A Combinatorial Proof of $log(e^z) = z$

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As he was rushing to his Complex Analysis graduate class, my colleague, Hector Sussmann, asked me whether I can find a formal-power-series-proof of $log(e^z) = z$, where

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad ,$$

and

$$log(1+z) := \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k} \quad .$$

Of course, Hector was aware that one way is to use 'calculus', of the formal kind (of course, he knows that I don't believe in any other versions!), as outlined, for example, in my classroom note

`http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/lag.pdf` .

Indeed $(\log(e^z))' = (e^z)'/e^z = e^z/e^z = 1$ and 'integrating' gives the desired result, but he asked for a *direct* calculus-free proof, even of the *good* (i.e. formal) kind.

Here goes

$$log(e^z) = log(1 + \sum_{k=1}^{\infty} \frac{z^k}{k!}) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (\sum_{a=1}^{\infty} \frac{z^a}{a!})^k$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \prod_{i=1}^{k} (\sum_{a_i=1}^{\infty} \frac{z_i^a}{a_i!})$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \sum_{n=1}^{\infty} \left(\sum_{\substack{(a_1,\dots,a_k) \in N^k \\ a_1+\dots+a_k=n}} \frac{1}{a_1! \cdots a_k!} \right) z^n$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \sum_{n=1}^{\infty} k! S(n,k) \frac{z^n}{n!} \quad ,$$

where $S(n,k)$, the Stirling numbers of the second kind, are the number of set partitions of $\{1, 2, \dots, n\}$ into k sets. It is well-known and trivial to see that

$$S(n,k) = S(n-1, k-1) + kS(n-1, k) \quad ,$$

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(element n may be either a loner, in which case the rest are partitioned into $k-1$ sets or it joins one of the k existing sets). Continuing, we get, by changing order of summation,

$$= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} (k-1)! S(n,k) \right) \frac{z^n}{n!}$$

The coefficient of $z^n/n!$ is

$$\sum_{k=1}^{n} (-1)^{k-1}(k-1)![S(n-1,k-1) + kS(n-1,k)] =$$

$$\sum_{k=1}^{n} (-1)^{k-1}(k-1)!S(n-1,k-1) + \sum_{k=1}^{n-1} (-1)^{k-1} k! S(n-1,k) =$$

$$\sum_{k=1}^{n-1} (-1)^{k} k! S(n-1,k) - \sum_{k=1}^{n-1} (-1)^{k} k! S(n-1,k) = 0 \quad ,$$

if $n > 1$. Of course, when $n = 1$ it is 1.