MATHEMATICS 300 — FALL 2002
INTRODUCTION TO MATHEMATICAL REASONING

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SOLUTIONS TO THE SECOND MIDTERM EXAM.

1. i. Define "union." **ANSWER.** See book, page 58.

ii. Define "symmmetric difference." **ANSWER.** See book, page 63. *NOTE: Several students mistakenly defined as "symmetric difference" what in reality is "set difference," which is a totally different concept.*

iii. Define "Cartesian product." **ANSWER.** See book, page 67.

iv. Define "equivalence relation." **ANSWER.** See book, page 79, for the definition of "equivalence relation," page 75 for the definition of what it means for a relation to be "reflexive," "symmetric," and "transitive," and pages 72, 73 for the definitions of "relation" and "relation on a set A."

v. Define "equivalence class." **ANSWER.** See book, page 82.

vi. Define "partition." **ANSWER.** See book, page 88.

2. (i) Prove: if A, B, C, D are sets such that $A \times B = C \times D$, then $A = C$ and $B = D$.

ANSWER. This cannot be proved because it is false. Here is a counterexample: let $A = \{1\}$, $C = \{2\}$, $B = C = \emptyset$. Then $A \times B = C \times D$, because $A \times B = \emptyset$ and $C \times D = \emptyset$. But it is not true that $A = C$.

(ii) Prove: if A, B, C, D are nonempty sets such that $A \times B = C \times D$, then $A = C$ and $B = D$.

PROOF. Let us first prove that $A = C$.
Let x be arbitrary.
Assume that $x \in A$.
Since $B \neq \emptyset$, we can pick $y \in B$.
Then $(x, y) \in A \times B$.
Therefore $(x, y) \in C \times D$, because $A \times B = C \times D$.
So $x \in C$.
Hence $x \in A \longrightarrow x \in C$.
A similar argument proves that $x \in C \longrightarrow x \in A$.
Therefore $x \in A \longleftrightarrow x \in C$.
Since x was arbitrary, we have shown that $\forall x, (x \in A \longleftrightarrow x \in C)$.
So $A = C$.
A similar argument proves that $B = D$.

3. (i) Prove: If R, S are equivalence relations on a set A, then $R \cap S$ is an equivalence relation on A.

PROOF. We show that $R \cap S$ is reflexive on A, symmetric, and transitive. To prove that $R \cap S$ is reflexive on A, we let x be an arbitrary member of A, and show that $(x, x) \in R \cap S$. Since R is reflexive on A, $(x, x) \in R$. Since S is reflexive on A, $(x, x) \in S$. Therefore $(x, x) \in R \cap S$, as desired. We now prove that $R \cap S$ is symmetric, i.e., that if $(x, y) \in R \cap S$ then $(y, x) \in R \cap S$. Let $(x, y) \in R \cap S$. Then $(x, y) \in R$ and $(x, y) \in S$. Since R is symmetric, $(y, x) \in R$. Since S is symmetric, $(y, x) \in S$. Therefore $(y, x) \in R \cap S$. Now let us show that $R \cap S$ is transitive. We have to prove that if $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$ then then $(x, z) \in R \cap S$. Let x, y, z be such that $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. Then $(x, y) \in R$, $(x, y) \in S$, $(y, z) \in R$, and $(y, z) \in S$. Since R is transitive, $(x, y) \in R$, and $(y, z) \in R$, we can conclude that $(x, z) \in R$. Since S is transitive, $(x, y) \in S$, and $(y, z) \in S$, we can conclude that $(x, z) \in S$. Therefore $(x, z) \in R \cap S$.

(ii) Prove: If R, S are equivalence relations on a set A, then $R \cup S$ is an equivalence relation on A.

ANSWER. This cannot be proved because it is false. To see this, here is a counterexample. Let $A = \mathbb{Z}$, let R be congruence modulo 2, let S be congruence modulo 3. Then the pairs $(2, 4)$ and $(4, 7)$ belong to $R \cup S$, because $(2, 4) \in R$ and $(4, 7) \in S$. But the pair $(2, 7)$ does not belong to $R \cup S$, because 2 is neither congruent to 7 modulo 2, nor congruent to 7 modulo 3. Hence $R \cup S$ is not transitive, so it is not an equivalence relation.

4. i. Let $A = \{(x, y) : x \in \mathbb{Z} \wedge y \in \mathbb{Z} \wedge y \neq 0\}$. Define a relation R on A by letting

$$R = \left\{ \Big((x, y), (x', y') \Big) : xy' = x'y \right\}.$$

Prove that R is an equivalence relation.

PROOF. We show that R is reflexive on A, symmetric, and transitive. To prove that R is reflexive on A, we let u be an arbitrary member of A, and show that $(u, u) \in R$. Since $u \in A$, u is an ordered pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $y \neq 0$. If we let $x' = x$, $y' = y$, then $xy' = x'y$, because both sides are equal to xy. Hence $\Big((x, y), (x', y')\Big) \in R$. That is, $(u, u) \in R$. So R is reflexive on A. Next, let us show that R is symmetric. Suppose $u \in A$, $u' \in A$, are such that $(u, u') \in R$. Since $u \in A$, u is an ordered pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $y \neq 0$. Since $u' \in A$, u' is an ordered pair $(x', y') \in \mathbb{Z} \times \mathbb{Z}$ such that $y' \neq 0$. Since $(u, u') \in R$, the equality $xy' = x'y$ holds. But then $x'y = xy'$, which shows that $(u'u) \in R$. Hence R is symmetric. Finally, let us prove that R is transitive. Suppose $u \in A$, $u' \in A$, $u'' \in A$ are such that $(u, u') \in R$ and $(u', u'') \in R$. Since $u \in A$, u is an ordered pair $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $y \neq 0$. Since $u' \in A$, u' is an ordered pair $(x', y') \in \mathbb{Z} \times \mathbb{Z}$ such that $y' \neq 0$. Since $(u, u') \in R$, the equality $xy' = x'y$ holds. Since $u'' \in A$, u'' is an ordered pair $(x'', y'') \in \mathbb{Z} \times \mathbb{Z}$ such that $y'' \neq 0$. Since $(u', u'') \in R$, the equality $x'y'' = x''y'$ holds. We now multiply $x'y'' = x''y'$ by y, and get $x'y''y = x''y'y$. Since $x'y = xy'$, we have $x''y'y = xy'y''$. Therefore $xy'y'' = x''y'y$. Since $y' \neq 0$, we can cancle the factor y' from both sides, and conclude that $x''y = xy''$. Therefore $(u, u'') \in R$. So R is transitive.

ii. Suppose that, instead of defining A as we did in Part i, we had defined $A = \{(x, y) : x \in \mathbb{Z} \wedge y \in \mathbb{Z}\}$. Would R still be an equivalence relation? (Do not just say "yes" or "no." Justify your answer.)

ANSWER. No, R would **not** be an equivalence relation. To see this, observe that if we take $u = (2, 3)$, $u' = (0, 0)$, and $u'' = (3, 7)$, then $(u, u') \in R$, because $2 \times 0 = 0 \times 3$, and $(u', u'') \in R$, because $0 \times 7 = 0 \times 3$, but $(u, u'') \notin R$, because $2 \times 7 \neq 3 \times 3$. So R is not transitive.

5. (i) Prove: if A, B are sets such that $A \Delta B = \emptyset$, then $A = B$.

PROOF. Let x be arbitrary.
Suppose $x \in A$. We prove $x \in B$ by contradiction.
Suppose $x \notin B$. Then $x \in A - B$. Hence $x \in (A - B) \cup (B - A)$.
So $x \in A \Delta B$. This contradicts the fact that $A \Delta B = \emptyset$.
Therefore $x \in B$.
Hence $x \in A \longrightarrow x \in B$.
A similar argument proves that $x \in B \longrightarrow x \in A$.
Therefore $x \in B \longleftrightarrow x \in A$.
Since x was arbitrary, we have shown that $\forall x, (x \in B \longleftrightarrow x \in A)$.
Hence $A = B$.

(ii) Prove: if A, B are sets such that $2^A = 2^B$, then $A = B$.

PROOF. Let x be arbitrary.
Suppose $x \in A$. Then $\{x\} \subseteq A$. So $\{x\} \in 2^A$. Since $2^A = 2^B$, we have $\{x\} \in 2^B$. This says that $\{x\} \subseteq B$. So every member of $\{x\}$ belongs to B. Hence $x \in B$, because $x \in \{x\}$.
Hence $x \in A \longrightarrow x \in B$.
A similar argument proves that $x \in B \longrightarrow x \in A$.
Therefore $x \in B \longleftrightarrow x \in A$.
Since x was arbitrary, we have shown that $\forall x, (x \in B \longleftrightarrow x \in A)$.
Hence $A = B$.

6. Let $\mathcal{P}$ be a partition of a set A. Define a relation R on A by letting

$$R = \{(x, y) : \exists X \in \mathcal{P}, (x \in X \wedge y \in X)\}.$$

Prove that R is an equivalence relation.

ANSWER. This is done in the book, page 89, Proposition 13.3.

7. A *telephone number* is a sequence of ten digits such that the leftmost digit is neither a 0 nor a 1. (The "digits" are the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.) Determine how many telephone numbers N there are such that (a) N consists of four even digits and six odd digits, (b) the even digits are all different and occur in increasing order from left to right. (For example, the number 3247533618 is O.K., but 3267533418 and 3024756318 are not.) Do **not** just write a number. Show how you get it.

ANSWER. Let us divide our numbers into two kinds. First, there are the numbers whose leftmost digit is odd. Second, there are the numbers whose leftmost digit is even. To construct a number of the first kind, we have to do the following:

1. Pick a set of four locations out of the nine other than the lefmost one, and allocate this set to even digits. This can be done in $\binom{9}{4}$ ways.
2. Pick which four even digits to use. Since there is a total of five even digits, picking a set of four amounts to picking one, namely, the one that will not be used. So this can be done in 5 ways. Once we know which four even digits we will use, and what are the four locations where we will put them, we have only one way to put them there, because they have to be in increasing order.
3. Pick which odd digits we will use for the other six slots. We have four choices for the leftmost slot (because 1 is not allowed) and five for each of the other five slots, so the total number of ways to do this is 4×5^5.

Hence the number of ways to form a number of the first kind is $\binom{9}{4} \times 5 \times 4 \times 5^5$, i.e., $\binom{9}{4} \times 4 \times 5^6$.

To construct a number of the second kind, we have to do the following:

1. Pick a set of three locations out of the nine other than the lefmost one, and allocate this set to even digits. (In addition, of course, the leftmost place is allocated to an even digit as well.) This can be done in $\binom{9}{3}$ ways.
2. Pick which four even digits to use. Since we are not allowed to use 0 in the leftmost place, the increasing sequence of four even digits has to be $2, 4, 6, 8$. So we have only one option here.
3. Pick which odd digits we will use for the other six slots. We have five choices for each slot, so the total number of ways to do this is 5^6.

Hence the number of ways to form a number of the second kind is $\binom{9}{3} \times 5^6$.

$$5^6\left(\binom{9}{3} + \binom{9}{4} \times 4\right) = 5^6\left(\frac{9!}{6! \times 3!} + \frac{4 \times 9!}{5! \times 4!}\right) = 5^6\left(\frac{9 \times 8 \times 7}{6} + \frac{4 \times 9 \times 8 \times 7 \times 6}{24}\right),$$

which is equal to $5^6\Big(3 \times 4 \times 7 + 4 \times 3 \times 7 \times 6\Big)$, i.e., to $5^6 \times 3 \times 4 \times 7^2$, that is, to $5^4 \times 3 \times 7^2 \times 100$, which equals $625 \times 3 \times 49 \times 100$, i.e., $1,875 \times 4,900$, that is, **9,187,500**.

8. State and prove the binomial theorem.

ANSWER. This is done in the book, pages 98-99, Theorem 14.8. A different proof, using induction, was given in class.