MATHEMATICS 311 — FALL 2005

Advanced Calculus

H. J. Sussmann

INSTRUCTOR'S NOTES

November 10, 2005

1 Homework assignment due on Thursday, November 17

Do the proofs of Examples 6, 7, 8, 9, 10, 11, 12 , 13 and 14 below. (Note: this may seem like a lot of work, but it isn't, really, because *for each of the proofs, you are allowed to use, without having to reprove them, all the results of the previous examples*. For example, to do the proof of Example 9 you can use the results of Examples 1 through 8.)

2 Continuous functions

Definition 1. Assume that S is a subset of $\mathbb{R}$ and $f : S \mapsto \mathbb{R}$ is a function. We say that f is ***continuous*** if

- *whenever $(x_n)_{n=1}^{\infty}$ is a sequence of members of S that converges to an $x \in S$ then $\lim_{n\to\infty} f(x_n) = f(x)$.* $\diamondsuit$

Example 1. *Assume that $S \subseteq \mathbb{R}$, and $c \in \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$ be defined by*

$$f(x) = c \qquad \text{for} \qquad x \in S\,.$$

Then f is continuous. (In other words: a constant function is continuous.)

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence of members of S that converges to an $x \in S$. We want to prove that $\lim_{n\to\infty} f(x_n) = f(x)$. But $f(x_n) = c$ for all n, and $f(x) = c$. So $\lim_{n\to\infty} f(x_n) = f(x)$, as desired. $\diamondsuit$

Example 2. *Assume that $S \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$ be defined by*

$$f(x) = x \qquad \text{for} \qquad x \in S\,.$$

Then f is continuous. (In other words: the identity function id_S of any set S is continuous.)

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence of members of S that converges to an $x \in S$. We want to prove that $\lim_{n\to\infty} f(x_n) = f(x)$. But $f(x_n) = x_n$ for all n, and $f(x) = x$. So $\lim_{n\to\infty} f(x_n) = f(x)$, as desired, because $\lim_{n\to\infty} x_n = x$. $\diamondsuit$

Example 3. *Assume that $S \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$, $g : S \mapsto \mathbb{R}$, be continuous functions. Let $h : S \mapsto \mathbb{R}$ be defined by*

$$h(x) = f(x) + g(x) \qquad \text{for} \qquad x \in S\,.$$

Then h is continuous. (In other words: the sum of two continuous functions is continuous.)

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence of members of S that converges to an $x \in S$. We want to prove that $\lim_{n\to\infty} h(x_n) = h(x)$. But $h(x_n) = f(x_n) + g(x_n)$ for all n, and $h(x) = f(x) + g(x)$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(x)$. Since g is continuous, $\lim_{n\to\infty} g(x_n) = g(x)$. It then follows from the Algebraic Limit Theorem that $\lim_{n\to\infty} \Big(f(x_n) + g(x_n)\Big) = f(x) + g(x)$, i.e., that $\lim_{n\to\infty} h(x_n) = h(x)$, as desired. $\diamondsuit$

Example 4. *Assume that $S \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$, $g : S \mapsto \mathbb{R}$, be continuous functions. Let $h : S \mapsto \mathbb{R}$ be defined by*

$$h(x) = f(x) \cdot g(x) \qquad \text{for} \qquad x \in S\,.$$

Then h is continuous. (In other words: the product of two continuous functions is continuous.)

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence of members of S that converges to an $x \in S$. We want to prove that $\lim_{n\to\infty} h(x_n) = h(x)$. But $h(x_n) = f(x_n) \cdot g(x_n)$ for all n, and $h(x) = f(x) \cdot g(x)$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(x)$. Since g is continuous, $\lim_{n\to\infty} g(x_n) = g(x)$. It then follows from the Algebraic Limit Theorem that $\lim_{n\to\infty} \Big(f(x_n) \cdot g(x_n)\Big) = f(x) \cdot g(x)$, i.e., that $\lim_{n\to\infty} h(x_n) = h(x)$, as desired. $\diamondsuit$

Example 5. *Assume that $S \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$, $g : S \mapsto \mathbb{R}$, be continuous functions. Assume in addition that $g(x) \neq 0$ for all $x \in S$. Let $h : S \mapsto \mathbb{R}$ be defined by*

$$h(x) = \frac{f(x)}{g(x)} \qquad \text{for} \qquad x \in S\,.$$

Then h is continuous. (In other words: the quotient of a continuous function by a nowhere vanishing continuous function is continuous.)

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence of members of S that converges to an $x \in S$. We want to prove that $\lim_{n\to\infty} h(x_n) = h(x)$. But $h(x_n) = \frac{f(x_n)}{g(x_n)}$ for all n, and $h(x) = \frac{f(x)}{g(x)}$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(x)$. Since g is continuous, $\lim_{n\to\infty} g(x_n) = g(x)$. It then follows from the Algebraic Limit Theorem (using the fact that $g(x) \neq 0$) that $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(x)}{g(x)}$, i.e., that $\lim_{n\to\infty} h(x_n) = h(x)$, as desired. $\diamondsuit$

Example 6. *Assume that $S \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$, $g : S \mapsto \mathbb{R}$, be continuous functions. Assume in addition that $g(x) \neq 0$ for all $x \in S$. Let $h : S \mapsto \mathbb{R}$ be defined by*

$$h(x) = \max\Big(\, f(x)\,,\, g(x)\Big) \qquad \text{for} \qquad x \in S\,.$$

Then h is continuous. (In other words: the maximum of two continuous functions is continuous.)

Proof. Homework problem. $\diamondsuit$

Example 7. *Assume that $S \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$, $g : T \mapsto \mathbb{R}$, be continuous functions. Assume in addition that $f(x) \in T$ for all $x \in S$. Let $h : S \mapsto \mathbb{R}$ be defined by*

$$h(x) = g\Big(\, f(x)\Big) \qquad \text{for} \qquad x \in S\,.$$

Then h is continuous. (In other words: the composite of two continuous functions is continuous.)

Proof. Homework problem. $\diamondsuit$

Example 8. *Assume that $S \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$ be given by*

$$f(x) = |x| \qquad \text{for} \qquad x \in S\,.$$

Then f is continuous. (In other words: the "absolute value function" is continuous.)

Proof. Homework problem. (*It would be nice if you could do this by applying the results of previous examples, without having to write sequences and limits.*) $\diamondsuit$

Example 9. *Assume that $S \subseteq \mathbb{R}$. Let $f : S \mapsto \mathbb{R}$ be a continuous function. Let $g : S \mapsto \mathbb{R}$ be given by*

$$g(x) = |f(x)| \qquad \text{for} \qquad x \in S\,.$$

Then g is continuous. (In other words: the absolute value of a continuous function is continuous.)

Proof. Homework problem. $\diamondsuit$

Example 10. *Let* $S = \{x \in \mathbb{R} : x \geq 0\}$. *Let* $f : S \mapsto \mathbb{R}$ *be given by*

$$f(x) = \sqrt{x} \qquad \text{for} \qquad x \in S\,.$$

Then f is continuous. (In other words: the "square root function" is continuous.)

Proof. Homework problem. $\diamondsuit$

Example 11. *Assume that* $S \subseteq \mathbb{R}$. *Let* $f : S \mapsto \mathbb{R}$ *be a continuous function. Assume that* $f(x) \geq 0$ *whenever* $x \in S$. *Define a function* $g : S \mapsto \mathbb{R}$ *by letting*

$$g(x) = \sqrt{f(x)} \qquad \text{for} \qquad x \in S\,.$$

Then g is continuous. (In other words: the square root of a continuous function is continuous.)

Proof. Homework problem. $\diamondsuit$

Example 12. *Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *be given by*

$$f(x) = \frac{1 + 3x - 7x^2 + 23x^3 + 6x^4}{1 + x^2 + x^4} \qquad \text{for} \qquad x \in \mathbb{R}\,.$$

Then f is continuous.

Proof. Homework problem. $\diamondsuit$

Example 13. *Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *be given by*

$$f(x) = \sqrt{1 + 3\,x^2 + x^4} \qquad \text{for} \qquad x \in \mathbb{R}\,.$$

Then f is continuous.

Proof. Homework problem. $\diamondsuit$

Example 14. *Let* $f : \mathbb{R} \mapsto \mathbb{R}$ *be given by*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \geq 0\,, \\ 0 & \text{if} \quad x < 0\,, \end{cases}$$

for $x \in \mathbb{R}$. *Then f is not continuous.*

Proof. Homework problem. $\diamondsuit$