Construction of minimal cocycles arising from specific differential equations

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Abstract

Blending methods of Topological Dynamics and Control Theory, we develop a new technique to construct compact-Lie-group-valued minimal cocycles arising as fundamental matrix solutions of linear differential equations with recurrent coefficients subject to a given constraint. The precise requirement on the coefficients is that they belong to a specified closed convex subset S of the Lie algebra L of the Lie group. Our result is proved for a very thin class of cocycles, since the dimension of S is allowed to be much smaller than that of L, and the only assumption on S is that $L_0(S) = L$, where $L_0(S)$ is the ideal of $L(S)$ generated by the difference set $S - S$, and $L(S)$ is the Lie subalgebra of L generated by S. This covers a number of differential equations arising in Mathematical Physics, and applies in particular to the widely studied example of the Rabi oscillator.

§1. Introduction

We develop a technique to construct cocycles that have desired dynamical properties and arise as the fundamental matrix solutions to linear differential equations of a given specific form. Our motivation comes from recent developments in Mathematical Physics regarding stability questions in the evolution of quantum systems, which turn out to be intimately related to the dynamical properties of certain flows [10]. As an example of the general situation to be studied here, consider the so called "Rabi oscillator," i.e. the system governed by the equation

$$i\frac{d\psi}{dt} = \begin{pmatrix} \lambda & f(t) \\ f(t)^* & -\lambda \end{pmatrix}\psi, \qquad \psi \in \mathbf{C}^2. \tag{1.1}$$

This is the Schrödinger equation for the dynamics of a "two level atom" or a spin $1/2$ particle moving under external magnetic field $f(t)$. The function $f\colon \mathbf{R} \to \mathbf{C}$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX is a complex-valued potential, typically quasiperiodic in t, and $\lambda \in \mathbf{R}$ is a fixed parameter. In the past few years this system has been extensively investigated by physicists using numerical techniques, cf. [1], [5], [15], [16]. First rigorous results regarding existence of quasi-periodic solutions have recently been proved by P. Bleher, H. Jauslin and J. Lebowitz [3], using the K.A.M. technique. In contrast, we shall prove that the $SU(2,\mathbf{C})$-valued cocycle generated by (1.1) is minimal for generic f.

Our result will in fact be valid for time-dependent differential equations far more general than (1.1), of the form $x' = B(t)x$, where x takes values in $\mathbf{R}^n$ or $\mathbf{C}^n$, and $t \to B(t)$ is a real or complex matrix-valued function. The time-dependence of $B(t)$ will be "recurrent," in the sense that we will think of the matrix functions $t \to B(t)$ as arising from some given continuous matrix-valued map A, defined on a space Ω where some recurrent flow T is given, by evaluating A along trajectories of T. Even more generally, the maps A will in fact take values in the Lie algebra L of a compact connected Lie group G, and will generate cocycles $X_A\colon \Omega \times \mathbf{R} \to G$. We will try to prove that, for "very thin" classes C of maps A, the corresponding cocycles are minimal for generic $A \in C$. (Various results about generic behavior of cocycles within suitable classes have been known for some time, cf. [6], [12], [13]. However, the classes where all the known results hold are cohomology invariant, in the sense that all cocycles cohomologous to a cocycle in the class are again in the class. In our case the classes are *never* cohomology invariant since the differential equation from which the cohomologous cocycles arise may fail to be of the special form we desire. Furthermore, in our situation there is a very stringent condition on cocycles, namely, they must arise as solutions to a differential equation of a given specific form, such as (1.1).)

Our "thin classes" are described via a constraint on the values of the map $A\colon \Omega \to L$, and arise from maps taking values in a fixed given closed convex subset S of L. Typically S will be "thin" in L, in the sense that the dimension $\dim G$ of G will in general be considerably higher than the number $\dim S$ of free parameters that we are allowed to vary to generate our minimal cocycle. (For example, in the case of the Rabi oscillator the cocycle takes values in a three-dimensional Lie group, but we only have two real degrees of freedom, since the map f which is our perturbation takes values in a two-dimensional space of complex numbers.)

Precisely, we will start by specifying

(1) a compact connected Lie group G with Lie algebra L,
(2) a subset S of L,
(3) a flow (Ω, T), where Ω is a compact metric space and $T = \{T_t\}_{t\in\mathbf{R}}$ is a one-parameter group of homeomorphisms $T_t\colon \Omega \to \Omega$ of Ω.

Remark 1.1. Equation (1.1) is a special case of the above situation, corresponding to $G = SU(2,\mathbf{C})$ (so that $L = su(2,\mathbf{C})$, the set of all 2×2 skew-hermitian matrices) and $S = S_\lambda$, the set of all $M \in su(2,\mathbf{C})$ of the form $\begin{pmatrix} -i\lambda & ia+b \\ ia-b & i\lambda \end{pmatrix}$, where $a, b \in \mathbf{R}$, so S is in fact a two-dimensional affine subspace of the three-dimensional Lie algebra L. $\square$

The functions $t \to B(t)$ will then be those of the form $t \to A(T_t\omega)$, for $\omega \in \Omega$. This gives rise to a family of differential equations

$$x'(t) = A(T_t\omega)x(t), \quad x \in G, \tag{1.2.ω}$$

parametrized by points $\omega \in \Omega$. (We use the following notational convention: we think of L as the tangent space of G at the identity element e_G of G. The effect on a $w \in L$ of the differential dR_x of the right translation $G \ni z \to R_x(z) = zx \in G$ is written wx (rather than $(dR_x)(w)$ or $(R_x)_*(w)$, as is often done in Differential Geometry), so wx is a tangent vector at x, and then the map $V_w: x \to wx$ is a right-invariant vector field. The map $w \to V_w$ enables us to identify L with the space of all right-invariant vector fields on G. The right-hand side of (1.2.ω) is then the value at $x(t)$ of the right-invariant vector field whose value at e_G is $A(T_t\omega)$, so (1.2.ω) is equivalent to $x'(t) = V_{A(T_t\omega)}(x(t))$. If G is a matrix Lie group, then $A(T_t\omega)x(t)$ can be interpreted as an ordinary matrix product. The reader who so prefers can assume throughout the paper that G is a matrix Lie group and $A(T_t\omega)x(t)$ is an ordinary product.)

Let $A: \Omega \to L$ be a continuous map. For each $\omega \in \Omega$, let $t \to X_A(\omega, t)$ be the fundamental matrix solution to (1.2.ω), i.e. the solution $x(\cdot)$ of (1.2.ω) such that $x(0) = e_G$. Then the map $X_A: \Omega \times \mathbf{R} \to G$ is continuous and satisfies the cocycle identity

$$X_A(\omega, t+s) = X_A(T_t(\omega), s)X_A(\omega, t) \qquad \text{for all } \omega \in \Omega,\ t, s \in \mathbf{R}. \tag{1.3}$$

The proper analogue of the flow generated by the differential equation (1.1) is now the skew-product flow $(G \times \Omega, T^A)$ generated by the cocycle X_A on $G \times \Omega$, where $T^A = \{T_t^A\}_{t \in \mathbf{R}}$, and $T^A: G \times \Omega \to G \times \Omega$ is the map given by

$$T_t^A(g, \omega) = (X_A(\omega, t)g, T_t(\omega)). \tag{1.4}$$

We will prove that if (Ω, T) is minimal then the skew-product flow T^A defined on $G \times \Omega$ by (1.4) is minimal for a generic S-valued A, provided that S is convex and not too small. We recall that a flow is *minimal* if every orbit is dense or, equivalently, if there are no proper closed invariant subsets.

To state the condition on S, let us first define $L(S)$ to be the Lie subalgebra of L generated by S, and let $L_0(S)$ be the ideal of $L(S)$ generated by the difference set

$$S - S = \{x - y : x \in S,\ y \in S\}.$$

We now introduce two fundamental concepts of Nonlinear Control Theory (cf. for example [17]):

Definition 1.1. A subset S of the Lie algebra L of a Lie group G has the **accessibility property** if $L(S) = L$, and the **strong accessibility property** if $L_0(S) = L$. $\square$

Remark 1.2. In Control Theory, the accessibility properties of Definition 1.1 are used for general sets S of smooth vector fields on a smooth manifold M. In that case, one lets $L(S)$ be the Lie algebra of vector fields generated by S, and defines $L_0(S)$ as above. Then S is said to have the **accessibility property** (resp. the **strong accessibility property**) at a point $p \in M$ if $\{X(p) : X \in L(S)\} = T_pM$ (resp. $\{X(p) : X \in L_0(S)\} = T_pM$), where T_pM is the tangent space of M at p. When M is a Lie group and S consists of translation-invariant vector fields, then these conditions do not depend on p, and are equivalent to those of Definition 1.1. $\square$

We now state the main theorem, whose proof will be given in the next section.

Theorem 1.1. *Let $(\Omega, \{T_t\}_{t\in\mathbf{R}})$ be a flow on a compact metric space Ω. Let G be a compact connected Lie group with Lie algebra L, and let S be a subset of L. Let $C(\Omega, S)$ denote the metric space of all continuous maps from Ω to S with the supremum metric. Assume that*

1. *the flow $(\Omega, \{T_t\}_{t\in\mathbf{R}})$ is aperiodic and minimal,*
2. *S is closed and convex,*
3. *S has the strong accessibility property.*

For each $A \in C(\Omega, S)$, consider the the skew-product flow $(G \times \Omega, T^A)$ defined by $T^A = \{T_t^A\}_{t\in\mathbf{R}}$, where the T_t^A are given by (1.4). Let

$$C_{min}(\Omega, S) = \{A \in C(\Omega, S) : (G \times \Omega, T^A) \text{ is minimal}\}.$$

Then $C_{min}(\Omega, S)$ is a residual subset of $C(\Omega, S)$.

Remark 1.3. Notice that our condition on the set S is a little stronger than demanding that S generate L as a Lie algebra over $\mathbf{R}$. It is easy to see that Theorem 1.1 is no longer true if in Hypothesis (3) strong accessibility is replaced by accessibility. To see this, let $\Omega = \mathbf{T}^2 = S^1 \times S^1$ be the 2-torus, and let T be given by $T_t(e^{i\theta_1}, e^{i\theta_2}) = (e^{i(\theta_1+t)}, e^{i(\theta_2+\alpha t)})$, where α is an irrational number. Then T is minimal and aperiodic. Let $G = S^1$, so $L = \mathbf{R}$. Let $S = \{1\}$. Then S has the accessibility property, but the strong accessibility condition fails. Since S consists of a single element, the map $A : \Omega \to L$ is unique. It is easy to see that T^A is not minimal, since

$$T_t^A(e^{i\theta_0}, e^{i\theta_1}, e^{i\theta_2}) = \left(e^{i(\theta_0+t)}, e^{i(\theta_1+t)}, e^{i(\theta_2+\alpha t)}\right),$$

so every point (z_0, z_1, z_2) in the T^A-orbit of $(1, 1, 1)$ satisfies $z_0 = z_1$, showing that the orbit is not dense. $\square$

Example. For the Rabi oscillator, $G = SU(2,\mathbf{C})$ and $L = su(2,\mathbf{C})$, as explained before, and the set S is given by

$$S = \left\{ \begin{pmatrix} -i\lambda & z \\ -\overline{z} & i\lambda \end{pmatrix} : z \in \mathbf{C} \right\}.$$

Then S is closed convex and $S - S$ contains the matrices $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which generate L as a Lie algebra. Then $L_0(S) = L$, and the conditions of Theorem 1.1 are satisfied. □

To prove Theorem 1.1, we develop a new technique to construct cocycles with desired dynamical properties in "thin" classes. Our technique involves a blend of methods of Dynamical Systems and Control Theory. The main idea is to think of a perturbation needed to get the desired dynamical properties for the cocycle as a control function that steers the state of a certain control process to the desired target. We shall prove an accessibility result (Proposition 2.1) for a system of non-autonomous right-invariant vector fields on a Lie group. (See [11], [17]. Results of this type are of independent interest in Control Theory.)

In Theorem 1.1 minimality of the skew product can be replaced by topological weak-mixing (see [14] for the needed modifications). Furthermore, using arguments of [13] one can also show the generic absence of almost periodic solutions to the differential equation for any $\omega \in \Omega$. These results can be obtained by minor modifications of our technique and are left to the reader.

We would like to conjecture that in fact for a generic A the above flow is ergodic. However at this point there are non-trivial technical difficulties in proving ergodicity. In the example of the Rabi oscillator, the stability properties of quantum evolution are studied through the spectral properties of the one-parameter unitary group $V = \{V_t\}_{t\in\mathbf{R}}$ defined on $L^2(\Omega, \mathbf{C}^2, \mu)$ by

$$V_t f(\omega) = X_A(\omega, t)^{-1} f(T_t(\omega)), \tag{1.5}$$

where μ is a given invariant Borel probability measure on Ω. The above conjecture would imply that generically these operators have only purely continuous spectrum.

§2. Proof of Theorem (1.1)

Fix a point $(e_G, \omega_0) \in G \times \Omega$, where $\omega_0 \in \Omega$ and e_G is the identity element of G. Given nonempty open sets $U_1 \subseteq G$ and $U_2 \subseteq \Omega$ define the set:

$$E(U_1, U_2) = \{A \in C(\Omega, S)\colon \text{the orbit of } (e_G, \omega_0) \text{ under } T^A \text{ intersects } U_1 \times U_2\}.$$

We will show that $E(U_1, U_2)$ is open and dense in $C(\Omega, S)$. This will imply our desired conclusion. Indeed, by varying U_1 and U_2 over a countable base of the topology of Ω and considering intersections of the corresponding sets $E(U_1, U_2)$, it will follow that the set $\{A\colon \text{the orbit of } (e_G, \omega_0) \text{ under the flow } T^A \text{ is dense in } \Omega\}$ is residual. Since $G \times \Omega$ is a compact group extension of a minimal flow, existence of one dense orbit implies minimality [7], so $C_{min}(\Omega, S)$ is residual as stated.

Openness of $E(U_1, U_2)$ follows easily from the continuous dependence of solutions of $(1.2.\omega)$ with respect to A. We now proceed to prove the density.

Let $A_0 \in C(\Omega, S)$ and $\varepsilon > 0$ be given. From now on, we choose an inner product $\langle \cdot, \cdot \rangle$ on L, and let $||\cdot||$ be the corresponding norm. We want to construct $A \in C(\Omega, S)$ that is ε-close (in the chosen norm) to A_0 and such that the orbit of (e_G, ω_0) under T^A intersects $U_1 \times U_2$. We now sketch this construction.

The minimality of T enables us to find arbitrarily large $r > 0$ such that $T_r(\omega_0) \in U_2$. Naturally, the corresponding points $X_{A_0}(\omega_0, r)$ need not belong to U_1. However, we will show that if r is large enough then a suitably chosen S-valued ε-perturbation A of A_0 will satisfy $X_A(\omega_0, r) \in U_1$. The property that $X_A(\omega_0, r) \in U_1$ only depends on the values of A on the set

$$K(r) = \{T_t(\omega_0)\colon 0 \le t \le r\},$$

which is homeomorphic to the interval $[0, r]$ via the map $t \to T_t(\omega_0)$, due to the aperiodicty of T. So all we need is to construct the desired perturbation A on $K(r)$, and then extend it to all of Ω using Tietze's Extension Theorem. (A simple argument will show that the extension can be kept S-valued and ε-close to A_0 if these properties hold on $K(r)$.) We therefore have to show that if r is large enough, and $\gamma_{0,r}$ denotes the restriction to $[0, r]$ of the curve $\gamma_0\colon \mathbf{R} \to S$ given by $\gamma_0(t) = A_0(T_t(\omega_0))$, then $\gamma_{0,r}$ can be ε-perturbed in the space $C([0, r], S)$ of continuous S-valued functions on $[0, r]$ so that for the resulting curve γ, if we solve the Cauchy problem $g'(t) = \gamma(t)g(t)$, $g(0) = e_G$, then the solution will satisfy $g(r) \in U_1$. Now, an equation such as

$$g' = \gamma(t)g\,, \tag{2.1}$$

in which γ is an arbitrary curve with values in some subset S of the Lie algebra L of a Lie group G, is known in Control Theory (cf., e.g., [11]), as a **right-invariant control system on a Lie group**. The curves γ are the **controls**, and each control gives rise to a **trajectory** for each initial condition $g(0)$. If $\bar{g} \in G$, then the set of all points of the form $g(r)$, for all trajectories $g(\cdot)$ —corresponding to all possible controls— that satisfy $g(0) = \bar{g}$, is known as the **time r reachable set** from $\bar{g}$. More generally, one can consider instead of a fixed "control constraint" such as $\gamma(t) \in S$, a time-varying constraint of the form $\gamma(t) \in S(t)$, where the set $S(t)$ depends on time. In particular, our problem can now be stated as follows: given a compact connected Lie group G, a subset S of the Lie algebra L of G, an $\varepsilon > 0$, and a curve $\gamma_0\colon \mathbf{R} \to S$, we want to show —under suitable hypotheses on S— that for sufficiently large r the time r reachable set from e_G for the control system $g' = \gamma(t)g$, with time-varying constraint $\gamma(t) \in S$, $||\gamma(t) - \gamma_0(t)|| < \varepsilon$, is the whole group G. To achieve this, we first pick an arbitrary $\tau > 0$ —which in our proof below will be taken equal to 1— and show that the time τ reachable set *by means of piecewise continuous controls* has nonempty interior. This will turn out to be true for an arbitrary curve γ_0, so in particular we can apply it to all the translates $t \to \gamma_0(t + k\tau) = \gamma_0^k(t)$, for $k = 0, 1, 2, \ldots$. In each case, we get a nonempty open subset V_k of G such that every point g_k of V_k can be reached from e_G in time τ by means of a trajectory of $g' = \gamma(t)g$ corresponding to a piecewise continuous —g_k-dependent— control $\gamma^k: [0, \tau] \to S$ that is ε-close to γ_0^k on $[0, \tau]$. If we translate the γ^k's back to the intervals $[k\tau, (k+1)\tau]$, and concatenate them, we get, for each $m \in \mathbf{N}$, a piecewise continuous control $\hat{\gamma}^m: [0, m\tau] \to S$ that is ε-close to γ_0 and gives rise to a trajectory going from e_G to $g_m g_{m-1} \ldots g_1$ in time $m\tau$. Now suppose that for some m we could guarantee that

$$V_m V_{m-1} \ldots V_1 = G\,. \tag{2.2}$$

In that case we can arbitrarily specify an element h of G, and then choose $\gamma = \hat{\gamma}^m$ so that the solution of (2.1) that goes through e_G at time 0 will satisfy $g(m\tau) = h$. If we then pick $r > m\tau$ such that $T_r(\omega_0) \in U_2$, and then choose h such that $X_{A_0}(T_{m\tau}(\omega_0), r - m\tau)h \in U_1$, and extend γ to $[0, r]$ by letting it equal γ_0 on $(m\tau, r]$, then this γ will be our desired perturbation, except only for the minor detail that γ is only piecewise continuous, and we need it to be continuous. This last point, however, is easily taken care of: one can approximate γ by continuous S-valued maps γ^ν that are ε-close to γ_0, in such a way that the corresponding trajectories g^ν converge uniformly to $g(\cdot)$. (Naturally, this approximation is not possible in the uniform topology, but it is possible, for example, in L^1 norm, and this suffices to get uniform convergence of the solutions.) Then $g^\nu(r)$ will be in U_1 for large enough ν, and the function γ^ν will be our desired perturbation.

The crucial question is therefore whether the V_k can be chosen so that (2.2) holds for some m. Notice that each V_k corresponds to a different control γ_0^k. (More precisely, V_k depends on the restriction $\bar{\gamma}_0^k$ of γ_0^k to $[0, \tau]$.) If we could guarantee that every V_k contains a translate of some fixed neighborhood W of e_G, then the validity of (2.2) for some m would be a consequence of the fact that a compact connected topological group has the following Property A^*:

Definition 2.1. A topological group G has **Property** A^* if given any neighborhood V of the identity there exists a $m \in \mathbf{N}$ (depending on V) such that, if $W_1, W_2, \ldots, W_m$ are any right translates of V, then $W_m W_{m-1} \cdots W_1 = G$. □

The proof that G has Property A^* is given at the end of the paper. In view of this property, our conclusion will follow if we show that the V_k's can be chosen in a uniform way, in the sense that each contains a translate of some fixed neighborhood W of e_G.

The crucial point is that, although the neighborhoods V_k depend on the curves $\bar{\gamma}_0^k \in C([0, \tau], S)$, the desired uniformity can be achieved *because the set of curves*

$\bar{\gamma}_0^k$ *that interest us is precompact.* The precompactness follows easily from the fact that the curves $t \to A_0(T_t(\omega))$, $0 \leq t \leq \tau$, form a compact set as ω varies over Ω. The fact that this implies the existence of W is the content of our main accessibility result, Proposition 2.1 below.

We now introduce some notations so as to be able to state and prove Proposition 2.1. For a bounded or unbounded closed interval I, let $C^{pc}(I,S)$ denote the set of all piecewise continuous maps $A: I \to S$, so $A \in C^{pc}(I,S)$ if A is an S-valued map on I that is continuous at all points of I except for a finite set of jump discontinuities. If $0 \in I$, $A \in C^{pc}(I,S)$, let Γ_A denote the unique G-valued absolutely continuous solution curve of the initial value problem

$$g' = A(t)g(t)\,, \quad t \in I\,,$$
$$g(0) = e_G.$$

If $A \in C^{pc}(I,S)$ and $\delta > 0$, define the set $N_S(A,\delta)$ as follows:

$$N_S(A,\delta) = \{B \in C^{pc}(I,S): \sup_{t\in I} ||A(t) - B(t)|| < \delta\}.$$

Proposition 2.1. *Let G be a Lie group with Lie algebra L. Let $S \subseteq L$ be as in Theorem 1.1. Let $\tau > 0$, and let $\mathcal{F} \subseteq C([0,\tau],S)$ be a compact subset. Then given $\delta > 0$ there exist a neighborhood W of the identity e_G of G, depending on δ but independent of $A \in \mathcal{F}$, such that for each $A \in \mathcal{F}$ the set $\{\Gamma_B(\tau): B \in N_S(A,\delta)\}$ contains some right translate of W.*

Before proving Proposition 2.1, we show how it allows us to prove Theorem 1.1 by following the strategy outlined above.

Recall that $A_0 \in C(\Omega,S)$ and $\varepsilon > 0$ are given and we want to find $A \in C(\Omega,S)$ that is ε-close to A_0 and such that the orbit of (e_G,ω_0) under T_t^A intersects $U_1 \times U_2$. We now list the steps in the construction of A in complete detail.

Step 1. Let $\delta = \frac{\varepsilon}{2}$.

Step 2. Apply Proposition 2.1 with $\tau = 1$, $\mathcal{F} = \{(A_0)_\omega: \omega \in \Omega\} \subseteq C([0,1],S)$, where $(A_0)_\omega(t) = A_0(T_t(\omega))$, $t \in [0,1]$. We then get a neighborhood W of e_G, depending only on δ, such that for each $\omega \in \Omega$ there exists some $h(\omega) \in G$ satisfying

$$Wh(\omega) \subseteq \{\Gamma_B(1): B \in N_S((A_0)_\omega,\delta)\}. \tag{2.3}$$

Step 3. Corresponding to W, pick $m \in \mathbf{N}$ according to Property A^*.

Step 4. Set $\xi_j = T_j(\omega_0)$, $0 \leq j \leq m$. Consider $A_j \in \mathcal{F}$ defined by

$$A_j(t) = A_0(T_t(\xi_{j-1})), \qquad t \in [0,1],\ 1 \leq j \leq m.$$

Letting $h_j = h(\xi_{j-1})$, $1 \leq j \leq m$, (2.3) yields;

$$Wh_j \subseteq \{\Gamma_B(1): B \in N_S(A_j,\delta)\}. \tag{2.4}$$

Step 5. By Property A^* we have,

$$Wh_mWh_{m-1}\cdots Wh_1 = G. \tag{2.5}$$

Step 6. Using the minimality of the base flow, select $r > m$ such that $T_r(\omega_0) \in U_2$.

Step 7. Pick $g \in G$ such that

$$X_{A_0}(T_m(\omega_0), r-m)g \in U_1\,. \tag{2.6}$$

Step 8. Using (2.5), write

$$g = g_m g_{m-1} \cdots g_1\,,\ \ g_j \in Wh_j \text{ for } j=1,\cdots,m\,. \tag{2.7}$$

Step 9. Using (2.4), select $\hat{A}_j \in N_S(A_j,\delta)$ such that $g_j = \Gamma_{\hat{A}_j}(1)$.

Step 10. Define

$$\tilde{A}(t) = \hat{A}_j(t-(j-1)) \text{ if } j-1 \le t < j,\ 1 \le j \le m\,,$$

$$\tilde{A}(t) \equiv A_0(T_t(\omega_0)) \text{ if } t \in \mathbf{R}\backslash[0,m).$$

Then $\tilde{A} \in C^{pc}(\mathbf{R},S)$, $\Gamma_{\tilde{A}}(r) \in U_1$, and $||\tilde{A}(t) - A_0(T_t(\omega_0))|| < \delta$ for all $t \in \mathbf{R}$.

Step 11. Let $\{\tilde{A}^\nu\}$ be a sequence of continuous S-valued functions on $\mathbf{R}$ that converges to $\tilde{A}$ in L^1 norm on each compact interval, and satisfies

$$||\tilde{A}^\nu(t) - A_0(T_t(\omega_0))|| < 2\delta \text{ for all } t \in \mathbf{R}\,. \tag{2.8}$$

(To see that such a sequence exists, define $\tilde{A}^\nu(t) = \nu\int_t^{t+\frac{1}{\nu}} \tilde{A}(s)ds$. Then $\tilde{A}^\nu$ is S-valued, because S is closed convex and $\tilde{A}$ is S-valued. If we define

$$A_0^\nu(t) = \nu \int_t^{t+\frac{1}{\nu}} A_0(T_s(\omega_0))ds,$$

then it is clear that $A_0^\nu(t) \to A_0(T_t(\omega_0))ds$ uniformly as $\nu \to \infty$, because the function $t \to A_0(T_t(\omega_0))ds$ is uniformly continuous. Since $||\tilde{A}^\nu(t) - A_0^\nu(t)|| \le \delta$ for each t, we may assume —by taking ν sufficiently large— that (2.8) holds. Finally, it is clear that $\tilde{A}^\nu(t) \to \tilde{A}(t)$ as $\nu \to \infty$ whenever t is a point of continuity of $\tilde{A}$. Since the $\tilde{A}^\nu$ are clearly uniformly bounded, the L^1 convergence on compact intervals follows trivially.)

Step 12. It follows trivially from Gronwall's inequality that $\Gamma_{\tilde{A}^\nu}(r) \to \Gamma_{\tilde{A}}(r)$ as $\nu \to \infty$. Since $\Gamma_{\tilde{A}}(r) \in U_1$ and U_1 is open, we conclude that $\Gamma_{\tilde{A}^\nu}(r) \in U_1$ for some ν. Pick a ν for which this is true, and let $A^*\colon [0,r] \to S$ be the restriction of $\tilde{A}^\nu$ to $[0,r]$. Then A^* is continuous and S-valued on $[0,r]$, satisfies $\Gamma_{A^*}(r) \in U_1$, and is ε-close to $t \to A_0(T_t(\omega_0))$ in supremum norm on $[0,r]$.

Step 13. Using the fact that the flow T is aperiodic, we may identify the interval $[0, r]$ with the compact subset $K = \{T_t(\omega_0): t \in [0, r]\}$ of Ω. So we may regard A^* as defined on K. It is then easy to see that A^* has a continuous extension $A: \Omega \to L$ that satisfies

$$A(\omega) \in S \text{ for all } \omega \in \Omega\,, \tag{2.9.i}$$

$$||A(\omega) - A_0(\omega)|| < \varepsilon, \text{ for all } \omega \in \Omega\,. \tag{2.9.ii}$$

Indeed, the Tietze Extension Theorem, applied to the map $A^* - A_0$, which is defined on K and takes values in the open ball $\mathbf{B}(\varepsilon) = \{x \in L: ||x|| < \varepsilon\}$, yields a $\mathbf{B}(\varepsilon)$-valued extension θ of $A^* - A_0$ to all of Ω, and if we let $A = A_0 + \theta$ we get an extension of A^* to Ω that satisfies (2.9.ii). If A is not S valued, then we can modify A by composing with π_S, the projection map from L to S. (Recall that, if $x \in L$, then $\pi_S(x)$ is, by definition, the point of S closest to x. Since the norm $||\cdot||$ arises from an inner product, π_S is well defined and continuous.) Since A_0 is S-valued, it is clear that $\pi_S \circ A$ is pointwise closer to A_0 than A is, so (2.9.ii) remains true after this modification. (Let $\omega \in \Omega$, and write $x = A(\omega)$, $y = \pi_S(x)$, $z = A_0(\omega)$. Let $z_s = y + s(z - y)$ for $s \in \mathbf{R}$. Then $||x - z_s||^2 = ||x - y||^2 + s^2||y - z||^2 + 2s\langle x - y, y - z\rangle$. If $\langle x - y, y - z\rangle < 0$, it would follow that $||x - z_s||^2 < ||x - y||^2$ for small positive s. Since $z_s \in S$ for $0 \le s \le 1$, this would contradict the fact that $y = \pi_S(x)$. So $\langle x - y, y - z\rangle \ge 0$. But then, setting $s = 1$, we get $||x - z||^2 \ge ||y - z||^2$, so y is closer to z than x is, as desired.)

Notice that by construction $X_A(\omega_0, r) = \Gamma_A(r) \in U_1$. Then

$$T_r^A(e_G, \omega_0) = (X_A(\omega_0, r), T_r(\omega_0)) \in U_1 \times U_2\,.$$

So the orbit of (e_G, ω_0) under T^A intersects $U_1 \times U_2$, and the proof of Theorem 1.1 is complete, modulo Proposition 2.1. □

We now turn to the proof of Proposition 2.1. We will use a well known accessibility result from Control Theory, whose short proof will be given in full. The result says that, given a generating subset S of the Lie algebra L, the set of points of the Lie group G that can be reached from the identity by a finite polygonal path consisting of integral curves of vector fields in S has nonempty interior. To state it precisely, we first introduce some notation.

If M is a smooth manifold and $x \in M$, we use T_xM to denote the tangent space of M at x. We recall that if G is a Lie group then the Lie algebra of G is, by definition, the tangent space $L = T_{e_G}G$, and we agree to identify L with the space of right-invariant vector fields on G. With this identification, the exponential map $L \ni X \to e^X \in G$ satisfies $\frac{d}{dt}(e^{tX}) = Xe^{tX}$.

Given $k \in \mathbf{N}$ and $X = (X_1, \cdots, X_k) \in S^k \equiv S \times \cdots \times S$, define a map $E_X: \mathbf{R}^k \to G$ by

$$E_X(\bar{t}) \equiv E_X(t_1, \cdots, t_k) = e^{t_1X_1} \cdots e^{t_kX_k}.$$

Let $(D(E_X))_{\bar{t}}: \mathbf{R}^k \to T_{E_X(\bar{t})}(G) \equiv L$ be the derivative of E_X at $\bar{t}$. Let

$$\rho(\bar{t}) = \mathrm{rank}(D(E_X))_{\bar{t}} \quad \text{and} \quad \rho_X = \mathrm{Max}\{\rho_X(\bar{t}) : \bar{t} \in \mathbf{R}^k\}.$$

Lemma 2.1. *Let G be an m-dimensional Lie group with Lie algebra L, and let S be a subset of L that has the accessibility property (i.e. S generates L as a Lie algebra). Then there exists an m-tuple $X = (X_1, \cdots, X_m) \in S^m$ such that for every $\eta > 0$ the set $\{e^{s_1 X_1} \cdots e^{s_m X_m} : 0 < s_j, \sum_{j=1}^{m} s_j < \eta\}$ has nonempty interior.*

Proof. We first prove by induction on k that

(2.1.I) *for $1 \leq k \leq m$ there exists $X \in S^k$ such that $\rho_X = k$.*

This is trivially true for $k = 1$. Assume that it is true for a $k \in \{1, \ldots, m-1\}$. Let $X \in S^k$ be such that $\rho_X = k$, and find $\bar{t} = (t_1, \cdots, t_k) \in \mathbf{R}^k$ such that $\rho_X(\bar{t}) = k$. The implicit function theorem implies that there exists an $\varepsilon > 0$ such that E_X maps a cubic $\varepsilon-$neighborhood $V(\bar{t}, \varepsilon)$ of $\bar{t}$ diffeomorphically onto a k-dimensional submanifold Σ of G. Recall that if two vector fields are tangent to Σ then so is their Lie bracket. Thus, since S generates L, there must exist $Y \in S$ and $g \in G$ such that $Yg \notin T_g\Sigma$. (Otherwise, every $X \in L$ would be tangent to Σ, contradicting the fact that $k < m$.) Let $g = E_X(\tilde{t})$ for some $\tilde{t} = (\tilde{t_1}, \cdots, \tilde{t_k}) \in V(\bar{t}, \varepsilon)$. Let $\hat{X} = (Y, X_1, \cdots, X_k) \in S^{k+1}$. Then

$$E_{\hat{X}}(\tau, s_1, \cdots, s_k) = e^{\tau Y} e^{s_1 X_1} \cdots e^{s_k X_k}, \quad (s_1, \cdots, s_k) \in \mathbf{R}^k, \ \tau \in \mathbf{R}.$$

Since $\rho_X(\bar{t}) = k$, it is clear that $\rho_{\hat{X}}(0, \bar{t}) = k + 1$. This completes the proof of (2.1.I).

If we now apply (2.1.I) with $k = m$, we find a $\bar{t} = (t_1, \cdots, t_m) \in \mathbf{R}^m$ and $X \in S^m$ such that $\rho_X(\bar{t}) = m$. Since the map E_X is real-analytic, the set U of points $\bar{t}$ for which $\rho_X(\bar{t}) = m$ is dense in $\mathbf{R}^k$. In particular, given any $\eta > 0$, U contains points $\bar{t} = (t_1, \cdots, t_m)$ such that $0 < t_j$ for $j = 1, \ldots, m$ and $\sum_{j=1}^{m} t_j < \eta$. The desired result then follows. $\square$

To prove Proposition 2.1, we need to sharpen the above lemma, and show that in fact the set $\{e^{s_1 X_1} \cdots e^{s_m X_m} : 0 < s_j, \sum_{j=1}^{m} s_j = \eta\}$ has nonempty interior for all $\eta > 0$. This is where the strong accessibility condition on S will be decisive. To discuss this in detail, we need to introduce some more notations.

First, let us write $L^*(S)$ to denote the linear span of all brackets

$$v = [v_1, [v_2, \ldots, [v_{k-1}, v_k] \ldots]], \quad v_j \in S, \quad k \geq 2.$$

Let $\Lambda(S)$ be the linear span of S, and use $\Lambda_0(S)$ to denote the linear span of all the differences $v - w$, for $v, w \in S$. Then $\Lambda_0(S)$ can also be characterized as the set of all finite linear combinations $\sum_j \lambda_j v_j$, $v_j \in S$, $\sum_j \lambda_j = 0$. It is then clear that $L_0(S) = L^*(S) + \Lambda_0(S)$ and $L(S) = L^*(S) + \Lambda(S)$. (The first equality follows from the facts that

(1) $L^*(S) \subseteq L_0(S)$, which is true because, whenever a bracket v is of the form $[v_1,[v_2,\ldots,[v_{k-1},v_k]\ldots]]$, then $v = [v_1,[v_2,\ldots,[v_{k-1},v_k - v_{k-1}]\ldots]]$,
(2) $S - S \subseteq \Lambda_0(S) \subseteq L_0(S)$,
(3) $L^*(S) + \Lambda_0(S)$ is an ideal of $L(S)$.

The second equality is trivial.) This implies in particular that if $L_0(S) \neq L(S)$ then $\dim(L(S)) = 1 + \dim(L_0(S))$. We summarize these observations as follows:

(I) *If $v \in L$, then $v \in L_0(S)$ if and only if v can be expressed as*

$$v = v_0 + \sum_{j=1}^{k} \lambda_j v_j, \quad \sum_{j=1}^{k} \lambda_j = 0, \quad v_0 \in L^*(S), \quad v_j \in S \text{ for } j > 0. \tag{2.10}$$

(II) *Either $L(S) = L_0(S)$ or $L_0(S)$ has codimension one in $L(S)$.*

Now let $\tilde{G} = G \times \mathbf{R}$ and $\hat{S} = \{(v,1): v \in S\}$. Let $\tilde{L}$ be the Lie algebra of $\tilde{G}$. Let $L(\hat{S})$ be the Lie subalgebra of $\tilde{L}$ generated by $\hat{S}$.

Lemma 2.2. *With the above notations, $L_0(S) = L$ if and only if $L(\hat{S}) = \tilde{L}$.*

Proof. For each $v \in L$, let $\tilde{v} = (v,0) \in \tilde{L}$, and write $\hat{v} = (v,1)$. Then $v \to \tilde{v}$ is a Lie algebra homomorphism. In particular, if $v \in L^*(S)$ then $\tilde{v} \in L^*(\hat{S})$ because, if $v = [v_1,[v_2,\ldots,[v_{k-1},v_k]\ldots]]$, then

$$\tilde{v} = [\tilde{v}_1,[\tilde{v}_2,\ldots,[\tilde{v}_{k-1},\tilde{v}_k]\ldots]] = [\hat{v}_1,[\hat{v}_2,\ldots,[\hat{v}_{k-1},\hat{v}_k]\ldots]],$$

since $k \geq 2$.

Suppose $L_0(S) = L$. Since $\{\tilde{v}: v \in L\}$ and $(0,1)$ span $\tilde{L}$, it is enough to show that these vectors belong to $L(\hat{S})$. Let $v \in L$. Since $L_0(S) = L$,

$$v = v_0 + \sum_{j=1}^{k} \lambda_j v_j, \quad \sum_{j=1}^{k} \lambda_j = 0,$$

where the v_j are as in (2.10). But then $\tilde{v} = \tilde{v}_0 + \sum_{j=1}^{k} \lambda_j \tilde{v}_j$, and $\sum_{j=1}^{k} \lambda_j \tilde{v}_j = \sum_{j=1}^{k} \lambda_j \hat{v}_j$, since $\sum_{j=1}^{k} \lambda_j = 0$. Since we know that $\tilde{v}_0 \in L^*(\hat{S})$, we conclude that $\tilde{v} \in L(\hat{S})$. Therefore $\{\tilde{v}: v \in L\} \subseteq L(\hat{S})$. Now pick any $v \in S$. Then $(0,1) = \hat{v} - \tilde{v}$. Since $\hat{v} \in \hat{S}$ and $\tilde{v} \in L(\hat{S})$, $(0,1)$ belongs to $L(\hat{S})$ as well. So $L(\hat{S}) = \tilde{L}$.

Conversely, suppose that $\tilde{L} = L(\hat{S})$. Then every $w \in \tilde{L}$ can be written as

$$w = w_0 + \sum_{j=1}^{k} \lambda_j w_j, \tag{2.11}$$

where $w_0 \in L^*(\hat{S})$ and w_j belong to $\hat{S}$ for $1 \leq j \leq k$. In particular, given any $v \in L$, we can take $w = \tilde{v}$, and express it in the form (2.11). Equating second components, we conclude that $\sum_{j=1}^{k} \lambda_j = 0$. Now, it is easy to see that $w_0 = \tilde{v}_0$ for some $v_0 \in L^*(S)$. If we write $w_j = \hat{v}_j$ for $j = 1, \ldots, k$, we see that $v = v_0 + \sum_{j=1}^{k} \lambda_j v_j$ with $\sum_{j=1}^{k} \lambda_j = 0$, $v_0 \in L^*(S)$, and $v_j \in S$ for $1 \leq j \leq k$. So $v \in L_0(S)$. This shows that $L = L_0(S)$, concluding our proof. $\square$

We are now ready to prove the desired sharper version of Lemma 2.1.

Lemma 2.3. *Let G be an m-dimensional Lie group with Lie algebra L, and let S be a subset of L that has the strong accessibility property. Then there exists an m-tuple $X = (X_1, \cdots, X_m) \in S^m$ such that for every $\eta > 0$ the set*

$$\{e^{s_1 X_1} \cdots e^{s_m X_m} : 0 < s_j,\ \sum_{j=1}^{m} s_j \ = \ \eta\}$$

has nonempty interior.

Proof. Apply Lemma 2.1 to the Lie group $\tilde{G} = G \times \mathbf{R}$, the Lie algebra $\tilde{L} = L \times \mathbf{R}$ of $\tilde{G}$, and the subset $\hat{S} = \{(v,1) : v \in S\}$ of $\tilde{L}$. Lemma 2.2 ensures that the hypothesis of Lemma 2.1 is satisfied. Thus we get $\hat{X}_j = (X_j, 1) \in \hat{S}$, $1 \leq j \leq m$ such that the set $\{e^{s_1 \hat{X}_1} \cdots e^{s_m \hat{X}_m} : 0 < s_j,\ \sum_{j=1}^{m} s_j \ < \eta\}$ contains a nonempty open subset $\tilde{U}$ of $\tilde{G}$. Fix any point $q \in \tilde{U}$. Write $q = (p, r)$, $p \in G$, $r \in \mathbf{R}$. Let $U = \{g \in G : (g, r) \in \tilde{U}\}$. Then U is a nonempty open subset of G. If $g \in U$, then $(g, r) = e^{s_1 \hat{X}_1} \cdots e^{s_m \hat{X}_m}$ for some $s_1, \ldots, s_m$ such that $0 < s_j$ and $\sum_j s_j < \eta$. Since

$$e^{s_1 \hat{X}_1} \cdots e^{s_m \hat{X}_m} = \left(e^{s_1 X_1} \cdots e^{s_m X_m},\ \sum_{j=1}^{m} s_j\right),$$

we have in particular $\sum_{j=1}^{m} s_j = r$. So every $g \in U$ is of the form $e^{s_1 X_1} \cdots e^{s_m X_m}$ with $0 < s_j$, $\sum_j s_j < \eta$, and $\sum_j s_j = r$. It then follows that $r < \eta$. Now let $U^* = \{g e^{(\eta - r) X_m} : g \in U\}$. Then U^* is a nonempty open subset of G, and every member of U^* is of the form $e^{s_1 X_1} \cdots e^{(s_m + (\eta - r)) X_m}$ with $0 < s_j$, $\sum_{j=1}^{m} s_j = r$, i.e. of the form $e^{s_1 X_1} \cdots e^{s_m X_m}$ with $0 < s_j$, $\sum_{j=1}^{m} s_j = \eta$. This proves the lemma. $\square$

Proof of Proposition 2.1. Let $m = \dim(G)$. Recall that we are given a compact set $\mathcal{F} \subseteq C([0,\tau], S)$, $\tau > 0$, and a $\delta > 0$. Let $K = \{A(t) : t \in [0,\tau],\ A \in \mathcal{F}\}$. Then $K \subseteq S$ is compact. Let $\{U_j : 1 \leq j \leq q\}$ be a cover of K by relatively open subsets of S with diameter less than δ. Using a Lebesgue number for this cover and the fact that $\mathcal{F}$ is compact and hence equicontinuous, we can find p such that $0 < p < \tau$ with the property that for each $A \in \mathcal{F}$ there is a $j = j(A)$ such that

$$\{A(t): 0 \leq t \leq p\} \subseteq U_j. \tag{2.12}$$

We now let Σ denote the set of all m-tuples $s = (s_1, \ldots, s_m) \in \mathbf{R}^m$ such that $s_k > 0$ for $k = 1, \ldots, m$ and $\sum_k s_k = p$. If $X = (X_1, \ldots, X_m) \in L^m$ and $s = (s_1, \ldots, s_m) \in \mathbf{R}^m$, write $e^{sX} = e^{s_m X_m} \cdots e^{s_1 X_1}$.

Since is S convex, it is easy to verify that every nonempty relatively open subset of S also has the strong accessibility property. Then, applying Lemma 2.3 with U_j in the role of S, we get, for every $j \in \{1, \ldots, q\}$, an m-tuple

$$X^j = (X_1^j, \cdots, X_m^j) \in U_j \times \cdots \times U_j$$

such that the set $Q_j = \{e^{sX^j}: s \in \Sigma\}$ contains a translate $W_j h_j$ of some neighborhood W_j of the identity. Let

$$\tilde{W} = \cap_{j=1}^q W_j\,, \quad W = \cap_{g \in G} g \tilde{W} g^{-1}\,.$$

Then W is a neighborhood of e_G, since G is compact. Moreover, $Wg = gW$ for all $g \in G$. Clearly, $Wh_j \subseteq Q_j$ for every j. (Notice that W depends only on p, which in turn depends only on ε and not on individual $A \in \mathcal{F}$.) Now, given any $s \in \Sigma$, $1 \leq j \leq q$, define $H_{s,j}: [0,p] \to S$ to be the piecewise constant map whose value on the interval $[s_1 + \ldots + s_{k-1}, s_1 + \ldots + s_k)$ is X_k^j. Given $A \in \mathcal{F}$, $s \in \Sigma$, define $B_{s,A}: [0,\tau] \to S$ by letting $B_{s,A}(t) = H_{s,j(A)}(t)$ for $0 \leq t < p$, $B_{s,A}(t) = A(t)$ for $p \leq t \leq \tau$.

Then, if $A \in \mathcal{F}$, it is clear that $B_{s,A} \in N_S(A, \delta)$ for every $s \in \Sigma$. Moreover,

$$W h_{j(A)} \subseteq Q_{j(A)} = \{\Gamma_{H_{s,j(A)}}(p): s \in \Sigma\}\,,$$

and then

$$W \tilde{h}_A \subseteq \{\Gamma_{B_{s,A}}(\tau): s \in \Sigma\} \subseteq \{\Gamma_B(\tau): B \in N_S(A, \delta)\}\,,$$

where $\tilde{h}_A = \Gamma_A(\tau)\Gamma_A(p)^{-1} h_{j(A)}$, and we use the fact that

$$\Gamma_A(\tau)\Gamma_A(p)^{-1} W h_{j(A)} = W \Gamma_A(\tau)\Gamma_A(p)^{-1} h_{j(A)} = W \tilde{h}_A\,.$$

This shows that W has the desired property. $\square$

Proof that a compact connected group has property A^*. Let V be a given neighborhood of e_G. Let $U = \cap\{gVg^{-1}: g \in G\}$. Then U is a neighborhood of e_G as well, $U \subseteq V$, and $gU = Ug$ for all $g \in G$. Since G is compact connected and U is a neighborhood of e_G, $G = U^m$ for some $m \in \mathbf{N}$. Hence, given any m right translates $W_1, W_2, \cdots, W_m$ of V, say $W_j = V g_j$, we have

$$G \subseteq U^m g_m \cdots g_1 = (U g_m) \cdots (U g_1) \subseteq W_m \cdots W_1.$$

This completes the proof. $\square$

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