MATHEMATICS 252 — FALL 2006

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FORCED OSCILLATIONS

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Forced harmonic oscillators. A *forced hamrmonic oscillator* is a system governed by a second-order inhomogeneous linear differential equation

$$A\frac{d^2y}{dt}(t) + B\frac{dy}{dt}(t) + Cy(t) = u(t)\,. \tag{1}$$

Here, A, B, C are real constants such that $A > 0$, $B \geq 0$ and $C > 0$, and $u(t)$ is a function. The function u is called the *forcing term*, or the *input*. For any given input u, any function $y(t)$ which is a solution of (1) is called a *response*, or *output*, for u.

Equation (1) can also be written as

$$A\frac{d^2y}{dt} + B\frac{dy}{dt} + Cy = u\,. \tag{2}$$

The superposition principle (also called the "linearity principle".

> THEOREM: If y_1 is a response of the system (1) for an input u_1, and y_2 is a response for an input u_2, then the sum $y_1 + y_2$ is a response for the input $u_1 + u_2$.

PROOF. Let $y = y_1 + y_2$, $u = u_1 + u_2$, We just verify that (2) is true. We have

$$A\frac{d^2y_1}{dt} + B\frac{dy_1}{dt} + Cy_1 = u_1\,,$$

and

$$A\frac{d^2y_2}{dt} + B\frac{dy_2}{dt} + Cy_2 = u_2\,.$$

So, adding these two equalities, we get

$$\Big(A\frac{d^2y_1}{dt} + B\frac{dy_1}{dt} + Cy_1\Big) + \Big(A\frac{d^2y_2}{dt} + B\frac{dy_2}{dt} + Cy_2\Big) = u_1 + u_2\,,$$

and then

$$A\Big(\frac{d^2y_1}{dt} + \frac{d^2y_2}{dt}\Big) + B\Big(\frac{dy_1}{dt} + \frac{dy_2}{dt}\Big) + +C(y_1 + y_2) = u_1 + u_2\,,$$

so

$$A\frac{d^2(y_1+y_2)}{dt} + B\frac{d(y_1+y_2)}{dt} + C(y_1 + y_2 = u_1 + u_2\,,$$

that is,

$$A\frac{d^2y}{dt} + B\frac{dy}{dt} + Cy = u\,.$$

END OF PROOF

The general solution. Suppose we are given an input function u, and we want to find all the solutions of (2). Let us fix one solution y_{part} of (2). Let y_{hom} be a solution of the *homogeneous equation*

$$A\frac{d^2y}{dt} + B\frac{dy}{dt} + Cy = 0\,. \tag{3}$$

Then

- y_{hom} is a response for the input 0,
- y_{part} is a response for the input u,

so the superposition principle tells us that $y_{hom} + y_{part}$ is a response for the input u.

If we now look at all possible solutions y_{hom} of (3)—that is, we consider the *general solution* $y_{hom,gen}$ of (3), which involves two arbitrary constants that are to be selected when an initial condition is given—and fix a particular solution y_{part} of (2), then $y_{hom,gen} + y_{part}$ gives us all the solutions of (2), so $y_{hom,gen} + y_{part}$ is the *general solution* of (2).

EXAMPLE. *Find the general solution of the forced harmonic oscillator equation*

$$\frac{d^2y}{dt}(t) + 4\frac{dy}{dt}(t) + 8y(t) = \cos\, t\,. \tag{4}$$

SOLUTION. Let us first find the general solution of the homogeneous equation

$$\frac{d^2y}{dt}(t) + 4\frac{dy}{dt}(t) + 8y(t) = 0\,. \tag{5}$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 8 = 0\,,$$

whose solutions are given by

$$\lambda = \frac{1}{2}\Big(-4 \pm \sqrt{4^2 - 4\times 8}\Big) = \frac{1}{2}\Big(-4 \pm \sqrt{-16}\Big) = \frac{1}{2}\Big(-4 \pm 4i\Big) = -2 \pm 2i\,.$$

Hence the general solution of (5) is

$$y(t) = e^{-2t}\Big(k_1\, \cos\, 2t + k_2\, \sin\, 2t\Big)\,,$$

where k_1, k_2 are arbitrary real constants. If we write the vector (k_1, k_2) in polar coordinates, $k_1 = r\, \cos\, \theta$, $k_2 = r\, \sin\, \theta$, we get

$$y(t) = r\, e^{-2t}\Big(\cos\, \theta\, \cos\, 2t + \sin\, \theta\, \sin\, 2t\Big) = r\, e^{-2t}\cos(2t - \theta)\,.$$

If we write $\theta = 2\varphi$, so $2t - \theta = 2(t - \varphi)$, we see that *the general solution of (5) is*

$$y(t) = r\, e^{-2t}\cos\, 2(t - \varphi)\,, \quad r \in \mathbb{R}\,, \varphi \in \mathbb{R}\,, r \geq 0\,.$$

Now let us find a particular solution y_{part} of the inhomogeneous equation (4). We make the educated guess

$$y_{part}(t) = c_1\, \cos\, t + c_2\, \sin\, t\,, \tag{6}$$

and seek to find c_1, c_2 such that y_{part} satisfies (4).

Differentiation of both sides of (6) yields

$$y'_{part}(t) = c_2\, \cos\, t - c_1\, \sin\, t\,, \tag{7}$$

and then a second differentiation produces the result

$$y''_{part}(t) = -c_1\, \cos\, t - c_2\, \sin\, t\,. \tag{8}$$

Then

$$\begin{aligned}
&y''_{part}(t) + 4y'_{part}(t) + 8y_{part}(t) \\
&= \Big(-c_1 \cos t - c_2 \sin t\Big) + 4\Big(c_2 \cos t - c_1 \sin t\Big) + 8\Big(c_1 \cos t + c_2 \sin t\Big) \\
&= (-c_1 + 4c_2 + 8c_1) \cos t + (-c_2 - 4c_1 + 8c_2) \sin t \\
&= (7c_1 + 4c_2) \cos t + (7c_2 - 4c_1) \sin t\,.
\end{aligned}$$

Since we want $y''_{part}(t) + 4y'_{part}(t) + 8y_{part}(t) = \cos t$, we have to choose c_1 and c_2 such that

$$7c_1 + 4c_2 = 1 \qquad \text{and} \qquad 7c_2 - 4c_1 = 0\,.$$

This means that $c_2 = \frac{4c_1}{7}$, and then

$$1 = 7c_1 + 4c_2 = 7c_1 + 4 \times \frac{4c_1}{7} = \Big(7 + \frac{16}{7}\Big)c_1 = \frac{65}{7}c_1\,,$$

so $c_1 = \frac{7}{65}$, and then $c_2 = \frac{4c_1}{7} = \frac{4}{65}$. Therefore our particular solution is

$$y_{part}(t) = \frac{7}{65} \cos t + \frac{4}{65} \sin t\,.$$

Finally, *the general solution of (5) is*

$$y(t) = r\, e^{-2t} \cos 2(t - \varphi) + \frac{7}{65} \cos t + \frac{4}{65} \sin t\,, \quad r \in \mathbb{R}\,, \varphi \in \mathbb{R}\,, r \geq 0\,.$$

An alternative form is obtained by writing the vector $\left(\frac{7}{65}, \frac{4}{65}\right)$ in polar coordinates: if

$$\frac{7}{65} = R \cos \psi\,, \quad \frac{4}{65} = R \sin \psi\,,$$

then $R = \frac{1}{\sqrt{65}}$ (because $7^2 + 4^2 = 65$), and $\psi = \arctan\frac{4}{7}$. So *the general solution of (5) is*

$$y(t) = r\, e^{-2t} \cos 2(t - \varphi) + \frac{1}{\sqrt{65}} \cos \Big(t - \arctan\frac{4}{7}\Big)\,, \quad r \in \mathbb{R}\,, \varphi \in \mathbb{R}\,, r \geq 0\,.$$

Sinusoids. A *sinusoid* is a function $f(t)$ of the form

$$f(t) = \alpha \cos \omega(t - \varphi)\,,$$

where α is ≥ 0, ω is > 0, and φ is a real number. The number α is the *amplitude* of the sinusoid, the number ω is the *frequency*, and the number φ is the *phase*.

THEOREM: If ω is any positive number, then every function $f(t)$ of the form $f(t) = a \cos \omega t + b \sin \omega t$, with a, b real numbers, is a sinusoid $\alpha \cos \omega(t - \varphi)$ of frequency ω, with amplitude $\boxed{\alpha = \sqrt{a^2 + b^2}}$.

PROOF. We write the vector (a, b) in polar coordinates: if $a = \alpha \cos \psi$, and $b = \alpha \sin \psi$, then $\alpha = \sqrt{a^2 + b^2}$, and ψ is the angle from the x axis to the vector from $(0, 0)$ to (a, b). Then

$$f(t) = a \cos \omega t + b \sin \omega t = \alpha(\cos \psi \cos \omega t + \sin \psi \sin \omega t = \alpha \cos(\omega t - \psi)\,.$$

If we define φ by $\omega\varphi = \psi$, then

$$f(t) = \alpha \cos \omega(t - \varphi)\,,$$

and our proof is complete.

REMARK. How is the phase φ determined? To answer this question, it suffices to show how ψ is determined, because once we know ψ we can find φ using $\varphi = \frac{\psi}{\omega}$.

Since $a = \alpha \cos \psi$ and $b = \alpha \sin \psi$, we can conclude that

$$\cos \psi = \frac{a}{\alpha} \quad \text{and} \quad \sin \psi = \frac{b}{\alpha}\,, \tag{9}$$

provided that $\alpha \neq 0$. Therefore

$$\tan \psi = \frac{b}{a} \quad \text{if} \quad a \neq 0 \qquad \text{and} \quad \cot \psi = \frac{a}{b} \quad \text{if} \quad b \neq 0\,. \tag{10}$$

Now, if $\alpha \neq 0$, then .one of the numbers a, b is $\neq 0$, and then we can use one of the formulas of (10). This, however, is not enough, because if you take an angle ψ for which (10), and let $\tilde{\psi} = \psi + \pi$, then $\tilde{\psi}$ also satisfies (10). Which is one is right? The answer is: the one that gives the correct values, including the sign, of a and b, using (9).

Let us look at two examples. First suppose $a = 1$ and $b = \sqrt{3}$. Then $a^2 + b^2 = 4$, so $\alpha = 2$. Then $\tan \psi = \sqrt{3}$, so ψ could be $\frac{\pi}{3}$ (that is, 60 degrees). However, ψ could also be $\frac{4\pi}{3}$ (that is, 240 degrees), because $\tan \frac{4\pi}{3}$ is also equal to $\sqrt{3}$. Which one is the correct ψ? The answer is $\frac{\pi}{3}$, because $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, so $\alpha \cos \frac{\pi}{3} = 1 = a$ and $\alpha \sin \frac{\pi}{3} = \sqrt{3} = b$.

Now suppose, instead, that $a = -1$ and $b = -\sqrt{3}$. Then $a^2 + b^2 = 4$, so $\alpha = 2$. Again, we get $\tan \psi = \sqrt{3}$, so ψ could be $\frac{\pi}{3}$ or $\frac{4\pi}{3}$, as before. Which one is the correct ψ? The answer this time is $\frac{4\pi}{3}$, because $\cos \frac{4\pi}{3} = -\frac{1}{2}$ and $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, so $\alpha \cos \frac{4\pi}{3} = -1 = a$ and $\alpha \sin \frac{4\pi}{3} = -\sqrt{3} = b$.

Summarizing,

> The phase φ of the sinusoid $a \cos \omega t + b \sin \omega t$ is determined by the formula $\varphi = \frac{\psi}{\omega}$, where ψ is determined by (10), together with the specification that, of the two values of ψ that satisfy (10), we choose the one for which (9) holds as well.

The system responses to a sinusoidal input. Consider a *forced harmonic oscillator with a sinusoidal input*, obeying the equation

$$A\frac{d^2y}{dt}(t) + B\frac{dy}{dt}(t) + Cy(t) = \alpha \cos \omega(t - \varphi)\,. \tag{11}$$

Let us find the general solution of (11).

First, let us find a particular solution y_{part}. We make the educated guess

$$y_{part}(t) = c_1 \cos \omega(t - \varphi) + c_2 \sin \omega(t - \varphi)\,, \tag{12}$$

and seek to find c_1, c_2 such that y_{part} satisfies (11). To simplify matters, let us write $\tau = t - \varphi$, so our educated guess is

$$y_{part} = c_1 \cos \omega\tau + c_2 \sin \omega\tau\,, \tag{13}$$

Differentiation of both sides of (13) yields

$$y'_{part} = \omega c_2 \cos \omega\tau - \omega c_1 \sin \omega\tau\,, \tag{14}$$

and then a second differentiation gives

$$y''_{part} = -\omega^2 c_1 \cos \omega\tau - \omega^2 c_2 \sin \omega\tau\,. \tag{15}$$

Then

$$\begin{aligned}
&Ay''_{part} + By'_{part} + Cy_{part} \\
&= A\Big(- \omega^2 c_1 \cos \omega\tau - \omega^2 c_2 \sin \omega\tau\Big) + B\Big(\omega c_2 \cos \omega\tau - \omega c_1 \sin \omega\tau\Big) + C\Big(c_1 \cos \omega\tau + c_2 \sin \omega\tau\Big) \\
&= (-A\omega^2 c_1 + B\omega c_2 + Cc_1) \cos \omega\tau + (-A\omega^2 c_2 - B\omega c_1 + Cc_2) \sin \omega\tau \\
&= \Big((C - A\omega^2)c_1 + B\omega c_2\Big) \cos \omega\tau + \Big((C - A\omega^2)c_2 - B\omega c_1\Big) \sin \omega\tau\,.
\end{aligned}$$

Since we want $Ay''_{part} + By'_{part} + Cy_{part} = \alpha \cos \omega\tau$, we have to choose c_1 and c_2 such that

$$(C - A\omega^2)c_1 + B\omega c_2 = \alpha \qquad \text{and} \qquad (C - A\omega^2)c_2 - B\omega c_1 = 0\,.$$

This means that

$$c_2 = \frac{B\omega}{C - A\omega^2} c_1\,,$$

and then

$$\alpha = (C - A\omega^2)c_1 + B\omega c_2 = (C - A\omega^2)c_1 + B\omega \frac{B\omega}{C - A\omega^2} c_1\,,$$

so

$$\alpha = \left(\frac{(C - A\omega^2)^2}{C - A\omega^2} + \frac{B^2\omega^2}{C - A\omega^2} \right) c_1 = \frac{(C - A\omega^2)^2 + B^2\omega^2}{C - A\omega^2} c_1$$

and then

$$c_1 = \frac{C - A\omega^2}{(C - A\omega^2)^2 + B^2\omega^2}\,\alpha$$

and

$$c_2 = \frac{B\omega}{C - A\omega^2} c_1 = \frac{B\omega}{(C - A\omega^2)^2 + B^2\omega^2}\,\alpha\,.$$

Therefore our particular solution is

$$y_{part}(t) = \frac{\alpha}{(C - A\omega^2)^2 + B^2\omega^2} \Big((C - A\omega^2) \cos \omega\tau + B\omega \sin \omega\tau \Big)\,.$$

An even nicer way to write y_{part} is by writing the vector (c_1, c_2) in polar coordinates: if $c_1 = R\cos\psi$, and $c_2 = R\sin\psi$, then $R = \sqrt{c_1^2 + c_2^2}$, and ψ is the angle from the x axis to the vector from $(0, 0)$ to (c_1, c_2).

We can compute R quite explicitly:

$$\begin{aligned} R &= \sqrt{c_1^2 + c_2^2} \\ &= \sqrt{\frac{(C - A\omega^2)^2}{\Big((C - A\omega^2)^2 + B^2\omega^2\Big)^2}\alpha^2 + \frac{B^2\omega^2}{\Big((C - A\omega^2)^2 + B^2\omega^2\Big)^2}\alpha^2} \\ &= \alpha \sqrt{\frac{(C - A\omega^2)^2 + B^2\omega^2}{\Big((C - A\omega^2)^2 + B^2\omega^2\Big)^2}} \\ &= \frac{\alpha}{\sqrt{(C - A\omega^2)^2 + B^2\omega^2}} \end{aligned}$$

If we define θ by $\omega\theta = \psi$, then

$$y_{part}(t) = R\cos(\omega\tau - \psi) = R\cos\omega(\tau - \theta) = R\cos\omega(t - \varphi - \theta)\,,$$

so

$$\boxed{y_{part}(t) = \frac{\alpha}{\sqrt{(C - A\omega^2)^2 + B^2\omega^2}} \cos\omega(t - \varphi - \theta)\,.}$$

Finally,

The general solution of (11) is

$$y(t) = y_{gen,hom}(t) + \frac{\alpha}{\sqrt{(C - A\omega^2)^2 + B^2\omega^2}} \cos \omega(t - \varphi - \theta)\,,$$

where $y_{gen,hom}(t)$ is the general solution of the homogeneous equation

$$A\frac{d^2y}{dt}(t) + B\frac{dy}{dt}(t) + Cy(t) = 0\,. \qquad (16)$$

Analysis of the solutions in the damped case. Suppose our oscillator is *damped*, that is, $B > 0$. Then all the solutions of the homogeneous equation (16) go to zero as $t \to +\infty$.

Suppose we specify initial conditions $y(t_0) = y_0$, $y'(t_0) = \bar{y}_0$, and solve (11). Then the solution $y(t)$ will be a sum $\tilde{y}(t) + \frac{\alpha}{\sqrt{(C-A\omega^2)^2+B^2\omega^2}} \cos \omega(t - \varphi - \theta)$, where the function $\tilde{y}$ will be determined by the initial condition. (Recall that $y_{gen,hom}(t)$ contains two arbitrary constants. Using the initial condition, we find the constants, and obtain a function $y(t)$ with no constants at all.)

So our solution $y(t)$ is the sum of two parts:

- a function $\tilde{y}(t)$ that depends on the initial condition, and goes to zero as $t \to +\infty$,
- the function $\frac{\alpha}{\sqrt{(C-A\omega^2)^2+B^2\omega^2}} \cos \omega(t - \varphi - \theta)$, which does not depend on the initial condition, and does not go to zero as $t \to +\infty$.

The first part is called a *transient*, because it "goes away" as time passes. The second part is called a *steady-state solution*, and represents what one actually sees after a long time, when the transient part has practically disappeared, and all the effect of the initial condition is forgotten.

Furthermore, the steady-state solution is exactly a sinusoid of the same frequency as the input, and with an amplitude obtained by multiplying the input amplitude α by a number $\gamma(\omega)$ that depends on ω, and is given by

$$\gamma(\omega) = \frac{1}{\sqrt{(C - A\omega^2)^2 + B^2\omega^2}}\,. \qquad (17)$$

The number $\gamma(\omega)$ is called the *frequency response*, or *gain*, of our oscillator.

Summarizing,

Every solution of (11), for every initial condition, is the sum of a "transient part" that goes to zero as $t \to +\infty$ and a "steady-state part". The steady-state part is a sinusoid of the same frequency as the input, and has an amplitude equal to the amplitude of the input multiplied by the gain $\gamma(\omega)$ given by (17). The steady-state part does not depend on the initial condition at all. All the dependence on the initial condition is contained in the transient part, whose effect fades away as time passes. (In other words: the oscillator "forgets the initial condition.")

A tuning example.

PROBLEM: *Find the general solution and the steady-state solution of the forced harmonic oscillator equation*

$$\frac{d^2y}{dt}(t) + 4\frac{dy}{dt}(t) + 8y(t) = \cos\, t + \cos\, 5t\,. \tag{18}$$

SOLUTION. earlier, we found the general solution of the homogeneous equation (5). The result was

$$y_{hom,gen}(t) = r\, e^{-2t} \cos\, 2(t-\varphi)\,, \quad r \in \mathbb{R}\,, \varphi \in \mathbb{R}\,, r \geq 0\,.$$

To find the general solution of (18) we use the superposition principle, according to which the general solution of (18) is the sum of $y_{hom,gen}(t)$ plus a particular solution of the inhomogeneous equation with forcing term $\cos\, t$, plus a particular solution of the inhomogeneous equation with forcing term $5\cos\, 2t$.

To find the particular solutions with forcing terms $\cos\, t$ and $5\cos\, 2t$, we first compute the gain function γ. Clearly, γ is given by

$$\gamma(\omega) = \frac{1}{\sqrt{(8-\omega^2)^2 + 4^2\omega^2}}\,.$$

Then

$$\gamma(1) = \frac{1}{\sqrt{(8-1)^2+4^2}} = \frac{1}{\sqrt{7^2+4^2}} = \frac{1}{\sqrt{49+16}} = \frac{1}{\sqrt{65}} \sim \frac{1}{8.07} \sim 0.124\,.$$

Also,

$$\gamma(5) = \frac{1}{\sqrt{(8-5^2)^2+4^2\times 5^2}} = \frac{1}{\sqrt{13^2+16\times 25}} = \frac{1}{\sqrt{169+400}} = \frac{1}{\sqrt{569}} \sim \frac{1}{23.8} \sim 0.042\,.$$

So

- The steady-state response to the input $\cos t$ is $\frac{1}{\sqrt{65}}\cos\,(t-\varphi_1)$, for some angle φ_1. (This is exactly what we had found before.)
- The steady-state response to the input $\cos 5t$ is $\frac{1}{\sqrt{569}}\cos\, 5(t-\varphi_2)$, for some angle φ_2.
- The steady-state response to the input $\cos t + \cos 5t$ is $\frac{1}{\sqrt{65}}\cos\,(t-\varphi_1) + \frac{1}{\sqrt{569}}\cos\, 5(t-\varphi_2)$, that is, approximately,

$$0.124\, \cos\,(t-\varphi_1) + 0.042\, \cos\, 5(t-\varphi_2)\,.$$

Notice that

- The input is $\cos t + \cos 5t$, that is, a sum of two sinusoids with frequencies 1 and 5. In this input, both sinusoids have the same amplitude, so "both frequencies are represented with about the same strength."
- The steady-state response to the input is $0.124\, \cos\,(t-\varphi_1) + 0.042\, \cos\, 5(t-\varphi_2)$. This is also a sum of two sinusoids with frequencies 1 and 5. However, in this response the sinusoid with frequency 1 has an amplitude about three times larger than the sinusoid with frequency 5.

Now let us see what happens if we change one of the oscillator parameters, and take $C = 36$ instead of $C = 8$. That is, we will now look at the forced harmonic oscillator with equation

$$\frac{d^2y}{dt}(t) + 4\frac{dy}{dt}(t) + 36y(t) = \cos\, t + \cos\, 5t\,. \tag{19}$$

In this case, the gain function γ is given by

$$\gamma(\omega) = \frac{1}{\sqrt{(36-\omega^2)^2 + 4^2\omega^2}}\,.$$

Then

$$\gamma(1) = \frac{1}{\sqrt{(36-1)^2+4^2}} = \frac{1}{\sqrt{35^2+4^2}} = \frac{1}{\sqrt{1225+16}} = \frac{1}{\sqrt{1241}} \sim \frac{1}{35.2} \sim 0.028\,.$$

On the other hand,

$$\gamma(5) = \frac{1}{\sqrt{(36-5^2)^2+4^2\times 5^2}} = \frac{1}{\sqrt{11^2+16\times 25}} = \frac{1}{\sqrt{121+400}} = \frac{1}{\sqrt{521}} \sim \frac{1}{22.8} \sim 0.044\,.$$

Then

- The steady-state response to the input $\cos t + \cos 5t$ is $0.028\,\cos(t-\varphi_1) + 0.044\,\cos 5(t-\varphi_2)$. This is also a sum of two sinusoids with frequencies 1 and 5. However, in this response the sinusoid with frequency 5 has an amplitude about twice as large as that of the sinusoid with frequency 1.

In other words,

> *Our oscillator acts as a* **filter**, *or* **detector**. *If the input is a superposition of signals of different frequencies, the resulting output is also a superposition of signals of the same frequencies, but the actual proportions of the frequencies are changed, in such a way that some of them become relatively stronger.*
> *By changing the values of the oscillator parameters (that is, by "tuning" our receptor) we can change which frequencies are enhanced.*

Improving the filtering effect. If we want to use a forced harmonic oscillator such as the one of our example to separate out a signal we want from another one in which we are not interested, then the actual example we just gave does what we want, but very poorly.

If we use the oscillator with $C = 8$ to filter out the signal with frequency 5, then we get the desired sinusoid with frequency 1 with about 3 times the strength of the undesired sinusoid of frequency 5. This means that about 25% of the resulting response will still be in the unwanted frequency.

Similarly, if we use the oscillator with $C = 36$ to filter out the signal with frequency 1, then we get the desired sinusoid with frequency 5 with about twice the strength of the undesired sinusoid of frequency 1. This means that about 35% of the resulting response will still be in the unwanted frequency.

How can be improve the filtering effect? One obvious way is to design a more sophisticated system, by combining several oscillators. We take our forced oscillator, use its response as the input of another oscillator of the same time, then use the response as the input of a third oscillator of the same time, and so on. For example, if we do this three times, with $C = 8$, we get a system

$$\begin{aligned}
\frac{d^2y_1}{dt}(t) + 4\frac{dy_1}{dt}(t) + 8y_1(t) &= \cos t + \cos 5t\,,\\
\frac{d^2y_2}{dt}(t) + 4\frac{dy_2}{dt}(t) + 8y_2(t) &= y_1(t)\,,\\
\frac{d^2y_3}{dt}(t) + 4\frac{dy_3}{dt}(t) + 8y_3(t) &= y_2(t)\,.
\end{aligned}$$

(All this can be realized with electric circuits.) Now, in the steady-state response y_1 the frequency-1 part will be about three times larger than the frequency-5 part. And then in y_2 the frequency-1 part will be about nine times larger than the frequency-5 part. And, finally, y_3 the frequency-1 part will be about 27 times larger than the frequency-5 part.

A similar argument works with $C = 36$, and gives us a filter in which the frequency-5 part is about 16 times larger than the frequency-1 part.

Analysis of the gain function. The gain function γ is given by

$$\gamma(\omega) = \frac{1}{\sqrt{(C - A\omega^2)^2 + B^2\omega^2}} \,. \tag{20}$$

How does this function behave? What does its graph look like? For what frequency ω is the gain largest?

To understand the answers to these questions, we use the methods of Freshman Calculus. You might think that we have to compute the derivative $\frac{d\gamma}{d\omega}$, which looks awfully complicated. However, things are in fact much easier.

First of all, if we want to understand when $\gamma(\omega)$ is, for example, an increasing function of ω, it suffices to study the function

$$\eta(\omega) = (C - A\omega^2)^2 + B^2\omega^2 \,,$$

because γ increases when η decreases, γ decreases when η increases, and γ has a maximum when η has a minimum,

Again, you may think that studying η is hard, because it is a complicated plynomial in ω. However, if we use the variable $\xi = \omega^2$, then $\eta(\omega)$ is just $(C - A\xi)^2 + B^2\xi$, which is very easy to study.

Completing the square, we get

$$\begin{aligned}
(C - A\xi)^2 + B^2\xi &= C^2 + A^2\xi^2 - 2AC\xi + B^2\xi \\
&= C^2 + A^2\xi^2 + (B^2 - 2AC)\xi \\
&= C^2 + A^2\xi^2 + 2A\xi\Big(\frac{B^2 - 2AC}{2A}\Big) \\
&= C^2 + A^2\xi^2 + 2A\xi\Big(\frac{B^2}{2A} - C\Big) \\
&= C^2 + \Big(A\xi + \frac{B^2}{2A} - C\Big)^2 - \Big(\frac{B^2}{2A} - C\Big)^2 \\
&= C^2 + \Big(A\xi + \frac{B^2}{2A} - C\Big)^2 - \frac{B^4}{4A^2} - C^2 + \frac{CB^2}{A} \\
&= \frac{CB^2}{A} - \frac{B^4}{4A^2} + \Big(A\xi + \frac{B^2}{2A} - C\Big)^2 \\
&= \frac{CB^2}{A}\Big(1 - \frac{B^2}{4AC}\Big) + \Big(A\xi + \frac{B^2}{2A} - C\Big)^2 \,.
\end{aligned}$$

Now suppose that our oscillator is underdamped, so $B^2 < 4AC$. Then the number $\frac{CB^2}{A}\Big(1 - \frac{B^2}{4AC}\Big)$ is positive. Furthermore, the smallest possible value of $\eta(\omega)$ is obtained when

$$A\xi + \frac{B^2}{2A} - C = 0 \,,$$

that is, when

$$\xi = \frac{1}{A}C - \frac{B^2}{2A} \,,$$

or

$$\xi = \frac{C}{A} - \frac{B^2}{2A^2} \,,$$

which is equivalent to

$$\xi = \frac{2AC - B^2}{2A^2}\,.$$

Now suppose that $B^2 < 2AC$. Then the number ξ given by the previous equation is positive, and we can define

$$\omega_{max} = \frac{1}{\sqrt{2}A}\sqrt{2AC - B^2}\,.$$

This value of ω is the one for which $\eta(\omega)$ is as small as possible, so $\gamma(\omega)$ is as large as possible.

We have thus shown that

If $B^2 < 2AC$, then the frequency ω for which the $\gamma(\omega)$ has its maximum value is $\omega_{max} = \frac{1}{\sqrt{2}A}\sqrt{2AC - B^2}$.

We can now understand tuning better: if we want to tune our oscillator so that a particularly desired frequency $\bar{\omega}$ is selected, it suffices to change one of the parameters—for example C— so that ω_{max} will become $\bar{\omega}$. This can be done, for example, by choosing

$$C = A\bar{\omega}^2 + \frac{B^2}{2A}\,.$$

The graph of the gain function. Here is the graph of the gain function for $B^2 < 2AC$.

Two questiona for you to think about. What happens if $2AC < B^2 < 4AC$? What happens if $B^2 > 4AC$?