

  (***) since $x^2 = 36$, it follows that $x = 6$ or $x = -6$;
  but we know that $x \geq 0$, so $x = 6$.

  Notice that this will be a correct justification *if you have already established before that* $x \geq 0$. Otherwise, you need to explain *how we know* that x is nonnegative.

- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.

- Try to say things correctly, following all the rules, but *in your own words.* Please *no rote learning.*

- Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you've tried to read any complicated document.
- *Do not assume that the people reading your paper can read your mind. Do assume that they are intelligent, but also assume that they are busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.*

Effective written exposition will be worth at least 50% of your grade. Conversely, bad or unclear exposition may be penalized as much as 50% of the grade or even more.

- The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

2.2 Your written work

You should pay attention to presentation, especially for the homework:

- A nicely typed homework (e.g., using a word processor) is preferable to handwritten work. Handwritten work is acceptable too, but in that case:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**.
  - You should use a pen. Never use a pencil.
  - Use any color other than red (for example, black, blue, or green), but DO NOT USE RED. (Reason: The use of red is reserved for the instructor's and grader's comments.)
- Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more than one sheet then the sheets are **stapled** and the pages are **numbered**.
- If you tear off the sheets from a spiral notebook, please make sure before you hand them in that there are none of those ugly hanging shreds of paper at the margins. (Use scissors or a cutter, if necessary.)

2.3 Answering questions in this course

In this course, whenever you are given a problem where you are asked to do something, your answer should be either:

(a) doing what you were asked to do,

or

(b) showing—that is *proving*—that it cannot be done.

(See, for example, Problems 2, 3, 4, 5, 8, 10, 12 below, where the correct answer is "what you asked me to do cannot be done.")

Notice that, when the answer is that "it cannot be done," it is not enough for you to *say* that it cannot be done. *You have to tell me why.* In other words, you have to *prove* that it cannot be done.

This remark is very important, and will apply throughout the semester, not just during the first week. And it applies to all your work, to the homework, the quizzes, the midterm exams, and the final exam. $\diamondsuit$

2.4 Some examples of problems with correct answers

Here are some examples of problems with correct solutions:

PROBLEM 1: Express the number 26 as a sum of two odd natural numbers.

ANSWER: $26 = 3 + 23$. $\diamondsuit$

REMARK: *There are lots of other solutions, of course! For example, here are two solutions different from the one given above:* $26 = 7 + 19$, $26 = 13 + 13$.

PROBLEM 2: Prove that 2021 is prime.

ANSWER: This cannot be done, because 2021 is **not** prime. *Proof that 2021 is not prime*: $2021 = 47 \times 43$. $\diamondsuit$

PROBLEM 3: Express the number 27 as the sum of two odd natural numbers.

SOLUTION: This is impossible. REASON: the sum of two odd numbers is always even. Since 27 is odd, it cannot be the sum of two odd numbers. $\diamondsuit$

PROBLEM 4: Prove that $2+2=5$.

SOLUTION: This cannot be done. REASON: The statement "$2+2=5$" is false, and false statements cannot be proved. (How do we prove that "$2+2=5$ is false? If you don't care about this, just forget it, but if this issue worries you then you are asking a very good question. We will discuss that later.)

PROBLEM 5: Prove that for every positive integer x there is another positive integer y such that $0<y<x$.

SOLUTION: This cannot be done. REASON: The statement that we are being asked to prove is false, and a false statement cannot be proved.

The reason that the statement is false is as follows: 1 is a positive integer, and there is no positive integer y such that $0<y<1$, because 1 is the smallest positive integer. So if we take x to be 1 then the conclusion is not true. So it is not true that "for *every* positive integer x there is another positive integer y such that $0<y<x$." ♢

PROBLEM 6: Express the number 12 as the difference of the squares of two natural numbers.

SOLUTION: $12=16-4$, so $12=4^2-2^2$. ♢

PROBLEM 7: Express the number 20 as the sum of the squares of two natural numbers.

SOLUTION: $20=4+16=2^2+4^2$. ♢

PROBLEM 8: Express the number 4 as the sum of the squares of two positive integers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 4 as a sum of two positive integers:

$$4=1+3\,,\qquad 4=2+2\,.$$

In neither of these two equalities are the two terms on the right-hand side squares.

PROBLEM 9: Express the number 4 as the sum of the squares of two integers.

SOLUTION: $4=0+4=0^2+2^2$. ♢

PROBLEM 10: Express the number 12 as the sum of the squares of two integers.

SOLUTION: This cannot be done. REASON: here are all the ways to express 12 as a sum of two nonnegative integers:

$12 = 0{+}12$, $12 = 1{+}11$, $12 = 2{+}10$, $12 = 3{+}9$, $12 = 4{+}8$, $12 = 5{+}7$, $12 = 6{+}6$.

In none of these expressions are the two terms on the right-hand side squares of natural numbers. $\diamondsuit$

PROBLEM 11: Express the number 12 as the sum of the squares of three natural numbers.

SOLUTION: $12 = 4 + 4 + 4$, so $12 = 2^2 + 2^2 + 2^2$. $\diamondsuit$

PROBLEM 12: Express the number 15 as the sum of the squares of three integers.

SOLUTION: This cannot be done. REASON: If we are going to write 15 as the sum of three squares, then we should be able to write it as the sum of two nonnegative integers, one of which is a square, while the other is the sum of two squares.

So let us look at all the ways to write 15 as a sum of two nonnegative integers, see in which cases one of these numbers is square, and then see if the other number is the sum of two squares.

Here are all the ways to express 15 as a sum of two nonnegative integers:

$$\begin{array}{llll} 15 = 0 + 15, & 15 = 1 + 14, & 15 = 2 + 13, & 15 = 3 + 12, \\ 15 = 4 + 11, & 15 = 5 + 10, & 15 = 6 + 9, & 15 = 7 + 8\,. \end{array}$$

There are four expressions in the above list where one of the numbers is a square:

$$15 = 0 + 15, \quad 15 = 1 + 14, \quad 15 = 4 + 11, \quad 15 = 6 + 9\,.$$

In each case, we must ask whether the other number is a sum of two squares.

For "15=0+15," we have to see if 15 is the sum of two squares. The answer is "no," because we already have the list of all (eight) ways of writing 15 as a sum of two nonnegative integers, and in no case are both those numbers suqares.

For "15=1+14," we have to see if 14 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 14 are 0, 1, 4, and 9. So if are going to write 14 as a sum of two squares, the only possibilities are $14 = 0 + 14$, $14 = 1 + 13$, $14 = 4 + 10$, $14 = 9 + 6$. But in all these cases the other number—that is, 14, 13, 10, or 6—is not a square.)

For "15=4+11," we have to see if 11 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 11 are 0, 1, 4, and 9. So if are going to write 11 as a sum of two squares, the only possibilities are $11 = 0 + 11$, $11 = 1 + 10$, $11 = 4 + 7$, $14 = 9 + 5$. But in all these cases the other number (that is, 11, 10, 7, or 5) is not a square.)

Finally, for "15=6+9," we have to see if 6 is the sum of two squares. The answer is "no." (Reason: the only squares not greater than 6 are 0, 1, and 4. So if are going to write 6 as a sum of two squares, the only possibilities are $6 = 0 + 6$, $6 = 1 + 5$, $6 = 4 + 2$. But in all these cases the other number (that is, 6, 5, or 2) is not a square.) $\diamondsuit$

PROBLEM 13: Express the number 15 as the sum of the squares of four natural numbers.

SOLUTION: $15 = 1 + 1 + 4 + 9 = 1^2 + 1^2 + 2^2 + 3^2$. $\diamondsuit$

PROBLEM 14: Express the number 674 as the sum of the squares of four natural numbers.

SOLUTION: $674 = 144 + 400 + 121 + 9 = 12^2 + 20^2 + 11^2 + 3^2$. $\diamondsuit$

REMARK: You may be thinking *how on Earth did I figure out how to solve problem 14?* That's a good question. It turns out that there is a technique for doing this, but it's not easy to explain. Maybe we will talk about it later in the course. But keep in mind that the question "how did I find a solution to problem 14?" is quite different from the question "is $674 = 12^2 + 20^2 + 11^2 + 3^2$ a correct solution?" This second question is easy to answer: you just compute $12^2 + 20^2 + 11^2 + 3^2$ and verify that what you get is 674. $\diamondsuit$

REMARK: You may have noticed that I gave you examples of natural numbers that could be expressed as the sum of two squares, and of natural numbers that could not. Then I gave you examples of natural numbers that could be expressed as the sum of three squares, and of natural numbers that could not. But for *four* squares what I have done is different: I gave you examples of natural numbers that could be expressed as the sum of four squares, but I did not give you examples of natural numbers that could not. Is there a reason for this? The answer is **yes**. The mathematician Lagrange (1736-1813) proved that **every natural number can be expressed as the sum of the squares of four natural numbers.** This is a hard theorem, and I will try to say a few words about it later in the course. $\diamondsuit$

3 Homework assignment No. 1, due on Thursday January 27

Before you start writing your homework, read carefully the rest of this handout, in particular §2 on "writing mathematics and submitting homework." Pay attention to §2.3, on "answering questions in this course."

The problems marked (#) are harder. Do not worry too much if you cannot do them, but please give them a try.

1. Find integers x, y such that $x^2 - y^2 = 28$.

2. Find integers x, y such that $x^2 - y^2 = 29$.

3. Find integers x, y such that $x^2 - y^2 = 30$.

4. Prove that every prime number greater than 2 is odd.

5. (#) Prove that every year must have a Friday the 13th.

6. (#) In the planet of the Klingons, they have a calendar exactly like the one we use here on Earth (there are 12 months, January has 31 days, February has 28 or 29, March has 31, April has 30, etc.), except for one thing: the order of the months is different. We do not know what it is, but we know for sure that it is different from ours. (For example, it may be that June is the first month, September is the second one, etc.) Prove that it is still true that every year must have a Friday the 13th.

7. Book, §1.2 (pages 15, 16, 17), Problem 5, parts (b), (d), (e), (f), (g). (Do **not** use a calculator.)

8. Book, §1.2 (pages 15, 16, 17), Problem 14.

9. Book, §1.3 (pages 24, 25, 26), Problem 1, all the non-starred items. (Use "$x \in \mathbb{R}$" for "x is a real number," "$x \in \mathbb{C}$" for "x is a complex number.")

10. Do Parts (k) and (m) of the previous problem without using the unique existence quantifier $\exists!$.

11. Book, §1.3 (pages 24, 25, 26), Problem 5, all the non-starred items.

4 Review questions for the first week of classes

Here are some examples of questions that I wouldn't assign as homework problems (because if I do then you could just copy the answers from the book), but could be asked in a quiz or midterm exam.

1. Define "conjunction," "disjunction," "negation".

2. Define "equivalence" of propositional forms.

3. Define "denial."

4. Define "tautology."

5. Define "contradiction."

6. State and prove Pythagoras' theorem.

7. Prove that there exists a prime number greater than 1000000000000000000000000000000000000000000000000.

8. Prove that for every positive integer n there exists a prime number greater than n.