MATHEMATICS 311 — FALL 2005
Advanced Calculus

H. J. Sussmann — September 8, 2005

Updated September 22, 2005

1 Information on the course

1.1 Course schedule

Our class meets on **Tuesdays and Thursdays, 7th period (6:10pm to 7:30pm)**, and **Thursdays, 8th period (7:40pm to 9:00pm)**, in Hardenbergh Hall A4.

1.2 About the instructor

My *name* is **H.J. Sussmann.** My *office* is **Hill 538**.
My *Rutgers phone extension* is 5-5407.
My *e-mail address* is **sussmann@math.rutgers.edu**.

1.3 Web page

I will set up a *Web page* for our Math 311 section:

http://www.math.rutgers.edu/~sussmann/math311page.html

All the instructor's notes will be available there.

1.4 Office hours

My office hours will be:

- **Tuesday and Thursday, 1:00 pm to 3:00 pm** in my office, **Hill 538**;
- any other time, by appointment, in my office, Hill 538.

1.5 Homework and midterms

Homework will count for about one third of your grade. There will be *two midterms*, which will count—together—for another third of your grade. The *final exam* will count for the remaining third.

Midterm dates: Thursday, October 13 and **Thursday, December 1**.

1.6 The textbook

We will be following the book *Understanding Analysis*, by Stephen Abbott (Springer).

2 Writing mathematics & submitting homework

2.1 Write clearly in complete meaningful sentences

You should write so that you can be easily understood by a properly trained English-speaking individual. In particular, this means that you must

- Use *complete English sentences*, that make clearly identifiable *statements* with a *clear meaning* that can can be understood by anyone reading what you wrote. For example:
  - If you tell me that "she is very smart," but you haven't told me who "she" is, then I don't know who you are talking about, so you haven't made a statement with a clear meaning.
  - If you write "$x > 0$," but you haven't told me who "x" is, then I don't know what you are talking about, so you haven't made a statement.
  - If I ask you to state Pythagoras' theorem and your answer only says "$a^2 + b^2 = c^2$," then you are not specifying what you are talking about, because you have not said what "a," "b," and "c" are supposed to be.[1]
- Avoid exaggerated or incorrect use of cryptic mathematical notation.
- Explain what you are doing.
- Make sure that letter "variables" are only used if either: (i) it has been said before what these letters stand for, or (ii) they are "closed variables" (or "dummy variables," or "bound variables") in the sense that will be discussed in great detail in class, and will also be explained later in these notes.

  NOTE: Letter[2] variables function in mathematical writing very much like pronouns (he, she, they) in ordinary English. For example, the pronouns "he" and "she" stand for a person, but exactly which person this is we

[1] Here is a correct statement of Pyhtagoras' theorem: *Let T be a right triangle, let a, b be the lengths of the legs of T, and let c be the lenght of the hypothenuse. Then $a^2 + b^2 = c^2$.*

[2] By "letter" I mean any of the usual letters of the English alphabet, lower case or capital. (NOTE: Capitals and lower case letters are **different** symbols. So if in a text we use both x and X as variables, then they are not the same thing. In addition, mathematicians often use as variables Greek letters and other special symbols such as $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$. In principle, you could use any symbol (for example, $\diamondsuit$, $\Box$, $\clubsuit$) as a variable.)

don't know unless it is somehow announced in advance or clear from the context. Furthermore, the person that the pronoun stands for can **vary** from paragraph to paragraph or even within a single paragraph. (For example, "Senator Collins said she would vote in favor, but Senator Boxer said she would vote against.") So pronouns are **variables**, in the sense that they stand for a thing or person that can **vary**. Letter variables work the same way: we can say things like "if $x = 5$ then $x^2 + x + 3$ is equal to 33, but if $x = 6$ then $x^2 + x + 3 = 45$." Notice that each time we make an assertion about x (for example, "$x^2 + x + 3$ is equal to 33," or "$x^2 + x + 3 = 45$") a value for x is **declared** before the assertion is made. A declaration can be overruled by declaring a different value for x. (In our example, x is declared to be 5, so "x" stands for "5" until we make a new declaration by letting x equal 6.) A particular declaration can have a very long **scope** (i.e., it can be valid for a long text following it). And, as long as a particular declaration holds, the letter x stands for a name of a particular thing, which is **fixed**, i.e., **constant**, so x is a **constant**. In other words, when we use a variable, and declare a particular value for it, then we "treat the variable as a constant." Some symbols such as 0, 1, 2, are assigned a fixed value once and for all for ever, and this can happen even to letter variables. (For example,π, e.) We will have a lot more to say about variables and constants later on.

- Provide proper connectives between equations as well as between ideas.
- Make sure that all the rules of English grammar (including those of spelling and punctuation) are strictly obeyed.
- Try to say things correctly but *in your own words.* Please *no rote learning.*
- Please proofread carefully what you hand in. Ideally, you should read and reread and revise almost any formal communication. **Neatness and clarity count**, as you well know if you've tried to read any complicated document.
- *Do not assume that the people reading your paper can read your mind. Do assume that they are intelligent, but also assume that they are busy, and cannot and will not spend an excessive amount of time puzzling out your meaning. Communication is difficult, and written technical communication is close to an art.*

Effective written exposition will be worth at least 40% of your grade. Conversely, bad or unclear exposition may be penalized as much as 40% of the grade or even more.

- The best reference known to me on effective writing is *The Elements of Style* by Strunk and White, a very thin paperback published by Macmillan. It isn't expensive, and it is easy to read. I recommend it.

2.2 Your written work

You should pay attention to presentation, especially for the homework:

- A nicely typed homework (e.g., using a word processor) is preferable to handwritten work. Handwritten work is acceptable too, but in that case:
  - If you have to cross out lots of words, then you should rewrite the whole thing anew, **cleanly and neatly**.
  - You should use a pen. Never use a pencil.
  - Use any color other than red (for example, black, blue, or green), but DO NOT USE RED. (Reason: The use of red is reserved for the instructor's and grader's comments.)
- Make sure that **your name appears in every sheet of paper you hand in**, and that if you are handing in more than one sheet then the sheets are **stapled** and the pages are **numbered**.
- If you tear off the sheets from a spiral notebook, please make sure before you hand them in that there are none of those ugly hanging shreds of paper at the margins. (Use scissors or a cutter, if necessary.)

3 What we know about the real number system

The following statement tells us **all we know** about the real number system. **Everything else has to be proved**.

$$\boxed{\boxed{\text{The real number system } \mathbb{R} \text{ is a complete ordered field.}}}$$

Let us explain what this means.

Fields. A *field* is a set F of objects (often called "numbers") endowed with the following structure:

(1) a binary operation[3] $+$ on F, called "addition,"

(2) a binary operation[4] $\cdot$ on F, called "multipication,"

(3) two members of F, called 0 ("zero") and 1 ("one"),

such that

[3] A *binary operation* on a set S is a rule (or, if you prefer, function) B that for every pair x, y of members of S produces a member xRy of S.

[4] We often write "xy" rather than "$x \cdot y$", and sometimes we write "$x \times y$".

(f1) (commutativity) $(\forall x, y \in F)(x + y = y + x)$, and $(\forall x, y \in F)(x \cdot y = y \cdot x)$;

(f2) (associativity) $(\forall x, y, z \in F)((x + y) + z = x + (y + z))$, and $(\forall x, y, z \in F)((x \cdot y) \cdot z = x \cdot (y \cdot z))$;

(f3) (existence of identities) $(\forall x \in F)(x + 0 = x)$, and $(\forall x \in F)(x \cdot 1 = x)$;

(f4) (existence of inverses):

(f4.1) $0 \neq 1$,

(f4.2) $(\forall x \in F)(\exists y \in F)(x + y = 0)$,

(f4.3) $(\forall x \in F)\Big(x \neq 0 \Longrightarrow (\exists y \in F)(x \cdot y = y\Big)$;

(f5) (distributivity): $(\forall x, y, z \in F)(x \cdot (y + z) = x \cdot y + x \cdot z)$.

Ordered fields. An *ordered field* is a field F, which in addition to the field structure $(+, \cdot, 0, 1)$, is also endowed with a binary relation[5] $\leq$ such that

(o1) $(\forall x, y \in F)(x \leq y \vee y \leq x)$;

(o2) $(\forall x, y \in F)((x \leq y \wedge y \leq x) \Longrightarrow x = y)$;

(o3) $(\forall x, y, z \in F)((x \leq y \wedge y \leq z) \Longrightarrow x \leq z)$;

(o4) $(\forall x, y, z \in F)((y \leq z \Longrightarrow x + y \leq x + z)$;

(o5) $(\forall x, y \in F)((0 \leq x \wedge 0 \leq y) \Longrightarrow 0 \leq x \cdot y)$.

Completeness. Let F be an ordered field. Suppose S is a subset of F. An **upper bound** for S is a member b of F such that

(UB) $(\forall s \in S)(s \leq b)$.

A subset S of an ordered field F is ***bounded above*** it if has an upper bound.

A **least upper bound** for S is an upper bound b for S such that $b \leq b'$ for all upper bounds b' of S.

An ordered field F is **complete** if every nonempty subset S of F which is bounded above has a least upper bound.

How do we work with all this? We now know all the basic facts about $\mathbb{R}$, because we know that $\mathbb{R}$ is a complete ordered field, and that's all there is to know.

Everything else must be proved. And ***every concept that does not already appear in the properties listed above has to be defined***, for the

[5] A *binary relation* on a set S is a set of ordered pairs of members of S. If R is a binary relation on S, we write "xRy" instead of "$(x, y) \in R$," and read "xRy" as "x is R-related to y," or something similar. For example, if P is the set of all people, then the relation "father of" is the set F of all ordered pairs (x, y) such that $x \in P$, $y \in P$, and x is y's father. We write "xFy" instead of "$(x, y) \in F$," and read "xFy" and "x is y's father."

very simple reason that if we do not give a definition we do not know what it means, and if we do not know what it means then we cannot prove anything about it.

For example, the properties we have listed do not say anything about $-$, 2, 3, 4, $<$, $>$, $\geq$, or $|\cdots|$ (absolute value). So all these things have to be defined.

Definition 1. $2 = 1 + 1$.

Definition 2. $3 = 2 + 1$.

Definition 3. $4 = 3 + 1$.

Definition 4. Let $x, y \in \mathbb{R}$. Then

(i) "$x < y$" means "$x \leq y$ and $x \neq y$,"

(ii) "$x \geq y$" means "$y \leq x$,"

(iii) "$x > y$" means "$x \geq y$ and $x \neq y$."

Definition 5. Let $x \in \mathbb{R}$. Then $-x$ is the unique $y \in \mathbb{R}$ such that $x + y = 0$. ***Important remark:*** To be able to define "$-x$" this way, we need to know that the y such that $x + y = 0$ exists and is unique. The fact that it exists is part of Axiom (f4). The fact that it is unique needs proof, and ***the book cheats***[6] ***by not proving it.*** Here is the proof: suppose $x + y = 0$ and $x + z = 0$. We have to prove that $y = z$. But $y = y + 0 = y + (x + z) = (y + x) + z = (x + y) + z = 0 + z = z + 0 = z$. So $y = z$, and we are done.

Definition 6. Let $x \in \mathbb{R}$ be such that $x \neq 0$. Then x^{-1} is the unique $y \in \mathbb{R}$ such that $x \cdot y = 1$. ***Important remark:*** Once again, to be able to define "x^{-1}" this way, we need to know that the y such that $x \cdot y = 1$ exists and is unique. The fact that it exists is part of Axiom (f4). The fact that it is unique needs proof, and the book cheats by not proving it. Here is the proof: suppose $x \cdot y = 1$ and $x \cdot z = 1$. We have to prove that $y = z$. But $y = y \cdot 1 = y \cdot (x \cdot z) = (y \cdot x) \cdot z = (x \cdot y) \cdot z = 1 \cdot z = z \cdot 1 = z$. So $y = z$, and we are done.

Definition 7. Let $x \in \mathbb{R}$. Then

$$|x| = \begin{cases} x & \text{if} \quad 0 \leq x \\ -x & \text{if} \quad x < 0\,. \end{cases}$$

Definition 8. Let $x, y \in \mathbb{R}$. Then "$x - y$" stands for "$x + (-y)$."

Definition 9. Let $x, y \in \mathbb{R}$ be such that $y \neq 0$. Then "$\frac{x}{y}$" stands for "$x \cdot y^{-1}$."

Problem 1. Prove that if $x, y \in \mathbb{R}$ then one and only one of the following three statements is true: $x < y$, $x = y$, $x > y$.

Problem 2. Prove that if $x, y \in \mathbb{R}$ then $|x \cdot y| = |x| \cdot |y|$.

[6]not much, really, just a little biit, but this is a zero-tolerance course, and cheating is cheating!

Problem 3. (The triangle inequality.) Prove that if $x, y \in \mathbb{R}$ then $|x + y| \leq |x| + |y|$.

Problem 4. Prove that $2 + 2 = 4$.

Problem 5. Prove that $2 \cdot 2 = 4$.

Solved problem 1. Prove that $(\forall x \in \mathbb{R})(x \cdot 0 = 0)$. ***Proof.*** Let $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
x \cdot 0 &= x \cdot 0 + 0 \\
&= x \cdot 0 + (x \cdot 0 + (-x \cdot 0)) \\
&= (x \cdot 0 + x \cdot 0) + (-x \cdot 0) \\
&= x \cdot (0 + 0) + (-x \cdot 0) \\
&= x \cdot 0 + (-x \cdot 0) \\
&= 0\,.
\end{aligned}$$

Solved problem 2. Prove that $1 > 0$. ***Proof.*** We know from (o1) that $0 \leq 1$ or $1 \leq 0$. So we consider separately the two cases, namely, $0 \leq 1$ and $1 \leq 0$.

Case I: Suppose $0 \leq 1$. Then $1 > 0$, because (f4) tells us that $0 \neq 1$. (Recall that the definition of "$>$" says that "$x > y$" means "$x \geq y$ and $x \neq y$," so to conclude that $1 > 0$ we need to show that $1 \geq 0$ and $1 \neq 0$. But $1 \geq 0$ because we are assuming that $0 \leq 1$, and the definition of $\geq$ tells us that "$1 \geq 0$" means precisely "$0 \leq 1$." And $1 \neq 0$ because (f4) says that $0 \neq 1$.)

Case II: Suppose $1 \leq 0$. Then $(-1) + 1 \leq (-1) + 0$ by (o4). But $(-1) + 1 = 1 + (-1) = 0$ by (f1) and the definition of $-$. And $(-1) + 0 = 0 + (-1) = -1$ by (f1) and (f3). So $0 \leq -1$. But then (o5) tells us that $0 \leq (-1) \cdot (-1)$. Now we claim that $(-1) \cdot (-1) = 1$. Here is a proof:

$$\begin{aligned}
(-1) \cdot (-1) &= (-1) \cdot (-1) + 0 \\
&= (-1) \cdot (-1) + (1 + (-1)) \\
&= (-1) \cdot (-1) + ((-1) + 1) \\
&= ((-1) \cdot (-1) + (-1)) + 1 \\
&= ((-1) \cdot (-1) + (-1) \cdot 1) + 1 \\
&= (-1) \cdot ((-1) + 1) + 1 \\
&= (-1) \cdot (1 + (-1)) + 1 \\
&= (-1) \cdot 0 + 1 \\
&= 0 + 1 \\
&= 1\,.
\end{aligned}$$

Since we know that $0 \leq (-1) \cdot (-1)$, we conclude that $0 \leq 1$. Then we are back in Case I, and we have shown that in that case $1 > 0$.

So $1 > 0$ in both cases.

Problem 6. For the strings of equalities in Solved Problems 1 and 2, provide the justifications of all the steps.

Problem 7. Prove that $2 \neq 0$.

Problem 8. Prove that if $x \in \mathbb{R}$ then $-|x| \leq x \leq |x|$.

Problem 9. Prove that if $x, y \in \mathbb{R}$ then $|x - y| \geq ||x| - |y||$.

Problem 10. Prove that if $a, b \in \mathbb{R}$ then $a = b \Longleftrightarrow (\forall \varepsilon > 0)(|a - b| \leq \varepsilon)$.

Problem 11. book, Exercise 1.2.2, page 11.

Problem 12. book, Exercise 1.2.6, pages 11, 12.

Problem 13. book, Exercise 1.2.7, page 12.

4 What we know about the natural numbers, the integers, and the rational numbers

Our axioms do no say anything about $\mathbb{N}$, the set of natural numbers. So we have to define what $\mathbb{N}$ means, that is, we have to define what it means for a real number x to be a "natural number." Here is the definition.

First, we define the notion of ***inductive set***: A subset S of $\mathbb{R}$ is *inductive* if

- $1 \in S$,
- $(\forall x \in \mathbb{R})(x \in S \Longrightarrow x + 1 \in S)$.

Definition 10. A ***natural number*** is a real number x that belongs to every inductive subset of $\mathbb{R}$. We use $\mathbb{N}$ to denote the set of all natural numbers.

Solved problem 3. Prove that 1 is a natural number. *Proof.* We have to prove that 1 belongs to every inductive subset of $\mathbb{R}$. So let $S \subseteq \mathbb{R}$ be inductive. Then the definition of "inductive set" tells us that $1 \in S$, which is what we needed to prove. **END.**

Solved problem 4. Prove that if n is a natural number then $n+1$ is a natural number. *Proof.* Fix an $n \in \mathbb{N}$. We have to prove that $n + 1$ belongs to every inductive subset of $\mathbb{R}$. So let $S \subseteq \mathbb{R}$ be inductive. Then the definition of "inductive set" tells us that $(\forall x \in \mathbb{R})(x \in S \Longrightarrow x + 1 \in S)$. So in particular $n \in S \Longrightarrow n + 1 \in S$. But n is a natural number and S is inductive, so inS, and then $n + 1 \in S$. **END.**

Solved problem 5. Prove that 0 is not a natural number. *Proof.* We have to find an inductive set S such that $0 \notin S$. Let us take $S = \{x \in \mathbb{R} : x > 0\}$. Then $1 \in S$ because we have already proved that $1 > 0$. Furthermore, if $x \in S$ then $x + 1 \in S$. (Proof: if $x \in S$ then $x > 0$. So $x + 1 > 1$ and $1 > 0$, so $x + 1 > 0$. Hence $x + 1 \in S$.) So S is inductive. But it is not true that $0 > 0$, so $0 \notin S$. **END.**

Theorem 1. (The principle of mathematical induction, PMI.) *Let S be a subset of $\mathbb{N}$ such that S is inductive. Then $S = \mathbb{N}$.* Proof: To prove that $S = \mathbb{N}$ we have to prove that $S \subseteq \mathbb{N}$ and $\mathbb{N} \subseteq S$. The statement that $S \subseteq \mathbb{N}$ is true because we are assuming that S is a subset of $\mathbb{N}$. Now let us prove that $\mathbb{N} \subseteq S$. We have to prove that $(\forall n \in \mathbb{N})(n \in S)$. So let $n \in \mathbb{N}$ be arbitrary. Then n belongs to every inductive set, because n is a natural number. So $n \in S$, because S is inductive. **END.**

Solved problem 6. Prove that the sum of two natural numbers is a natural number. *Proof:* We have to show that, if $n \in \mathbb{N}$ and $m \in \mathbb{N}$ are arbitrary, then $n + m \in \mathbb{N}$. Let us fix an arbitrary $n \in \mathbb{N}$. We will prove that $n + m \in \mathbb{N}$ for every $m \in \mathbb{N}$ using the PMI. Let $S = \{m \in \mathbb{N} : n + m \in \mathbb{N}\}$. Then $S \subseteq \mathbb{N}$. Let us show that S is inductive. To prove this, we have to show, first of all, that $1 \in S$. That is, we have to show that $n + 1 \in \mathbb{N}$, but this is true because $n \in \mathbb{N}$, in view of Solved Problem 4. Next we have to prove that if $m \in S$ then $m+1 \in S$. So suppose $m \in S$. This means that $n+m \in \mathbb{N}$. By Solved Problem 4, $(n + m) + 1 \in \mathbb{N}$. But $(n + m) + 1 = n + (m + 1)$. So $n + (m + 1) \in \mathbb{N}$. This tells us that $m + 1 \in S$. So we have shown that S is inductive. By the PMI, $S = \mathbb{N}$. This says that $n + m \in \mathbb{N}$ for every $m \in \mathbb{N}$. **END.**

Problem 14. Prove that the product of two natural numbers is a natural number.

Problem 15. book, Exercise 1.2.10, page 12.

Definition 11. An ***integer*** is a real number x such that $x = n - m$ for some natural numbers n, m. We use $\mathbb{Z}$ to denote the set of all integers, so

$$\mathbb{Z} = \{x \in \mathbb{R} : (\exists n, m \in \mathbb{N})(x = n - m)\,.$$

Solved problem 7. Prove that every natural number is an integer. *Proof:* Let $n \in \mathbb{N}$. Then $n = (n + 1) - 1$, and both $n + 1$ and 1 are natural numbers. So n is an integer. **END.**

Solved problem 8. Prove that 0 is an integer. *Proof:* $0 = 1 - 1$. **END.**

Solved problem 9. Prove that if n is an integer then $-n$ is an integer. *Proof:* Let n be an integer. Then we can pick natural numbers p, q such that $n = p-q$. But then $-n = q - p$, so $-n$ is also an integer. **END.**

Solved problem 10. Prove that the sum of two integers is an integer. *Proof:* Let m, n be integers. Pick natural numbers p, q, r, s such that $n = p - q$ and $m = r - s$. Then $n + m = (p - q) + (r - s)$, so $n + m = (p + r) - (q + s)$. Since $p + r$ and $q + s$ are natural numbers, this proves that $n + m$ is an integer.

Problem 16. Prove that the product of two integers is an integer.

Definition 12. A ***rational number*** is a real number x such that $x = \frac{n}{m}$ for some integers n, m such that $m \neq 0$. We use $\mathbb{Q}$ to denote the set of all rational numbers, so

$$\mathbb{Q} = \{x \in \mathbb{R} : (\exists n, m \in \mathbb{Z})(m \neq 0 \wedge mx = n)\,.$$

Problem 17. Prove that the sum of two rational numbers is a rational number.

Problem 18. Prove that the product of two rational numbers is a rational number.

Problem 19. Prove that $0 < \frac{1}{2}$ and $\frac{1}{2} < 1$.

Problem 20. Prove that $\frac{1}{2}$ is not an integer.

5 Some problems on completeness

Solved problem 11. Prove that if S is a nonempty subset of $\mathbb{R}$ which is bounded above, then the least upper bound of S (whose existence is guaranteed by the fact that $\mathbb{R}$ is complete) is unique. *Proof:* To prove that the least upper bound is unique, we have to show that if u, v are least upper bounds of S then $u = v$. So let u, v be least upper bounds of S. Then, since u is an upper bound of S, and v is a least upper bound of S, it follows that $v \leq u$. Similarly, v is an upper bound of S, and u is a least upper bound of S, it follows that $u \leq v$. So $u = v$.

Problem 21. Is the empty set bounded above in $\mathbb{R}$? Does the empty set have a least upper bound in $\mathbb{R}$? (Additional question for those who are curious enough: *Why do you think that, all of a sudden, in this particular problem I am talking about "least upper bound in $\mathbb{R}$" rather than just "least upper bound"?*)

Problem 22. Book, Exercise 1.3.9, page 18.

Solved problem 12. Prove that $(\exists x \in \mathbb{R})(x^2 = 2)$. (The definition of "$x^2$" is "$x \cdot x$ for every $x \in \mathbb{R}$.")

Proof: Let $T = \{t \in \mathbb{R} : t^2 < 2\}$. We want to use the completeness of $\mathbb{R}$ to conclude that T has a least upper bound. For this purpose we need to show (a) that T is not empty, and (b) that T is bounded above. T *is not empty* because $1 \in T$, since $1^2 = 1 < 2$. Also, T *is bounded above* because 2 is an upper bound for T. (*Proof:* We have to show that if $t \in T$ then $t \leq 2$. Let $t \in T$. Then $t^2 < 2$. If $t \leq 0$, then $t \leq 2$ as well, since $0 < 2$. Next suppose that $t > 0$. Assume "$t \leq 2$" was not true. Then $t > 2$. Since $t > 0$, it follows that $t^2 = t \cdot t > 2 \cdot t$, and also—since $2 > 0$—that $2 \cdot t > 2 \cdot 2 = 4 > 2$. So $t^2 > 2$, contradicting the fact that $t^2 < 2$.)

Since $T \neq \emptyset$ and T is bounded above, it follows from the completeness property that T has a least upper bound. Call it L. We are going to prove that $L^2 = 2$. We know that either $L^2 = 2$ or $L^2 < 2$ or $L^2 > 2$. So to prove that $L^2 = 2$ it suffices to exclude the other two possibilities, i.e., to show that "$L^2 < 2$" and "$L^2 > 2$" are both false.

Suppose that $L^2 < 2$. We are going to get a contradiction by finding a real number M such that $M > L$ and $M^2 < 2$. If we do this, then it will follow that $M \in T$, so $M \leq L$ (because L is an upper bound for T) and this will contradict the fact that $M > L$. This contradiction will prove that "$L^2 < 2$" is false. To find M, we try $M = L + \varepsilon$, where ε is a positive number that we will have to choose judiciously. We have

$$M^2 = (L + \varepsilon)^2 = L^2 + 2L\varepsilon + \varepsilon^2 ,$$

and we want $M^2 < 2$. Since $L^2 < 2$, the number $2 - L^2$ is positive. Let us try to find ε such that $2L\varepsilon + \varepsilon^2 < 2 - L^2$. For this purpose, first observe that $L > 0$ (because $1 \in T$ and L is an upper bound for T, so $1 \leq L$, and then $0 < L$). Let us pick

$$\varepsilon = \min\Big(L\,,\ \frac{2 - L^2}{4L}\Big)\,.$$

Then $\varepsilon^2 \leq L\varepsilon$, because $\varepsilon \leq L$, so

$$2L\varepsilon + \varepsilon^2 \leq 3L\varepsilon < 4L\varepsilon \leq 4L \cdot \frac{2 - L^2}{4L} = 2 - L^2\,.$$

Hence $M^2 < 2$ and we have excluded the possibility that $L^2 < 2$.

Now suppose that $L^2 > 2$. We are going to get a contradiction by finding a real number M such that $0 < M < L$ and $M^2 > 2$. If we do this, then it will follow that M is an upper bound for T. (*Proof:* If $t \in T$, let us prove that $t \leq M$. Suppose $t > M$. Then $t^2 > M^2 > 2$ so $t^2 > 2$, contradicting the fact that $t \in T$.) But this will contradict the fact that L is the least upper bound of T, and this contradiction will prove that "$L^2 > 2$" is false. To find M, we try $M = L - \varepsilon$, where ε is a positive number that we will have to figure out how to choose. We have

$$M^2 = (L - \varepsilon)^2 = L^2 - 2L\varepsilon + \varepsilon^2\,,$$

and we want $M^2 > 2$. Since $L^2 > 2$, the number $L^2 - 2$ is positive. Let us try to find ε such that $2L\varepsilon - \varepsilon^2 < L^2 - 2$. (Why? Because if $2L\varepsilon - \varepsilon^2 < L^2 - 2$ then $-2L\varepsilon + \varepsilon^2 > 2 - L^2$, so $L^2 - 2L\varepsilon + \varepsilon^2 > 2$, i.e., $M^2 > 2$.) Clearly, it suffices to find ε such that $2L\varepsilon < L^2 - 2$. (Why? Because if $2L\varepsilon < L^2 - 2$ then *a fortiori* $2L\varepsilon - \varepsilon^2 < L^2 - 2$.) Let us pick

$$\varepsilon = \frac{L^2 - 2}{3L}\,.$$

Then

$$2L\varepsilon < 3L\varepsilon = 3L \cdot \frac{L^2 - 2}{3L} = L^2 - 2\,.$$

Hence $M^2 > 2$ and we have excluded the possibility that $L^2 < 2$. **END**

Problem 23. Prove that for every real number x such that $x \geq 0$ there exists a real number y such that $y^2 = x$.

Problem 24. Prove that for every real number x there exists a real number y such that $y^3 = x$. (Note: the definition of "y^3" is "$y^3 = y \cdot y \cdot y$ for every $y \in \mathbb{R}$.")

Homework assignment No. 1, due on Thursday September 8: Problems 3, 10 and 18.

Homework assignment No. 2, due on Thursday September 15: Problems 22, 23, and 24.

Homework assignment No. 3, due on Thursday September 22: Book, Exercises 1.4.4, 1.4.11, 1.5.4. (NOTE: An earlier version of this handout listed the following problems: 1.4.4, 1.4.13, 1.5.4. This was a mistake. I did *not* intend to assign Problem 1.4.13, on the Cantor-Schröder-Berstein Theorem. On the other hand, *those students who wish to do Problem 1.4.13 instead of 1.4.11 are welcome to do so, and if they do this problem well they will get extra bonus credit.*

Homework assignment No. 4, due on Thursday September 29: Book, Exercises 2.2.1, 2.2.2, 2.2.4.

Optional problem, for extra credit; also due on Thursday September 29: Prove that for every natural number n and every nonnegative real number x, there exists a real number y such that $y^n = x$. (The precise definition of u^n, for $u \in \mathbb{R}$, is as follows[7]: if $u \in \mathbb{R}$, then $u^1 = u$ and, for $n \in \mathbb{N}$, $u^{n+1} = u \cdot u^n$.)

Here are some hints for your proof. We have already done this for $n = 2$ and $n = 3$. What you have to do in the general case is very much the same, except that you need to make a few changes in the argument. The beginning of the proof, where you define the set T, and prove that T is nonempty (if $x > 0$) and bounded above, is easy, and essentially the same as what you did for $n = 2$ and $n = 3$. Then you get the least upper bound L of the set T, and the part of the proof that needs some work is to prove that $L^n = x$. To do this, you must show that the two other possibilities, namely, $L^n < x$ and $L^n > x$, cannot occur. And this is done very much as in the case of $n = 2$ or $n = 3$: to exclude $L^n < x$, you find a real number ε such that $\varepsilon > 0$ and $(L + \varepsilon)^n < x$, and then proceed as in the $n = 2$ and $n = 3$ proofs; to exclude $L^n > x$, you find a real number ε such that $0 < \varepsilon < L$ and $(L - \varepsilon)^n > x$, and then proceed as in the $n = 2$ and $n = 3$ proofs. The only problem is, how do you choose ε? I will give you a helpful hint for the first of the two εs, and I will then leave it up to you to figure out how to choose the other one.

So suppose that $L^n < x$, and let us find a real number ε such that $\varepsilon > 0$ and $(L + \varepsilon)^n < x$. To do this, it is convenient to start by proving a lemma:

LEMMA: *if* $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$, *and* $0 < \alpha \le \frac{1}{2n}$, *then* $(1 + \alpha)^n < 1 + 2n\alpha$.

You should prove this by induction.

Now that you have proved the lemma, you can use it. You want the inequality $(L + \varepsilon)^n < x$. Let $\theta = \frac{x}{L^n}$, so $\theta > 1$, because we are working under the assumption that $L^n < x$. Let $\alpha = \min\left(\frac{1}{2n}, \frac{\theta-1}{2n}\right)$. Then the lemma tells us that $(1 + \alpha)^n < 1 + 2n\alpha$, since $\alpha \le \frac{1}{2n}$. But

$$1 + 2n\alpha \le 1 + 2n \cdot \frac{\theta - 1}{2n} = 1 + \theta - 1 = \theta\,,$$

because $\alpha \le \frac{\theta-1}{2n}$. So $(1 + \alpha)^n < \theta$. Now multiply this by L^n, and get the inequality $L^n(1 + \alpha)^n < \theta L^n$. But $L^n(1+\alpha)^n = \Big(L(1+\alpha)\Big)^n = (L+L\alpha)^n$, and $\theta L^n = x$ So $(L + L\alpha)^n < x$. Then we may choose $\varepsilon = L\alpha$, and we are done.

[7] This is an example of a *recursive definition*.