Construction of ergodic cocycles that are fundamental solutions to linear systems of a special form

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Abstract.
If $T = \{T_t\}_{t\in\mathbb{R}}$ is an aperiodic measure-preserving jointly continuous flow on a compact metric space Ω endowed with a Borel probability measure m, and G is a compact Lie group with Lie algebra L, then to each continuous map $A : \Omega \mapsto L$ we can associate the fundamental matrix solution $\Omega \times \mathbb{R} \ni (\omega, t) \mapsto X^A(\omega, t) \in G$ of the family of time-dependent ordinary differential equations

$$X'(t) = A(T_t\omega)(X(t)), \quad x \in G, \ \omega \in \Omega.$$

The corresponding skew-product flow $T^A = \{T_t^A\}_{t\in\mathbb{R}}$ on $G \times \Omega$ is then defined by letting $T_t^A(g, \omega) = (X^A(\omega, t)g, T_t\omega)$ for $(g, \omega) \in G \times \Omega$, $t \in \mathbb{R}$. The flow T^A is measure-preserving on $(G \times \Omega, \nu_G \otimes m)$ (where ν_G is normalized Haar measure on G) and jointly continuous. For a given closed convex subset S of L, we study the set $C_{erg}(\Omega, S)$ of all continuous maps $A : \Omega \mapsto S$ for which the flow T^A is ergodic. We develop a new technique to determine a necessary and sufficient condition for the set $C_{erg}(\Omega, S)$ to be residual. It turns out that there is, associated to Ω, T, G, and S, a flow T^{torus} on the product $\mathbf{T} \times \Omega$—where $\mathbf{T}$ is a torus associated to S and G—which is a lift of T. The desired necessary and sufficient condition is then expressed in terms of this lift: $C_{erg}(\Omega, S)$ is residual if and only if the following three properties are satisfied: (a) G is connected, (b) S satisfies an algebraic controllability condition, known as the "dense accessibility property," and (c) the torus lift T^{torus} is ergodic. As a special case of this, $C_{erg}(\Omega, S)$ is always residual if T is ergodic, G is connected and S satisfies the "dense strong accessibility property," because in this situation the torus lift is just T. Since the dimension of S can be much smaller than that of L, our result proves that ergodicity is typical even within very "thin" classes of cocycles. This covers a number of differential equations arising in mathematical physics, and in particular applies to the widely studied example of the Rabi oscillator. As a consequence of our ergodicity theorem, it follows that the spectrum of the quasi-energy operator associated with the cocycle X^A is purely continuous for a typical A. In the case of the Rabi oscillator, this shows that the situation for a generic continuous map A is quite different from what had been found earlier, using the K.A.M. technique, for some highly non-generic classes of very special maps, for which the quasi-energy operator had been proved to have a pure point spectrum.

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§1. Introduction.

In this paper we develop a new technique to construct ergodic cocycles that arise as the fundamental matrix solutions to linear differential equations of a given form. These constructions are motivated by questions regarding spectral and stability of forced quantum oscillator systems (in particular the so called "Rabi oscillator," which models forced oscillations of a spin 1/2 particle moving under an external stationary stochastic field [BJL,DGP]). These questions lead to problems about ergodic properties of certain skew-product flows (see [NJ]). In [NS] we had established that the cocycle arising from a generic Rabi oscillator is minimal. In this paper we shall prove that in fact it is ergodic. This will help us establish that typically the spectrum of the "quasi-energy operator" associated with such systems is only continuous. This fact should be contrasted with a result of [BJL], where the authors prove the existence of only discrete pure point spectrum using the KAM technique.

As in [NS], our result will in fact be valid for a wide class of time dependent linear differential systems of a specific given form, where the time dependence is recurrent. Such systems are given by specifying data Ω, T, m, G, L, S, where

(1) Ω is a compact metric space, m is a Borel probability measure on Ω, and $T = \{T_t\}_{t\in\mathbb{R}}$ is a jointly continuous aperiodic m-preserving flow (cf. Defs. 2.3, 2.4, 2.5 below) on Ω,
(2) G is a compact connected Lie group and L is the Lie algebra of G;
(3) S is a subset of L.

(A particular case of the situation considered here arises when the flow is quasi-periodic.)

We use e_G, ν_G to denote, respectively, the identity element of G, and Haar measure on G, normalized so that $\nu_G(G) = 1$. We write $C^0(\Omega, S)$ to denote the space of continuous S-valued maps on Ω. Then every map $A \in C^0(\Omega, S)$ gives rise to a family of ordinary differential equations (parametrized by points $\omega \in \Omega$)

$$x' = A(T_t\omega)x, \quad x \in G,\ \omega \in \Omega\,. \tag{1.1}$$

Here the meaning of the product notation used in (1.1) is as follows. We identify L with the tangent space of G at the identity. If $g \in G$, and R_g denotes the right translation by g—i.e. the map $G \ni h \mapsto hg \in G$—then, if $v \in L$, the expression vg denotes the vector $(dR_g)(v)$, so vg is a tangent vector to G at g. It follows that the map $G \ni g \mapsto v^{rinv}(g) \stackrel{def}{=} vg \in TG$ (where TG is the tangent bundle of G) is a right-invariant vector field on G. Using the bijection $L \ni v \mapsto v^{rinv} \in L^{rinv}$ from L onto the set L^{rinv} of right-invariant vector fields on G, we can identify L with L^{rinv}. Then the right hand side of (1.1) is the value at x of the right invariant vector field $A(T_t\omega)^{rinv}$.

For each fixed $\omega \in \Omega$, we let $\mathbb{R} \ni t \mapsto X^A(\omega, t) \in G$ denote the fundamental matrix solution to (1.1), i.e., the solution $x(\cdot)$ of (1.1) such that $x(0) = e_G$. Then the map $X^A : \Omega \times \mathbb{R} \mapsto G$ is continuous and satisfies the *cocycle identity*

$$X^A(\omega, t+s) = X^A(T_t\omega, s)X^A(\omega, t) \qquad \text{for all } \omega \in \Omega,\ t, s \in \mathbb{R}. \tag{1.2}$$

Using X^A, we define the *skew-product flow* $T^A = \{T_t^A\}_{t\in\mathbb{R}}$ on $G \times \Omega$, by letting

$$T_t^A(g,\omega) = (X^A(\omega,t)g, T_t\omega). \tag{1.3}$$

Notice that T^A is jointly continuous, and the product measure $\nu_G \otimes m$ is invariant under T^A.

If G is a matrix Lie group, say $G \subseteq GL(n,\mathbb{C})$, then each map A gives rise to a one-parameter group $V^A = \{V_t^A\}_{t\in\mathbb{R}}$ of bounded operators on $L^2(\Omega,\mathbb{C}^n,m)$, defined by setting

$$V_t^A f(\omega) = X^A(\omega,t)^{-1} f(T_t\omega) \text{ for } f \in L^2(\Omega,\mathbb{C}^n,m)\,,\ \omega\in\Omega\,,\ t\in\mathbb{R}\,. \tag{1.4}$$

If G is a subgroup of the unitary group $U(n)$, then V^A is a strongly continuous one-parameter group of unitary operators, i.e., a unitary representation of $\mathbb{R}$. In the Physics literature the infinitesimal generator of this one-parameter group is called the *quasi-energy operator*, and the stability properties of the evolution of such systems are studied through the spectral properties of this representation. The evolution is regarded as *stable* if the spectrum of the quasi-energy operators is discrete pure point. Here we shall show, in Theorem B, that within very thin classes of linear differential systems of a special form, those for which the quasi-energy operator has only purely continuous spectrum are generic.

The absence of point spectrum for a quasi-energy operator is very closely related to the problem of "lifting ergodicity." The ergodicity lifting problem involves constructing ergodic skew-product flows T^A that are ergodic on $(G\times\Omega, \nu_G\otimes m)$, where the map A is constrained by the special form of the underlying family of differential equations. In Theorem A we will determine a necessary and sufficient condition for the flow T^A to be ergodic for maps A that are generic within the class of all maps satisfying the constraints. In particular, Theorem A will imply that, under suitable hypothesis on the nature of the constraints, the skew-product flow is ergodic for a generic map A that satisfies the constraints. The spectral result of Theorem B will then follow as a corollary.

The special class of maps A is described by specifying a fixed subset S of the Lie algebra L and requiring that the map A be S-valued. The conditions on S that will guarantee ergodicity of the generic lifts are all properties commonly encountered in control theory, under various names such as "accessibility" and "strong accessibility" (cf. [SJ], [JS], [S]).

An obvious necessary condition for ergodicity of the generic flows T^A is that G be connected. Assuming that G is connected, it turns out that a certain Lie algebraic condition, called *strong accessibility* (cf. Definition 2.1 below), is sufficient for the desired ergodicity. On the other hand, this condition is not necessary, since a weaker condition that we call "strong dense accessibility" (cf. Definition 2.2) suffices. But even strong dense accessibility fails to be necessary. It turns out that an even weaker condition, called "dense accessibility" (cf. Definition 2.2), is necessary for the desired ergodicity. With these observations, it is clear that he only case where it may be unclear whether generic ergodicity holds in when the dense accessibility condition is satisfied but the strong dense accessibility condition is not. In this case, the gap between the two conditions results in the existence of a compact, connected, Abelian quotient $\mathbf{T}$ of G and a lift of T to the product $\mathbf{T} \times \Omega$. The necessary and sufficient condition for generic ergodicity is then that this lift be ergodic.

If the strong dense accessibility condition holds, and *a fortiori* if strong accessibility holds, then $\mathbf{T}$ is trivial, so the lift of T is just T, and then ergodicity of T implies ergodicity of the generic lifts. With this condition, the set S can still be very "thin" in L, as in the following example.

Example. (The Rabi-oscillator.) Consider the system

$$i\frac{d\psi}{dt} = \begin{pmatrix} \lambda & f(t) \\ \overline{f(t)} & -\lambda \end{pmatrix}\psi, \qquad \psi \in \mathbb{C}^2. \tag{1.5}$$

(This is the Schrödinger equation which describes the dynamics of a two-level atom or a spin 1/2 particle moving under an external magnetic field $f(t)$.) The function $f : \mathbb{R} \to \mathbb{C}$ is a complex-valued potential, typically quasiperiodic in t, and $\lambda \in \mathbb{R}$ is a fixed parameter. Here the Lie algebra L is $su(2, \mathbb{C})$, the Lie algebra of all 2×2 skew-hermitian matrices, and $S \equiv S_\lambda$ is the set of matrices in L of the form $\begin{pmatrix} -i\lambda & ia+b \\ ia-b & i\lambda \end{pmatrix}$, where $a, b \in \mathbb{R}$. So S is a two-dimensional affine subspace of a three dimensional Lie algebra L.

It is easy to verify (see [NS]) that the set S_λ has the strong accessibility property in L.]

§2. Statement of the main results.

If G is a Lie group, and $g \in G$, we use R_g, L_g, A_g, to denote, respectively, the right translation by g, the left translation by g, and the inner automorphism determined by g, so that R_g, L_g and A_g are maps from G to G, given by

$$R_g(x) \stackrel{def}{=} xg\,, \quad L_g(x) \stackrel{def}{=} gxG\,, \quad A_g(x) \stackrel{def}{=} gxg^{-1} \in G\,, \quad \text{if} \quad x, g \in G\,.$$

Clearly, the identities

$$A_g = L_g R_{g^{-1}} = R_{g^{-1}} L_g\,,\ A_g A_{g'} = A_{gg'}\,,\ R_g R_{g'} = R_{g'g}\,,\ L_g L_{g'} = L_{gg'}\,,\ L_g R_{g'} = R_{g'} L_g$$

hold for every $g, g' \in G$

We use $L(G)$, e_G, T_gG, TG to denote, respectively, the Lie algebra of G, the identity element of G, the tangent space of G at a point $g \in G$, and the tangent bundle of G. As explained in §1, $L(G)$ is, by definition, the tangent space $T_{e_G}G$, and we identify $L(G)$ with the space $L^{rinv}(G)$ of right-invariant vector fields on G. Furthemore, if $v \in L(G) = T_{e_G}G$, then the map $G \ni g \mapsto vg \stackrel{def}{=} dR_g(v) \in TG$ is the right-invariant vector field that corresponds to v under the identification of $L(G)$ with $L^{rinv}(G)$.

If L is a real Lie algebra, and S is a subset of L, we use $Lie(S; L)$ to denote the Lie subalgebra of L generated by S. We write

$$S - S = \{x - y \mid x \in S,\ y \in S\}\,,$$

and let $Lie_0(S; L)$ be the ideal of $Lie(S; L)$ generated by $S - S$.

2.1 Definition. A subset S of the Lie algebra L has the *accessibility property in* L if $Lie(S;L) = L$, and the *strong accessibility property in* L if $Lie_0(S:L) = L$. $\diamondsuit$

If G is a Lie group and S is a subset of $L(G)$, we use $Gr(S;G)$, $Gr_0(S;G)$ to denote, respectively, the connected Lie subgroups of G whose corresponding Lie subalgebras are $Lie(S;L(G))$ and $Lie_0(S;L(G))$.

2.2 Definition. If G is a Lie group and S is a subset of the Lie algebra $L(G)$, we say that S has the *dense accessibility property in* G if $Gr(S;G)$ is dense in G, and that S has the *strong dense accessibility property in* G if $Gr_0(S;G)$ is dense in G. $\diamondsuit$

If L is a Lie algebra, $S \subseteq L$, and $\bar{s}$ is any member of S, then $Lie(S;L) = Lie_0(S;L) + \mathbb{R}\bar{s}$. (Indeed, let $\Lambda = Lie_0(S;L) + \mathbb{R}\bar{s}$. Then Λ is clearly a Lie subalgebra of L, because $[X + r\bar{s}, X' + r'\bar{s}] = [X, X'] + r[\bar{s}, X'] - r'[X, \bar{s}]$ whenever $X, X' \in Lie_0(S;L)$ and $r, r' \in \mathbb{R}$, and $[X, X']$, $[\bar{s}, X']$, $[X, \bar{s}]$ belong to $Lie_0(S;L)$ because $X, X' \in Lie_0(S;L)$, $\bar{s} \in Lie(S;L)$, and $Lie_0(S;L)$ is an ideal of $Lie(S;L)$. Furthemore, $S \subseteq \Lambda$, because if $s \in S$ then $s = s - \bar{s} + \bar{s}$, and $s - \bar{s} \in Lie_0(S;L)$. So $Lie(S;L) \subseteq \Lambda$, and then So $Lie(S;L) = \Lambda$.) It follows that

$$\text{(2.1)} \qquad \text{Either } Lie_0(S;L) = Lie(S;L) \quad \text{or} \quad \dim Lie(S;L)/Lie_0(S;L) = 1\,.$$

If G is a Lie group and $S \subseteq L(G)$, then

(A) *$Gr_0(S;G)$ is a normal subgroup of $Gr(S;G)$;*
(B) *if $\bar{s}$ is any member of S, then every $g \in Gr(S;G)$ can be expressed as a product $g = g_0 e^{t\bar{s}}$ for some $g_0 \in Gr_0(S;G)$, $t \in \mathbb{R}$.*

(Indeed, (A) and (B) are obviously true if $Lie_0(S;L) = Lie(S;L)$, so we may assume that $Lie(S;\mathbb{R}) = Lie_0(S;L) \oplus \mathbb{R}\bar{s}$. Let H be the set of those $g \in Gr(S;G)$ that have the desired expression. It then follows immediately from the implicit function theorem, applied to the map $Gr_0(S;G) \times \mathbb{R} \ni (g_0, t) \mapsto g_0 e^{t\bar{s}} \in Gr(S;G)$, that H contains some neighborhood U of e_G relative to $Gr(S;G)$. If $g, g' \in H$, then we can write $g = g_0 e^{t\bar{s}}$, $g' = g_0' e^{t'\bar{s}}$, with $g_0, g_0' \in Gr_0(S;G)$ and $t, t' \in \mathbb{R}$. Then $g^{-1} = \hat{g}_0 e^{t\bar{s}}$, where $\hat{g} = e^{t\bar{s}} g_0 e^{-t\bar{s}} \in Gr_0(S;G)$, and $gg' = g_0 e^{t\bar{s}} g_0' e^{t'\bar{s}} = \tilde{g}_0 e^{(t+t')\bar{s}}$, where $\tilde{g} = g_0(e^{t\bar{s}} g_0' e^{-t\bar{s}}) \in Gr_0(S;G)$. So $g^{-1} \in H$ and $gg' \in H$, showing that H is a subgroup of $Gr(S;G)$. But then $H = Gr(S;G)$, because $Gr(S;G)$ is connected and H contains a neighborhood of e_G in $Gr(S;G)$.)

Furthemore,

$$\text{(2.2)} \qquad \text{The class of } e^{t\bar{s}} \text{ modulo } Gr_0(S;G) \text{ is independent of the choice of } \bar{s} \in S\,.$$

(Indeed, if $s, \tilde{s} \in S$, and $\zeta(t) = e^{ts} e^{-t\tilde{s}}$, then

$$\zeta'(t) = s e^{ts} e^{-t\tilde{s}} - e^{ts} \tilde{s} e^{-t\tilde{s}} = \big(s - e^{ts} \tilde{s} e^{-ts}\big)\zeta(t) = \big(s - dA_{e^{ts}}(\tilde{s})\big)\zeta(t) \in Lie_0(S;L(G))\zeta(t)\,,$$

so the curve ζ is tangent to the foliation of $Gr(S;G)$ whose leaves are the translates of $Gr_0(S;G)$, and this implies that ζ is entirely contained in one leaf, so $\zeta(t) \in Gr_0(S;G)$ for all t, since $\zeta(0) = e_G$.)

Let $\overline{Gr(S;G)}$, $\overline{Gr_0(S;G)}$ denote, respectively, the closures of $Gr(S;G)$, $Gr_0(S;G)$ in G. Then $\overline{Gr(S;G)}$ and $\overline{Gr_0(S;G)}$ are closed, connected subgroups of G, and $\overline{Gr_0(S;G)}$ is a normal subgroup of $\overline{Gr(S;G)}$. Use $[g]$ to denote, for each $g \in \overline{Gr(S;G)}$, the class of g modulo $\overline{Gr_0(S;G)}$, so $[g]$ coincides with the right translate $\overline{Gr_0(S;G)}g$ and also with the left translate $g\overline{Gr_0(S;G)}$.

In view of (2.2), if $s, s' \in S$ then $e^{ts}e^{-ts'} \in Gr_0(S;G)$, so $e^{ts}e^{-ts'} \in \overline{Gr_0(S;G)}$ and then $[e^{ts}] = [e^{ts'}]$. It follows that if $s \in S$, then $\{[e^{ts}]\}_{t\in\mathbb{R}}$ is a one-parameter subgroup of the quotient $Q(S;G) \stackrel{def}{=} \overline{Gr(S;G)}/\overline{Gr_0(S;G)}$, which does not depend on the choice of s and is dense in $Q(S;G)$. (The density follows because if $[g] \in Q(S;G)$ then $g = \lim_{j\to\infty} g_j$, $g_j \in Gr(S;G)$, and $g_j = g_{0,j}e^{t_j s}$, $g_{0,j} \in Gr_0(S;G)$, so $[g_j] = [e^{t_j s}]$, and then $[g] = \lim_{j\to\infty}[g_j] = \lim_{j\to\infty}[e^{t_j s}]$.) Therefore the group $Q(S;G)$ is Abelian, because (a) if $S \neq \emptyset$ then $Q(S;G)$ has a dense one-parameter subgroup, and (b) if S is empty then $\overline{Gr(S;G)} = \overline{Gr_0(S;G)} = \{e_G\}$, so $Q(S;G)$ is trivial. In the special case when G is compact, the subgroup $\overline{Gr(S;G)}$ is compact as well, so $Q(S;G)$ is a compact, connected, Abelian Lie group, i.e., a torus.

As is customary when a group is Abelian, we use additive rather than multiplicative notation for the group operation on $Q(S;G)$. If $S \neq \emptyset$, we let $\mathbf{s}_{S;G}$ be the infinitesimal generator of the one-parameter subgroup $\{[e^{ts}]\}_{t\in\mathbb{R}}$ defined by any $s \in S$. (Equivalently: if $s \in S$, then s belongs to $L(\overline{Gr(S;G)})$, and then $\mathbf{s}_{S;G}$ is the class of s modulo $L(\overline{Gr_0(S;G)})$, so $\mathbf{s}_{S;G} \in L(\overline{Gr(S;G)})/L(\overline{Gr_0(S;G)}) \sim L(Q(S;G))$.) If $S = \emptyset$, so $L(Q(S;G)) = \{0\}$, we let $\mathbf{s}_{S;G} = 0$. Again, we use additive rather than multiplicative notation, and write $t\mathbf{s}_{S;G}$ rather than $e^{t\mathbf{s}_{S;G}}$.

If G is compact, we write $\mathbf{T}_{S;G}$ instead of $Q(S;G)$, and use $\mathbf{m}_{S;G}$ to denote the dimension of $\mathbf{T}_{S;G}$, so $\mathbf{T}_{S;G}$ is an $\mathbf{m}_{S;G}$-dimensional torus, i.e., a product $(\mathbb{R}/\mathbb{Z})^{\mathbf{m}_{S;G}}$ of $\mathbf{m}_{S;G}$ copies of the one-dimensional torus $\mathbb{R}/\mathbb{Z}$. If we identify $\mathbf{T}_{S;G}$ with $(\mathbb{R}/\mathbb{Z})^{\mathbf{m}_{S;G}}$ by choosing a basis of $L(\mathbf{T}_{S;G})$, thereby identifying $L(\mathbf{T}_{S;G})$ with $\mathbb{R}^{\mathbf{m}_{S;G}}$, then $\mathbf{s}_{S;G} = (s_1, \ldots, s_{\mathbf{m}_{S;G}})$, where the real numbers $s_1, \ldots, s_{\mathbf{m}_{S;G}}$ are linearly independent over $\mathbb{Q}$.

2.3 Definition. If X is a set, a *flow on* X is a one-parameter family $T = \{T_t\}_{t\in\mathbb{R}}$ of maps $X \ni x \mapsto T_t x \in X$ such that T_0 is the identity map of X and $T_{t+s} = T_t \circ T_s$ whenever $t, s \in \mathbb{R}$. If X is a topological space and $T = \{T_t\}_{t\in\mathbb{R}}$ is a flow on X, then T is *continuous* if every map T_t is continuous, and T is *jointly continuous* if the map $X \times \mathbb{R} \ni (x,t) \mapsto T_t x \in X$ is continuous. If $(X, \mathcal{B})$ is a measurable space (that is, X is a set and $\mathcal{B}$ is a σ-algebra of subsets of X), then T is *measurable* if $T_t(E) \in \mathcal{B}$ whenever $E \in \mathcal{B}$ and $t \in \mathbb{R}$. If $(X, \mathcal{B}, m)$ is a measure space, then an *m-preserving flow on* X (or a *measure-preserving flow on* $(X, \mathcal{B}, m)$) is a measurable flow $T = \{T_t\}_{t\in\mathbb{R}}$ on X maps $X \ni x \mapsto T_t x \in X$ such that $m(T_t A) = m(A)$ for every $A \in \mathcal{B}$, $t \in \mathbb{R}$. $\diamondsuit$

2.4 Definition. A measure-preserving flow $T = \{T_t\}_{t\in\mathbb{R}}$ on a probability space $(X, \mathcal{B}, m)$ is said to be *ergodic* if, whenever a set A belongs to $\mathcal{B}$ and is T-invariant (i.e., such that $m(A\Delta T_t A) = 0$ for all $t \in \mathbb{R}$, where Δ denotes symmetric difference), it follows that $m(A) = 0$ or $m(A) = 1$. $\diamondsuit$

2.5 Definition. A flow $T = \{T_t\}_{t\in\mathbb{R}}$ on a probability space $(X, \mathcal{B}, m)$ is said to be *aperiodic* if there exists a set $B \in \mathcal{B}$ (the "bad" set) such that $m(B) = 0$, having the property that $T_t x \neq x$ whenever $x \notin B$ and $t \neq 0$. $\diamondsuit$

If Ω is a topological space, we write $\mathcal{B}_\Omega$ to denote the Borel σ-algebra of Ω. A *Borel probability measure* on Ω, is a probability measure defined on the Borel σ-algebra $\mathcal{B}_\Omega$. A *Borel probability space* is a probability space $(\Omega, \mathcal{F}, m)$ such that Ω is a topological space and $\mathcal{F}$ is the Borel σ-algebra $\mathcal{B}_\Omega$, in which case, of course, m is a Borel probability measure on Ω. The phrase "the Borel probability space (Ω, m)" will mean "the probability space $(\Omega, \mathcal{B}_\Omega, m)$." A Borel probability space $(\Omega, \mathcal{B}_\Omega, m)$ will be said to be *metric* if Ω is a metric space, and *compact* if Ω is a compact.

If Ω is a metric space, then d_Ω will denote the distance function of Ω.

If Ω, U are topological spaces, then $C^0(\Omega, U)$ will denote the space of all continuous maps from Ω to U. If Ω is compact and U is complete metric then $C^0(\Omega, U)$ is a complete metric space, endowed with the supremum distance $d_{sup} : C^0(\Omega, U) \times C^0(\Omega, U) \mapsto \mathbb{R}$ given by $d_{sup}(f, g) = sup\{d_U(f(x), g(x)) : x \in \Omega\}$. Clearly, $C^0(\Omega, U)$ is a Banach space if Ω is compact and U is a Banach space.

If G is a compact Lie group and $S \subseteq L(G)$, then the torus $\mathbf{T}_{S;G}$ is a probability space, endowed with its normalized Haar measure $\nu_{\mathbf{T}_{S;G}}$, and we use $T^{\mathbf{T}_{S;G}}$ to denote the flow on $\mathbf{T}_{S;G}$ determined by the one-parameter group $\{t\mathbf{s}_{S;G}\}_{t\in\mathbb{R}}$ introduced above. So $T^{\mathbf{T}_{S;G}} = \{T_t^{\mathbf{T}_{S;G}}\}_{t\in\mathbb{R}}$ where, for each $t \in \mathbb{R}$, $T_t^{\mathbf{T}_{S;G}}$ is the map $\mathbf{T}_{S;G} \ni \tau \mapsto \tau + t\mathbf{s}_{S;G} \in \mathbf{T}_{S;G}$. It is clear that

(2.3) $\quad T^{\mathbf{T}_{S;G}}$ is a jointly continuous, $\nu_{\mathbf{T}_{S;G}}$-preserving, ergodic flow on $\mathbf{T}$.

2.6 Definition. Let $T = \{T_t\}_{t\in\mathbb{R}}$ be a flow on a set Ω. Let G be a compact Lie group, and let S be a subset of the Lie algebra $L(G)$. Define

$$T_t^{\mathbf{T}_{S;G}\times\Omega}(\tau, \omega) = (T_t^{\mathbf{T}_{S;G}}(\tau), T_t\omega) \quad \text{for} \quad \tau \in \mathbf{T}_{S;G}\,,\ \omega \in \Omega\,,\ t \in \mathbb{R}\,.$$

Then $T^{\mathbf{T}_{S;G}\times\Omega} = \{T_t^{\mathbf{T}_{S;G}\times\Omega}\}_{t\in\mathbb{R}}$, which is clearly a flow on $\mathbf{T}_{S;G} \times \Omega$, is called the *torus lift of T determined by the pair (G, S)*. $\diamondsuit$

It is clear that if T preserves a probability measure m on ω then $T^{\mathbf{T}_{S;G}\times\Omega}$ preserves the product measure $\nu_{\mathbf{T}_{S;G}} \otimes m$ on $\mathbf{T}_{S;G} \times \Omega$.

Our two main results are

Theorem A. *Let (Ω, m) be a compact metric Borel probability measure space, and let $T = \{T_t\}_{t\in\mathbb{R}}$ be a jointly continuous aperiodic m-preserving flow on Ω. Let G be a compact connected Lie group with Lie algebra L and let S be a nonempty closed convex subset of L. Let ν_G be Haar measure on G, normalized so that $\nu(G) = 1$. Then the following three conditions are equivalent:*

(1) *The set S has the dense accessibility property in G and the torus lift $T^{\mathbf{T}_{S;G}\times\Omega}$ is ergodic on the Borel probability space $(\mathbf{T}_{S;G}\times\Omega, \nu_{\mathbf{T}_{S;G}}\otimes m)$.*

(2) *The set*

$$C_{erg}(\Omega, S) = \{A \in C^0(\Omega, S) : T^A \text{ is ergodic on } (G\times\Omega, \nu_G \otimes m)\}, \tag{2.1}$$

is nonempty.

(3) *$C_{erg}(\Omega, S)$ is a residual subset of $C^0(\Omega, S)$.*

and

Theorem B. *Assume that the hypothesis of Theorem A, and condition (1) of the conclusion hold, and suppose in addition that*

(1) *n is a positive integer,*

(2) *G is a closed subgroup of the unitary group $U(n)$,*

and

(3) *G does not fix any ray in the complex projective n-space $P(\mathbb{C}^n)$.*

Let $C_{cont}(\Omega, S)$ be the set

$$\{A \in C^0(\Omega, S) : \text{the quasi-energy operator of } V^A \text{ has only continuous spectrum}\}.$$

Then $C_{cont}(\Omega, S)$ is a residual subset of $C^0(\Omega, S)$.

The proof of Theorem A will involve a series of constructions based on applying three general results. The first one is a corollary of the version of Rokhlin's lemma, due to D. Lind [L], for continuous-time flows. The second one is a slight generalization of a well known control theory result that relates the equal-time reachable sets to the accessibility Lie algebra of a control system. The third one is about approximating the integral of a function over a compact group by an average of translates of the function. Sections 4, 5 and 6 are devoted to the statements and proofs of these three background results. But first we turn to the easy proof that Theorem B follows from Thoerem A.

§3. Proof that Theorem A implies Theorem B.

We use the following lemma.

3.1 Lemma. *Suppose that the conditions of Theorem B are satisfied, and let $A \in C^0(\Omega, S)$ be such that T^A is ergodic on $(G \times \Omega, \nu_G \otimes m)$. Then the point spectrum of the one-parameter group V^A is empty.*

Proof. Suppose $f \in L^2(\Omega, \mathbb{C}^n, m)$ is an eigenvector of V^A. Since V^A is a unitary one-parameter group, our assumption implies the existence of a $\lambda \in \mathbb{R}$ such that the condition

$$(V_t^A f)(\omega) = e^{i\lambda t} f(\omega) \quad \text{for a.e. } \omega \in \Omega$$

holds for all $t \in \mathbb{R}$. Consider the map $\rho : U(n) \times \Omega \to P(\mathbb{C}^n)$, defined by

$$\rho(g, \omega) = \pi(g^{-1} f(\omega)),$$

where $\pi : \mathbb{C}^n \to P(\mathbb{C}^n)$ is the canonical projection onto the projective space. Then, if $t \in \mathbb{R}$ and $g \in U(n)$, the equalities

$$\begin{aligned} \rho(T_t^A(g, \omega)) &= \rho\big(X^A(\omega, t)g, T_t\omega\big) \\ &= \pi\big(g^{-1} X^A(\omega, t)^{-1} f(T_t\omega)\big) \\ &= \pi\big(g^{-1} V_t^A f(\omega)\big) \\ &= \pi\big(e^{i\lambda t} g^{-1} f(\omega)\big) \\ &= \pi\big(g^{-1} f(\omega)\big) \\ &= \rho(g, \omega) \end{aligned}$$

hold for a.e. ω. Thus $(\rho \circ T_t^A)(g, \omega) = \rho(g, \omega)$ for a.e. (g, ω). Thus ρ is T^A-invariant and hence, by the ergodicity assumption, it is a.e. constant. Thus there is a non-zero vector $v \in \mathbb{C}^n$ such that

$$\pi(g^{-1} f(\omega)) = \pi(v) \quad \text{for a.e. } (g, \omega).$$

In particular, we can pick ω such that

$$\pi(g^{-1} f(\omega)) = \pi(v)$$

for all g in a subset E of G of full measure. Then

$$\pi(hv) = \pi(hg^{-1} f(\omega)) = \pi((gh^{-1})^{-1} f(\omega))$$

for all $h \in G$ and all $g \in E$. Fix $g \in E$. Then the set $\tilde{E}_g = \{h \in G : gh^{-1} \in E\}$ is of full measure, and

$$\pi((gh^{-1})^{-1} f(\omega)) = \pi(v) \quad \text{whenever } h \in \tilde{E}_g\,,$$

because $gh^{-1} \in E$, and

$$\pi((gh^{-1})^{-1} f(\omega)) = \pi(hv) \quad \text{whenever } h \in \tilde{E}_g\,,$$

because $g \in E$. Therefore $\pi(hv) = \pi(v)$ for all $h \in \tilde{E}_g$, and then $\pi(hv) = \pi(v)$ for all $h \in G$, since $\tilde{E}_g$ is of full measure. Hence the stabilizer of this fixed ray is all of G, contradicting Assumption (6) of Theorem B. ♢

§4. Background results: Rokhlin towers and Lind's theorem.

The first background result—stated below as Proposition 4.2—will be a corollary of a theorem of D. Lind (see [L]) which extends the classical lemma of Rokhlin for aperiodic measure-preserving transformations to aperiodic, measure-preserving $\mathbb{R}$-actions. We first review Lind's theorem, for which purpose we need to introduce some more notations.

If $N \in \mathbb{R}$, $N > 0$, we use Bor_N, Leb_N, to denote, respectively, the σ-algebras of Borel- and Lebesgue-measurable subsets of the interval $[0, N]$, and bor_N, leb_N to denote the normalized Borel and Lebesgue measures on $[0, N]$, so that $bor_N : Bor_N \mapsto \mathbb{R}$, $leb_N : Leb_N \mapsto \mathbb{R}$, $bor_N(E) = \frac{|E|}{N}$ whenever $E \in Bor_N$, and $leb_N(E) = \frac{|E|}{N}$ whenever $E \in Leb_N$, where $|\cdot|$ is the usual Lebesgue measure on $\mathbb{R}$.

If $(X, \mathcal{B}, m)$ is a probability space, then $\hat{\mathcal{B}}_m$ is the *m-completion* of the σ-algebra $\mathcal{B}$, i.e., the set of all $E \subseteq X$ such that $E = B \cup N$ for some $B \in \mathcal{B}$ and some N such that $N \subseteq N'$ for an $N' \in \mathcal{B}$ for which $m(N') = 0$. We use $\hat{m}$ to denote the natural extension of m to $\hat{\mathcal{B}}_m$. The probability space $(X, \hat{\mathcal{B}}_m, \hat{m})$ is the *completion* of $(X, \mathcal{B}, m)$.

If $(X_1, \mathcal{B}_1, m_1)$, $(X_2, \mathcal{B}_2, m_2)$ are probability spaces, then $(X_1, \mathcal{B}_1, m_1) \times (X_2, \mathcal{B}_2, m_2)$ will denote the usual product probability space $(X_1 \times X_2, \mathcal{B}_1 \otimes \mathcal{B}_2, m_1 \otimes m_2)$, where $\mathcal{B}_1 \otimes \mathcal{B}_2$ is the σ-algebra of subsets of $X_1 \times X_2$ generated by the products $E_1 \times E_2$, for $E_1 \in \mathcal{B}_1$, $E_2 \in \mathcal{B}_2$, and $m_1 \otimes m_2$ is the product measure. We will then write $(X_1, \mathcal{B}_1, m_1) \hat{\times} (X_2, \mathcal{B}_2, m_2)$ to denote the *completed product*, i.e., the completion of $(X_1, \mathcal{B}_1, m_1) \times (X_2, \mathcal{B}_2, m_2)$. Therefore,

$$(4.1) \quad (X_1, \mathcal{B}_1, m_1) \hat{\times} (X_2, \mathcal{B}_2, m_2) \stackrel{def}{=} (X_1 \times X_2, \hat{\mathcal{B}}_m, \hat{m}), \text{ where } \mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2, \ m = m_1 \otimes m_2 .$$

If $(X, \mathcal{B})$ is a measurable space, and $E \in \mathcal{B}$, then $\mathcal{B}\lceil E$ is the *restriction of* $\mathcal{B}$ *to* E, i.e., the set $\{S \in \mathcal{B} : S \subseteq E\}$. Then $(E, \mathcal{B}\lceil E)$ is a measurable space as well.

If $(X, \mathcal{B}, m)$ is a probability space, $E \in \mathcal{B}$, and $m(E) > 0$, $m\lceil_{nrm} E$ denotes the *normalized restriction of* m *to* E, that is, the function $m\lceil_{nrm} E : \mathcal{B}\lceil E \mapsto \mathbb{R}$ given by $m\lceil_{nrm} E(S) = \frac{m(S)}{m(E)}$ for $S \in \mathcal{B}\lceil E$. Then $(E, \mathcal{B}\lceil E, m\lceil_{nrm} E)$ is a probability space as well. ◊

Recall that a *Lebesgue probability space* is a probability space $(X, \mathcal{B}, m)$ such that there exist

(a) an $X_0 \in \mathcal{B}$ such that $m(X_0) = 0$,

(b) a finite or countable subset S of X such that $\{s\} \in \mathcal{B}$ and $m(\{s\}) > 0$ whenever $s \in S$,

(c) a real number a such that $0 \le a$ and $a + \sum_{s \in S} m(\{s\}) = 1$,

(d) a subset I_0 of the interval $I = [0, a]$ such that $\lambda(I_0) = 0$,

and

(e) a bijective map φ from $X \backslash (X_0 \cup S)$ onto $I \backslash I_0$, such that φ and φ^{-1} are measurable and $\lambda(\varphi(E)) = m(E)$ for every E such that $E \subseteq X \backslash (X_0 \cup S)$ and $E \in \mathcal{B}$.

It was proved by von Neumann in [vN] (cf. also Billingsley [BI], p. 69), that if X is a complete separable metric space, $\mathcal{B}$ is the Borel σ-algebra of X, and m is a Borel probability measure on X, then $(X, \hat{\mathcal{B}}_m, \hat{m})$ is a Lebesgue space.

Lind's result is then as follows.

Theorem 4.1. *Assume that $(X, \mathcal{B}, m)$ is a Lebesgue space and $T = \{T_t\}_{t\in\mathbb{R}}$ is a jointly measurable, m-preserving, aperiodic flow on X. Then, given $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists a set $F \subseteq X$ such that*

1. *the sets F and $\mathcal{T}_{[0,N]}(F) \stackrel{def}{=} \cup\{T_t(F) : 0 \le t \le N\}$ belong to $\mathcal{B}$;*
2. *$T_tF \cap T_sF = \emptyset$ whenever $t \ne s$, $t, s \in [0, N]$;*
3. *$m(\mathcal{T}_{[0,N]}(F)) > 1 - \varepsilon$;*
4. *there exist a σ-algebra $\tilde{\mathcal{B}}$ of subsets of F, and a probability measure $\tilde{m}$ defined on $\tilde{\mathcal{B}}$, such that the bijective map $\varphi : F \times [0, N] \mapsto \mathcal{T}_{[0,N]}(F)$ defined by $\varphi(x, t) = T_tx$ is an isomorphism from the probability space $(F, \tilde{\mathcal{B}}, \tilde{m})\hat{\times}([0, N], Bor_N, bor_N)$ to the probability space $\Big(\mathcal{T}_{[0,N]}(F), \mathcal{B}\lceil\mathcal{T}_{[0,N]}(F), m\lceil_{nrm}\mathcal{T}_{[0,N]}(F)\Big)$.* ♢

We will use Lind's theorem in the form of the following corollary.

Proposition 4.2. *Assume that Ω is a compact metric space, m is a Borel probability measure on Ω, and $T = \{T_t\}_{t\in\mathbb{R}}$ is a jointly continuous, m-preserving, aperiodic flow on X. Then, given $N \in \mathbb{N}$ and $\varepsilon > 0$ there exists a compact subset $E \subseteq X$ such that*

1. *$T_tE \cap T_sE = \emptyset$ whenever $t \ne s$, $t, s \in [0, N]$;*
2. *$m(\mathcal{T}_{[0,N]}(E)) > 1 - \varepsilon$ (where $\mathcal{T}_{[0,N]}(E) \stackrel{def}{=} \cup\{T_t(E) : 0 \le t \le N\}$);*
3. *there exists a unique Borel probability measure $\check{m}$ on E having the property that the homeomorphism $\varphi : E \times [0, N] \mapsto \mathcal{T}_{[0,N]}(E)$ defined by $\varphi(x, t) = T_tx$ is an isomorphism from the probability space $(E \times [0, N], \check{m} \otimes bor_N)$ to the probability space $\Big(\mathcal{T}_{[0,N]}(E), m\lceil_{nrm}\mathcal{T}_{[0,N]}(E)\Big)$.*

Proof. We assume, as we clearly may without loss of generality, that $\varepsilon < 1$.

Let $\mathcal{B}$ be the Borel σ-algebra $\mathcal{B}_\Omega$. By von Neumann's theorem, the completed probability space $(\Omega, \hat{\mathcal{B}}_m, \hat{m})$ is a Lebesgue space. Let $\mathcal{B}^\#$ be the $\tilde{m}$-completion of $\tilde{\mathcal{B}}$.

Apply Theorem 5.2 with $\frac{\varepsilon}{2}$ instead of ε, and get F, $\tilde{\mathcal{B}}$, $\tilde{m}$ having the properties of the conclusion of that theorem. Let $S_1 = F \times [0, N]$, $S_2 = \mathcal{T}_{[0,N]}(F)$, and let $\mathcal{B}_1$, $\mathcal{B}_2$, be, respectively, the $\tilde{m} \otimes bor_N$-completion of the σ-algebra $\tilde{\mathcal{B}} \otimes Bor_N$, and the σ-algebra $\hat{\mathcal{B}}_m\lceil S_2$. Write m_1, m_1 to denote, respectively, the natural extension of $\tilde{m} \otimes bor_N$ to $\mathcal{B}_1$, and the measure $\hat{m}\lceil_{nrm}S_2$, so that $m_2(S) = \frac{\hat{m}(S)}{\hat{m}(S_2)}$ whenever $S \in \hat{\mathcal{B}}_m$ and $S \subseteq S_2$.

Then the bijective map $\varphi : S_1 \mapsto S_2$ is a $\mathcal{B}_1$-$\mathcal{B}_2$-isomorphism, in the sense that if $B \subseteq S_1$ then $B \in \mathcal{B}_1$ if and only if $\varphi(B) \in \mathcal{B}_2$.

We claim that the inclusion map $\iota : F \times [0, N] \mapsto \Omega \times [0, N]$ is $\mathcal{B}_1$-to-Borel measurable, in the sense that if E is a Borel subset of $\Omega \times [0, N]$, then $E \cap (F \times [0, N]) \in \mathcal{B}_1$. To see this, observe that $\iota = \Psi \circ \Theta \circ \Phi$, where

(a) Φ is the map $F \times [0, N] \ni (\omega, t) \mapsto (T_t\omega, t) \in S_2 \times [0, N]$,
(b) Θ is the inclusion map from $S_2 \times [0, N]$ to $\Omega \times [0, N]$,
and

(c) Ψ is the map $\Omega \times [0,N] \ni (\omega,t) \mapsto (T_{-t}\omega,t) \in \Omega \times [0,N]$.

The map Φ is given by $\Phi(\omega,t) = (\varphi(\omega,t),t)$. It then follows easily that

(A) *If* $E \in \mathcal{B}_2 \otimes Bor_N$ *then* $\Phi^{-1}(E) \in \mathcal{B}_1$.

(Indeed, it suffices to prove this if $E = E' \times E''$, $E' \in \mathcal{B}_2$, $E'' \in Bor_N$. But in this case

$$\Phi^{-1}(E) = \{(\omega,t) \in F \times [0,N] : \varphi(\omega,t) \in E',\, t \in E''\} = \varphi^{-1}(E') \cap F \times E'',$$

so $\Phi^{-1}(E) \in \mathcal{B}_1$.) Furthermore,

(B) *The inclusion map* Θ *is measurable, i.e., if* $E \in \mathcal{B} \otimes Bor_N$ *then* $E \cap (S_2 \times [0,N])$ *belongs to* $\mathcal{B}_2 \otimes Bor_N$.

(Again, it suffices to prove this if $E = E' \times E''$, $E' \in \mathcal{B}$, $E'' \in Bor_N$, and in this case $E \cap (S_2 \times [0,N]) = (E' \cap S_2) \times E''$, which clearly belongs to $\mathcal{B}_2 \otimes Bor_N$, since $E' \cap S_2 \in \mathcal{B}_2$, because $S_2 \in \hat{\mathcal{B}}_m$.) Finally, Ψ is continuous, so

(C) $\Psi^{-1}(E) \in \mathcal{B} \otimes Bor_N$ *whenever* $E \in \mathcal{B} \otimes Bor_N$.

Combining (A), (B), and (C), we find that ι is measurable, and our claim is proved.

Now, given a positive integer k, we can find a finite partition $W_{1,k}, \ldots, W_{\bar{s}(k),k}$ of $\Omega \times [0,N]$ into Borel subsets whose diameter—with respect to the distance function D on $\Omega \times [0,N]$ given by $D((\omega,t),(\omega',t')) = d_\Omega(\omega,\omega') + |t-t'|$—is not greater than 2^{-k}. After relabeling, we may assume that there exists $\hat{s}(k) \in \{1,\ldots,\bar{s}(k)\}$ such that $W_{s,k} \cap (F \times [0,N]) \neq \emptyset$ for $s = 1,\ldots,\hat{s}(k)$ and $W_{s,k} \cap (F \times [0,N]) = \emptyset$ for $s = \hat{s}(k)+1,\ldots,\bar{s}(k)$. If we define $V_{s,k} = W_{s,k} \cap (F \times [0,N])$ for $s = 1,\ldots,\hat{s}(k)$, then $(V_{1,k},\ldots,V_{\hat{s}(k),k})$ is a partition of $F \times [0,N]$ into nonempty sets of diameter not greater than 2^{-k}, and the fact that ι is measurable implies that the $V_{s,k}$ belong to $\mathcal{B}_1$. Let $X_{s,k} = \varphi(V_{s,k})$, so $(X_{1,k},\ldots,X_{\hat{s}(k),k})$ is a partition of S_2 for each k. Since φ is a $\mathcal{B}_1$-$\mathcal{B}_2$-isomorphism, the sets $X_{s,k}$ belong to $\mathcal{B}_2$. Using the regularity of the measure $\hat{m}$, we can find compact sets $K_{s,k}$ such that $K_{s,k} \subseteq X_{s,k}$ and $\hat{m}(X_{s,k} \backslash K_{s,k}) < \hat{s}(k)^{-1}2^{-1-k}\varepsilon$. Let $K_k^* = \cup_{s=1}^{\hat{s}(k)} K_{sk}$, so K_k^* is a compact subset of S_2 such that $\hat{m}(S_2 \backslash K_k^*) \leq 2^{-1-k}\varepsilon$. Let $K^* = \cap_{k=1}^{\infty} K_k^*$. Then K^* is a compact subset of S_2 such that $\hat{m}(S_2 \backslash K^*) \leq \frac{\varepsilon}{2}$. Since $\hat{m}(S_2) > 1 - \frac{\varepsilon}{2}$, we conclude that $\hat{m}(K^*) > 1 - \varepsilon$.

Pick points $v_{s,k} \in V_{s,k}$, and define $\psi_k : K_k^* \mapsto S_1$ by letting $\psi(\omega) = v_{s,k}$ if $\omega \in K_{s,k}$. Then ψ_k is continuous on K_k^*, and $D(\varphi^{-1}(\omega),\psi_k(\omega)) \leq 2^{1-k}$ for all $\omega \in K_k^*$. It follows that all the maps ψ_k are continuous on K^*. Since the ψ_k converge uniformly to φ^{-1} on K^*, we conclude that φ^{-1} is continuous on K^*. Hence $E^* \stackrel{def}{=} \varphi^{-1}(K^*)$ is a compact subset of $F \times [0,N]$. Furthermore, $E^* \in \mathcal{B}_1$ (because $K^* \in \mathcal{B}$ and $K^* \subseteq S_2$, so $K^* \in \mathcal{B}_2$, and then $E^* \in \mathcal{B}_1$, since φ is a $\mathcal{B}_1$-$\mathcal{B}_2$-isomorphism).

Let E be the image of E^* under the projection $(\omega,t) \mapsto \omega$. Then E is a compact subset of F, and $E^* \subseteq E \times [0,N]$. Furthermore, the set $K = \varphi(E \times [0,N]) = \mathcal{T}_{[0,N]}(E)$ is compact, since φ is continuous. Therefore $K \in \mathcal{B}_2$, and then $E \times [0,N] \in \mathcal{B}_1$. Since $K^* \subseteq K$, we have $m(K) = \hat{m}(K) \geq \hat{m}(K^*) > 1 - \varepsilon$. It is clear that the sets T_tE, for $0 \leq t \leq N$, are pairwise disjoint, since the T_tF are pairwise disjoint and $E \subseteq F$.

Finally, we construct the Borel measure $\check{m}$. For this purpose, we first show that the Borel σ-algebra $\mathcal{B}_E$ of E is contained in $\mathcal{B}^\#$. To see this, observe that if E' is a compact subset of E, then $E' \times [0,N]$ is compact, so $\varphi(E' \times [0,N])$ is compact, and then $\varphi(E' \times [0,N]) \in \mathcal{B}_2$, from which it follows that $\mathbf{E}' = E' \times [0,N]) \in \mathcal{B}_1$. It follows that for almost every t the section $\mathbf{E}'_t = \{\omega : (\omega,t) \in \mathbf{E}'$ belongs to $\mathcal{B}^\#$. But $\mathbf{E}'_t = E'$ for all t, and then $E' \in \mathcal{B}^\#$. It follows that every compact subset of E is in $\mathcal{B}^\#$, and then $\mathcal{B}_E \subseteq \mathcal{B}^\#$, as stated.

We can now define a finite Borel measure μ on E by restricting to $\mathcal{B}_E$ the natural extension $\tilde{m}^\#$ of $\tilde{m}$ to $\mathcal{B}^\#$. Then

$$\mu(E) = m_1(E \times [0,N]) = m_2(K) = \frac{\hat{m}(K)}{\hat{m}(S_2)} > \frac{1-\varepsilon}{\hat{m}(S_2)} \geq 1 - \varepsilon > 0\,.$$

It is therefore possible to normalize μ and define a Borel probability measure $\check{m}$ on E by letting $\check{m}(A) = \frac{\mu(A)}{\mu(E)}$ for $A \in \mathcal{B}_E$.

It is clear that φ is a homeomorphism from $E \times [0,N]$ onto K, since φ is a continuous bijection and $E \times [0,N]$ is compact. Furthermore, the Borel σ-algebra of $E \times [0,N]$ is the product $\mathcal{B}_E \otimes Bor_{_N}$. Hence the restriction ψ of φ to $E \times [0,N]$ is a $\mathcal{B}_E \otimes Bor_{_N}$-$\mathcal{B}_K$-isomorphism. To conclude our proof, we have to show that $(\check{m} \otimes bor_{_N})(A) = \frac{m(\psi(A))}{m(K)}$ whenever $A \in \mathcal{B}_E \otimes Bor_{_N}$. Clearly, it suffices to prove this if $A = X \times Y$, where X is a Borel subset of E and Y is a Borel subset of $[0,N]$. In that case, $X \in \mathcal{B}^\#$, and

$$(\check{m} \otimes bor_{_N})(A) = \frac{\lambda(Y)\mu(X)}{N\mu(E)} = \frac{\lambda(Y)\tilde{m}^\#(X)}{N\mu(E)} = \frac{(\tilde{m}^\# \otimes bor_{_N})(X \times Y)}{\mu(E)} = \frac{m_1(A)}{\mu(E)}\,.$$

On the other hand, $m_1(A) = m_2(\psi(A))$, and $\mu(E) = m_1(E \times [0,N]) = m_2(K)$. Therefore

$$(\check{m} \otimes bor_{_N})(A) = \frac{m_2(\psi(A)}{m_2(K)} = \frac{\frac{\hat{m}(\psi(A))}{\hat{m}(S_2)}}{\frac{\hat{m}(K)}{\hat{m}(S_2)}} = \frac{\hat{m}(\psi(A))}{\hat{m}(K)} = \frac{m(\psi(A))}{m(K)}\,,$$

and our proof is complete. ♢

The set F in Theorem is 4.1 a "transversal" to the flow and $\tilde{m}$ is a "transversal measure" induced by m on F. The family of sets $\{T_tF\}_{t\in[0,N]}$ is known as a *Rokhlin tower of height* N, and the sets F, $\mathcal{T}_{[0,N]}(F)$ are, respectively, the *base* and the *strip* of the tower. Proposition 4.2 says that in the case of a compact metric space the transversal (i.e., the base of the tower) can be chosen to be a compact set (in which case of course the strip will be compact as well), and the transversal measure can be chosen to be a Borel probability measure.

§5. Background results on accessibility.

In this section G is a connected Lie group, $\mathbf{L}$ is its Lie algebra, S is a nonempty subset of $\mathbf{L}$, $Lie(S;\mathbf{L})$, $Lie_0(S;\mathbf{L})$ are the Lie subalgebras of $\mathbf{L}$ defined in §2, and $Gr(S;G)$, $Gr_0(S;G)$ are the corresponding connected Lie subgroups of G. We assume $\mathbf{L}$ is endowed with a Euclidean inner product and its corresponding norm.

Then $Gr_0(S;G)$ is a normal subgroup of $Gr(S;G)$, and one of the following possibilities occurs:

(1) $Lie_0(S;\mathbf{L}) = Lie(S;\mathbf{L})$ and $Gr_0(S;G) = Gr(S;G)$,
(2) $\dim Lie_0(S;\mathbf{L}) = \dim Lie(S;\mathbf{L}) - 1$, and $Gr_0(S;G)$ has codimension one in $Gr(S;G)$ and is dense in $Gr(S;G)$,
(3) $\dim Lie_0(S;\mathbf{L}) = \dim Lie(S;\mathbf{L}) - 1$, $Gr_0(S;G)$ has codimension one in $Gr(S;G)$ and is closed in $Gr(S;G)$, and the quotient $Gr(S;G)/Gr_0(S;G)$ is isomorphic to $\mathbb{S}^1$,
(4) $\dim Lie_0(S;\mathbf{L}) = \dim Lie(S;\mathbf{L}) - 1$, $Gr_0(S;G)$ has codimension one in $Gr(S;G)$ and is closed in $Gr(S;G)$, and the quotient $Gr(S;G)/Gr_0(S;G)$ is isomorphic to $\mathbb{R}$.

5.1 Definition. Given $a, b \in \mathbb{R}$ such that $b > a$, an *S-valued control* on the interval $[a,b]$ is a Lebesgue integrable map $\eta : [a,b] \mapsto S$. ♢

Naturally, then, $L^1([a,b],S)$ is the set of all S-valued controls on $[a,b]$, and it is a metric space endowed with the distance obtained by restricting the distance function of the Banach space $L^1([,b],\mathbf{L})$. Clearly, if S is closed and convex in $\mathbf{L}$, then $L^1([a,b],S)$ is a closed convex subset of $L^1([a,b],\mathbf{L})$.

If $\eta \in L^1([a,b],S)$ is a control, we let Ξ^η denote the fundamental solution of the ordinary differential equation

$$g'(t) = \eta(t) \cdot g(t)\,. \tag{5.1}$$

In other words, $[a,b] \times [a,b] \ni (t,s) \mapsto \Xi^\eta(t,s) \in G$ is the map characterized by the fact that for each $s \in [a,b]$ the map $[a,b] \ni t \mapsto \Xi^\eta(t,s)$ is absolutely continuous and satisfies

$$\begin{cases} \frac{\partial \Xi^\eta}{\partial t}(t,s) & = & \eta(t)\Xi^\eta(t,s) \quad \text{for a.e. } t \in [a,b], \\ \Xi^\eta(s,s) & = & e_G\,. \end{cases} \tag{5.2}$$

The existence and uniqueness of Ξ^η follows from standard facts about ordinary differential equations. (Local existence and global uniqueness of the solutions of (5.2) follow because the right-hand side is measurable with respect to t and locally Lipschitz with respect to g with an integrable Lipschitz constant. Global existence is a consequence of the translation invariance of (5.2), as follows. We can fix a compact neighborhood K of e_G and a positive number α such that the solution ξ^η of (5.1) with initial condition $\xi(a) = e_G$ can never leave K as long as $\|\eta\|_{L^1} \le \alpha$. This implies that ξ^η exists globally on $[a,b]$ if $\|\eta\|_{L^1} \le \alpha$. Then, if $\eta : [a,b] \mapsto S$ is an arbitrary control, we can divide the interval $[a,b]$ into subintervals $[t_0,t_1]$, $[t_1,t_2]$, $\ldots$, $[t_{k-1},t_k]$, with $t_0 = a$, $t_k = b$ such that the restriction η_i of η to $[t_{i-1},t_i]$ has L^1 norm not greater than α. If we then define $\xi : [a,b] \mapsto G$ inductively by letting $\xi(t) = \xi^{\eta_i}(t)\xi(t_{i-1})$ for $t_{i-1} < t \le t_i$, starting with $\xi(t_0) = e_G$, we see that ξ is a solution of (5.3) on $[a,b]$ such that $\xi(a) = e_G$.)

Given $s, r \in [a,b]$, both functions $[a,b] \ni t \mapsto \Xi^\eta(t,s)\Xi^\eta(s,r)$ and $[a,b] \ni t \mapsto \Xi^\eta(t,r)$ are solutions of (5.1) which take the value $\Xi^\eta(s,r)$ for $t = s$. Hence these two solutions coincide, which means that

$$\Xi^\eta(t,r) = \Xi^\eta(t,s)\Xi^\eta(s,r) \quad \text{whenever} \quad t,s,r \in [a,b]\,. \tag{5.3}$$

In particular, if we take $r = t$ and use the fact that $\Xi^\eta(t,t) = e_G$, we find

$$\Xi^\eta(s,t) = \Xi^\eta(t,s)^{-1} \quad \text{whenever} \quad t,s \in [a,b]\,. \tag{5.4}$$

Also, $\Xi^\eta(t,s) = \Xi^\eta(t,a)\Xi^\eta(a,s) = \Xi^\eta(t,a)\Xi^\eta(s,a)^{-1} = \xi^\eta(t)\xi^\eta(s)^{-1}$, so

$$\Xi^\eta(t,s) = \xi^\eta(t)\xi^\eta(s)^{-1} \quad \text{whenever} \quad t,s \in [a,b]\,. \tag{5.5}$$

Furthermore, since η is S-valued,

$$\Xi^\eta(t,s) \in Gr(S;G) \quad \text{whenever} \quad t,s \in [a,b]\,. \tag{5.6}$$

In addition, we can compare points accessible at a given time, and show that

$$\text{if } \eta_1,\eta_2 \in L^1([a,b],S) \text{ then} \quad \xi^{\eta_1}(t)\Big(\xi^{\eta_2}(t)\Big)^{-1} \in Gr_0(S;G) \text{ whenever } t \in [a,b]\,. \tag{5.7}$$

To see this, we first remark that

$$\text{if } g \in Gr(S;G) \text{ and } s \in S\,, \text{ then } gsg^{-1} - s \in Lie_0(S;\mathbf{L})\,. \tag{5.8}$$

Indeed, $e^{tu}se^{-tu} - s \in Lie_0(S;\mathbf{L})$ if $u \in S$, because (a) $e^{tu}se^{-tu} - s = \int_0^t e^{ru}[u,s]e^{-ru}dr$, (b) $[u,s] = [u,s-u]$, so $[u,s] \in Lie_0(S;\mathbf{L})$ because $s - u \in Lie_0(S;\mathbf{L})$ and $Lie_0(S;\mathbf{L})$ is an ideal of $Lie(S;\mathbf{L})$, and (c) $e^{ru}[u,s]e^{-ru} \in Lie_0(S;\mathbf{L})$ for each r, because $[u,s] \in Lie_0(S;\mathbf{L})$, $Lie_0(S;\mathbf{L})$ is an ideal of $Lie(S;\mathbf{L})$, and $e^{ru}[u,s]e^{-ru} = e^{r\,\mathrm{ad}_u}[u,s]$. It then follows by induction that $gsg^{-1} - s \in Lie_0(S;\mathbf{L})$ if $s \in S$, $g = e^{t_1u_1}e^{t_2u_2}\cdots e^{t_mu_m}$, $u_1,u_2,\ldots,u_m \in S$ and $t_1,t_2,\ldots,t_m \in \mathbb{R}$, because

$$gsg^{-1} - s = e^{t_1u_1}(hsh^{-1} - s)e^{-t_1u_1} + e^{t_1u_1}se^{-t_1u_1} - s\,, \quad \text{where} \quad h = e^{t_2u_2}\cdots e^{t_mu_m}\,.$$

Since every $g \in Lie(S;\mathbf{L})$ can be expressed as a product $e^{t_1u_1}e^{t_2u_2}\cdots e^{t_mu_m}$ as above, (5.8) follows.

Now fix $s \in [a,b]$, and let $\zeta_s(t) = \Xi^{\eta_1}(t,s)\Xi^{\eta_2}(t,s)^{-1}$. Then $\Xi^{\eta_1}(t,s) = \zeta_s(t)\Xi^{\eta_2}(t,s)$, so

$$\begin{aligned}
\eta_1(t)\zeta_s(t)\Xi^{\eta_2}(t,s) &= \eta_1(t)\Xi^{\eta_1}(t,s)\\
&= \frac{\partial \Xi^{\eta_1}}{\partial t}(t,s)\\
&= \frac{d\zeta_s}{dt}(t)\Xi^{\eta_2}(t,s) + \zeta_s(t)\frac{\partial \Xi^{\eta_2}}{\partial t}(t,s)\\
&= \frac{d\zeta_s}{dt}(t)\Xi^{\eta_2}(t,s) + \zeta_s(t)\eta_2(t)\Xi^{\eta_2}(t,s)\\
&= \frac{d\zeta_s}{dt}(t)\Xi^{\eta_2}(t,s) + \zeta_s(t)\eta_2(t)\zeta_s(t)^{-1}\zeta_s(t)\Xi^{\eta_2}(t,s)\\
&= \frac{d\zeta_s}{dt}(t)\Xi^{\eta_2}(t,s) + \zeta_s(t)\eta_2(t)\zeta_s(t)^{-1}\zeta_s(t)\Xi^{\eta_2}(t,s)
\end{aligned}$$

from which it follows that

$$\frac{d\zeta_s}{dt}(t) = \theta_s(t)\zeta_s(t)\,, \text{ where } \theta_s(t) = \eta_1(t) - \zeta_s(t)\eta_2(t)\zeta_s(t)^{-1}\,. \tag{5.9}$$

Since $\theta_s(t) = \eta_1(t) - \eta_2(t) - \Big(\zeta_s(t)\eta_2(t)\zeta_s(t)^{-1} - \eta_2(t)\Big)$, (5.8) implies that $\theta_s(t) \in Lie_0(S;G)$ for every t. It then follows from (5.9) that $\zeta_s(t) \in Gr_0(S;G)$ for every t, since $\zeta_s(s) = e_G$, and (5.7) is proved.

A trivial consequence of (5.7) is the fact that $e^{ts}e^{-ts'} \in Gr_0(S;G)$ whenever $s, s' \in S$ and $t \in \mathbb{R}$. Hence $Gr_0(S;G)e^{ts}e^{-ts'} = Gr_0(S;G)$, so $Gr_0(S;G)e^{ts} = Gr_0(S;G)e^{ts'}$. Furthermore, $Gr_0(S;G)e^{ts} = e^{ts}Gr_0(S;G)$ because $Gr_0(S;G)$ is a normal subgroup of $Gr(S;G)$. Hence *the left translate* $e^{ts}Gr_0(S;G)$ *coincides with the right translate* $Gr_0(S;G)e^{ts}$ *and does not depend on* s. We will use $Gr_t(S;G)$ to denote the translate $Gr_0(S;G)e^{ts}$ for $s \in S$. It then follows from (5.7) (taking $\eta_1 = \eta$, and letting η_2 be a constant control) that

$$\text{if } \eta \in L^1([a,b],S) \text{ then } \quad \xi^\eta(t) \in Gr_{t-a}(S;G) \text{ whenever } t \in [a,b]\,. \tag{5.10}$$

Finally, we will need the fact that

$$\text{the map } \quad L^1([a,b],S) \times [a,b] \times [a,b] \ni (\eta,t,s) \mapsto \Xi^\eta(t,s) \in G \quad \text{is continuous.} \tag{5.11}$$

(To prove this, it suffices to show that if $\{\eta^k\}_{k=1}^\infty$ is a sequence of controls on $[a,b]$ that converges in L^1 to a control η^∞, then the trajectories ξ^{η^k} converge uniformly to ξ^{η^∞}, since once this is proved the uniform convergence of Ξ^{η^k} to Ξ^{η^∞} on $[a,b]$ follows from the formula $\Xi^{\eta^k}(t,s) = \xi^{\eta^k}(t)\xi^{\eta^k}(s)^{-1}$. Using right invariance as before, it suffices to consider the case when the L^1 norms of all the η^k are bounded by a constant α such that all the trajectories ξ^{η^k} are contained in a compact neighborhood K of e_G which is a subset of the domain U of a cubic coordinate chart. If $X_1, \ldots, X_n$ is a basis of L, then the vector fields X_j can be regarded, on U, as smooth $\mathbb{R}^n$-valued functions of $g \in [-1,1]^n$. Then $\eta^k(t) = \sum_{i=1}^n \eta_i^k(t)X_i$, where the functions η_i^k are integrable, and each trajectory ξ^{η^k} has a time derivative bounded by $C\|\eta^k(t)\|$, for some constant C. To prove that $\xi^{\eta^k} \to \xi^{\eta^\infty}$ uniformly, it suffices to take an arbitrary subsequence $\{\xi^{\eta^{k(\ell)}}\}$ and prove that it has a subsequence $\{\xi^{\eta^{k(\ell(j))}}\}$ that converges uniformly to ξ^{η^∞}. Given the sequence $\{\xi^{\eta^{k(\ell)}}\}$, pick a subsequence $\{\xi^{\eta^{k(\ell(j))}}\}$ such that $\|\eta^{k(\ell(j))} - \eta^\infty\| \le 2^{-j}$. Then the function

$$[a,b] \ni t \mapsto \psi(t) \stackrel{def}{=} \|\eta^\infty(t)\| + \sum_{j=1}^\infty \|\eta^{k(\ell(j))}(t) - \eta^\infty(t)\|$$

is integrable, and $\|\eta^{k(\ell(j))}(t)\| \le \psi(t)$ for all t and all j. Therefore the sequence $\{\xi^{\eta^{k(\ell(j))}}\}_{j=1}^\infty$ is bounded and equicontinuous, so we can extract a uniformly convergent subsequence, and then it follows by standard arguments that the limit ξ of this subsequence must be a solution of (5.1) for $\eta = \eta^\infty$. Hence $\xi = \xi^{\eta^\infty}$, completing the proof.)

We now study perturbations of continuous controls, If $\eta \in C^0([a,b],S)$—i.e., if η is a continuous control—and $\delta > 0$, we use $N_S(\eta,\delta)$ to denote the open δ-neighborhood of η in $C^0([a,b],S)$, that is,

$$N_S(\eta,\delta) = \{\tilde{\eta} \in C^0([a,b],S) : \sup_{t\in[a,b]} \|\tilde{\eta}(t) - \eta(t)\| < \delta\},$$

so the sets $N_S(\eta,\delta)$, as δ varies over all positive numbers, form a fundamental system of neighborhoods of the control η in $C^0([a,b],S)$. We use $N_S^0(\eta,\delta)$ to denote the set of all $\tilde{\eta}$ in $N_S(\eta,\delta)$ such that $\tilde{\eta} - \eta$ vanishes on $[a, a+\varepsilon] \cup [b-\varepsilon, b]$ for some positive ε.

5.2 Definition. Given a set $\mathcal{N}$ of controls defined on a fixed interval of the form $[0,T]$, and a $g \in G$, the *$\mathcal{N}$-reachable set* from g is the subset $\mathcal{R}(\mathcal{N},g)$ of G defined by

$$\mathcal{R}(\mathcal{N},g) \stackrel{def}{=} \{\xi^\eta(T)g : \eta \in \mathcal{N}\}\,. \tag{5.12} \quad \diamondsuit$$

Clearly, the family $\{\mathcal{R}(\mathcal{N},g)\}_{g\in G}$ of $\mathcal{N}$-reachable sets satisfies the following *right-invariance property*:

$$\mathcal{R}(\mathcal{N},gg') = \mathcal{R}(\mathcal{N},g)g' \quad \text{whenever} \quad g,g' \in G\,. \tag{5.13}$$

In particular, the $\mathcal{N}$-reachable set from a given $g \in G$ is the right translate by g of the $\mathcal{N}$-reachable set from e_G.

Furthermore, (5.10) implies that

$$\mathcal{R}(\mathcal{N},e_G) \subseteq Gr_T(S;G) \quad \text{whenever} \quad \mathcal{N} \subseteq L^1([0,T],S)\,. \tag{5.14}$$

The following assertion is then the first of the two main results of this section.

Proposition 5.3. *Let G be a Lie group with Lie algebra $\mathbf{L}$, and let S be a nonempty closed convex subset of $\mathbf{L}$. Let $T > 0$, and let $\mathcal{F}$ be a compact subset of $C^0([0,T],S)$. Then given $\bar{\delta} > 0$ there exist a neighbourhood W in $Gr_0(S;G)$ of the identity e_G of G, depending on $\bar{\delta}$ but independent of $\eta \in \mathcal{F}$, such that for every $\eta \in \mathcal{F}$ the reachable set $\mathcal{R}(N_S^0(\eta,\bar{\delta}),e_G)$ contains some right translate of W by a member of $Gr(S;G)$.*

Proof. For each $k \in \mathbb{N}$, define a function $\nu_k : \mathbb{R}^k \mapsto \mathbb{R}$ by letting $\nu_k(t_1,\dots,t_k) = t_1 + \dots + t_k$. For each $\bar{\mathbf{t}} \in \mathbb{R}^k$, let $A_k(\bar{\mathbf{t}}) = \{\mathbf{t} \in \mathbb{R}^k : \nu_k(\mathbf{t}) = \nu_k(\bar{\mathbf{t}})\}$. Then $A_k(\bar{\mathbf{t}})$ is an affine subspace of $\mathbb{R}^k$ of dimension $k-1$. For each k-tuple $\mathbf{s} = (s_1,\dots,s_k)$ of members of S, let $\mu_{k,\mathbf{s}}$ be the map from $\mathbb{R}^k$ to $Gr(S;G)$ given by

$$\mu_{k,\mathbf{s}}(t_1,\dots,t_k) = e^{t_1 s_1} e^{t_2 s_2} \cdots e^{t_k s_k}\,. \tag{5.15}$$

For each $k \in \mathbb{N}$, $\mathbf{s} \in S^k$, $\mathbf{t} \in \mathbb{R}^k$, we let $\rho(k, \mathbf{s}, \mathbf{t})$, $\rho_0(k, \mathbf{s}, \mathbf{t})$ be, respectively, the rank at $\mathbf{t}$ of the differential $d\mu_{k,\mathbf{s}}$, and the rank at $\mathbf{t}$ of the differential $d\mu_{k,\mathbf{s},0,\mathbf{t}}$ of the restriction $\mu_{k,\mathbf{s},0,\mathbf{t}}$ of $\mu_{k,\mathbf{s}}$ to $A_k(\mathbf{t})$. Let $\bar{\rho}$ (resp. $\bar{\rho}_0$) be the maximum of all the numbers $\rho(k, \mathbf{s}, \mathbf{t})$ (resp. $\rho_0(k, \mathbf{s}, \mathbf{t})$) for all possible $(k, \mathbf{s}, \mathbf{t})$. We will show that

$$\bar{\rho} = \dim Gr(S; G) \qquad \text{and} \qquad \bar{\rho}_0 = \dim Gr_0(S; G) \,. \tag{5.16}$$

To prove that $\bar{\rho} = \dim Gr(S; G)$, we pick $(\bar{k}, \bar{\mathbf{s}}, \bar{\mathbf{t}})$ such that $\rho(\bar{k}, \bar{\mathbf{s}}, \bar{\mathbf{t}}) = \bar{\rho}$. Then in particular the map $\mathbb{R}^k \ni \mathbf{t} \mapsto \mu_{\bar{k},\bar{\mathbf{s}}}(\mathbf{t}) \in Gr(S; G)$ has constant rank $\bar{\rho}$ near $\bar{\mathbf{t}}$, so $\mu_{\bar{k},\bar{\mathbf{s}}}(t)$ maps some open neighborhood N of $\bar{\mathbf{t}}$ submersively onto a $\bar{\rho}$-dimensional embedded submanifold M of $Gr(S; G)$. It follows that every member of $Lie(S; \mathbf{L})$—regarded as a right-invariant vector field on $Gr(S; G)$—is tangent to M at every point of M. (Indeed, the set of vector fields X on $Gr(S; G)$ that are tangent to M is a Lie algebra, and the Lie subalgebra of $Lie(S; \mathbf{L})$ generated by S is $Lie(S; \mathbf{L})$. So it suffices to prove that every member of S is tangent to M. If $s \in S$ is not tangent to M, we may pick $\tilde{\mathbf{t}} \in N$ such that the vector $s(\mu_{\bar{k},\bar{\mathbf{s}}}(\tilde{\mathbf{t}}))$ is not tangent to M at $\mu_{\bar{k},\bar{\mathbf{s}}}(\tilde{\mathbf{t}})$. But then, if $\bar{\mathbf{s}} = (\bar{s}_1, \dots, \bar{s}_{\bar{k}})$, and $\tilde{\mathbf{t}} = (\tilde{t}_1, \dots, \tilde{t}_{\bar{k}})$ the map

$$\mathbb{R}^{\bar{k}+1} \ni (t_1, \dots, t_k, t) \mapsto e^{t_1 \bar{s}_1} e^{t_2 \bar{s}_2} \cdots e^{t_k \bar{s}_k} e^{ts}$$

has rank $\bar{\rho} + 1$ at $(\tilde{t}_1, \dots, \tilde{t}_{\bar{k}}, 0)$, contradicting the maximality of $\bar{\rho}$.) It follows that M is open in $Gr(S; G)$, and $\bar{\rho} = \dim Gr(S; G)$.

In order to prove that $\bar{\rho}_0 = \dim Gr_0(S; G)$, we first observe that $\bar{\rho}_0 \leq \dim Gr_0(S; G)$, because for every k, $\mathbf{t}$, $\mathbf{s}$, the map $\mu_{k,\mathbf{s},0,\mathbf{t}}$ takes values in $Gr_{\nu_k(\mathbf{t})}(S; G)$, since

$$e^{u_1 s_1} e^{u_2 s_2} \cdots e^{u_k s_k} \in Gr_0(S; G) e^{u_1 s_1} e^{u_2 s_2} \cdots e^{u_k s_k} = Gr_{u_1 + u_2 + \ldots + u_k}(S; G) \,.$$

Let $n = \dim Gr_0(S; G)$. Then either $\dim Gr(S; G) = n + 1$ or $\dim Gr(S; G) = n$. If $\dim Gr(S; G) = n + 1$ then $\bar{\rho} = n + 1$, and this easily implies that $\bar{\rho}_0 = n$, since we know that $\bar{\rho}_0 \leq n$, and the rank of the differential $d\mu_{\bar{k},\bar{\mathbf{s}},0,\bar{\mathbf{t}}}(\bar{\mathbf{t}})$ cannot possibly be smaller than n.

We now consider the case when $\dim Gr(S; G) = n$, i.e., when $Lie_0(S; G) = Lie(S; G)$. In that case, we augment our system, by writing $\mathcal{G} = Gr(S; G) \times \mathbb{R}$ and $\mathcal{L} = Lie(S; \mathbf{L}) \times \mathbb{R}$, so $\mathcal{G}$ is a connected Lie group with Lie algebra $\mathcal{L}$, and the Lie bracket in $\mathcal{L}$ is given by $[(X, r), (Y, s)] = ([X, Y], 0)$ whenever (X, r) and (Y, s) belong to $\mathcal{L}$. Let $\mathcal{S} = \{(X, 1) : X \in S\}$, and let $\mathcal{L}(\mathcal{S})$, $\mathcal{L}_0(\mathcal{S})$ be, respectively, the Lie subalgebra of $\mathcal{L}$ generated by $\mathcal{S}$, and the smallest ideal of $\mathcal{L}(\mathcal{S})$ that contains $\mathcal{S} - \mathcal{S}$. Then $\mathcal{S}$ is a closed convex subset of $\mathcal{L}$.

Furthermore, the fact that $Lie_0(S; \mathbf{L}) = Lie(S; \mathbf{L})$ implies that $\mathcal{L}(\mathcal{S}) = \mathcal{L}$. (Indeed, let U be the set of those $X \in Lie(S; \mathbf{L})$ such that $(X, 0) \in \mathcal{L}(\mathcal{S})$. Then U is a Lie subalgebra of $Lie(S; \mathbf{L})$, and $X - Y \in U$ whenever $X, Y \in S$. Let V be the set of all $X \in Lie(S; \mathbf{L})$ such that $\mathrm{ad}_X(U) \subseteq U$. Then V is a Lie subalgebra of $Lie(S; \mathbf{L})$, and $S \subseteq V$, because if $X \in S$ and $Y \in U$ then $(X, 1) \in \mathcal{L}(\mathcal{S})$ and $(Y, 0) \in \mathcal{L}(\mathcal{S})$, so $([X, Y], 0) \in \mathcal{L}(\mathcal{S})$, and then $[X, Y] \in U$. So $Lie(S; \mathbf{L}) \subseteq V$, and then $V = Lie(S; \mathbf{L})$, because $Lie(S; \mathbf{L}) \subseteq V$. But then U is an ideal of $Lie(S; \mathbf{L})$. Since $S - S \subseteq U$, $U = Lie_0(S; \mathbf{L})$, and then $U = Lie(S; \mathbf{L})$ because we are assuming that $Lie_0(S;\mathbf{L}) = Lie(S;\mathbf{L})$. Hence $(X,0) \in \mathcal{L}(\mathcal{S})$ for all $X \in Lie(S;\mathbf{L})$, so $Lie(S;\mathbf{L}) \times \{0\} \subseteq \mathcal{L}(\mathcal{S})$. Furthermore, if we fix an $X \in S$, then $(X,1) \in \mathcal{S}$, so $(X,1) \in \mathcal{L}(\mathcal{S})$. On the other hand, $(X,0) \in \mathcal{L}(\mathcal{S})$ because $X \in U$. Hence $(0,1) \in \mathcal{L}(\mathcal{S})$. Therefore $Lie(S;\mathbf{L}) \times \{0\} \subseteq \mathcal{L}(\mathcal{S})$ and $\{0\} \times \mathbb{R} \subseteq \mathcal{L}(\mathcal{S})$. So $\mathcal{L}(\mathcal{S}) = \mathcal{L}$.)

The set $Lie(S;\mathbf{L}) \times \{0\}$ is clearly an ideal of $\mathcal{L}(\mathcal{S})$ and contains $\mathcal{S} - \mathcal{S}$. Therefore $\mathcal{L}_0(\mathcal{S}) \subseteq Lie(S;\mathbf{L}) \times \{0\}$, and then $\mathcal{L}_0(\mathcal{S}) = L(S) \times \{0\}$, because $\mathcal{L}(\mathcal{S}) = \mathcal{L} = Lie(S;\mathbf{L}) \times \mathbb{R}$, and the codimension of $\mathcal{L}_0(\mathcal{S})$ in $\mathcal{L}(\mathcal{S})$ is 0 or 1. It follows that if we replace G by $\mathcal{G}$ and S by $\mathcal{S}$ then we are in the situation considered earlier, i.e., $\dim \mathcal{L}(\mathcal{S}) = 1 + \dim \mathcal{L}_0(\mathcal{S})$. Hence, if for each positive integer k and each k-tuple $\mathbf{s} = (s_1, \ldots, s_k)$ of members of S we let $\hat{\mu}_{k,\mathbf{s}}$ be the map from $\mathbb{R}^k$ to $\mathcal{G}$ given by

$$\hat{\mu}_{k,\mathbf{s}}(t_1, \ldots, t_k) = e^{t_1\sigma_1} e^{t_2\sigma_2} \cdots e^{t_k\sigma_k} ,$$

where $\sigma_i = (s_i, 1)$ for $i = 1, \ldots, k$, and $\mathcal{L} \ni u \mapsto e^u \in \mathcal{G}$ is the exponential map of $\mathcal{G}$, we can conclude that the maximum rank of the differentials of the maps $\hat{\mu}_{k,\mathbf{s}}$ is the dimension of $\dim \mathcal{L}(\mathcal{S})$, i.e., $n+1$. On the other hand, it is easy to see that

$$\hat{\mu}_{k,\mathbf{s}}(t_1, \ldots, t_k) = (e^{t_1 s_1} e^{t_2 s_2} \cdots e^{t_k s_k}, t_1 + t_2 + \cdots + t_k) = (\mu_{k,\mathbf{s}}(t_1, \ldots, t_k), t_1 + t_2 + \cdots + t_k) .$$

Therefore, if we pick $(\bar{k}, \bar{\mathbf{s}}, \bar{\mathbf{t}})$ such that $d\hat{\mu}_{k,\bar{\mathbf{s}}}(\bar{\mathbf{t}})$ has rank $n+1$, then $d\hat{\mu}_{k,\bar{\mathbf{s}},0,\bar{\mathbf{t}}}(\bar{\mathbf{t}})$ has rank n, completing the proof of (5.16).

If we now fix $s^* \in S$, pick $(\bar{k}, \bar{\mathbf{s}}, \bar{\mathbf{t}})$ such that $\rho_0(\bar{k}, \bar{\mathbf{s}}, \bar{\mathbf{t}}) = n$, and write $\bar{\mathbf{s}} = (\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_{\bar{k}})$, we can let

$$\bar{\mathbf{s}}(h) = ((1-h)s^* + h\bar{s}_1, (1-h)s^* + h\bar{s}_2, \ldots, (1-h)s^* + h\bar{s}_{\bar{k}}) .$$

It is clear that the map $\mathbb{R} \times \mathbb{R}^{\bar{k}} \ni (h, \mathbf{t}) \mapsto \mu_{\bar{k},\bar{\mathbf{s}}(h)}(\mathbf{t})$ is real-analytic. Hence the fact that for $h = 1$ the map $A_{\bar{k}}(\bar{\mathbf{t}}) \ni \mathbf{t} \mapsto \mu_{\bar{k},\bar{\mathbf{s}},0,\bar{\mathbf{t}}}(\mathbf{t})$ has rank n at $\bar{\mathbf{t}}$ implies that for arbitrarily small positive ε there exist points $\bar{\mathbf{t}}_\varepsilon = (t_{1,\varepsilon}, \ldots, t_{\bar{k},\varepsilon}) \in]0,\varepsilon[^{\bar{k}}$ and $h_\varepsilon \in]0,\varepsilon[$ such that the map $A_{\bar{k}}(\bar{\mathbf{t}}_\varepsilon) \ni \mathbf{t} \mapsto \mu_{\bar{k},\bar{\mathbf{s}}(h_\varepsilon),0,\bar{\mathbf{t}}_\varepsilon}(\mathbf{t})$ has rank n at $\bar{\mathbf{t}}_\varepsilon$. Since the convexity of S implies that the points $(1-h)s^* + h\bar{s}_j$ belong to S if $0 \le h \le 1$, we can conclude that

(#) *for every positive ε there exist $\bar{\mathbf{s}}_\varepsilon = (\bar{s}_{1,\varepsilon}, \bar{s}_{2,\varepsilon}, \ldots, \bar{s}_{\bar{k},\varepsilon}) \in S^k$, $\bar{\mathbf{t}}_\varepsilon \in]0,\varepsilon[^{\bar{k}}$, such that $\|\bar{s}_{j,\varepsilon} - s^*\| < \varepsilon$ and the map $A_{\bar{k}}(\bar{\mathbf{t}}_\varepsilon) \ni \mathbf{t} \mapsto \mu_{\bar{k},\bar{\mathbf{s}}_\varepsilon,0,\bar{\mathbf{t}}_\varepsilon}(\mathbf{t})$ has rank n at $\bar{\mathbf{t}}_\varepsilon$.*

Now let ε = *whatever*. Using the fact that $\mathcal{F}$ is compact and hence equicontinuous, choose a positive α such that $\|\eta(t) - \eta(t')\| < \varepsilon$ whenever $t, t' \in [0,T]$, $|t - t'| \le \alpha$, and $\eta \in \mathcal{F}$.

Now fix an $\eta_* \in \mathcal{F}$, and a T_0 such that $0 < T_0 < T$, and then choose T_-, T_+ such that $0 < T_- < T_0 < T_+ < T$ and $\theta \stackrel{def}{=} T_+ - T_- < \alpha$. Then, using (#) with $\eta_*(T_0)$ in the role of s^*, choose $\bar{k} \in \mathbb{N}$, $\bar{\mathbf{t}} = (\bar{t}_1, \ldots, \bar{t}_{\bar{k}})$ such that $\bar{t}_j > 0$ for all j and $\bar{t}_1 + \ldots + \bar{t}_{\bar{k}} < \theta$, and $\bar{\mathbf{s}} = (\bar{s}_1, \ldots, \bar{s}_{\bar{k}}) \in S^k$ such that $\|\bar{s}_j - \eta_*(T_0)\| < \varepsilon$ for all j, having the property that the map

$$A_{\bar{k}}(\bar{\mathbf{t}}) \ni (t_1, \ldots, t_{\bar{k}}) \mapsto e^{t_1\bar{s}_1} e^{t_2\bar{s}_2} \cdots e^{t_k\bar{s}_k}$$

has rank n at $\bar{\mathbf{t}}$.

Now fix a compact subinterval $I = [\bar{T}, \bar{T}+\bar{\theta}]$ of the open interval $]T_-, T_+[$, and a neighborhood N of $\bar{\mathbf{t}}$ relative to $A_{\bar{k}}(\bar{\mathbf{t}})$ all whose members $(t_1, \ldots, t_{\bar{k}})$ have positive coordinates. For each $(t_1, \ldots, t_{\bar{k}}) \in N$ and each $\eta \in \mathcal{F}$, let $\eta^{t_1,\cdots,t_{\bar{k}}}$ be the control defined by

$$(5.17) \quad \eta^{t_1,\cdots,t_{\bar{k}}}(t) = \begin{cases} \eta(t) & \text{if} \quad t<\bar{T} \ \text{ or } \ t \geq \bar{T}+\bar{\theta} \\ s_j & \text{if} \quad \bar{T}+t_1+\ldots+t_{j-1} \leq t < \bar{T}+t_1+\ldots+t_j\,, \ j=1,\ldots,\bar{k}\,. \end{cases}$$

Then the controls $\eta^{t_1,\cdots,t_{\bar{k}}}$ satisfy

$$(5.18) \ \|\eta^{t_1,\cdots,t_{\bar{k}}}(t)-\eta(t)\| \leq 3\varepsilon \ \text{ if } \ t \in [0,T], \eta \in \mathcal{F}, \|\eta-\eta_*\|_{sup} < \varepsilon\,, \ \text{and } (t_1,\ldots,t_{\bar{k}}) \in N\,,$$

since $\eta^{t_1,\cdots,t_{\bar{k}}}(t) = \eta(t)$ if $t < \bar{T}$ or $t \geq \bar{T}+\bar{\theta}$, and $\eta^{t_1,\cdots,t_{\bar{k}}}(t) = \bar{s}_j$ for some j if $\bar{T} \leq t < \bar{T}+\bar{\theta}$, while $\|\eta(t)-\eta(T_0)\| < \varepsilon$, $\|\eta(T_0)-\eta_*(T_0)\| < \varepsilon$, and $\|\bar{s}_j - \eta_*(T_0)\| < \varepsilon$.

Clearly, if $(t_1, \ldots, t_{\bar{k}}) \in N$ and $\eta \in \mathcal{F}$, we have

$$\xi^{\eta^{t_1,\ldots,t_{\bar{k}}}}(1) = \Xi^\eta(\bar{T}+\bar{\theta}, T)e^{t_{\bar{k}}\bar{s}_{\bar{k}}} \cdots e^{t_2\bar{s}_2}e^{t_1\bar{s}_1}\xi^\eta(\bar{T})$$

from which it follows that for each $\eta \in \mathcal{F}$ the map $N \ni (t_1, \ldots, t_{\bar{k}}) \mapsto \xi^{\eta^{t_1,\ldots,t_{\bar{k}}}}(1) \in G$ has rank n at $\bar{\mathbf{t}}$.

We now regularize the controls $\eta^{t_1,\cdots,t_{\bar{k}}}$ using a positive real regularization parameter ρ. First, we extend each η and its corresponding controls $\eta^{t_1,\cdots,t_{\bar{k}}}$—all of which agree with η near T—to maps $\tilde{\eta}$, $\tilde{\eta}^{t_1,\cdots,t_{\bar{k}}}$, defined on the whole half-line $[0,\infty[$, by letting $\tilde{\eta}(t) = \tilde{\eta}^{t_1,\cdots,t_{\bar{k}}}(t) = \eta(T)$ if $t > T$, $\eta \in \mathcal{F}$. Then the inequality of (5.18) is true for all $t \in [0,+\infty[$. We then define

$$(5.20) \qquad \eta^{t_1,\cdots,t_{\bar{k}};\rho}(t) = \frac{1}{\rho}\int_t^{t+\rho} \tilde{\eta}^{t_1,\cdots,t_{\bar{k}}}(s)\,ds \quad \text{for} \quad t \in [0,T]\,, \ \rho > 0.$$

It is clear that the functions $\eta^{t_1,\cdots,t_{\bar{k}};\rho}$ take values in S, because the $\tilde{\eta}^{t_1,\cdots,t_{\bar{k}}}$ take values in S, and S is closed and convex. It is also clear that the $\eta^{t_1,\cdots,t_{\bar{k}};\rho}$ are continuous. Furthermore, if $\|\eta-\eta_*\|_{sup} < \varepsilon$, then $\|\tilde{\eta}^{t_1,\cdots,t_{\bar{k}}}(t)-\eta(t)\| \leq 3\varepsilon$ for each t, and $\|\eta(t)-\eta(t')\| \leq \varepsilon$ if $t \leq s \leq t+\alpha$. Hence, if we define $\eta^{t_1,\cdots,t_{\bar{k}};0} \stackrel{def}{=} \eta^{t_1,\cdots,t_{\bar{k}}}$, the inequalities

$$(5.21) \qquad \|\eta^{t_1,\cdots,t_{\bar{k}};\rho}(t)-\eta(t)\| \leq 4\varepsilon$$

hold for all $(t_1, \ldots, t_{\bar{k}}) \in N$, all $t \in [0,T]$, and all $\eta \in \mathcal{F}$ such that $\|\eta-\eta_*\|_{sup} < \varepsilon$, as long as $0 \leq \rho \leq \alpha$.

Next, we pick a smooth function $\varphi : [0,T] \mapsto \mathbb{R}$ such that $0 \leq \varphi(t) \leq 1$ for all t, $\varphi(t) = 0$ for $0 \leq t \leq \frac{T_-}{2}$ and for $\frac{T_++T}{2} \leq t \leq T$, and $\varphi(t) = 1$ for $T_- \leq t \leq T_+$. We then define

$$(5.22) \qquad \eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}(t) = \big(1-\varphi(t)\big)\eta(t) + \varphi(t)\eta^{t_1,\cdots,t_{\bar{k}};\rho}(t)$$

for $\eta \in \mathcal{F}$, $t \in [0,T]$, $(t_1,\dots,t_{\bar{k}}) \in N$, $\rho \in [0,\alpha]$. Then the functions $\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}$ are also S-valued (because their value at each t is a convex combination of $\eta(t)$ and $\eta^{t_1,\cdots,t_{\bar{k}};\rho}(t)$. both of which belong to S). Also, the $\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}$ are continuous if $\rho > 0$. Furthermore, (5.21) implies

$$(5.23)\quad \|\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho} - \eta\|_{sup} \le 4\varepsilon \text{ if } \eta \in \mathcal{F}\,,\ \|\eta - \eta_*\|_{sup} < \varepsilon\,, (t_1,\dots,t_{\bar{k}}) \in N\,,\ \rho \in [0,\alpha]\,,$$

and it follows from our construction of the $\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}$ that, if we let $\beta = \min\left(\frac{T_-}{2}, \frac{T-T_+}{2}\right)$, then

$$(5.24)\qquad \eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}(t) = \eta(t) \quad \text{if} \quad t \in [0,\beta] \cup [T-\beta, T]$$

whenever $\eta \in \mathcal{F}$, $\|\eta - \eta_*\|_{sup} < \varepsilon$, $(t_1,\dots,t_{\bar{k}}) \in N$, and $\rho \in [0,\alpha]$.

We will also need the fact that

(5.25) the map $\mathcal{F}\times N\times[0,\alpha] \ni (\eta,(t_1,\dots,t_{\bar{k}}),\rho) \mapsto \eta^{\#,t_1,\cdots,t_{\bar{k}};\rho} \in L^1([0,T],S)$ is continuous.

(Indeed, let $C = \max(\|\bar{s}_1\|, \|\bar{s}_2\|,\dots,\|\bar{s}_{\bar{k}}\|) + \sup\{\|\eta\|_{sup} : \eta \in \mathcal{F}\}$. Then the bounds $\|\eta^{t_1,\cdots,t_{\bar{k}}} - \zeta^{t'_1,\cdots,t'_{\bar{k}}}\|_{L^1} \le \|\eta - \zeta\|_{L^1} + C\sum_{j=1}^{\bar{k}} |t_i - t'_i|$ hold for $\eta,\zeta \in \mathcal{F}$, and this implies that

$$\begin{aligned}\|\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho} - \zeta^{\#,t'_1,\cdots,t'_{\bar{k}};\rho}\|_{L^1} &\le \|\eta - \zeta\|_{L^1} + \|\eta - \zeta\|_{sup} + C\sum_{j=1}^{\bar{k}} |t_i - t'_i| \\ &\le (1+T)\|\eta - \zeta\|_{sup} + C\sum_{j=1}^{\bar{k}} |t_i - t'_i|\end{aligned}$$

for $\eta,\zeta \in \mathcal{F}$, $\rho \ge 0$. Hence, if we define maps $\Psi_\rho : \mathcal{F} \times N \mapsto L^1([0,\tau],S)$ by letting $\Psi_\rho(\eta;t_1,\dots,t_{\bar{k}}) = \eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}$, the Ψ_ρ are continuous for each $\rho \in [0,\alpha]$, and the family $\{\Psi_\rho\}_{0\le\rho\le\alpha}$ is uniformly bounded and equicontinuous on N. Let $\{\rho_j\}_{j=1}^\infty$ be a sequence in $[0,\alpha]$ that converges to a limit ρ. Then for each fixed $\eta \in \mathcal{F}$, $(t_1,\dots,t_{\bar{k}}) \in N$, the functions $\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}$, for $0 \le \rho \le \alpha$, are bounded by a fixed constant. In addition, it is clear that as $j \to \infty$ the functions $\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho_j}$ converge uniformly to $\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}$ if $\rho > 0$, and pointwise at every point of continuity t of $\eta^{\#,t_1,\cdots,t_{\bar{k}};0}$ if $\rho = 0$. Since $\eta^{\#,t_1,\cdots,t_{\bar{k}};0}$ is piecewise continuous, the $\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho_j}$ converge to $\eta^{*,t_1,\cdots,t_{\bar{k}};\rho}$ in L^1 (even if $\rho = 0$). Hence the maps Ψ_{ρ_j} converge pointwise to Ψ_ρ as as $j \to \infty$, and the equicontinuity of $\{\Psi_{\rho_j}\}_{0<\rho\le\alpha}$ implies that the convergence is uniform of compact subsets of $\mathcal{F} \times N$. Since this is true for every convergent sequence $\{\rho_j\}_{j=1}^\infty$ in $[0,\alpha]$, (5.25) follows immediately.)

Let $\mathcal{M}$ be the map from $\mathcal{F} \times N \times [0,\alpha]$ to G given by

$$(5.26)\qquad \mathcal{M}(\eta,(t_1,\dots,t_{\bar{k}}),\rho) = \xi^{\eta^{\#,t_1,\cdots,t_{\bar{k}};\rho}}(T)\,.$$

Then $\mathcal{M}$ takes values in $Gr_T(S;G)$, and (5.11) and (5.25) imply that $\mathcal{M}$ is continuous. Furthermore, (5.19) implies that

$$\mathcal{M}(\eta_*,(t_1,\dots,t_{\bar{k}}),0)=\xi^{\eta_*^{\#,t_1,\dots,t_{\bar{k}};\rho}}(T)=\Xi^\eta(\bar{T}+\bar{\theta},T)e^{t_{\bar{k}}\bar{s}_{\bar{k}}}\cdots e^{t_2\bar{s}_2}e^{t_1\bar{s}_1}\xi^\eta(\bar{T}),$$

from which it follows that the map

$$N\ni\mathbf{t}\mapsto\mathcal{M}(\eta_*,\mathbf{t},0)\in Gr_T(S;G)$$

has rank n at $\bar{\mathbf{t}}$. Hence (if we write $\mathcal{B}_n(r)=\{x\in\mathbb{R}^n:\|x\|\le r\}$), the implicit function theorem implies that there exist (a) a neighborhood B_1 of $\mathcal{M}(\eta_*,\mathbf{t},0)$ in $Gr_T(S;G)$ which is diffeomorphic by means of a map Φ to the closed unit ball $\mathcal{B}_n(1)$, (b) an r such that $0<r<1$, (c) a smooth map $\Theta:\mathcal{B}_n(r)\mapsto N$ such that

$$\Phi\big(\mathcal{M}(\eta_*,\Theta(x),0)=x\qquad\text{for all}\qquad x\in\mathcal{B}_n(r).$$

The continuity of $\mathcal{M}$ then implies that there exists a positive number γ such that

$$\mathcal{M}(\eta,\Theta(x),\rho)\in B_1\qquad\text{and}\qquad\|\Phi\big(\mathcal{M}(\eta,\Theta(x),\rho)\big)-x\|\le\frac{r}{2}\tag{5.27}$$

whenever $x\in\mathcal{B}_n(r)$, $\eta\in\mathcal{F}$, $\|\eta-\eta_*\|_{sup}\le\gamma$, and $0\le\rho\le\gamma$. It follows that

$$\text{if }x\in\mathcal{B}_n\Big(\frac{r}{2}\Big),\ \eta\in\mathcal{F},\ \|\eta-\eta_*\|_{sup}\le\gamma,\ \rho\in[0,\gamma],\ \text{ then }(\exists\mathbf{t}\in N)\Big(\Phi\big(\mathcal{M}(\eta,\mathbf{t},\rho)\big)=x\Big)$$

(Indeed, suppose that $x\in\mathbb{R}^n$, $\|x\|\le\frac{r}{2}$, $\eta\in\mathcal{F}$, $\|\eta-\eta_*\|_{sup}\le\gamma$, and $\rho\in[0,\gamma]$. Define $H(y)=x+y-\Phi\big(\mathcal{M}(\eta,\Theta(y),\rho)\big)$, for $y\in\mathcal{B}_n(r)$. Then ((5.27)) implies that $\|H(y)\|\le r$ whenever $\|y\|\le r$, so H is a continuous map from $\mathcal{B}_n(r)$ to $\mathcal{B}_n(r)$. By Brouwer's fixed point theorem, there exists $y\in\mathcal{B}_n(r)$ such that $H(y)=y$. If we let $\mathbf{t}=\Phi(y)$, then $\mathbf{t}\in N$ and $\Phi\big(\mathcal{M}(\eta,\mathbf{t},\rho)\big)=x$.)

Therefore, if we write $\mathcal{M}^{\eta,\rho}(\mathbf{t})\overset{def}{=}\mathcal{M}(\eta,\mathbf{t},\rho)$, and let $\tilde{W}=\Phi^{-1}\big(\{x\in\mathbb{R}^n:\|x\|<\frac{r}{2}\}\big)$, then $\tilde{W}$ is a nonempty relatively open subset of $Gr_T(S;G)$, and $\tilde{W}\subseteq\mathcal{M}^{\eta,\rho}(N)$ whenever $\eta\in\mathcal{F}$, $\|\eta-\eta_*\|_{sup}\le\gamma$, and $\rho\in[0,\gamma]$. In particular, suppose we fix $\eta\in\mathcal{F}$ such that $\|\eta-\eta_*\|_{sup}\le\min(\gamma,\varepsilon)$, and ρ such that $0<\rho\le\min(\gamma,\alpha)$. Now, if $\mathbf{t}\in N$, then the control $\eta^{\#,\mathbf{t};\rho}$ satisfies $\|\eta^{\#,\mathbf{t};\rho}-\eta\|_{sup}\le 4\varepsilon=\bar{\delta}$, and in addition $\eta^{\#,\mathbf{t};\rho}$ is continuous and $\eta^{\#,\mathbf{t};\rho}\equiv\eta$ on the set $[0,\beta]\cup[T-\beta,T]$. Therefore $\eta^{\#,\mathbf{t};\rho}\in N_S^0(\eta,\bar{\delta})$. Hence the set $\mathcal{R}(N_S^0(\eta,\bar{\delta}),e_G)$ contains the set $\{\xi^{\eta^{\#,\mathbf{t};\rho}}(T):\mathbf{t}\in N\}$, which is equal to $\mathcal{M}^{\eta,\rho}(N)$. So $\mathcal{R}(N_S^0(\eta,\bar{\delta}),e_G)$ contains $\tilde{W}$. If we fix a member g of $\tilde{W}$, and let $W=\tilde{W}g^{-1}$, then W is a neighborhood of e_G in $Gr_0(S;G)$, and

$$Wg\subseteq\mathcal{R}(N_S^0(\eta,\bar{\delta}),e_G)\quad\text{whenever}\quad\eta\in\mathcal{F}\quad\text{and}\quad\|\eta-\eta_*\|_{sup}\le\min(\gamma,\varepsilon).$$

So we have found, for each $\eta_* \in \mathcal{F}$, a neighborhood W_{η_*} of e_G in $Gr_0(S;G)$, a member g_{η_*} of $Gr_T(S;G)$, and a positive number γ_{η_*}, such that

$$W_{\eta_*} g_{\eta_*} \subseteq \mathcal{R}(N^0_S(\eta,\bar\delta), e_G) \quad \text{whenever} \quad \eta \in \mathcal{F} \quad \text{and} \quad \|\eta - \eta_*\|_{sup} \leq \gamma_{\eta_*} \,.$$

Now find a finite subset $\{\eta^1_*, \dots, \eta^m_*\}$ of $\mathcal{F}$ such that the sets

$$\mathcal{F}^j \stackrel{def}{=} \{\eta \in \mathcal{F} : \|\eta - \eta^j_*\|_{sup} \leq \gamma_{\eta^j_*}\} \,,$$

for $j = 1, \dots, m$, cover $\mathcal{F}$. Let $W = \bigcap_{j=1}^m W_{\eta^j_*}$. Then W is a neighborhood of e_G in $Gr_0(S;G)$. Furthermore, if $\eta \in \mathcal{F}$ then we can find $j \in \{1, \dots, m\}$ such that $\eta \in \mathcal{F}^j$. Then $\|\eta - \eta^j_*\|_{sup} \leq \gamma_{\eta^j_*}\}$, so $W_{\eta^j_*} g_{\eta^j_*} \subseteq \mathcal{R}(N^0_S(\eta,\bar\delta), e_G)$. Therefore

$$W g_{\eta^j_*} \subseteq \mathcal{R}(N^0_S(\eta,\bar\delta), e_G) \,,$$

since $W \subseteq W_{\eta^j_*}$. Hence the set $\mathcal{R}(N^0_S(\eta,\bar\delta), e_G)$ contains a right translate of W, and our proof is complete. ♢

The second main result of the section is a simple corollary of the first one. In order to state it, we need to introduce some notations.

If η_i, for $i = 1, 2$, are continuous functions on intervals $[0, T_i]$ with values in a topological space X, such that $\eta_1(T_1) = \eta_2(0)$, then the *concatenation* $\eta_2 \# \eta_1$ of η_1 and η_2 is the function $\eta_2 \# \eta_1 : [0, T_1 + T_2] \mapsto X$ defined by

$$(\eta_2 \# \eta_1)(t) = \begin{cases} \eta_1(t) & \text{if} \quad t \in [0, T_1] \\ \eta_2(t - T_1) & \text{if} \quad t \in [T_1, T_1 + T_2] \,. \end{cases}$$

If $T > 0$, X is a topological space, $\mathcal{F}$ is a subset of $C^0([0,T], X)$, and N is a positive integer, we use $\mathcal{F}^{(N)}$ to denote the set of all $\eta \in C^0([0, NT], X)$ that are concatenations $\eta_1 \# \eta_2 \# \cdots \# \eta_N$ of N members of $\mathcal{F}$.

Proposition 5.4. *Let G be a compact Lie group with Lie algebra $\mathbf{L}$, and let S be a nonempty closed convex subset of $\mathbf{L}$. Let $T > 0$, and let $\mathcal{F}$ be a compact subset of $C^0([0,T], S)$. Let K b e a compact subset of $Gr_0(S;G)$. Then given $\bar\delta > 0$ there exists a positive integer N such that for every $\eta \in \mathcal{F}^{(N)}$ the reachable set $\mathcal{R}(N^0_S(\eta,\bar\delta), e_G)$ contains some right translate of K by a member of $Gr(S;G)$.*

Proof. Using the fact that G is compact, we endow G with a bi-invariant Riemannian metric Γ_G. Then Γ_G induces bi-invariant Riemannian metrics $\Gamma_{Gr(S;G)}$, $\Gamma_{Gr_0(S;G)}$, on the Lie subgroups $Gr(S;G)$, $Gr_0(S;G)$ of G. It is then clear that the automorphisms $Gr_0(S;G) \ni g \mapsto hgh^{-1} \in Gr_0(S;G)$, for $h \in Gr(S;G)$, are isometric maps relative to the metric $\Gamma_{Gr_0(S;G)}$.

We then apply Proposition 5.4 and find an open neighborhood W of e_G in $Gr_0(S;G)$ such that $\mathcal{R}(N_S^0(\eta,\bar\delta),e_G)$ contains some right translate of W by a member of $Gr(S;G)$ whenever $\eta\in\mathcal{F}$. By making W smaller, if necessary, we assume that W is the open ball $\{g\in Gr_0(S;G):\mathrm{dist}_{\Gamma_{Gr_0(S;G)}}(x,e_G)<\gamma\}$, for some positive γ. Then $hW=Wh$ whenever $h\in Gr(S;G)$, because $hWh^{-1}=W$.

The sets $W^k=\{g_1g_2\cdots g_k\,:\,g_1\in W,g_2\in W,\ldots,g_k\in W\}$ cover $Gr_0(S;G)$, since $Gr_0(S;G)$ is connected. Hence we can find $N\in\mathbb{N}$ such that $K\subseteq W^N$. Now, if $\eta\in\mathcal{F}^{(N)}$, and we write η as a concatenation $\eta_1\#\eta_2\#\cdots\#\eta_N$ of N members of $\mathcal{F}$, it follows that the set $\mathcal{R}(N_S^0(\eta,\bar\delta),e_G)$ contains the product

$$P=\mathcal{R}(N_S^0(\eta_N,\bar\delta),e_G)\cdot\mathcal{R}(N_S^0(\eta_{N-1},\bar\delta),e_G)\cdot\;\cdots\;\mathcal{R}(N_S^0(\eta_2,\bar\delta),e_G)\cdot\mathcal{R}(N_S^0(\eta_1,\bar\delta),e_G)\,.$$

For each of the factors $\mathcal{R}(N_S^0(\eta_j,\bar\delta),e_G)$ we can choose a member g_j of $Gr(S;G)$ such that $Wg_j\subseteq\mathcal{R}(N_S^0(\eta_j,\bar\delta),e_G)$. Then

$$\mathcal{R}(N_S^0(\eta,\bar\delta),e_G)\supseteq P\supseteq Wg_N\cdot Wg_{N-1}\cdot\;\cdots\;\cdot Wg_2\cdot Wg_1=W^Ng\supseteq Kg\,,$$

where $g=g_N\cdot g_{N-1}\cdot\;\cdots\;\cdot g_2\cdot g_1$. So $\mathcal{R}(N_S^0(\eta,\bar\delta),e_G)$ contains a right-translate of K by a member of $Gr(S;G)$, as stated. ♢

§6. Background results: discrete approximations of Haar measures.

The third background result is about approximating the integral of a function over a compact group by the average of its translates. This result basically restates, in a way that will be particularly convenient for us to use, the evident facts that (a) the Haar measure ν_G is a weak*-limit—in the space of finite Borel measures on G, regarded as the dual $C^0(G,\mathbb{C})^*$ of $C^0(G,\mathbb{C})$—of measures ν_j that are averages of finitely many Delta functions, and (b) since the ν_j weak*-converge to ν_G, they are uniformly bounded in norm, so on any bounded subset $\mathcal{K}$ of $C^0(G,\mathbb{C})$ the sequence $\{\nu_j\}_{j=1}^{\infty}$ is uniformly bounded, equicontinuous, and pointwise convergent to ν_G, from which it follows that the convergence has to be uniform on $\mathcal{K}$ if $\mathcal{K}$ is compact. For completeness, we include a direct proof.

Recall that $C^0(G,\mathbb{C})$ denotes the Banach space of all continuous complex-valued functions on G with the supremum metric. If $h\in C^0(G,\mathbb{C})$ and $g\in G$, we define functions $\tau_gh\in C^0(G,\mathbb{C})$ by setting

$$(\tau_gh)(x)=h(g^{-1}x),\qquad\text{for }x\in G.\tag{6.1}$$

Clearly, each τ_g is an isometric linear map from $C^0(G,\mathbb{C})$ onto $C^0(G,\mathbb{C})$. Furthermore, τ_{e_G} is the identity map of $C^0(G,\mathbb{C})$, and $\tau_g\circ\tau_{g'}=\tau_{gg'}$ whenever $g,g'\in G$, so the map $g\mapsto\tau_g$ is a homomorphism from G to the group of isometric linear maps from $C^0(G,\mathbb{C})$ onto $C^0(G,\mathbb{C})$.

Proposition 6.1. *Let G be a compact metric group. If $\mathcal{K} \subseteq C^0(G, \mathbb{C})$ is compact and $\beta > 0$, then there exist a positive integer P, and members $g_1, \ldots, g_P$ of G, such that*

$$\Big\| \langle h \rangle - \frac{1}{P} \sum_{p=1}^{P} \tau_{g_p^{-1}} h \Big\|_{sup} \leq \beta \text{ for all } h \in \mathcal{K}\,. \tag{6.2}$$

Proof. Since $\mathcal{K}$ is a compact subset of $C^0(G, \mathbb{C})$, it is bounded and equicontinuous. Hence there exist (a) a constant C such that er$|h(x)| \leq C$ whenever $x \in G$ and $h \in \mathcal{K}$, and (b) a neighborhood W of e_G such that $|h(x) - h(y)| < \frac{\beta}{2}$ whenever $xy^{-1} \in W$ and $h \in \mathcal{K}$. Partition G into measurable subsets $G_1, \ldots, G_R$ such that each G_r is contained in a left translate $\xi_r W$ of W. Let $\tilde{p}_r = \nu_G(G_r)$.

If $h \in \mathcal{K}$ and $x \in G$, then

$$\begin{aligned}\langle h \rangle &= \int_G h(gx)\, d\nu_G(g) = \sum_{r=1}^{R} \int_{G_r} h(gx)\, d\nu_G(g) \\ &= \sum_{r=1}^{R} \int_{G_r} (h(gx) - h(\xi_r x))\, d\nu_G(g) + \sum_{r=1}^{R} \tilde{p}_r (\tau_{\xi_r^{-1}} h)(x)\,.\end{aligned}$$

If $g \in G_r$, then $(gx)(\xi_r x)^{-1} = g\xi_r^{-1} \in W$, so $|h(gx) - h(\xi_r x)| \leq \frac{\beta}{2}$. Therefore

$$\Big| \langle h \rangle - \sum_{r=1}^{R} \tilde{p}_r \tau_{\xi_r^{-1}} h(x) \Big| \leq \frac{\beta}{2}$$

for all $h \in \mathcal{K}$, $x \in G$. Now, let M be a positive integer, and write

$$\tilde{p}_r = \frac{p_r}{M} + \hat{p}_r\,,$$

where $0 \leq \hat{p}_r < \frac{1}{M}$. Let $P = p_1 + \ldots + p_R$. Then $1 - \frac{R}{M} \leq \frac{P}{M} \leq 1$. Therefore $0 \leq 1 - \frac{P}{M} \leq \frac{R}{M}$. Moreover,

$$\Big| \sum_{r=1}^{R} \hat{p}_r \tau_{\xi_r^{-1}} h(x) \Big| \leq \frac{CR}{M}\,.$$

So

$$\Big| \langle h \rangle - \sum_{r=1}^{R} \frac{p_r}{M} \tau_{\xi_r^{-1}} h(x) \Big| \leq \frac{\beta}{2} + \frac{CR}{M}\,.$$

Therefore

$$\begin{aligned}\Big|\langle h\rangle - \sum_{r=1}^{R}\frac{p_r}{P}\tau_{\xi_r^{-1}}h(x)\Big| &\le \Big|\langle h\rangle - \sum_{r=1}^{R}\frac{p_r}{M}\tau_{\xi_r^{-1}}h(x)\Big| + \Big|\sum_{r=1}^{R}\Big(\frac{p_r}{M}-\frac{p_r}{P}\Big)\tau_{\xi_r^{-1}}h(x)\Big|\\ &\le \frac{\beta}{2}+\frac{CR}{M}+\Big(1-\frac{P}{M}\Big)\Big|\sum_{r=1}^{R}\frac{p_r}{P}\tau_{\xi_r^{-1}}h(x)\Big|\\ &\le \frac{\beta}{2}+\frac{CR}{M}+\frac{CR}{M}\sum_{r=1}^{R}\frac{p_r}{P}\\ &\le \frac{\beta}{2}+\frac{2CR}{M}\,.\end{aligned}$$

Now choose M such that $4CR < M\beta$. Then

$$\Big\|\langle h\rangle - \sum_{r=1}^{R}\frac{p_r}{P}\tau_{\xi_r^{-1}}h\Big\|_{sup} \le \beta \ \text{ for all } h\in\mathcal{K}\,.$$

If we now define

$$\begin{aligned} g_1 = g_2 = \cdots = g_{p_1} &= \xi_1\,,\\ g_{p_1+1} = g_{p_1+2} = \cdots = g_{p_1+p_2} &= \xi_2\,,\\ \cdots\\ g_{p_1+p_2+\cdots+p_{R-1}+1} = g_{p_1+p_2+\cdot+p_{R-1}+2} = \cdots = g_P &= \xi_R\,,\end{aligned}$$

then

$$\Big\|\langle h\rangle - \frac{1}{P}\sum_{r=1}^{P}\tau_{g_r^{-1}}h\Big\|_{sup} \le \beta \ \text{ for all } h\in\mathcal{K}\,,$$

and our proof is complete. ◊

§7. Introduction to the proof of Theorem A.

It is clear that condition (3) of Theorem A implies condition (2). The implication (2)⇒(1) is trivial, because (a) every right coset $\overline{Gr(S;G)}g$ is invariant under the flow of the differential equation (1.1), from which it follows that if E is any union of cosets then $E\times\Omega$ is invariant under T^A; therefore T^A cannot be ergodic unless $\overline{Gr(S;G)} = G$; (b) if $\overline{Gr(S;G)} = G$, then the map $G\times\Omega \ni (g,\omega)\mapsto \pi(g,\omega)\stackrel{def}{=}([g],\omega)\in \mathbf{T}_{S;G}\times\Omega$—where $[g]$ is the class of g modulo $\overline{Gr_0(S;G)}$—satisfies

$$T_t^{\mathbf{T}_{S;G}\times\Omega}(\pi(g,\omega)) = \pi(T_t^A(g,\omega)) \ \text{ whenever } \ g\in G\,,\ \omega\in\Omega\,,\ t\in\mathbb{R}$$

and

$$(\nu_G \otimes m)(\pi^{-1}(E)) = (\nu_{\mathbf{T}_{S;G}} \otimes m)(E) \text{ whenever } E \in \mathcal{B}_{\mathbf{T}_{S;G}\times\Omega}\,;$$

therefore, if E is a $T^{\mathbf{T}_{S;G}\times\Omega}$-invariant Borel subset of $\mathbf{T}_{S;G} \times \Omega$ for which the inequalities $0 < (\nu_{\mathbf{T}_{S;G}} \otimes m)(E) < 1$ hold, then $\pi^{-1}(E)$ is a T^A-invariant Borel subset of $G \times \Omega \ni (g, \omega)$, and $0 < (\nu_G \otimes m)(\pi^{-1}(E)) < 1$; hence $T^{\mathbf{T}_{S;G}\times\Omega}$ must be ergodic if T^A is ergodic.

The remainder of the paper is devoted to the proof of the implication (1)⇒(3). The first step will be to introduce some notational conventions.

To begin with, we will work from now with a fixed Lie group G, that we assume to be compact and connected, a fixed compact metric Borel probability space (Ω, m), and a flow $T = \{T_t\}_{t\in\mathbb{R}}$ on Ω, that we assume to be m-preserving and aperiodic. In addition, we assume that a nonempty, closed, convex subset S of the Lie algebra of G is fixed as well, and that the pair (G, S) has the dense accessibility property, i.e., $\overline{Gr(S;G)} = G$. We use $\mathbf{F}$ to denote the subgroup $\overline{Gr_0(S;G)}$ of G, and $\mathbf{T}$ to denote the quotient group $\mathbf{T}_{S;G} = G/\mathbf{F}$. Then $\mathbf{F}$ is a closed normal subgroup of G, and $\mathbf{T}$ is a compact connected Abelian group, i.e., $\mathbf{T}$ a torus. We let $\mathbf{m}$ be the dimension of $\mathbf{T}$. We identify $\mathbf{T}$ with the product $(\mathbb{R}/\mathbb{Z})^{\mathbf{m}} = \mathbb{R}/\mathbb{Z}\times\cdots\times\mathbb{R}/\mathbb{Z}$ of $\mathbf{m}$ copies of the circle $\mathbb{R}/\mathbb{Z}$, and we use coordinates $\tau_1, \ldots, \tau_m$, (which are real numbers modulo $\mathbb{Z}$) for $\mathbf{T}$, so the symbol τ will denote a typical point of $\mathbf{T}$, regarded as a member of $(\mathbb{R}/\mathbb{Z})^{\mathbf{m}}$.

We use $[g]$ to denote the equivalence class modulo $\mathbf{F}$ of a member g of G, so $[g]$ is the right translate $\mathbf{F}g$, which coincides with the left translate $g\mathbf{F}$, since $g\mathbf{F}g^{-1} = \mathbf{F}$, because $\mathbf{F}$ is a normal subgroup. Then $[g]$ is a member of $\mathbf{T}$, and we use $\tau_i(g)$, $i = 1, \ldots, \mathbf{m}$ to denote its τ coordinates, and $\vec{\tau}(g)$ to denote the coordinate vector of $[g]$, which is a member of $(\mathbb{R}/\mathbb{Z})^{\mathbf{m}}$. With these conventions and identifications, if $g \in G$ then $[g]$, $\mathbf{F}g$, and $g\mathbf{F}$, are really three different names of the same object, and the three notations for it will be used interchangeably.

We use $\mathbf{L}(G)$, $\mathbf{L}(\mathbf{F})$, $\mathbf{L}(\mathbf{T})$ to denote, respectively, the Lie algebras of the Lie groups G, $\mathbf{T}$, $\mathbf{F}$, regarded as the tangent spaces $T_{e_G}G$, $T_{e_G}\mathbf{F}$, $T_{e_G}\mathbf{T}$ to G, $\mathbf{T}$, $\mathbf{F}$ at the identity element e_G of G, and identified in the standard way with the spaces of right-invariant vector fields on G, $\mathbf{T}$, $\mathbf{F}$. Then $\mathbf{L}(\mathbf{F})$ is an ideal of $\mathbf{L}(G)$, $\mathbf{L}(\mathbf{T})$ is the quotient $\mathbf{L}(G)/\mathbf{L}(\mathbf{F})$, and $\mathbf{L}(\mathbf{T})$ is naturally identified with $\mathbb{R}^{\mathbf{m}}$ using the coordinates $\tau_1, \ldots, \tau_m$ introduced above. The identity element $e_{\mathbf{T}}$ of $\mathbf{T}$ then just becomes the $\mathbf{m}$-tuple $\mathbf{0^m} = (0, \ldots, 0) \in (\mathbb{R}/\mathbb{Z})^{\mathbf{m}}$.

We fix an inner product $\langle\cdot,\cdot\rangle_{e_G}$ on the Lie algebra $\mathbf{L}(G)$ which is invariant under the inner automorphisms dA_g of $\mathbf{L}(G)$ induced, for each $g \in G$, by the inner autmorphism A_g of G. We let $\|\cdot\|$ be the corresponding norm.

We define an inner product $\langle\cdot,\cdot\rangle_g$ at each tangent space T_gG by right-translating $\langle\cdot,\cdot\rangle_{e_G}$ via the differential dR_g. Then the family $\Gamma_G = \{\langle\cdot,\cdot\rangle_g\}_{g\in G}$ is a bi-invariant Riemannian metric on G. We let d_G be the distance function associated to this metric. Then the right translations $G \ni h \mapsto R_g(h) \stackrel{def}{=} hg \in G$ and the left translations $G \ni h \mapsto L_g(h) \stackrel{def}{=} gh \in G$ are isometries.

We view G as a fiber bundle $\mathbf{B}$ over the base space $\mathbf{T}$ with fiber $\mathbf{F}$. (Actually, $\mathbf{B}$ is a principal bundle, but we will not use this fact.) The fibers of $\mathbf{B}$ are then the cosets $\mathbf{F}g$, for $g \in G$.

More precisely, given any $\tau \in \mathbf{T}$, the fiber $\mathbf{F}_\tau$ over τ is the coset $\mathbf{F}g$, where g is any member of G such that $\vec{\tau}(g) = \tau$. With these notations, $\mathbf{F}_{\vec{\tau}(g)} = \mathbf{F}g = g\mathbf{F} = [g]$ for every $g \in G$. Clearly, $\mathbf{F}_{\mathbf{0}^{\mathbf{m}}} = \mathbf{F}$.

The bi-invariant Riemannian metric Γ_G induces, by restriction, Euclidean inner products $\langle\cdot,\cdot\rangle_{\mathbf{F},g} : T_g[g] \times T_g[g] \mapsto \mathbb{R}$ on the tangent spaces $T_g[g]$ of the fibers. These tangent spaces are of course mapped to one another by the differentials dR_g, dL_g, of the right translations and of the left translations of G, and these maps are isometries, so the family of inner products $\langle\cdot,\cdot\rangle_{\mathbf{F},g}$, $g \in G$, is bi-invariant.

In particular, the inner products $\langle\cdot,\cdot\rangle_{\mathbf{F},g}$ for $g \in \mathbf{F}$ define a Riemannian metric $\Gamma_{\mathbf{F}}$ on $\mathbf{F}$ which is $\mathbf{F}$-bi-invariant, i.e., invariant under right and left translations of $\mathbf{F}$ by members of $\mathbf{F}$. More generally, for each $\tau \in \mathbf{T}$ the fiber $\mathbf{F}_\tau$ has a Riemannian metric $\Gamma_{\mathbf{F}_\tau}$—consisting of the inner products $\langle\cdot,\cdot\rangle_g$ for $g \in \mathbf{F}_\tau$—which is invariant under right and left translations of $\mathbf{F}_\tau$ by members of $\mathbf{F}$.

The Riemannian metrics Γ_G, $\Gamma_{\mathbf{F}_\tau}$ on the compact manifolds G, $\mathbf{F}_\tau$, give rise to volume forms which, normalized so that the total volume is 1, define probability measures ν_G, $\nu_{\mathbf{F}_\tau}$, on G and the fibers $\mathbf{F}_\tau$. We write $\nu_{\mathbf{F}} = \nu_{\mathbf{F}_{\mathbf{0}^{\mathbf{m}}}}$. Then ν_G, $\nu_{\mathbf{F}}$ are the normalized Haar measures on G, $\mathbf{F}$. For each fiber $\mathbf{F}_\tau = \mathbf{F}g$, the measure $\nu_{\mathbf{F}_\tau}$ is the image of $\nu_{\mathbf{F}}$ under the translations R_g, L_g.

The normalized Haar measure on $\mathbf{T} \sim (\mathbb{R}/\mathbb{Z})^{\mathbf{m}}$ is $\nu_{\mathbf{T}}$, the product of $\mathbf{m}$ copies of the standard Borel measure on $\mathbb{R}/\mathbb{Z}$. We will write $\int \ldots d\tau$—rather than $\int \ldots d\nu_{\mathbf{T}}(\tau)$– to indicate integration with respect to this measure.

The measures ν_G, $\nu_{\mathbf{F}_\tau}$, $\nu_{\mathbf{T}}$ are related by the formula

$$\int_G f(g)\, d\nu_G(g) = \int_{\mathbf{T}} \Big(\int_{\mathbf{F}_\tau} f(g) d\nu_{\mathbf{F}_\tau}(g) \Big) d\tau$$

for every bounded Borel measurable function f on G.

If $s \in S$, then we know that the one-parameter subgroup $\{[e^{ts}]\}_{t\in\mathbb{R}}$ of $\mathbf{T}$ does not depend on the choice of s and is dense in $\mathbf{T}$. Let $\mathbf{s}$ be the infinitesimal generator of this subgroup. Then $\mathbf{s} \in \mathbf{L}(\mathbf{T}) = \mathbb{R}^{\mathbf{m}}$, so $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_{\mathbf{m}})$, where the real numbers $\mathbf{s}_1, \ldots, \mathbf{s}_{\mathbf{m}}$ are linearly independent over $\mathbb{Q}$. Clearly,

$$\vec{\tau}(e^{ts}) = (t\mathbf{s}_1, \ldots, t\mathbf{s}_{\mathbf{m}}) \in (\mathbb{R}/\mathbb{Z})^{\mathbf{m}} \quad \text{for each} \quad t \in \mathbb{R}\,.$$

We write $T^{\mathbf{T}}$ rather than $T^{\mathbf{T}_{S;G}}$ to denote the flow on $\mathbf{T}$ determined by the one-parameter group $\{[e^{ts}]\}_{t\in\mathbb{R}}$ introduced above, and use $T^{\mathbf{T}\times\Omega}$, rather than $T^{\mathbf{T}_{S;G}\times\Omega}$ for the torus lift of T determined by (G,S). Then $T^{\mathbf{T}\times\Omega}$ is jointly continuous and $\nu_{\mathbf{T}} \otimes m$-preserving. The assumption that T is aperiodic implies that $T^{\mathbf{T}\times\Omega}$ is aperiodic as well.

If $f \in L^2(G,\nu_G)$, we use $\langle f\rangle$ to denote the average of f over G with respect to ν_G, and $\langle f\rangle_{\mathbf{F}_\tau}$ to denote the average of f over the fiber $\mathbf{F}_\tau$ with respect to $\nu_{\mathbf{F}_\tau}$, so

$$\langle f\rangle = \int_G f(g)\, d\nu_G(g) \qquad \text{and} \qquad \langle f\rangle_{\mathbf{F}_\tau} = \int_{\mathbf{F}_\tau} f(g)\, d\nu_{\mathbf{F}_\tau}(g)\,.$$

If f belongs to $L^2(G\times\Omega,\nu_{_G}\otimes m)$, $\tau\in\mathbf{T}$, and $\omega\in\Omega$, we use f^ω, $f^{\tau;\omega}$ to denote, respectively, the functions

$$G\ni g\mapsto f^\omega(g)\stackrel{def}{=}f(g,\omega)\in\mathbb{C}\,,\qquad \mathbf{F}_\tau\ni g\mapsto f^{\tau,\omega}(g)\stackrel{def}{=}f(g,\omega)\in\mathbb{C}\,.$$

Then $f^\omega\in L^2(G,\nu_{_G})$ for m-almost all $\omega\in\Omega$, $f^{\tau;\omega}\in L^2(\mathbf{F}_\tau;\mathbb{C})$ for $\nu_{\mathbf{T}}\otimes m$-almost all $(\tau,\omega)\in\mathbf{T}\times\Omega$, and

$$\|f\|_{L^2}^2=\int_\Omega\|f^\omega\|_{L^2}^2dm(\omega)=\int_{\mathbf{T}\times\Omega}\|f^{\tau;\omega}\|_{L^2}^2d(\nu_{\mathbf{T}}\otimes m)(\tau,\omega)\,.$$

For $f\in L^2(G\times\Omega,\nu_{_G}\otimes m)$, set

$$\hat{f}(g,\omega)=\langle f^{\vec{\tau}(g)},\omega\rangle_{\mathbf{F}_\tau}\stackrel{def}{=}\int_{\mathbf{F}_{\vec{\tau}(g)}}f(g,\omega)d\nu_{\mathbf{F}_\tau}(g)\,.$$

Then $\hat{f}$ is in fact the orthogonal projection of f on the space of square-integrable functions on $G\times\Omega$ that are functions of (τ,ω) only, i.e., constant on each fiber $\mathbf{F}_\tau\times\{\omega\}$.

We use

$$\begin{aligned}\mathcal{H}&=L^2(G\times\Omega,\nu_{_G}\otimes m)\,,\\ \mathcal{H}_0&=\{f\in\mathcal{H}:\int_{G\times\Omega}f(g,\omega)\,d(\nu_{_G}\otimes m)(g,\omega)=0\ \}\,,\\ \mathcal{H}_{0,av}&=\{f\in\mathcal{H}:\hat{f}\equiv 0\}\,.\end{aligned}$$

Given $A\in C^0(\Omega,\mathbf{L})$ and $t\in\mathbb{R}$, define a unitary operator U_t^A on $\mathcal{H}$ by setting

$$U_t^Af=f\circ T_t^A. \tag{7.1}$$

Next, given a positive real number M, define the time-averaging operator $W_{M,A}$ on $\mathcal{H}$ by setting y

$$W_{M,A}f=\frac{1}{M}\int_0^M U_t^Af\,dt. \tag{7.2}$$

We will then prove the following.

Theorem 7.1. *Under the hypotheses of Theorem A, if S has the dense accessibility property in G, then the set*

$$C_{erg,av}(\Omega,S)=\{A\in C^0(\Omega,S):\lim_{M\to\infty}W_{M,A}f=0\ \ \textit{for all}\ \ f\in\mathcal{H}_{0,av}\},$$

is a residual subset of $C^0(\Omega, S)$.

PROOF THAT THEOREM 7.1 IMPLIES THAT (1)⇒(3) IN THEOREM A. Assume that the hypotheses of Theorem A hold, and condition (1) of that theorem holds as well. Then S has the dense accessibilit property in G, so we can apply Theorem 4.1 and conclude that $C_{erg,av}(\Omega, S)$ is a residual subset of $C^0(\Omega, S)$. Furthemore, the torus lift $T^{\mathbf{T}\times\Omega}$ is ergodic.

Let $A \in C_{erg,av}(\Omega, S)$. If $f \in \mathcal{H}_0$, then $\tilde{f} \stackrel{def}{=} f - \hat{f} \in \mathcal{H}_{0,av}$, so $W_{M,A}\tilde{f} \to 0$ as $M \to \infty$. Moreover, since $T^{\mathbf{T}\times\Omega}$ is ergodic on $(\mathbf{T} \times \Omega, \nu_{\mathbf{T}} \otimes m)$, the time averages

$$W_M h \stackrel{def}{=} \frac{1}{M}\int_0^M (h \circ T_t)dt$$

converge to 0, if we let $h(\tau, \omega) \stackrel{def}{=} \langle f^\omega \rangle_{\mathbf{F}_\tau}$, because

$$\begin{aligned}\int_{\mathbf{T}\times\Omega} h(\tau,\omega)\, d(\nu_{\mathbf{T}} \otimes m)(\tau,\omega) &= \int_{\mathbf{T}\times\Omega} \Big(\int_{\mathbf{F}_\tau} f(g,\omega) d\nu_{\mathbf{F}_\tau}(g)\Big)\, d(\nu_{\mathbf{T}} \otimes m)(\omega) \\ &= \int_{G\times\Omega} f(g,\omega)\, d(\nu_G \otimes m)(g,\omega) \\ &= 0\,.\end{aligned}$$

Observe that

$$W_M h = W_{M,A}\hat{f}.$$

Therefore $W_{M,A} f = W_{M,A}\tilde{f} + W_{M,A}\hat{f} \to 0$ as $M \to \infty$. Since this is true for all $f \in \mathcal{H}_0$, we have shown that T^A is ergodic on $(G \times \Omega, \nu_G \otimes m)$. Therefore $A \in C_{erg}(\Omega, S)$.

Hence $C_{erg,av}(\Omega, S) \subseteq C_{erg}(\Omega, S)$. Since $C_{erg,av}(\Omega, S)$ is residual, the set $C_{erg}(\Omega, S)$ is residual as well. ♢

For $f \in \mathcal{H}_{0,av}$, $\varepsilon > 0$ and $\bar{n} \in \mathbb{N}$, define a set $\mathcal{E}(f, \varepsilon, \bar{n})$ as follows:

$$\mathcal{E}(f, \varepsilon, \bar{n}) = \{A \in C^0(\Omega, S) : \|W_{n,A} f\|_{L^2} < \varepsilon \text{ for some } n \in \mathbb{N} \text{ such that } n > \bar{n}\}\,. \tag{7.3}$$

Lemma 7.2. *Let* $A \in C^0(\Omega, S)$, *and let* $\mathcal{F}$ *be a dense subset of* $\mathcal{H}_{0,av}$. *If* $A \in \mathcal{E}(f, \frac{1}{n}, \overline{n})$ *for all* $f \in \mathcal{F}$, $n \in \mathbb{N}$, $\overline{n} \in \mathbb{N}$, *then* $A \in C_{erg,av}(\Omega, S)$.

Proof. Fix an $f \in \mathcal{F}$. By the L^2 ergodic theorem, the sequence $W_{n,A} f$ converges in $\mathcal{H}$ as $n \to \infty$ to some $f^* \in \mathcal{H}$. Since $A \in E(f, \frac{1}{n}, \overline{n})$ for all $n, \overline{n} \in \mathbb{N}$, f^* must be the zero function. So $W_{n,A} f \to 0$ for every $f \in \mathcal{F}$. Since $\mathcal{F}$ is dense in $\mathcal{H}_{0,av}$, and $\|W_{n,A}\|_{L^2} \leq 1$ for all n, it follows that $W_{n,A} f \to 0$ as $n \to \infty$ for all $f \in \mathcal{H}_{0,av}$, so $A \in C_{erg,av}(\Omega, S)$. ♢

In view of Lemma 7.2, Theorem 7.1 will follow from the Baire category theorem—together with the facts that (a) $C^0(\Omega, S)$ is a complete metric space, (b) $C^0(G \times \Omega; \mathbb{C}) \cap \mathcal{H}_{0,av}$ is dense in $\mathcal{H}_{0,av}$, and (c) $C^0(G \times \Omega; \mathbb{C})$ is separable— if we prove

Lemma 7.3. *Let* $f \in C^0(G\times\Omega;\mathbb{C})\cap\mathcal{H}_{0,av}$*. Then* $\mathcal{E}(f,\frac{1}{n},\overline{n})$ *is open and dense in* $C^0(\Omega,S)$*.*

The openness of the sets $\mathcal{E}(f,\varepsilon,\bar{n})$ is easily checked. We shall establish their density in the next section.

§8. Density of $E(f,\varepsilon,\bar{n})$.

We now turn to the proof of the density of $E(f,\varepsilon,\bar{n})$. For this purpose, we fix the following objects:

(1) a function $f \in C^0(G\times\Omega;\mathbb{C})\cap\mathcal{H}_{0,av}$;
(2) a map $A_0 \in C^0(\Omega,S)$;
(3) a positive number ε;
(4) a positive number δ and
(5) a positive integer $\bar{n}$.

To prove the density of $E(f,\varepsilon,\bar{n})$ we shall construct a map $A \in C^0(\Omega,S)$ such that

(1) $\|A_0 - A\|_{sup} < \delta$ and
(2) $A \in E(f,\varepsilon,\bar{n})$.

The construction of the desired map A will be carried out in several steps..

Step 1 : Recall that if $g\in G$ then τ_g is the map from $C^0(G;\mathbb{C})$ to $C^0(G;\mathbb{C})$ given by $(\tau_g h)(x) = h(g^{-1}x)$ for $x\in G$. It is clear that $\tau_g\circ\tau_{\tilde{g}} = \tau_{g\tilde{g}}$ whenever $g,\tilde{g}\in G$.

Similarly, we define $\lambda_g : C^0(G;\mathbb{C}) \mapsto C^0(G;\mathbb{C})$ by letting $\lambda_g h(x) = h(xg)$ for $x\in G$. Then $\lambda_g\circ\lambda_{\tilde{g}} = \lambda_{g\tilde{g}}$ whenever $g,\tilde{g}\in G$. It is clear that if $g,\tilde{g}\in G$, then the maps τ_g and $\lambda_{\tilde{g}}$ commute.

The map $G\times G\times\Omega \ni (\tilde{g},g,\omega)\mapsto \lambda_{\tilde{g}}\tau_g(f^\omega)\in C^0(G;\mathbb{C})$ is continuous. Therefore the set

$$\mathcal{K}_* = \{\lambda_{\tilde{g}}\tau_g(f^\omega) : \omega\in\Omega\,,\ g,\tilde{g}\in G\}\subseteq C^0(G;\mathbb{C}) \tag{8.1}$$

is compact. If $\mathrm{rest}_{\mathbf{F}}$ denotes the restriction map from $C^0(G;\mathbb{C})$ to $C^0(\mathbf{F};\mathbb{C})$ that sends each continuous function $f\in C^0(G;\mathbb{C})$ to its restriction to the subgroup $\mathbf{F}$, then $\mathrm{rest}_{\mathbf{F}}$ is continuous, so the set

$$\mathcal{K}^{\mathbf{F}}_* = \mathrm{rest}_{\mathbf{F}}(\mathcal{K}_*) = \{\mathrm{rest}_{\mathbf{F}}(\lambda_{\tilde{g}}\tau_g(f^\omega)) : \omega\in\Omega\,,\ g,\tilde{g}\in G\}\subseteq C^0(\mathbf{F};\mathbb{C}) \tag{8.2}$$

is compact as well. Furthermore, $\mathcal{K}_*$ is invariant under τ_g and λ_g for all $g\in G$, and $\mathcal{K}^{\mathbf{F}}_*$ is invariant under τ_g and λ_g for all $g\in\mathbf{F}$. (The invariance of $\mathcal{K}_*$ under the λ_g's is obvious, and the invariance under the τ_g's follows from the fact that τ_g and $\lambda_{\tilde{g}}$ commute. The invariance of $\mathcal{K}^{\mathbf{F}}_*$ under λ_g for $g\in\mathbf{F}$ follows because, if $h\in\mathcal{K}^{\mathbf{F}}_*$ and $h = \mathrm{rest}_{\mathbf{F}}(\lambda_{\tilde{\gamma}}\tau_\gamma(f^\omega))$, then $\lambda_g h = \mathrm{rest}_{\mathbf{F}}(\lambda_g\lambda_{\tilde{\gamma}}\tau_\gamma(f^\omega)) = \mathrm{rest}_{\mathbf{F}}(\lambda_{g\tilde{\gamma}}\tau_\gamma(f^\omega))\in\mathcal{K}^{\mathbf{F}}_*$. The invariance under τ_g for $g\in\mathbf{F}$ follows because, if $h\in\mathcal{K}^{\mathbf{F}}_*$ and $h = \mathrm{rest}_{\mathbf{F}}(\lambda_{\tilde{\gamma}}\tau_\gamma(f^\omega))$, then $\tau_g h = \mathrm{rest}_{\mathbf{F}}(\tau_g\lambda_{\tilde{\gamma}}\tau_\gamma(f^\omega)) = \mathrm{rest}_{\mathbf{F}}(\lambda_{\tilde{\gamma}}\tau_{g\gamma}(f^\omega))\in\mathcal{K}^{\mathbf{F}}_*$.)

We let $\mathcal{K}$ be the closed convex hull of $\mathcal{K}_*$, so $\mathcal{K}$ is a compact convex subset of $C^0(G;\mathbb{C})$ such that $\tau_g h\in\mathcal{K}$ and $\lambda_g h\in\mathcal{K}$ whenever $h\in\mathcal{K}$ and $g\in G$. Similarly, we let $\mathcal{K}^{\mathbf{F}}$ be the closed convex hull of $\mathcal{K}^{\mathbf{F}}_*$, so $\mathcal{K}^{\mathbf{F}}$ is a compact convex subset of $C^0(\mathbf{F};\mathbb{C})$ such that $\tau_g h\in\mathcal{K}^{\mathbf{F}}$ and $\lambda_g h\in\mathcal{K}^{\mathbf{F}}$ whenever $h\in\mathcal{K}^{\mathbf{F}}$ and $g\in\mathbf{F}$.

Step 2 : The equality $\langle h\rangle_{\mathbf{F}} = 0$ holds for every $h \in \mathcal{K}^{\mathbf{F}}$. Indeed, it clearly suffices to prove the equality for $h \in \mathcal{K}_*^{\mathbf{F}}$. But in that case $h = \mathrm{rest}_{\mathbf{F}}(\lambda_{\tilde{g}}\tau_g(f^\omega))$ for some $\omega \in \Omega$, $g, \tilde{g} \in G$, so

$$\begin{aligned}
\langle h\rangle_{\mathbf{F}} &= \int_{\mathbf{F}} h(u)d\nu_{\mathbf{F}}(u) \\
&= \int_{\mathbf{F}} f^\omega(g^{-1}u\tilde{g})d\nu_{\mathbf{F}}(u) \\
&= \int_{g^{-1}\mathbf{F}} f^\omega(v\tilde{g})d\nu_{g^{-1}\mathbf{F}}(v) \\
&= \int_{g^{-1}\mathbf{F}\tilde{g}} f^\omega(w)d\nu_{g^{-1}\mathbf{F}\tilde{g}}(w) \\
&= \int_{\mathbf{F}g^{-1}\tilde{g}} f^\omega(w)d\nu_{\mathbf{F}g^{-1}\tilde{g}}(w) \\
&= \langle f^\omega\rangle_{\mathbf{F}g^{-1}\tilde{g}} \\
&= 0\,,
\end{aligned}$$

where we have used the change of variables $v = g^{-1}u$ and the fact that the left translation $L_{g^{-1}}$ maps $\nu_{\mathbf{F}}$ to $\nu_{g^{-1}\mathbf{F}}$ to go from the integration over $\mathbf{F}$ to that over $g^{-1}\mathbf{F}$, and then the change of variables $w = v\tilde{g}$ and the fact that the right translation $R_{\tilde{g}}$ maps $\nu_{g^{-1}\mathbf{F}}$ to $\nu_{g^{-1}\mathbf{F}\tilde{g}}$ to go from the integration over $g^{-1}\mathbf{F}$ to that over $g^{-1}\mathbf{F}\tilde{g}$.

Step 3 : We fix a finite subset $\mathcal{K}_0$ of $\mathcal{K}$ such that every $h \in \mathcal{K}$ satisfies the inequality

$$\|h - h_0\|_{sup} < \frac{\varepsilon}{16} \tag{8.3}$$

for some $h_0 \in \mathcal{K}_0$.

Step 4 : We let $\hat{\kappa}$ be the number of members of $\mathcal{K}_0$, and define

$$\beta = \frac{\varepsilon}{8(1+\hat{\kappa})}\,. \tag{8.4}$$

Step 5 : We pick δ_1 such that $\delta_1 > 0$ and

$$|h(g_1) - h(g_2)| < \beta \quad \text{whenever } g_1, g_2 \in \mathbf{F}\,,\ h \in \mathcal{K}\,,\ \text{and } d_G(g_1, g_2) < \delta_1\,. \tag{8.5}$$

Step 6 : We apply Proposition 6.1 to the set $\mathcal{K}^{\mathbf{F}}$ and the number β. We get a positive integer R and members $\bar{g}^0, \ldots, \bar{g}^{R-1}$ of $\mathbf{F}$ such that

$$\Big\|\frac{1}{R}\sum_{r=0}^{R-1}\tau_{(\bar{g}^r)^{-1}}h\Big\|_{sup} \le \beta \text{ for all } h \in \mathcal{K}^{\mathbf{F}}\,. \tag{8.6}$$

It follows from (8.6) that

$$\Big\|\frac{1}{R}\sum_{r=0}^{R-1}\tau_{(\bar g^r)^{-1}}h\Big\|_{sup}\le\beta\ \text{ for all } h\in\mathcal{K}\,. \tag{8.7}$$

Indeed, if $h\in\mathcal{K}$ and $g\in G$, then, if we let $\hat h=\mathrm{rest}_{\mathbf{F}}(\lambda_g h)$, we have

$$\Big(\sum_{r=0}^{R-1}\tau_{(\bar g^r)^{-1}}h\Big)(g)=\sum_{r=0}^{R-1}h(\bar g^r g)=\sum_{r=0}^{R-1}(\lambda_g h)(\bar g^r)=\sum_{r=0}^{R-1}\hat h(\bar g^r)=\sum_{r=0}^{R-1}(\tau_{(\bar g^r)^{-1}}\hat h)(e_{\mathbf{F}})\,.$$

Since the function $\lambda_g h$ belongs to $\mathcal{K}$, its restriction $\hat h$ belongs to $\mathcal{K}^{\mathbf{F}}$. We can then apply (8.6) and conclude that $\Big\|\frac{1}{R}\sum_{r=0}^{R-1}\tau_{(\bar g^r)^{-1}}\hat h\Big\|_{sup}\le\beta$. In particular, $\frac{1}{R}\Big|\sum_{r=0}^{R-1}(\tau_{(\bar g^r)^{-1}}\hat h)(e_{\mathbf{F}})\Big|\le\beta$, so $\frac{1}{R}\Big|\Big(\sum_{r=0}^{R-1}\tau_{(\bar g^r)^{-1}}h\Big)(g)\Big|\le\beta$. Since g is an arbitrary member of G, we have shown that $\Big\|\frac{1}{R}\sum_{r=0}^{R-1}\tau_{(\bar g^r)^{-1}}h\Big\|_{sup}\le\beta$, concluding the proof of (8.7).

Step 7: We pick a compact subset $\mathcal{G}$ of $G_0(S)$ which is δ_1-dense in $\mathbf{F}$ (relative to the distance d_G restricted to $\mathbf{F}$).

Step 8: Each point ω of Ω gives rise, for each positive number a, to a continuous control $\eta^{*,\omega,a}:[0,a]\mapsto S$, defined by

$$\eta^{*,\omega,a}(t)=A_0(T_t\omega)\quad\text{for }\ t\in[0,a]\,. \tag{8.8}$$

In particular, if we take $a=1$, the map $\Omega\ni\omega\mapsto\eta^{*,\omega,1}\in C^0([0,1],S)$ is continuous. Therefore, if we let

$$\mathcal{F}=\{\eta^{*,\omega,1}:\omega\in\Omega\}\,, \tag{8.9}$$

then $\mathcal{F}$ is a compact subset of $C^0([0,1],S)$. Hence Proposition 5.4 enables us to pick, for this $\mathcal{F}$, and with $\tau=1$, $\bar\delta=\frac{\delta}{2}$, a positive integer κ and a $\hat g^\omega\in G$ for each $\omega\in\Omega$, such that

$$\mathcal{G}\hat g^\omega\subseteq\mathcal{R}\Big(N_S(\eta^{*,\omega,\kappa},\delta/2),e_G\Big)\,. \tag{8.10}$$

Step 9: For every $g\in G$ the set $\mathcal{G}g$ is δ_1-dense in $\mathbf{F}g$. (Indeed, $\mathcal{G}g\subseteq\mathbf{F}g$ because $\mathcal{G}\subseteq\mathbf{F}$. If $g'\in\mathbf{F}g$ then $g'g^{-1}\in\mathbf{F}$, and the fact that $\mathcal{G}$ is δ_1-dense in $\mathbf{F}$ implies that there exists $h\in\mathcal{G}$ such that $d_G(h,g'g^{-1})<\delta_1$. Then $hg\in\mathcal{G}g$, and $d_G(hg,g')<\delta_1$, because R_g is an isometry of G.)

Step 10: We choose $\mu\in\mathbb{N}$ such that

$$\frac{\|f\|_{sup}}{\mu}\le\frac{\varepsilon}{16}\,. \tag{8.11}$$

and let

$$c_1 = \frac{7\varepsilon^2}{128\|f\|_{sup}^2}\,. \tag{8.12}$$

We then choose $N = (1+\mu)\kappa\Lambda$ where Λ is a large positive integer. The precise choice of Λ will be made later, in Step 22.

Step 11 : Using Proposition 4.2, we pick a compact subset $\check{E}$ of Ω and a Borel probability measure $\check{m}$ on $\check{E}$ with the following properties:

(1) $T_s(\check{E}) \cap T_t(\check{E}) = \emptyset$ for all $t, s \in [0, N]$ such that $t \neq s$;
(2) $m(\mathcal{T}_{[0,N]}(\check{E})) \geq 1 - c_1$.
(3) the map $\varphi : \check{E} \times [0, N] \mapsto \mathcal{T}_{[0,N]}(\check{E})$ defined by $\varphi(x, t) = T_t x$ (which is clearly a homeomorphism in view of property (1)—because the set $\check{E}$ is compact and φ is continuous—and is therefore an isomorphism of measurable spaces from the product $(\check{E} \times [0, N], \mathcal{B}_{\check{E}} \otimes Bor_N)$ to $(\mathcal{T}_{[0,N]}(\check{E}), \mathcal{B}_{\mathcal{T}_{[0,N]}(\check{E})}))$ is in fact an isomorphism of probability spaces from the product $(\check{E} \times [0, N], \mathcal{B}_{\check{E}} \otimes Bor_N, \check{m} \otimes bor_N)$ to the space $(\mathcal{T}_{[0,N]}(\check{E}), \mathcal{B}_{\mathcal{T}_{[0,N]}(\check{E})}, m\lceil_{nm}\mathcal{T}_{[0,N]}(\check{E}))$, where $m\lceil_{nm}\mathcal{T}_{[0,N]}(\check{E})$ is the normalized restriction of m to the set $\mathcal{T}_{[0,N]}(\check{E})$, and bor_N is the usual Lebesgue measure on $[0, N]$, restricted to the Borel subsets of $[0, N]$, and normalized in such a way that $bor_N([0, N]) = 1$.

Step 12 : Using the uniform continuity of f on $G \times \Omega$, we choose a positive number c_2 such that

$$|f(g, \omega_1) - f(g, \omega_2)| \leq \frac{\varepsilon}{8} \quad \text{whenever} \quad g \in G\,,\ \omega_1, \omega_2 \in \Omega\,,\ \text{and } d_\Omega(\omega_1, \omega_2) \leq c_2\,. \tag{8.13}$$

Step 13 : Using the uniform continuity of the map $[0, N] \times \Omega \ni (t, \omega) \mapsto A_0(T_t\omega) \in S$, we choose a positive constant c_3 such that

$$\|A_0(T_t\omega) - A_0(T_t\omega')\| < \frac{\delta}{2} \quad \text{whenever } d_\Omega(\omega, \omega') < c_3 \text{ and } t \in [0, N]\,. \tag{8.14}$$

Step 14 : Using the uniform continuity of the maps $[0, N] \times \Omega \ni (t, \omega) \mapsto T_t\omega \in \Omega$ and $[0, N] \times \mathbf{T} \ni (t, \tau) \mapsto T_t^{\mathbf{T}}\tau \in \mathbf{T}$, we choose a positive constant c_4 such that

(1) $d_\Omega(T_t\omega, T_t\omega') \leq c_2$ whenever $d_\Omega(\omega, \omega') \leq c_4$ and $t \in [0, N]$,
(2) $d_{\mathbf{T}}(T_t^{\mathbf{T}}\tau, T_t^{\mathbf{T}}\tau') \leq c_2$ whenever $d_{\mathbf{T}}(\tau, \tau) \leq c_4$ and $t \in [0, N]$,
(3) $c_4 < c_3$.

Step 15 : We partition $\check{E}$ into finitely many Borel-measurable sets $\check{E}_1, \ldots, \check{E}_{\check{I}}$ of diameter less than c_4. Write $\check{\mathbf{I}} = \{1, \ldots, \check{I}\}$. Using the fact that $\check{m}$ is regular, we pick for each $i \in \check{\mathbf{I}}$ a compact set E_i such that $E_i \subseteq \check{E}_i$ and $\check{m}(\check{E}_i \backslash E_i) < \frac{c_1}{\check{s}}$. After making a permutation of the set $\check{\mathbf{I}}$, if neccessary, we assume that $\{i \in \check{\mathbf{I}} : \check{m}(E_i) = 0\} = \{i \in \check{\mathbf{I}} : i > I\}$ for an $I \in \check{\mathbf{I}}$. We then let $\mathbf{I} = \{1, \dots, I\}$, $E = E_1 \cup \dots \cup E_I$, and write

$$\mathbf{E}^r = \mathcal{T}_{[0,r]}(E) \quad \Big(\text{i.e.,} \quad \mathbf{E}^r = \varphi(E \times [0,r]) = \bigcup_{0 \le t \le r} T_t E\Big) \qquad \text{for} \quad 0 \le r \le N\,.$$

Then $m(\mathbf{E}^r) = \frac{r}{N} \cdot \check{m}(E) = \Big(1 - \frac{N-r}{N}\Big)\check{m}(E)$, and

$$\check{m}(E) = \sum_{i=1}^{I} \check{m}(E_i) = \sum_{i=1}^{\check{I}} \check{m}(E_i) \ge \sum_{i=1}^{\check{I}} \Big(\check{m}(\check{E}_i) - \frac{c_1}{\check{s}}\Big) = \Big(\sum_{i=1}^{\check{I}} \check{m}(\check{E}_i)\Big) - c_1 = 1 - c_1\,.$$

Therefore

$$m(\mathbf{E}^r) \ge \Big(1 - \frac{N-r}{N}\Big)(1 - c_1) \ge 1 - c_1 - \frac{N-r}{N}\,. \tag{8.15}$$

Step 16: Let $\mathcal{J} \overset{def}{=} \{0, 1, \dots, (1+\mu)\Lambda - 1\}$, so $\mathcal{J}$ is an integer interval with $(1+\mu)\Lambda$ members whose leftmost point is 0. We pick points $\bar{\omega}_i \in E_i$ for each $i \in \mathbf{I}$, and then set

$$\bar{\omega}_{i,j} = T_{j\kappa}(\bar{\omega}_i) \qquad \text{for} \qquad i \in \mathbf{I}\,, \quad j \in \mathcal{J}\,. \tag{8.16}$$

We associate to each index $j \in \mathcal{J}$ the real interval $\mathcal{I}_{(j)} = [j\kappa, (1+j)\kappa]$, so the $\mathcal{I}_{(j)}$ constitute a partition of the interval $[0, N]$ into $(1+\mu)\Lambda$ intervals of length κ.

Step 17: We divide the integer interval $\mathcal{J}$ into Λ blocks $J_0, J_1, \dots, J_{\Lambda-1}$ of length $1 + \mu$, given by

$$J_\ell = \{j \in \mathbb{N} : \ell(1+\mu) \le j \le (1+\ell)(1+\mu) - 1\} \ \text{for} \ \ell \in \mathcal{L}\,,$$

where $\mathcal{L} = \{0, 1, \dots, \Lambda - 1\}$.

In addition, we also associate to each $\ell \in \mathcal{L}$ the $\mu + 1$ real intervals of length κ given by

$$\mathcal{I}_{\ell,k} = [\ell\kappa(1+\mu) + k\kappa, \ell\kappa(1+\mu) + (1+k)\kappa] \quad \text{for} \quad k = 0, 1, \dots, \mu\,, \tag{8.17}$$

and the real interval

$$\mathcal{I}_\ell = [\ell\kappa(1+\mu), (1+\ell)\kappa(1+\mu)] = \bigcup_{k=0}^{\mu} \mathcal{I}_{\ell,k}\,. \tag{8.18}$$

Clearly,

$$\mathcal{I}_{\ell,k} = \mathcal{I}_{(j)} \quad \text{if} \quad j = \ell(1+\mu) + k\,. \tag{8.19}$$

(The reader should think of the members of $\mathcal{J}$ as $(1+\mu)\Lambda$ integer indices whose values j range from 0 to $(1+\mu)\Lambda-1$. Each j gives rise to a real interval $\mathcal{I}_{(j)}=[j\kappa,(j+1)\kappa]$. As the index j grows from 0 to $(1+\mu)\Lambda-1$, it goes successively through the intervals $J_0, J_1, \ldots, J_{\Lambda-1}$. The index k is like the index j, except that at the beginning of each J_ℓ we reset k to 0, so that in fact k is j modulo $1+\mu$. On J_ℓ, we have $j=\ell(1+\mu)+k$, and k varies from 0 to μ. For each ℓ, each index k gives rise to a real interval of length κ, namely, the interval $\mathcal{I}_{\ell,k}$ defined by (8.17). Since, for a given ℓ, k corresponds to $j=\ell(1+\mu)+k$, the interval $\mathcal{I}_{\ell,k}$ can also be labelled by j, as $\mathcal{I}_{(j)}$, as indicated in (8.19).)

Step 18 : In a way to be described later, in Steps 23.x and 23.xi, we select, for each $i\in\{1,\ldots,I\}$, $j\in\mathcal{J}$, a point $g^{\#,\bar{\omega}_{i,j}}$ belonging to $\mathcal{G}\hat{g}^{\bar{\omega}_{i,j}}$ and a control $\eta^{\#,\bar{\omega}_{i,j},\kappa}$ such that

$$\eta^{\#,\bar{\omega}_{i,j},\kappa}\in N_S^0(\eta^{*,\bar{\omega}_{i,j},\kappa},\delta/2) \quad \text{and} \quad \xi^{\eta^{\#,\bar{\omega}_{i,j},\kappa}}(\kappa)=g^{\#,\bar{\omega}_{i,j}}\,. \tag{8.20}$$

We then let

$$\begin{aligned}
\eta^{*,i,(j),\kappa}&=\eta^{*,\bar{\omega}_{i,j},\kappa}\,,\\
\eta^{\#,i,(j),\kappa}&=\eta^{\#,\bar{\omega}_{i,j},\kappa}\,,\\
g^{\#,i,(j)}&=g^{\#,\bar{\omega}_{i,j}}=\xi^{\eta^{\#,i,(j),\kappa}}(\kappa)\,.
\end{aligned}$$

We also use the indices ℓ,k as labels, as an alternative to using j, and write

$$\begin{aligned}
&\eta^{*,i;\ell,k;\kappa}=\eta^{*,i,(j),\kappa}\,,\quad \eta^{\#,i;\ell,k;\kappa}=\eta^{\#,i,(j),\kappa}\,,\quad\text{and}\quad g^{\#,i;\ell,k}=g^{\#,i,(j)}\\
&\text{if}\quad j=\ell(1+\mu)+k\,,\ \ell\in\mathcal{L}\,,\ k\in\{0,\ldots,\mu\}\,.
\end{aligned}\tag{8.21}$$

Step 19 : For each $\omega\in E=\cup_{i=1}^{I}E_i$, we define $\eta^{\#,\omega,N}:[0,N]\mapsto S$ by setting

$$\eta^{\#,\omega,N}(t)=\eta^{\#,i,(j),\kappa}(t-j\kappa)\quad\text{if}\quad \omega\in E_i\,,\ j\in\mathcal{J}\,,\quad\text{and}\quad t\in\mathcal{I}_{(j)}\,. \tag{8.22}$$

Clearly, the controls $\eta^{\#,\omega,N}$ satisfy

$$\eta^{\#,\omega,N}\in C^0([0,N],S)\quad\text{for each}\quad \omega\in E=\cup_{i=1}^{I}E_i\,, \tag{8.23}$$

$$\|\eta^{\#,\omega,N}(t)-\eta^{*,\omega,N}(t)\|<\delta\quad\text{whenever}\quad \omega\in E\text{ and }t\in[0,N]\,, \tag{8.24}$$

$$\eta^{\#,\omega,N}\equiv\eta^{\#,\bar{\omega}_i,N}\quad\text{for}\quad \omega\in E_i\,,\ i\in\{1,\ldots,I\}\,. \tag{8.25}$$

(Indeed, (8.25) follows trivially from (8.22). To prove (8.24), we observe that if $\omega \in E_i$, $j \in \mathcal{J}$, and $t \in [j\kappa, (1+j)\kappa]$, then

$$\begin{aligned}
\|\eta^{\#,\omega,N}(t) - \eta^{*,\omega,N}(t)\| &= (\|\eta^{\#,i,(j),\kappa}(t-j\kappa) - A_0(T_t\omega)\| \\
&< \|\eta^{\#,i,(j),\kappa}(t-j\kappa) - A_0(T_t\bar{\omega}_i)\| + \|A_0(T_t\bar{\omega}_i) - A_0(T_t\omega)\| \\
&\leq \|\eta^{\#,i,(j),\kappa}(t-j\kappa) - A_0(T_t\bar{\omega}_i)\| + \frac{\delta}{2} \\
&= \|\eta^{\#,i,(j),\kappa}(t-j\kappa) - A_0(T_{t-j\kappa}T_{j\kappa}\bar{\omega}_i)\| + \frac{\delta}{2} \\
&= \|\eta^{\#,i,(j),\kappa}(t-j\kappa) - A_0(T_{t-j\kappa}\bar{\omega}_{i,j})\| + \frac{\delta}{2} \\
&= \|\eta^{\#,i,(j),\kappa}(t-j\kappa) - \eta^{*,\bar{\omega}_{i,j},\kappa}(t-j\kappa)\| + \frac{\delta}{2} \\
&= \|\eta^{\#,\bar{\omega}_{i,j},\kappa}(t-j\kappa) - \eta^{*,\bar{\omega}_{i,j},\kappa}(t-j\kappa)\| + \frac{\delta}{2} \\
&< \delta\,,
\end{aligned}$$

where the second inequality follows from the fact that $\|A_0(T_t\omega_i) - A_0(T_t\omega)\| \leq \frac{\delta}{2}$, which is true in view of (8.14), because both $\bar{\omega}_i$ and ω belong to E_i and $\text{diameter}(E_i) < c_3$, and the last inequality follows from (8.20). Furthermore, (8.20) also implies that the functions $\eta^{\#,i,(j),\kappa}$ are continuous on $[0,\kappa]$, from which it follows that $\eta^{\#,\omega,N}$ is continuous on $[0,N]$, except possibly at the points $j\kappa$, $j = 0, 1, \ldots, (1+\mu)\Lambda$. On the other hand, if $\omega \in E_i$, then the function $\eta^{\#,\omega,N}$ coincides with $\eta^{*,\bar{\omega}_i,N}$ near $j\kappa$ for every $j \in \{0, \ldots (1+\mu)\Lambda\}$, because

$$\begin{aligned}
\eta^{\#,\omega,N}(t) &= \eta^{\#,i,(j),\kappa}(t-j\kappa) \\
&= \eta^{\#,\bar{\omega}_{i,j}}(t-j\kappa) \\
&= \eta^{*,\bar{\omega}_{i,j},\kappa}(t-j\kappa) \\
&= A_0(T_{t-j\kappa}\bar{\omega}_{i,j}) \\
&= A_0(T_t\bar{\omega}_i) \\
&= \eta^{*,\bar{\omega}_i,N}(t)
\end{aligned}$$

if $t - j\kappa$ is nonnegative and sufficiently small, whereas

$$\begin{aligned}
\eta^{\#,\omega,N}(t) &= \eta^{\#,i,(j-1),\kappa}(t-(j-1)\kappa) \\
&= \eta^{\#,\bar{\omega}_{i,j-1}}(t-(j-1)\kappa) \\
&= \eta^{*,\bar{\omega}_{i,j-1},\kappa}(t-(j-1)\kappa) \\
&= A_0(T_{t-(j-1)\kappa}\bar{\omega}_{i,j-1}) \\
&= A_0(T_t\bar{\omega}_i) \\
&= \eta^{*,\bar{\omega}_i,N}(t)
\end{aligned}$$

if $t - j\kappa$ is negative and sufficiently small. Hence $\eta^{\#,\omega,N}$ is also continuous near $j\kappa$, because $\eta^{*,\bar{\omega}_i,N}$ is. So $\eta^{\#,\omega,N}$ is continuous on $[0, N]$, as stated, and (8.23) is proved.)

Step 20 : We let $\tilde{A}_{\#}$ be the map

$$E \times [0, N] \ni (\omega, t) \mapsto \tilde{A}_{\#}(\omega, t) \stackrel{def}{=} \eta^{\#,\omega,N}(t) \in S \,.$$

It is clear that $\tilde{A}_{\#}$ is continuous. Using the bijection

$$E \times [0, N] \ni (\omega, t) \mapsto \varphi(\omega, t) \stackrel{def}{=} T_t \omega \in \mathcal{T}_{[0,N]}(E) \,,$$

we associate to $\tilde{A}_{\#}$ the continuous map $A_{\#} \stackrel{def}{=} \tilde{A}_{\#} \circ \varphi^{-1} : \mathcal{T}_{[0,N]}(E) \mapsto S$.

Step 21 : Using the Tietze extension theorem, we extend $A_{\#}$ to a continuous map $A : \Omega \to S$ such that

$$\|A(\omega) - A_0(\omega)\| < \delta \quad \text{for all} \quad \omega \in \Omega \,. \tag{8.26}$$

In principle, Tietze's theorem yields an extension A_1 of $A_{\#}$ to a continuous map from Ω to $\mathbf{L}(G)$. To get an S-valued extension that satisfies (8.26), we modify A_1 as follows. First, we define $A_2 = \pi \circ A_1$, where π is a continuous projection from $\mathbf{L}(G)$ to S. (Precisely, using the distance function $d_{\mathbf{L}(G)} : \mathbf{L}(G) \times \mathbf{L}(G) \mapsto \mathbb{R}$ arising from a Euclidean inner product on $\mathbf{L}(G)$, we let $\pi(x)$ be, for $x \in \mathbf{L}(G)$, the point of S closest to x. Since S is closed and convex, it is well known that π is well defined—i.e., that the point of S closest to x exists and is unique for every $x \in \mathbf{L}(G)$— and continuous.) Since $\pi(x) = x$ when $x \in S$, and $A_1(\omega) = A_{\#}(\omega) \in S$ when $\omega \in \mathcal{T}_{[0,N]}(E)$, we see that A_2 agrees with $A_{\#}$ on $\mathcal{T}_{[0,N]}(E)$, while in addition A_2 is S-valued. We then define $\tilde{\delta} = \max\{\|A_{\#}(\omega) - A_0(\omega)\| : \omega \in \mathcal{T}_{[0,N]}(E)\}$, so $\tilde{\delta} < \delta$, and we let $A(\omega)$ be, for each $\omega \in \Omega$, the point of the closed ball $B_\omega = \{x \in L : d_L(x, A_0(\omega)) \leq \tilde{\delta}\}$ that is closest to $A_2(\omega)$. Then A is continuous, because $A(\omega) = A_0(\omega) + \Pi(A_2(\omega) - A_0(\omega))$, where Π is the projection on the closed ball $\{x \in \mathbf{L}(G) : \|x\| \leq \tilde{\delta}\}$. In addition, A agrees with $A_{\#}$ on $\mathcal{T}_{[0,N]}(E)$, because if ω belongs to $\mathcal{T}_{[0,N]}(E)$ then $A_2(\omega) \in B_\omega$, so $A(\omega) = A_2(\omega) = A_{\#}(\omega)$. Furthermore, A takes values in S, because S is convex, and for every $\omega \in \Omega$ the point $A(\omega)$ is a convex combination of $A_0(\omega)$ and $A_2(\omega)$, both of which belong to S. Finally, the fact that $A(\omega) \in B_\omega$ for each ω implies that the bound (8.26) holds.

Step 22 : We choose $\lambda \in \mathbb{N}$ such that

$$\kappa\lambda(1+\mu) > \bar{n} \,, \qquad \frac{\|f\|_{sup}}{\lambda} \leq \frac{\varepsilon}{16} \,, \quad \text{and} \quad \frac{R\hat{\kappa}\|f\|_{sup}}{\lambda} \leq \frac{\varepsilon}{16} \,, \tag{8.27}$$

and then define $n = \kappa\lambda(1+\mu)$. Then

$$\frac{2\kappa(1+\mu)}{n}\|f\|_{sup} < \frac{\varepsilon}{8} \,. \tag{8.28}$$

Next, we choose Λ so that

$$(c_1 + \frac{\lambda}{\Lambda})\|f\|_{sup}^2 \leq \frac{7\varepsilon^2}{32}\,. \tag{8.29}$$

Step 23 : We are now going to estimate the norm $\|W_{n,A}f\|_{L_2}$.

Step 23.i : We estimate the contribution to $\|W_{n,A}f\|_{L_2}$ of the points ω that do not belong to the set $\mathbf{E}^{N-n} = \bigcup_{t\in[0,N-n]} T_t(E)$.

In view of (8.15), $\mathbf{E}^{N-n}$ satisfies

$$m(\mathbf{E}^{N-n}) \geq 1 - c_1 - \frac{n}{N}\,, \qquad m(\Omega\backslash\mathbf{E}^{N-n}) \leq c_1 + \frac{n}{N}\,.$$

Outside $G\times\mathbf{E}^{N-n}$, the function $W_{n,A}f$ is bounded pointwise by $\|f\|_{sup}$, so

$$\int_{G\times(\Omega\backslash\mathbf{E}^{N-n})} |W_{n,A}f(g,\omega)|^2 d(\nu_G\otimes m)(g,\omega) \leq \Big(\frac{n}{N} + c_1\Big)\|f\|_{sup}^2\,. \tag{8.30}$$

Step 23.ii : In order to estimate $W_{n,A}f(g,\omega)$ for a point $(g,\omega)\in G\times\mathbf{E}^{N-n}$, we first reduce that task to that of estimating a time average of f along a lift of a trajectory $t\mapsto T_t\bar{\omega}_i$.

Assume that (g,ω) belongs to $G\times\mathbf{E}^{N-n}$. Then there exist unique $i\in\{1,\dots,I\}$, $t\in[0,N-n]$, $\tilde{\omega}\in E_i$, such that $\omega = T_t\tilde{\omega}$. The time t belongs to $\mathcal{I}_{(j)}$ for a $j\in\mathcal{J}$, which is unique, except if t is an integer multiple of κ. Equivalently, $t\in\mathcal{I}_{\ell,k}$ for an $\ell\in\{0,\dots,\Lambda-1\}$ and a $k\in\{0,\dots,\mu\}$. Clearly, j,ℓ, and k are related by $j=\ell(1+\mu)+k$, and $t = j\kappa + t' = \ell\kappa(1+\mu) + k\kappa + t'$, where $0\leq t'\leq\kappa$. The fact that $t\leq N-n$ implies that

$$\ell\in\{0,\dots,\Lambda-\lambda-1\} \quad \text{and} \quad j\in\mathcal{J}'\,,$$

where

$$\mathcal{J}' \stackrel{def}{=} \{0,\dots,(\Lambda-\lambda)(1+\mu)-1\}\,.$$

Furthermore, $\omega = T_{j\kappa+t'}\tilde{\omega}$, from which it follows, if we write "$a \stackrel{\varepsilon}{\approx} b$" to mean "$|a-b|<\varepsilon$," that

$$\begin{aligned} W_{n,A}f(g,\omega) &= \frac{1}{n}\int_0^n f(X^A(\omega,s)g, T_s\omega)ds \\ &= \frac{1}{n}\int_0^n f(X^A(\omega,s)g, T_{s+j\kappa+t'}\tilde{\omega})ds \\ &\stackrel{\frac{\varepsilon}{8}}{\approx} \frac{1}{n}\int_0^n f(X^A(\omega,s)g, T_{s+j\kappa+t'}\bar{\omega}_i)ds\,, \end{aligned} \tag{8.31}$$

where, in the last step, we use the facts that

(a) $d_\Omega(\tilde\omega,\bar\omega_i) < c_4$, since both $\tilde\omega$ and $\bar\omega_i$ belong to E_i,

(b) $d_\Omega(T_{s+j\kappa+t'}\tilde\omega, T_{s+j\kappa+t'}\bar\omega_i) < c_2$ by (1) of Step 17, because $s+j\kappa+t' \in [0,N]$, and $d_\Omega(\tilde\omega,\bar\omega_i) < c_3$ by (2) of Step 17,

and

(c) $|f(X^A(\omega,s)g, T_{s+j\kappa+t'}\tilde\omega) - f(X^A(\omega,s)g, T_{s+j\kappa+t'}\bar\omega_i)| \le \frac{\varepsilon}{8}$ by Step 15.

Thus, writing $\bar\omega_{i,j} \stackrel{def}{=} T_{j\kappa}\bar\omega_i$ as before, we get

$$W_{n,A}f(g,\omega) \stackrel{\frac{\varepsilon}{8}}{\approx} \frac{1}{n}\int_0^n f(X^A(\omega,s)g, T_{s+t'}\bar\omega_{i,j})ds = \frac{1}{n}\int_{t'}^{n+t'} f(X^A(\omega,s'-t')g, T_{s'}\bar\omega_{i,j})ds' .$$

Then (8.28) yields $\frac{2t'}{n}\|f\|_{sup} \le \frac{2\kappa}{n}\|f\|_{sup} < \frac{\varepsilon}{8}$ and hence

$$\frac{1}{n}\int_{t'}^{n+t'} f(X^A(\omega,s'-t')g, T_{s'}\bar\omega_{i,j})ds' \stackrel{\frac{\varepsilon}{8}}{\approx} \frac{1}{n}\int_0^n f(X^A(\omega,s'-t')g, T_{s'}\bar\omega_{i,j})ds' .$$

We then obtain

$$\begin{aligned} W_{n,A}f(g,\omega) &\stackrel{\frac{\varepsilon}{4}}{\approx} \frac{1}{n}\int_0^n f(X^A(\omega,s'-t')g, T_{s'}\bar\omega_{i,j})ds' \\ &= \frac{1}{n}\int_0^n f(X^A(T_{j\kappa}\tilde\omega,s)X^A(T_{j\kappa}\tilde\omega,t')^{-1}g, T_s\bar\omega_{i,j})ds , \end{aligned} \tag{8.32}$$

because the cocycle identity $X^A(\omega,s_1+s_2) = X^A(T_{s_1}\omega,s_2)X^A(\omega,s_1)$ implies, if we use $-t'$ in the role of s_1 and s' in that of s_2, the identity $X^A(\omega,s'-t') = X^A(T_{-t'}(\omega),s')X^A(\omega,-t')$, while on the other hand $X^A(\omega,-t')^{-1} = X^A(T_{-t'}(\omega),t')^{-1}$, and $T_{-t'}(\omega) = T_{j\kappa}\tilde\omega$.

Next, we claim that

$$X^A(T_{j\kappa}\omega',s) = X^A(\bar\omega_{i,j},s) \quad \text{if} \quad s\in[0,n],\ \omega'\in E_i,\ j\in\mathcal{J}' . \tag{8.33}$$

To prove (8.33), we first observe that

$$A(T_\tau\omega') = \eta^{\#,\omega',N}(\tau) = \eta^{\#,\bar\omega_i,N}(\tau) = A(T_\tau\bar\omega_i) ,$$

for all $\tau\in[0,N]$. Thus, $X^A(\omega',\tau) = X^A(\bar\omega_i,\tau)$ for all $\tau\in[0,N]$. Now the claim follows by using the cocycle identity $X^A(\omega',s+j\kappa) = X^A(T_{j\kappa}\omega',s)X^A(\omega',j\kappa)$ to conclude that

$$\begin{aligned} X^A(T_{j\kappa}\omega',s) &= X^A(\omega',s+j\kappa)X^A(\omega',j\kappa)^{-1} \\ &= X^A(\bar\omega_i,s+j\kappa)X^A(\bar\omega_i,j\kappa)^{-1} \qquad (\text{since } s+j\kappa\in[0,N]) \\ &= X^A(T_j\bar\omega_i,s) \\ &= X^A(\bar\omega_{i,j},s) . \end{aligned}$$

It follows from (8.32) and (8.33) that

$$W_{n,A}f(g,\omega) \overset{\frac{\varepsilon}{4}}{\approx} \frac{1}{n}\int_0^n f(X^A(\bar{\omega}_{i,j},s)X^A(\bar{\omega}_{i,j},t')^{-1}g, T_s\bar{\omega}_{i,j})ds\,,$$

i.e., that

$$W_{n,A}f(g,\omega) \overset{\frac{\varepsilon}{4}}{\approx} \frac{1}{n}W_{n,A;i,j,t'}f(g)\,, \tag{8.34}$$

where

$$W_{n,A;i,j,t'}f(g) \overset{def}{=} \int_0^n f(X^A(\bar{\omega}_{i,j},s)X^A(\bar{\omega}_{i,j},t')^{-1}g, T_s\bar{\omega}_{i,j})ds\,. \tag{8.35}$$

Step 23.iii : We break up the integral over $[0,n]$ that occurs in (8.35) into $\lambda(1+\mu)$ integrals over $[0,\kappa]$ (recall that $n = \lambda\kappa(1+\mu)$) by writing

$$\begin{aligned} W_{n,A;i,j,t'}f(g) &= \sum_{q=0}^{\lambda(1+\mu)-1} \int_{q\kappa}^{(q+1)\kappa} f(X^A(\bar{\omega}_{i,j},s')X^A(\bar{\omega}_{i,j},t')^{-1}g, T_{s'}\bar{\omega}_{i,j})ds' \\ &= \sum_{q=0}^{\lambda(1+\mu)-1} \int_0^{\kappa} f(X^A(\bar{\omega}_{i,j},s+q\kappa)X^A(\bar{\omega}_{i,j},t')^{-1}g, T_s\bar{\omega}_{i,j+q})ds\,, \end{aligned} \tag{8.36}$$

where we have used the fact that if $s' = s + q\kappa$ then

$$T_s\bar{\omega}_{i,j+q} = T_sT_{(j+q)\kappa}\bar{\omega}_i = T_{s+j\kappa+q\kappa}\bar{\omega}_i = T_{s+q\kappa}T_{j\kappa}\bar{\omega}_i = T_{s+q\kappa}\bar{\omega}_{i,j} = T_{s'}\bar{\omega}_{i,j}\,.$$

Step 23.iv : As q varies from 0 to $\lambda(1+\mu)-1$, the index $j+q$ that occurs in the integrals of (8.36) takes values in the integer interval

$$\mathcal{Q}_n(j) = \{j, j+1, \dots, j+\lambda(1+\mu)-1\}\,,$$

which is a subset of $\mathcal{J}$ because $j \le (\Lambda-\lambda)(1+\mu)-1$. We group the $\lambda(1+\mu)$ indices $j+q$ into the blocks J_ℓ (of length $1+\mu$) for those ℓ such that $J_\ell \subseteq \mathcal{Q}_n(j)$, and separate out the remaining values of $j+q$.

For this purpose, we write $\mathcal{Q}_n(j) = \mathcal{Q}_n^c(j) \cup \mathcal{Q}_n^b(j)$, where the "central part" $\mathcal{Q}_n^c(j)$ is a disjoint union of intervals J_ℓ and the "boundary part" $\mathcal{Q}_n^b(j)$ has at most $2(1+\mu)$ members. Let $\Delta_n(j)$ be the set of indices ℓ such that $J_\ell \subseteq \mathcal{Q}_n(j)$. Then

$$W_{n,A;i,j,t'}f(g) = W^c_{n,A;i,j,t'}f(g) + W^b_{n,A;i,j,t'}f(g)\,, \tag{8.37}$$

where $W^c_{n,A;i,j,t'}f(g)$, $W^b_{n,A;i,j,t'}f(g)$ are respectively the contributions to the right-hand side of (8.36) of the terms for which $j+q\in\mathcal{Q}^c_n(j)$ and those for which $j+q\in\mathcal{Q}^b_n(j)$. Thus, in particular,

$$\begin{aligned} W^c_{n,A;i,j,t'}f(g) &= \sum_{j+q\in\mathcal{Q}^c_n(j)} \int_0^\kappa f(X^A(\bar\omega_{i,j},s+q\kappa)X^A(\bar\omega_{i,j},t')^{-1}g,T_s\bar\omega_{i,j+q})ds \\ &= \sum_{\ell\in\Delta_n(j)} W^c_{n,A;i,j,t';\ell}f(g)\,, \end{aligned} \tag{8.38}$$

where

$$W^c_{n,A;i,j,t';\ell}f(g)\stackrel{def}{=}\sum_{j+q\in J_\ell}\int_0^\kappa f(X^A(\bar\omega_{i,j},s+q\kappa)X^A(\bar\omega_{i,j},t')^{-1}g,T_s\bar\omega_{i,j+q})ds\,, \tag{8.39}$$

and

$$W^b_{n,A;i,j,t'}f(g)=\sum_{j+q\in\mathcal{Q}^b_n(j)}\int_0^\kappa f(X^A(\bar\omega_{i,j},s+q\kappa)X^A(\bar\omega_{i,j},t')^{-1}g,T_s\bar\omega_{i,j+q})ds\,. \tag{8.40}$$

Step 23.v : We estimate the contribution of the terms in the "boundary part."

Since $\mathcal{Q}^b_n(j)$ has at most $2(1+\mu)$ indices, (8.40) implies the bound

$$\frac{1}{n}\left|W^b_{n,A;i,j,t'}f(g)\right|\le\frac{2\kappa(1+\mu)\|f\|_{sup}}{n}=\frac{2\|f\|_{sup}}{\lambda}\,. \tag{8.41}$$

Step 23.vi : In order to estimate $W^c_{n,A;i,j,t'}f(g)$, we get a bound for each of the terms $W^c_{n,A;i,j,t';\ell}f(g)$ of the summation in (8.38).

Let a_ℓ be the smallest member of J_ℓ, so $a_\ell=\ell(1+\mu)$, and $J_\ell=\{a_\ell,a_\ell+1,\ldots,a_\ell+\mu\}$. Then in (8.39) we can rewrite the summation with $j+q=a_\ell+v$, and get

$$W^c_{n,A;i,j,t';\ell}f(g)=\sum_{v=0}^{\mu}\int_0^\kappa f(X^A(\bar\omega_{i,j},s+(a_\ell-j+v)\kappa)X^A(\bar\omega_{i,j},t')^{-1}g,T_s\bar\omega_{i,a_\ell+v})ds. \tag{8.42}$$

Using the cocycle identity we get

$$\begin{aligned} X^A(\bar\omega_{i,j},s+(a_\ell+v-j)\kappa) &= X^A(T_{(a_\ell-j)\kappa}\bar\omega_{i,j},v\kappa+s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa) \\ &= X^A(T_{(a_\ell-j)\kappa}T_{j\kappa}\bar\omega_i,v\kappa+s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa) \\ &= X^A(T_{a_\ell\kappa}\bar\omega_i,v\kappa+s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa) \\ &= X^A(\bar\omega_{i,a_\ell},v\kappa+s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa)\,. \end{aligned}$$

Therefore

$$W^{c}_{n,A;i,j,t';\ell}f(g) = \tag{8.43}$$
$$\sum_{v=0}^{\mu}\int_0^{\kappa} f(X^A(\bar\omega_{i,a_\ell}, v\kappa+s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa)X^A(\bar\omega_{i,j},t')^{-1}g, T_s\bar\omega_{i,a_\ell+v})ds\,.$$

Step 23.vii : We split the sum of (8.43) by separating out the first term from the μ remaining ones. We get

$$W^{c}_{n,A;i,j;t';\ell}f(g) = W^{c,-}_{n,A;i,j,t';\ell}f(g) + W^{c,+}_{n,A;i,j,t';\ell}f(g)\,, \tag{8.44}$$

where

$$W^{c,-}_{n,A;i,j,t';\ell}f(g) = \int_0^{\kappa} f(X^A(\bar\omega_{i,a_\ell}, s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa)X^A(\bar\omega_{i,j},t')^{-1}g, T_s\bar\omega_{i,a_\ell})ds\,,$$

$$W^{c,+}_{n,A;i,j,t';\ell}f(g) = \sum_{v=1}^{\mu}\int_0^{\kappa} \bar W^{c,+;f,g;s}_{n,A;i,j,t';\ell;v}\,ds\,, \tag{8.45}$$

and

$$\bar W^{c,+;f,g;s}_{n,A;i,j,t';\ell;v} = f(X^A(\bar\omega_{i,a_\ell}, v\kappa+s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa)X^A(\bar\omega_{i,j},t')^{-1}g, T_s\bar\omega_{i,a_\ell+v})\,. \tag{8.46}$$

Step 23.viii : We estimate the contribution of $W^{c,-}_{n,A;i,j,t';\ell}f$ by observing that each function $W^{c,-}_{n,A;i,j,t';\ell}f$ is pointwise bounded by $\kappa\|f\|_{sup}$. Since the number of members of $\Delta_n(j)$ is at most λ, we have the estimate

$$\frac{1}{n}\Big|\sum_{\ell\in\Delta_n(j)} W^{c,-}_{n;A,i,j,t';\ell}f(g)\Big| \le \frac{\lambda\kappa\|f\|_{sup}}{n} = \frac{\|f\|_{sup}}{1+\mu}\,. \tag{8.47}$$

Step 23.ix : We now do some preliminary work towards the key step of our construction, namely, getting bounds for $W^{c,+}_{n,A;i,j,t';\ell}f(g)$, This will be done by estimating $\bar W^{c,+;f,g;s}_{n,A;i,j,t';\ell;v}$ and using (8.45), and our first step is to rewrite $\bar W^{c,+;f,g;s}_{n,A;i,j,t';\ell;v}$ is a convenient way.

We use the cocycle identity to compute the factor by which g is left-multiplied in the first of the two arguments of f in (8.46).

$$\begin{aligned}
&X^A(\bar\omega_{i,a_\ell}, v\kappa+s)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa)X^A(\bar\omega_{i,j},t')^{-1}\\
&= X^A(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)X^A(\bar\omega_{i,a_\ell},\kappa)X^A(\bar\omega_{i,j},(a_\ell-j)\kappa)X^A(\bar\omega_{i,j},t')^{-1}\\
&= X^A(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)X^A(\bar\omega_{i,a_\ell},\kappa)X^A(\bar\omega_{i},a_\ell\kappa)X^A(\bar\omega_{i},j\kappa)^{-1}X^A(\bar\omega_{i,j},t')^{-1}\\
&= X^A(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)X^A(\bar\omega_{i,a_\ell},\kappa)X^A(\bar\omega_{i},a_\ell\kappa)\big(X^A(\bar\omega_{i,j},t')X^A(\bar\omega_{i},j\kappa)\big)^{-1}\\
&= X^A(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)X^A(\bar\omega_{i,a_\ell},\kappa)X^A(\bar\omega_{i},a_\ell\kappa)X^A(\bar\omega_{i},j\kappa+t')^{-1}\,.
\end{aligned}$$

Therefore,

$$\bar{W}^{c,+;f,g;s}_{n,A;i,j,t';\ell;v}$$
$$= f(X^A(\bar{\omega}_{i,a_\ell+1},(v-1)\kappa+s)X^A(\bar{\omega}_{i,a_\ell},\kappa)X^A(\bar{\omega}_i,a_\ell\kappa)X^A(\bar{\omega}_i,j\kappa+t')^{-1}g,T_s\bar{\omega}_{i,a_\ell+v})\,.$$

Step 23.x : We still have to choose the points $g^{\#,\bar{\omega}_{i,j}} \in G$ and the controls $\eta^{\#,\bar{\omega}_{i,j},\kappa}$, as indicated in Step 18. We begin by making the choice

$$\eta^{\#,\bar{\omega}_{i,j},\kappa} = \eta^{*,\bar{\omega}_{i,j},\kappa} \quad g^{\#,\bar{\omega}_{i,j}} = \xi^{\eta^{*,\bar{\omega}_{i,j},\kappa}}(\kappa) \quad \text{if} \quad j \notin \{\ell(1+\mu) : \ell \in \mathcal{L}\}\,.$$

It then follows from (8.22) that

$$\begin{aligned} A(T_t\bar{\omega}_i) &= A_\#(\bar{\omega}_i,t) \\ &= \eta^{\#,\bar{\omega}_i,N}(t) \\ &= \eta^{\#,i,(j),\kappa}(t-j\kappa) \\ &= \eta^{\#,\bar{\omega}_{i,j},\kappa}(t-j\kappa) \\ &= \eta^{*,\bar{\omega}_{i,j},\kappa}(t-j\kappa) \\ &= A_0(T_{t-j\kappa}\bar{\omega}_{i,j}) \\ &= A_0(T_{t-j\kappa}T_{j\kappa}\bar{\omega}_i) \\ &= A_0(T_t\bar{\omega}_i) \end{aligned}$$

whenever $i \in \mathbf{I}$, $j \in \mathcal{J}$, $j \notin \{\ell(1+\mu) : \ell \in \mathcal{L}\}$, and $t \in [j\kappa,(1+j)\kappa]$. Therefore

$$(8.48) \qquad A(T_t\bar{\omega}_i) = A_0(T_t\bar{\omega}_i) \quad \text{whenever} \quad t \in \bigcup_{\ell=0}^{\Lambda-1}[a_\ell\kappa+\kappa,a_{\ell+1}\kappa]\,.$$

For each i,ℓ, we let $F_{i,\ell}$ be the function on G defined by

$$(8.49) \qquad F_{i,\ell}(g) = \frac{1}{\kappa\mu}\sum_{v=1}^{\mu}\int_0^\kappa f(X^A(\bar{\omega}_{i,a_\ell+1},(v-1)\kappa+s)g,T_s\bar{\omega}_{i,a_\ell+v})ds\,.$$

Notice that $X^A(\bar{\omega}_{i,a_\ell+1},\tau)$, for $\tau \in [0,\kappa\mu]$, is obtained by solving the differential equation $\xi'(\tau) = A(T_\tau\bar{\omega}_{i,a_{\ell+1}})\xi(\tau)$ with initial condition $\xi(0) = e_G$, whereas $X^{A_0}(\bar{\omega}_{i,a_\ell+1},\tau)$ is obtained by solving the same initial value problem, with A replaced by A_0. On the other hand, (8.48) tells us that $A(T_{\tau+a_\ell\kappa+\kappa}\bar{\omega}_i) = A_0(T_{\tau+a_\ell\kappa+\kappa}\bar{\omega}_i)$ whenever $0 \le \tau \le a_{\ell+1}\kappa-(a_\ell+1)\kappa$, i.e., whenever $0 \le \tau \le \kappa\mu$. In other words, $A(T_\tau\bar{\omega}_{i,a_\ell+1}) = A_0(T_\tau\bar{\omega}_{i,a_\ell+1})$ whenever $0 \le \tau \le \kappa\mu$. It follows that $X^A(\bar{\omega}_{i,a_\ell+1},\tau) = X^{A_0}(\bar{\omega}_{i,a_\ell+1},\tau)$ when $\tau \in [0,\kappa\mu]$. Hence

$X^A(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)=X^{A_0}(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)$ whenever $v=1,\dots,\mu$, $s\in[0,\kappa]$. So we can rewrite (8.49) as

$$F_{i,\ell}(g)=\frac{1}{\kappa\mu}\sum_{v=1}^{\mu}\int_0^\kappa f(X^{A_0}(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)g,T_s\bar\omega_{i,a_\ell+v})ds\,. \tag{8.50}$$

We observe that

$$W^{c,+}_{n,A;i,j,t';\ell}f(g)=\kappa\mu F_{i,\ell}\big(X^A(\bar\omega_{i,a_\ell},\kappa)X^A(\bar\omega_i,a_\ell\kappa)X^A(\bar\omega_i,j\kappa+t')^{-1}g\big). \tag{8.51}$$

Step 23.xi : We have already chosen the $g^{\#,\bar\omega_{i,j}}$ and the controls $\eta^{\#,\bar\omega_{i,j},\kappa}$ when j is not one of the a_ℓ. We will now choose $g^{\#,\bar\omega_{i,a_\ell}}$ and $\eta^{\#,\bar\omega_{i,a_\ell},\kappa}$ for each index $i\in\mathbf{I}=\{1,\dots,I\}$ and each $\ell\in\mathcal{L}=\{0,1,\dots,\Lambda-1\}$.

For each $v\in\{1,\dots,\mu\}$ and each $s\in[0,\kappa]$, the function

$$g\mapsto f(X^{A_0}(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s)g,T_s\bar\omega_{i,a_\ell+v}) \tag{8.52}$$

is equal to $\tau_h(f^{\omega'})$, if $h=(X^{A_0}(\bar\omega_{i,a_\ell+1},(v-1)\kappa+s))^{-1}$ and $\omega'=T_s\bar\omega_{i,a_\ell+v}$. Therefore this function belongs to the set $\mathcal{K}_*$ defined in Step 2. It clearly follows from (8.50) that $F_{i,\ell}$ is an average of functions of the type (8.52). Hence $F_{i\ell}$ belongs to the closed convex hull $\mathcal{K}$ of $\mathcal{K}_*$. Consequently, each $F_{i,\ell}$ also belongs to $\mathcal{K}$. Thus, for each i,ℓ we can choose $H_{i,\ell}\in\mathcal{K}_0$ such that

$$\|F_{i,\ell}-H_{i,\ell}\|_{sup}\le\frac{\varepsilon}{8}.$$

For each $i\in\mathbf{I}$, $H\in\mathcal{K}_0$, let $\mathcal{A}(i,H)$ be the set of those indices $\ell\in\mathcal{L}$ such that $H_{i,\ell}=H$, and let $\mathbf{a}(i,H)$ be the number of members of $\mathcal{A}(i,H)$. Enumerate the members of $\mathcal{A}(i,H)$, from left to right, so that

$$\mathcal{A}(i,H)=\{\alpha_{i,H,1},\dots,\alpha_{i,H,\mathbf{a}(i,H)}\}\quad\text{and}\quad\alpha_{i,H,1}<\alpha_{i,H,2}<\dots<\alpha_{i,H,\mathbf{a}(i,H)}\,.$$

Then for each $i\in\mathbf{I}$, $\ell\in\mathcal{L}$, there exists a unique $\theta=\theta(i,\ell)\in\{1,\dots,\mathbf{a}(i,H_{i,\ell})\}$ such that $\ell=\alpha_{i,H_{i,\ell},\theta}$. We then define $\sigma(i,\ell)$, $\rho(i,\ell)$ to be, respectively, the quotient and remainder of $\theta(i,\ell)$ modulo R, so that

$$\theta(i,\ell)=\sigma(i,\ell)R+\rho(i,\ell)\,,\quad\sigma(i,\ell)\in\mathbb{Z}\,,\ \rho(i,\ell)\in\mathbb{Z}\,,\ \sigma(i,\ell)\ge0\,,\ \text{and}\ 0\le\rho(i,\ell)<R\,.$$

We are now ready to define the $g^{\#,\bar\omega_{i,a_\ell}}$ and the $\eta^{\#,\bar\omega_{i,a_\ell},\kappa}$ for $i\in\mathbf{I}$, $\ell\in\mathcal{L}$. We will do this by induction with respect to ℓ.

Fix an ℓ, and assume that the $g^{\#,\bar\omega_{i,a_{\ell'}}}$ and the $\eta^{\#,\bar\omega_{i,a_{\ell'}},\kappa}$ have already been chosen for all ℓ' such that $\ell'<\ell$. Then the point $X^A(\bar\omega_i,a_\ell\kappa)$ is determined, because the curve $[0,a_\ell\kappa]\ni\tau\mapsto X^A(\bar\omega_i,a_\ell\kappa)\in G$ is the unique solution $\tilde\xi$ of the initial value problem

$$\begin{cases}\xi'(\tau) &= A(T_\tau\bar\omega_i)\xi(\tau)\,,\\ \xi(0) &= e_G\,,\end{cases}$$

so $\tilde{\xi}$ is determined by the function $[0, a_\ell\kappa] \ni \tau \mapsto A(T_\tau\bar{\omega}_i) \in S$ which, by construction, is the restriction to $[0, a_\ell\kappa]$ of the control $\eta^{\#,\bar{\omega}_i,N}$.

On the other hand, $X^A(\bar{\omega}_{i,a_\ell}, \kappa)$ is not determined before we choose $g^{\#,\bar{\omega}_{i,a_\ell}}$, and the $\eta^{\#,\bar{\omega}_{i,a_\ell},\kappa}$, because the curve $[0,\kappa] \ni \tau \mapsto X^A(\bar{\omega}_{i,a_\ell}, \tau) \in G$ is the solution $\hat{\xi}$ of the initial value problem

$$\begin{cases} \xi'(\tau) &= A(T_\tau\bar{\omega}_{i,a_\ell})\xi(\tau)\,, \\ \xi(0) &= e_G\,, \end{cases}$$

and, if we let $j = \ell(1+\mu) = a_\ell$, then

$$\begin{aligned} A(T_\tau\bar{\omega}_{i,a_\ell}) &= A(T_\tau T_{a_\ell\kappa}\bar{\omega}_i) \\ &= A(T_{\tau+a_\ell\kappa}\bar{\omega}_i) \\ &= \eta^{\#,\bar{\omega}_i,N}(\tau + a_\ell\kappa) \\ &= \eta^{\#,i,(j),\kappa}((\tau + a_\ell\kappa) - j\kappa) \\ &= \eta^{\#,i,(j),\kappa}(\tau)\,, \end{aligned}$$

and

$$X^A(\bar{\omega}_{i,a_\ell}, \kappa) = \hat{\xi}(\kappa) = \xi^{\eta^{\#,i,(j),\kappa}}(\kappa) = g^{\#,\bar{\omega}_{i,j}} = g^{\#,\bar{\omega}_{i,a_\ell}}\,,$$

so the function $[0,\kappa] \ni \tau \mapsto A(T_\tau\bar{\omega}_{i,a_\ell}) \in D$ and the point $X^A(\bar{\omega}_{i,a_\ell}, \kappa) \in G$ depend very much on the choice of $g^{\#,\bar{\omega}_{i,a_\ell}}$ and $\eta^{\#,\bar{\omega}_{i,a_\ell},\kappa}$.

If we apply the result of Step 9 with $\hat{g}^{\bar{\omega}_{i,a_\ell}}X^A(\bar{\omega}_i, a_\ell\kappa)$ in the role og g, we can conclude that that the set $G\hat{g}^{\bar{\omega}_{i,a_\ell}}X^A(\bar{\omega}_i, a_\ell\kappa)$ is δ_1-dense in $\mathbf{F}\hat{g}^{\bar{\omega}_{i,a_\ell}}X^A(\bar{\omega}_i, a_\ell\kappa)$. We can therefore choose a member $\gamma_{i,\ell}$ of this set such that

(8.53) $$d_G(\gamma_{i,\ell}, \bar{g}^{\rho(i,\ell)}) < \delta_1\,.$$

We then let

$$g^{\#,\bar{\omega}_{i,a_\ell}} = \gamma_{i,\ell}X^A(\bar{\omega}_i, a_\ell\kappa)^{-1}\,.$$

It follows that

$$g^{\#,\bar{\omega}_{i,a_\ell}} \in \mathcal{G}\hat{g}^{\bar{\omega}_{i,a_\ell}}\,.$$

Then (8.10) implies that we can pick a control $\eta^{\#,\bar{\omega}_{i,a_\ell},\kappa} \in N_S^0(\eta^{*,\bar{\omega}_{i,a_\ell},\kappa}, \delta/2)$ such that $\xi^{\eta^{\#,\bar{\omega}_{i,a_\ell},\kappa}}(\kappa) = g^{\#,\bar{\omega}_{i,a_\ell}}$.

It follows from our choice of $g^{\#,\bar{\omega}_{i,a_\ell}}$ and $\eta^{\#,\bar{\omega}_{i,a_\ell},\kappa}$ that

(8.54) $$X^A(\bar{\omega}_{i,a_\ell}, \kappa)X^A(\bar{\omega}_i, a_\ell\kappa) = \gamma_{i,\ell}\,.$$

Therefore,

(8.55) $$W^{c,+}_{n,A;i,j,t';\ell}f(g) = \kappa\mu F_{i,\ell}\big(\gamma_{i,\ell}X^A(\bar{\omega}_i, j\kappa + t')^{-1}g\big).$$

Then (8.55) implies that

$$d_G(\gamma_{i,\ell} X^A(\bar{\omega}_i, j\kappa + t')^{-1} g, \bar{g}^{\rho(i,\ell)} X^A(\bar{\omega}_i, j\kappa + t')^{-1} g) < \delta_1 \quad \text{for all} \quad g \in G\,. \tag{8.56}$$

Let

$$\tilde{W}^{c,+}_{n,A;i,j,t';\ell} f(g) = \kappa\mu F_{i,\ell}\big(\bar{g}^{\rho(i,\ell)} X^A(\bar{\omega}_i, j\kappa + t')^{-1} g\big). \tag{8.57}$$

Then (8.5), (8.55), and the fact that $F_{i,\ell} \in \mathcal{K}$, imply that

$$|W^{c,+}_{n,A;i,j,t';\ell} f(g) - \tilde{W}^{c,+}_{n,A;i,j,t';\ell} f(g)| \leq \kappa\mu\beta \quad \text{for all} \quad g \in G\,. \tag{8.58}$$

For any i, j,

$$\begin{aligned}\sum_{\ell \in \Delta_n(j)} \tilde{W}^{c,+}_{n,A;i,j,t';\ell} f(g) &= \kappa\mu \sum_{\ell \in \Delta_n(j)} F_{i,\ell}\big(\bar{g}^{\rho(i,\ell)} X^A(\bar{\omega}_i, j\kappa + t')^{-1} g\big) \\ &= \kappa\mu \sum_{H \in \mathcal{K}_0} \sum_{\ell \in A(i,H) \cap \Delta_n(j)} \tau_{(\bar{g}^{\rho(i,\ell)})^{-1}} F_{i,\ell}\big(X^A(\omega_i, j\kappa + t')^{-1} g\big).\end{aligned}$$

Let

$$\mathcal{W}_{n,A;i,j}(g) = \kappa\mu \sum_{H \in \mathcal{K}_0} \sum_{\ell \in A(i,H) \cap \Delta_n(j)} \tau_{(\bar{g}^{\rho(i,\ell)})^{-1}} H(g). \tag{8.59}$$

Since $\|H - F_{i,\ell}\|_{sup} \leq \frac{\varepsilon}{4}$ whenever $\ell \in A(i,H)$, we have the bound

$$\Big|\mathcal{W}_{n,A;i,j}(X^A(\omega_i, j\kappa + t')^{-1} g) - \sum_{\ell \in \Delta_n(j)} \tilde{W}^{c,+}_{n,A;i,j,t';\ell} f(g)\Big| \leq \frac{\kappa\mu\lambda\varepsilon}{4}\,, \tag{8.60}$$

using the fact that $\Delta_n(j)$ has at most λ members. Then (8.58) implies

$$\Big|\mathcal{W}_{n,A;i,j}(X^A(\omega_i, j\kappa + t')^{-1} g) - \sum_{\ell \in \Delta_n(j)} W^{c,+}_{n,A;i,j,t';\ell} f(g)\Big| \leq \kappa\mu\lambda\big(\beta + \frac{\varepsilon}{4}\big)\,. \tag{8.61}$$

Observing that $\frac{\kappa\mu\lambda}{n} \leq 1$, we get

$$\frac{1}{n}\Big|\mathcal{W}_{n,A;i,j}(X^A(\omega_i, j\kappa + t')^{-1} g) - \sum_{\ell \in \Delta_n(j)} W^{c,+}_{n,A;i,j,t';\ell} f(g)\Big| \leq \beta + \frac{\varepsilon}{4}\,. \tag{8.62}$$

We now turn to the task of estimating $\mathcal{W}_{n,A;i,j}$. We write

$$\mathcal{W}_{n,A;i,j}(g) = \kappa\mu \sum_{H \in \mathcal{K}_0} \mathcal{W}_{n,A;i,j;H}(g),$$

where

$$\mathcal{W}_{n,A;i,j;H}(g) = \sum_{\ell \in A(i,H)\cap\Delta_n(j)} \tau_{(\bar{g}^{\rho(i,\ell)})^{-1}} H(g). \tag{8.63}$$

Now suppose that $A(i,H)\cap\Delta_n(j)$ has $q_{i,j,H}R + r_{i,j,H}$ members, where $q_{i,j,H}$ and $r_{i,j,H}$ are integers such that $q_{i,j,H} \geq 0$ and $0 \leq r_{i,j,H} < R$. Then the sum of the first $q_{i,j,H}P$ terms of (8.63) is equal to $q_{i,j,H}\sum_{r=0}^{R-1} \tau_{(\bar{g}^{\rho(i,\ell)})^{-1}} H(g)$, whose absolute value is bounded by $R\beta q_{i,j,H}$, because $H \in \mathcal{K}$ (cf. (8.7)).

The sum of the remaining $p_{i,j,H}$ terms is bounded by $p_{i,j,H}\|f\|_{sup}$. Thus

$$\Big|\mathcal{W}_{n,A;i,j;H}(g)\Big| \leq R\beta q_{i,j,H} + p_{i,j,H}\|f\|_{sup}\,.$$

Each number $q_{i,j,H}$ is bounded by $\frac{\lambda}{R}$, and $p_{i,j,H} \leq R$. So

$$\Big|\mathcal{W}_{n,A;i,j;H}(g)\Big| \leq \lambda\beta + R\|f\|_{sup}\,.$$

Therefore

$$\|\mathcal{W}_{n,A;i,j}\|_{sup} \leq \kappa\mu\hat{\kappa}(\lambda\beta + R\|f\|_{sup})\,. \tag{8.64}$$

If we combine (8.62) and (8.64), we find

$$\frac{1}{n}\Big\|\sum_{\ell\in\Delta_n(j)} W^{c,+}_{n,A;i,j,t';\ell} f(g)\Big\|_{sup} \leq \beta + \frac{\varepsilon}{4} + \frac{\kappa\mu\hat{\kappa}\lambda\beta}{n} + \frac{\kappa\mu\hat{\kappa}R}{n}\|f\|_{sup}\,, \tag{8.65}$$

We now use (8.44) and (8.47) and get

$$\frac{1}{n}\Big|W^c_{n,A;i,j,t'}\Big|_{sup} = \frac{1}{n}\Big|\sum_{\ell\in\Delta_n(j)} W^c_{n,A;i,j,t';\ell}\Big|_{sup} \leq \frac{\|f\|_{sup}}{1+\mu} + \beta + \frac{\varepsilon}{4} + \frac{\kappa\mu\hat{\kappa}\lambda\beta}{n} + \frac{\kappa\mu\hat{\kappa}R}{n}\|f\|_{sup}\,.$$

Then (8.37) and (8.41) imply

$$|W_{n,A,i,j,t'}(g)| \leq \frac{2\|f\|_{sup}}{\lambda} + \frac{\|f\|_{sup}}{1+\mu} + \beta + \frac{\varepsilon}{4} + \frac{\kappa\mu\hat{\kappa}\lambda\beta}{n} + \frac{\kappa\mu\hat{\kappa}R}{n}\|f\|_{sup}\,. \tag{8.66}$$

Finally, we use (8.34) and get the pointwise estimate

$$|W_{n,A}f(g,\omega)| \leq \frac{2\|f\|_{sup}}{\lambda} + \frac{\|f\|_{sup}}{1+\mu} + \beta + \frac{\varepsilon}{2} + \frac{\kappa\mu\hat{\kappa}\lambda\beta}{n} + \frac{\kappa\mu\hat{\kappa}R}{n}\|f\|_{sup}\,, \tag{8.67}$$

valid whenever $g \in G$ and $\omega \in \mathbf{E}^{N-n}$.

Since $n = \lambda\kappa(1+\mu)$, (8.67) implies

$$|W_{n,A}f(g,\omega)| \leq \frac{2\|f\|_{sup}}{\lambda} + \frac{\|f\|_{sup}}{1+\mu} + \beta + \frac{\varepsilon}{2} + \hat{\kappa}\beta + \frac{\hat{\kappa}R}{\lambda}\|f\|_{sup}\,, \tag{8.68}$$

that is,

$$|W_{n,A}f(g,\omega)| \leq \frac{\varepsilon}{2} + (1+\hat{\kappa})\beta + \Big(\frac{2}{\lambda} + \frac{1}{1+\mu} + \frac{\hat{\kappa}R}{\lambda}\Big)\|f\|_{sup}\,, \tag{8.69}$$

In view of (8.4), $(1+\hat{\kappa})\beta = \frac{\varepsilon}{8}$. Clearly, (8.11) and (8.27) imply that the last of the three terms of the right-hand side of (8.69) is bounded by $\frac{\varepsilon}{4}$. Therefore

$$|W_{n,A}f(\bar{g},\omega)| \leq \frac{7\varepsilon}{8} \quad \text{whenever} \quad g \in G \text{ and } \omega \in \mathbf{E}^{N-n}\,. \tag{8.70}$$

This, together with (8.30), gives

$$\int_{G\times\Omega} |W_{n,A}f|^2 \leq \frac{49\varepsilon^2}{64} + \Big(c_1 + \frac{n}{N}\Big)\|f\|^2_{sup} = \frac{49\varepsilon^2}{64} + \Big(c_1 + \frac{\lambda}{\Lambda}\Big)\|f\|^2_{sup}$$

because $\frac{n}{N} = \frac{\lambda}{\Lambda}$. Since $\big(c_1 + \frac{\lambda}{\Lambda}\big)\|f\|^2_{sup} \leq \frac{7\varepsilon^2}{32}$ in view of (8.29), we can conclude that $\|W_{n,A}f\|_{L^2} < \varepsilon$.

Since $n > \bar{n}$, we have proved that $A \in E(f,\varepsilon,\bar{n})$. By construction, $\|A - A_0\|_{sup} < \delta$. This concludes the proof of Lemma 7.3, and then the proof of Theorem 7.1 is complete. ◇

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