# MATHEMATICS 311 — FALL 2005

*Advanced Calculus*

*H. J. Sussmann*

## INSTRUCTOR'S NOTES

*November 29, 2005*

## 1 Homework assignment due on Thursday, December 1

Book, Exercises 4.3.4, 4.3.6(a), 4.3.7, 4.4.4, 4.5.7

## 2 Continuous functions

**Definition 1**. Assume that $S$ is a subset of $\mathbb{R}$ and $f : S \mapsto \mathbb{R}$ is a function. We say that $f$ is ***continuous*** if

- *whenever $(x_n)_{n=1}^{\infty}$ is a sequence of members of $S$ that converges to an $x \in S$ then $\lim_{n\to\infty} f(x_n) = f(x)$.* $\diamondsuit$

## 3 Compact sets and the Extreme Value Theorem

**Definition 2**. Assume that $S$ is a subset of $\mathbb{R}$. We say that $S$ is ***compact*** if every sequence $(x_n)_{n=1}^{\infty}$ of points of $S$ has a subsequence that converges to an $x \in S$.

**Example 1**. *Let $a, b$ be real numbers such that $a \leq b$. Let $S = \{x \in \mathbb{R} : a \leq x \leq b\}$. Then $S$ is compact.*

*Proof.* Let $(x_n)_{n=1}^{\infty}$ be a sequence of members of $S$. We want to extract a subsequence that converges to an $x \in S$. First of all, the sequence $(x_n)_{n=1}^{\infty}$ is bounded, because $S$ is bounded. So by the Bolzano-Weierstrass theorem $(x_n)_{n=1}^{\infty}$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to an $x \in \mathbb{R}$. We want to prove that $x \in S$. Since $x_{n_k} \in S$, we have $a \leq x_{n_k} \leq b$. Then $a \leq \lim_{k\to\infty} x_{n_k} \leq b$. Therefore $a \leq x \leq b$, so $x \in S$. $\diamondsuit$

**Theorem 1**. *Assume that $S$ is a nonempty compact subset of $\mathbb{R}$, and $f : S \mapsto \mathbb{R}$ is a continuous function. Then $f$ has a maximum and a minimum on $S$. That is, there exist points $\alpha, \beta \in S$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for all $x \in S$.*

*Proof.* Let $V$ be the set of values taken on by $f$, that is, $V = \{f(x) : x \in S\}$. Then $V$ is nonempty, because $S$ is nonempty, so we may pick a $\xi \in S$, and then $f(\xi) \in V$.

Let us prove that $V$ is bounded above. Suppose $V$ is not bounded above. Then for every $n \in \mathbb{N}$ we can pick $x_n \in S$ such that $f(x_n) > n$ (because if $x_n$ did not exist then $n$ would be an upper bound for $V$). Then $(x_n)_{n=1}^{\infty}$ is a sequence of members of $S$. Since $S$ is compact there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ that converges to an $x \in S$. Since $f$ is continuous, $\lim_{k\to\infty} f(x_{n_k}) = f(x)$. But this is impossible, because $f(x_{n_k}) > n_k \geq k$. So we have arrived at a contradiction, showing that $V$ is bounded above.

Since $V$ is bounded above and nonempty, $V$ has a least upper bound $v$. Then, if $n \in \mathbb{N}$ is arbitrary, the number $v - \frac{1}{n}$ is not an upper bound of $V$, so we may pick $x_n \in S$ such that $f(x_n) > v - \frac{1}{n}$. Then $(x_n)_{n=1}^{\infty}$ is a sequence of members of $S$. Since $S$ is compact there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ that converges to a point $\beta \in S$. Since $f$ is continuous, $\lim_{k\to\infty} f(x_{n_k}) = f(\beta)$. Since $f(x_{n_k}) > v - \frac{1}{n_k} \geq v - \frac{1}{k}$, we have $f(\beta) \geq v$. But $v$ is an upper bound of $V$, and $f(\beta) \in V$, so $f(\beta) \leq v$. Then $f(\beta) = v$. Given any $x \in S$, the number $f(x)$ belongs to $V$, so $f(x) \leq v$. Hence $f(x) \leq f(\beta)$ for every $x \in S$, so $f$ attains its maximum value at $\beta$.

The proof of the existence of a minimum is similar. $\diamondsuit$

# 4 Connected sets and the Intermediate Value Theorem

**Definition 3**. Assume that $S$ is a subset of $\mathbb{R}$. We say that $S$ is ***disconnected*** if there exist nonempty subsets $A, B$ of $S$ such that $S = A \cup B$, $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$. $\diamondsuit$

**Definition 4**. Assume that $S$ is a subset of $\mathbb{R}$. We say that $S$ is ***connected*** if $S$ is not disconnected. $\diamondsuit$

**Theorem 2**. *Assume that $S$ is a nonempty connected subset of $\mathbb{R}$, and $f : S \mapsto \mathbb{R}$ is a continuous function. Let $\alpha, \beta, c$ be real numbers such that $\alpha \in S$, $\beta \in S$, and $f(\alpha) < c < f(\beta)$. Then there exists a $\gamma \in S$ such that $f(\gamma) = c$.*

*Proof.* Suppose $\gamma$ does no exist. We are going to prove that $S$ is disconnected, contradicting the assumption that $S$ is connected.

Let $A = \{x \in S : f(x) < c\}$, $B = \{x \in S : f(x) > c\}$. Then every $x \in S$ is in $A$ or in $B$, so $S = A \cup B$. We now show that $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$. The two proofs are similar, so we will just do the first one.

Suppose $\bar{A} \cap B \neq \emptyset$. Pick a $\xi \in \bar{A} \cap B$. Then $\xi \in B$, so $\xi \in S$ and $f(\xi) > c$. On the other hand, $\xi \in \bar{A}$, so either $\xi \in A$ or there exists a sequence $(x_n)_{n=1}^{\infty}$ of points of $A$ that converges to $\xi$. If $\xi \in \bar{A}$ then $\xi < c$. If there exists a sequence $(x_n)_{n=1}^{\infty}$ of points of $A$ that converges to $\xi$, then $f(x_n) < c$ for each $n$, and $f(\xi) = \lim_{n\to\infty} f(x_n)$, so $f(\xi) \leq c$. Hence we have shown that $f(\xi) \leq c$ and $f(\xi) > c$. This is a contradiction, showing that $\bar{A} \cap B = \emptyset$.

Similarly, $A \cap \bar{B} = \emptyset$. Hence $S$ is disconnected, and we have derived a contradiction. $\diamondsuit$

# 5 Intervals

**Definition 5**. Assume that $S$ is a subset of $\mathbb{R}$. We say that $S$ is an ***interval*** if $(\forall a \in S)(\forall b \in S)(\forall c \in \mathbb{R})(a < c < b \Longrightarrow c \in S)$. $\diamondsuit$

**Theorem 3**. *Every interval is connected.*

*Proof.* Let $S$ be an interval. We will assume that $S$ is disconnected and derive a contradiction.

Suppose $S$ is disconnected. Pick nonempty subsets $A, B$ of $S$ such that $S = A \cup B$, $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$.

Pick members $a_1$, $b_1$ of $A, B$, and let $I_1$ be the interval $[a_1, b_1]$ if $a_1 < b_1$, and the interval $[b_1, a_1]$ if $b_1 < a_1$. Then $I_1$ is a closed interval one of whose endpoints is in $A$ while the other one is in $B$. Let $c$ be the midpoint of $I_1$. Then $c \in S$, because $S$ is an interval. So either $c \in A$ or $c \in B$. If $c \in A$, let $a_2 = c$, $b_2 = b_1$. If $c \in B$, let $a_2 = a_1$, $b_2 = c$. Let $I_2$ be the interval $[a_2, b_2]$ if $a_2 < b_2$, and the interval $[b_2, a_2]$ if $b_2 < a_2$. Then $I_2$ is a closed interval one of whose endpoints is in $A$ while the other one is in $B$, and in addition the length of $I_2$ is half the length of $I_1$.

Repeat this procedure, and construct a sequence $(I_n)_{n=1}^{\infty}$ of closed intervals such that $I_{n+1} \subseteq I_n$ for all $n$, $\text{lenght}(I_n) = 2^{1-n}\text{lenght}(I_1)$, and one of the endpoints of $I_n$ is in $A$ while the other one is in $B$. Let $a_n$ be the endpoint of $I_n$ that belongs to $A$, and let $b_n$ be the endpoint of $I_n$ that belongs to $B$.

Since the intervals $I_n$ are closed, bounded and nonempty, and form a decreasing sequence, there exists a $c \in \bigcap_{n=1}^{\infty} I_n$. Then, for each $n$, either $a_n \leq c \leq b_n$ or $b_n \leq c \leq a_n$. Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$. If $c \in A$, then the fact that $\lim_{n\to\infty} b_n = c$ shows that $c \in \bar{B}$, contradicting the fact that $A \cap \bar{B} = \emptyset$. If $c \in B$, then the fact that $\lim_{n\to\infty} a_n = c$ shows that $c \in \bar{A}$, contradicting the fact that $\bar{A} \cap B = \emptyset$.

So we have reached a contradiction, proving that $S$ is connected. $\diamondsuit$