Commentary on Norman Levinson's paper:

[L 91] Minimax, Liapunov and "Bang-Bang," *J. Differential Equations* **2** (1966), 218–241. (MR 33 #2447 (H. Halkin)).

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In this paper, his only venture into control theory, Levinson solves the following basic problem of fundamental importance in the theory of linear control systems. Use $\mathbb{R}^m$ to denote real m-dimensional Euclidean space, thought of as a space of column vectors. Let P be a compact convex polyhedron in $\mathbb{R}^m$, and let the set of vertices of P be denoted by V. Let $[a,b]$ be a given compact subinterval of $\mathbb{R}$, and suppose we fix a piecewise analytic $n \times m$ matrix-valued function $[a,b] \ni t \to F(t) \in \mathbb{R}^{n\times m}$. Write $BB([a,b],V)$ to denote the set of all maps $[a,b] \ni t \to w(t) \in V$ having a finite number of discontinuities on $[a,b]$. Since V is a finite set, the members of $BB([a,b],V)$ are exactly the piecewise constant V-valued maps on $[a,b]$. In control theory, such maps w are called *bang-bang controls.* The basic questions are:

(i) *Which vectors* $\gamma \in \mathbb{R}^n$ *can be represented in the form* $\gamma = \beta_{w,a,b}$ *, where*

$$\beta_{w,a,b} := \int_a^b F(t)w(t)dt\,, \tag{1}$$

and $w \in BB([a,b],V)$*?*

(ii) *How does the minimum number of discontinuities of the piecewise constant control function* w *on the given interval* $[a,b]$ *depend on the vector* γ*?*

(iii) *For a given* a*,* γ*, is there, among all the pairs* (b,w) *such that* $\gamma = \beta_{w,a,b}$ *, one with a minimum value of* b*? (In other words, is there a minimum-time bang-bang control?)*

THE CONTROL THEORY CASE: In control theory, one studies linear control systems in $\mathbb{R}^n$ of the form

$$\dot{x} = A(t)x + B(t)w\,, \tag{2}$$

where the real n-vector $x(t)$ represents the state of the system at time t, the real m-vector $w(t)$ is the control, and A, B are integrable functions on an interval $[a,b]$, with values in the spaces $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ of $n\times n$ and $n\times m$ matrices, respectively.

For a given initial condition $x(a) = \bar{x}$, and a given control function $t \to w(t)$, the solution $\xi_{w,\bar{x}}$ of (2) is given by

$$\xi_{w,\bar{x}}(t) = M(t,a)\bar{x} + \int_a^t M(t,s)B(s)w(s)ds\,,$$

where $[a,b]\times[a,b] \ni (t,s) \to M(t,s) \in \mathbb{R}^{n\times n}$ is the fundamental matrix solution of the linear system $\dot{X}(t) = A(t)X(t)$. The set $Reach_{A,B,\bar{x},a,b}$ of all possible vectors $\xi_{w,\bar{x}}(b)$, for all possible choices of the control w (for a fixed $\bar{x}$), is the "reachable set," or "attainable set," from $\bar{x}$ for the system (2), over the interval $[a,b]$.

The situation considered by Levinson is a special case of the linear control problem (2), corresponding to $A(t) \equiv 0$, $B(t) = F(t)$, $\bar{x} = 0$. In particular, the range $\mathcal{R}_{\beta,a,b}$ of the map $w \to \beta_{w,a,b}$ studied by Levinson is the same as the reachable set $Reach_{0,F,0,a,b}$. Conversely, for a control system (2), we have $\xi_{w,\bar{x}}(b) = \hat{x} + \int_a^b F(t)w(t)dt$, if we define $\hat{x} := M(b,a)\bar{x}$ and $F(t) := M(b,t)B(t)$, and then the reachable set $Reach_{A,B,\bar{x},a,b}$ is exactly $\mathcal{R}_{\beta,a,b}$, up to a translation. Since F is piecewise analytic if A and B are, we see that a general answer to Question (i) for Levinson's special situation immediately yields an answer for general reachable sets of linear control systems with piecewise analytic functions A and B.

In the special case studied by Levinson, for the vector $\beta_{w,a,b}$ given by (1) to be well defined it is not necessary that the control w take values in V and be piecewise constant. In fact, w can take arbitrary values in $\mathbb{R}^m$ as long as it is integrable. Under the constraint that w takes values in the polyhedron P (in which case integrability is of course equivalent to measurability), one wants to compare the bang-bang range $\mathcal{R}_{\beta,a,b}$ defined earlier with the range $\hat{\mathcal{R}}_{\beta,a,b}$ of the map $w \to \beta_{w,a,b}$ as w varies over the space $M([a,b],P)$ of all P-valued measurable controls w. Since $M([a,b],P)$ is much larger than $BB([a,b],V)$, it is natural to expect that $\hat{\mathcal{R}}_{\beta,a,b}$ would be much larger than $\mathcal{R}_{\beta,a,b}$.

It turns out, however, that *the two sets are equal.* In other words, *if a point γ admits an expression of the form $\gamma = \beta_{w,a,b}$ with a measurable P-valued w, then w can always be chosen to be V-valued and piecewise constant.* This statement is the so-called *bang-bang theorem* of linear system theory. To put this result in proper perspective, let us first point out that (i) $\hat{\mathcal{R}}_{\beta,a,b}$ is compact and convex, and (ii) $\mathcal{R}_{\beta,a,b}$ is dense in $\hat{\mathcal{R}}_{\beta,a,b}$. (These two observations follow trivially from the facts that (i') $M([a,b],P)$ is convex and weak*-compact, and (ii') $BB([a,b],V)$ is weak*-dense in $M([a,b],P)$.)

On the other hand, it is not at all obvious that $\mathcal{R}_{\beta,a,b}$ is convex or compact. In view of Facts (i) and (ii) above, the bang-bang theorem is clearly equivalent to the assertion that $\mathcal{R}_{\beta,a,b}$ is compact, and also to the stronger statement that $\mathcal{R}_{\beta,a,b}$ is compact and convex.

So one can prove the bang-bang theorem and the fact that $\mathcal{R}_{\beta,a,b}$ is compact and convex by means of either one of the following two basic strategies:

(a) Establish first the equality of $\mathcal{R}_{\beta,a,b}$ and $\hat{\mathcal{R}}_{\beta,a,b}$, and then conclude that $\mathcal{R}_{\beta,a,b}$ is compact and convex.

(b) Prove directly that $\mathcal{R}_{\beta,a,b}$ is compact, without studying $\hat{\mathcal{R}}_{\beta,a,b}$, and then conclude from the density of $\mathcal{R}_{\beta,a,b}$ in $\hat{\mathcal{R}}_{\beta,a,b}$ that $\mathcal{R}_{\beta,a,b} = \hat{\mathcal{R}}_{\beta,a,b}$.

Levinson's contribution was to answer Question (i) above by proposing a proof of the bang-bang theorem along the lines of strategy (b). His work yields a further result that in later years has turned out to be of the utmost importance in control theory, namely, an answer to Question (ii), in the form of a **bound on the number of switchings**.

Precisely, Levinson answers Questions (i) and (ii) as follows (cf. Theorems 1 and 2 in [L 91]):

Let P be a convex polyhedron in $\mathbb{R}^m$, and let V be the set of vertices of P. Assume that $[a,b] \ni t \to F(t) \in \mathbb{R}^{n\times m}$ is a piecewise analytic real $n \times m$ matrix function on the compact interval $[a,b]$, and define the n-component vector $\beta_{w,a,b}$ by (1), for every bang-bang control function $w \in BB([a,b],V)$. *Then the range $\mathcal{R}_{\beta,a,b}$ of the map $w \to \beta_{w,a,b}$, from $BB([a,b],V)$ to $\mathbb{R}^n$, is compact and convex in $\mathbb{R}^n$. Assume further that V has N members $v^{(j)}, j = 1,\ldots,N$. Then, for a real n-vector $\gamma \in \mathbb{R}^n$ to belong to $\mathcal{R}_{\beta,a,b}$, it is necessary and sufficient that the minimax condition*

$$J(\gamma,b) = \min\left\{\int_a^b \max_j (r'F(t)v^j)dt - r'\gamma : r \in \mathbb{R}^n, |r| = 1\right\} \geq 0 \tag{3}$$

(where, for $r \in \mathbb{R}$, r' denotes the transpose of r) hold.
Under his assumptions, Levinson then proves the *bound on the number of switchings*, a result that is perhaps the most important contribution of his paper, but has been overlooked in many subsequent works on reachable sets and time-optimal control: *There exists a positive integer ν, independent of γ and t, such that for every $t \in [a,b]$ and every $\gamma \in \mathcal{R}_{\beta,a,t}$ the bang-bang control $w : [a,t] \to V$ such that $\gamma = \beta_{w,a,t}$ can be chosen so that w has at most ν discontinuities.*

Since J in (3) is obviously continuous as a function of γ and b and since $J(\gamma, a) = -|\gamma| \leq 0$, it follows immediately that *if the minimax condition $J(\gamma, b) \geq 0$ holds (i.e. if $\gamma \in \mathcal{R}_{\beta,a,b}$), then there exists a unique $T \in [a,b]$ such that*

$$J(\gamma, T) = 0\,, \quad \text{and} \quad J(\gamma, \tau) < 0\,, \quad \text{for} \quad a \leq \tau < T\,. \tag{4}$$

Thus there is a $T_\gamma \in [a,b]$ such that $\gamma \in \mathcal{R}_{\beta,a,T_\gamma}$ but $\gamma \notin \mathcal{R}_{\beta,a,\tau}$ for $a \leq \tau < T_\gamma$. In other words, Levinson's result provides the following complete answer to Question (iii): *For every $\gamma \in \cup_{a \leq \tau \leq b} \mathcal{R}_{\beta,a,\tau}$ there is a time-optimal bang-bang control. That is, the set of those $\tau \in [a,b]$ such that $\gamma \in \mathcal{R}_{\beta,a,\tau}$ has a minimum element T_γ. Moreover, the bang-bang control $w : [a, T_\gamma] \to V$ such that $\gamma = \beta_{w,a,T_\gamma}$ can be chosen to have at most ν discontinuities.*

In the modern terminology of convex analysis, Levinson's minimax theorem says that $\mathcal{R}_{\beta,a,b}$ is compact and convex, and the real-valued function φ on $\mathbb{R}^n$ given by

$$\varphi(r) := \int_a^b \max_j (r' F(t) v^j) dt \tag{5}$$

is the *supporting functional* of $\mathcal{R}_{\beta,a,b}$, since $\mathcal{R}_{\beta,a,b}$ is the set of all $\gamma \in \mathbb{R}^n$ such that $r'\gamma \leq \varphi(r)$ for all $r \in \mathbb{R}^n$.

Liapunov's original theorem [2] (cf. LaSalle [4] for an explanation of the theorem and its relevance to control theory) is restated by Levinson in the context of his paper in the following equivalent form: Regard the closed interval $[0,1]$ as the one-dimensional polyhedron P with vertices at 0 and 1 . Let $[a,b] \ni t \to F(t) \in \mathbb{R}^n$ be an integrable vector-valued function on $[a,b]$. If the controls $t \to w(t)$ are scalar functions, measurable on $[a,b]$ and with values in the set of vertices 0 and 1 , then the range $\tilde{\mathcal{R}}_{\beta,a,b}$ of the map $w \to \beta_{w,a,b}$ defined by Eq. (1) is closed and convex. A more general version of Liapunov's result can be found in [8] or in [9].

Using this theorem, LaSalle had proved a bang-bang principle for optimal controls for general linear systems (2), yielding a minimum-time optimal control which is *measurable* —but not necessarily piecewise constant— and with values in V. As Levinson notes and others before him had realized, notably Halkin [7], the condition that a bang-bang control $t \to w(t)$ be merely measurable is not suitable for engineering applications. For these, bang-bang controls $t \to w(t)$ that are piecewise constant with a finite number of jumps are far more reasonable. So it is desirable to have theorems stating, in Halkin's words ([7], p. 201), that "anything which can be done with an arbitrary admissible control can also be done with a relay control with a finite number of switching times." Halkin [7] established this property in 1965 (by completely different methods) for *optimal controls* in real, linear time-varying systems of the form (2), where A and B are each piecewise analytic on $[a,b]$. As Levinson notes in [L 91], p. 219, it had been shown earlier in the book [5] that for linear systems of the form (2) "satisfying a rather complicated condition, called the general position condition," all time-optimal controls are bang-bang, with a finite number of discontinuities.

The results of Halkin and Levinson are fundamentally different from that of [5], in that they do not require any general position condition, and do not state that all optimal controls are bang-bang, but only that *among the optimal controls there is always one that is bang-bang.* But note that when the "general position condition" of [5] is satisfied, then the optimal control is unique. Thus the results of Halkin and Levinson imply that it must be bang-bang.

To see the difference between the conclusions of [5] and those of Halkin and Levinson, consider the two-dimensional linear system

$$\dot{x} = u\,, \qquad \dot{y} = v\,, \tag{6}$$

where the values of the two-dimensional control vector (u, v) are restricted to lie in the square $P = [-1, 1] \times [-1, 1]$ in $\mathbb{R}^2$. (This is a special case of (2), with $n = m = 2$, $A(t) \equiv 0$, and $B(t)$ equal to the identity matrix.) Suppose we want to transfer the point $(0, 0)$ to $(0, 1)$ in minimum time. Since $\dot{y}(t) \leq 1$ along any trajectory, the transfer cannot be achieved in time T unless $T \geq 1$. On the other hand, *any control* $[0,1] \ni t \to (u(t), v(t)) \in P$ *for which* $v(t) \equiv 1$ *and* $\int_0^1 u(t)dt = 0$ *achieves the desired transfer.* Thus *the optimal time is* 1, *but there are many optimal controls that are not bang-bang*, such as, for example, $u(t) \equiv 0$, or $u(t) \equiv t - \frac{1}{2}$. So the conclusion of the theorem in [5] fails for this example, but the results of Halkin and Levinson still apply, because among the many optimal controls one can always *choose* one that is bang-bang, for example by taking $u(t) = 1$ for $0 \leq t \leq \frac{1}{2}$ and $u(t) = -1$ for $\frac{1}{2} < t \leq 1$. Observe that the general position condition of [5] is violated in this case.

Levinson's treatment in [L 91], like that of Halkin in [7], goes beyond the result [5] in that it does not impose any general position requirement and involves *choice*, as illustrated by our preceding example. On the other hand, as observed independently by Z. Artstein and L. Berkovitz in recent correspondence pertaining to (2), *Levinson's analysis is not restricted to optimal controls, and achieves a true bang-bang solution satisfying a uniform bound on the number of switchings.*

Levinson's proofs are highly original and strikingly different from all other presentations. They do not depend at all on measure theory but rather on his solution of the minimax problem stated above. The uniform bound on the number of discontinuities of the piecewise constant function $t \to w(t)$ in his main theorem —also implying the uniform bound on the number of switchings of the control in (2)— is obtained by an elegant application of Rouché's theorem (cf. Lemma 2.6 in [L 91]).

Regarding Theorem 1 in his paper [L 91], Levinson thanks the referee for pointing out that "except for the last statement" (i.e. the uniform bound on the number of discontinuities), "Theorem 1 could be proved by the methods of Halkin [7]." A similar statement is also made by the reviewer of [L 91] (Halkin, in MR 33 #2447). The results of Levinson and Halkin pertaining to the control problem (2) are referenced in the 1975 monograph by Fleming & Rishel [16], p. 78, but with no mention of Levinson's uniform bound. Regrettably, Levinson's results are not referenced in any other books on control theory in which bang-bang controls are discussed (*e.g.*, by Berkovitz [15] or by Neustadt [17]). This omission leads to the conclusion that the significance of Levinson's bound on the number of switchings in a more applicable theory of reachable sets than now exists in print, *e.g.*, the paper of Cz. Olech [11], or Berkovitz [15], Ch. IV, Theorems 6.3 and 6.4, among others, has been missed by control theorists of the classical "real-variables-and-measure-theory" tradition.

On the other hand, subsequent developments in real-analytic geometry and nonlinear differential-geometric control theory reveal that *Levinson's uniform bound on the number of* *switchings is a critical result with deep consequences about the structure of reachable sets and the regularity properties of the value function.* This is because Levinson's bound makes it possible to apply to linear system theory the powerful machinery of *subanalytic sets*, which did not yet exist when Levinson wrote his paper, but was developed later by Gabrielov, Hironaka, Hardt, and others (cf. [12, 44], [13], [18], and also [38] for a recent elementary introduction).

Roughly speaking, subanalytic sets are proper real-analytic images of semianalytic sets. Precisely, a subset S of a real-analytic manifold M is *semianalytic* if every point p in M has a neighborhood U such that $S \cap U$ belongs to the Boolean algebra of subsets of U generated by the sets of the form $\{x \in U : f(x) = 0\}$ and $\{x \in U : f(x) > 0\}$, for all possible real-analytic functions $f : U \to \mathbb{R}$. One then calls S *subanalytic* if there exist a real-analytic manifold M', a semianalytic subset S' of M', and a real-analytic map $\Phi : M' \to M$, such that $\Phi(S') = S$ and Φ is proper on the closure of S' (that is, if $K \subseteq M$ and K is compact, then $\Phi^{-1}(K) \cap S'$ is relatively compact in M').

It is not difficult to show, using the above definition together with Levinson's theorem, that the sets $\mathcal{R}_{\beta,a,b}$ introduced above are finite unions of subanalytic sets, and hence $\mathcal{R}_{\beta,a,b}$ *is a subanalytic subset of* $\mathbb{R}^n$. It is also easy to show that the sets $\mathcal{R}^{\#}_{\beta,a,b} = \cup_{a \le t \le b} \mathcal{R}_{\beta,a,t}$ are subanalytic.

Naturally, if the application of the theory of subanalytic sets to Levinson's problem merely amounted to proving that the sets studied by Levinson are subanalytic, this would not be of much interest. It turns out, however, that *knowing that a set is subanalytic provides a great deal of information about its structure*, because subanalytic sets have very nice properties. For example: a subanalytic set is a locally finite union of pairwise disjoint connected embedded real-analytic submanifolds that are themselves subanalytic sets. So, once we know that $\mathcal{R}_{\beta,a,b}$ is subanalytic and compact, we can conclude that $\mathcal{R}_{\beta,a,b}$ is a finite union of pairwise disjoint connected embedded real-analytic submanifolds which can be taken to be subanalytic sets.

The subanalyticity of the reachable sets $\mathcal{R}^{\#}_{\beta,a,b}$ implies an important regularity result, namely, *that the minimum-time function is piecewise analytic*, in a sense that will be made precise below.

Suppose that we are given a subinterval I of $\mathbb{R}$ of the form $[a,b]$ or $[a,+\infty[$ and a piecewise analytic map $F : I \to \mathbb{R}^{n\times m}$. (If $I = [a,+\infty[$, then we say that F is piecewise analytic on I if the restriction of F to $[a,b]$ is piecewise analytic in the sense defined before for every $b \in I$.) Define a function $\tau : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by letting $\tau(\gamma)$ be, for $\gamma \in \mathbb{R}^n$, the infimum of all the times $t \in I$ such that $\gamma \in \mathcal{R}_{\beta,a,t}$. Levinson's result tells us, first of all, that if the set $Q(\gamma) := \{t \in I : \gamma \in \mathcal{R}_{\beta,a,t}\}$ is nonempty then its infimum is in fact a minimum, so $\tau(\gamma)$ is the minimum of the times t such that $t \in I$ and $\gamma \in \mathcal{R}_{\beta,a,t}$. (If $Q(\gamma) = \emptyset$, then of course $\tau(\gamma) = +\infty$.)

The subanalyticity of the reachable sets then implies that the set

$$\hat{G} = \{(\gamma,t) : \gamma \in \mathbb{R}^n\,,\, t \in I\,,\, \gamma \in \mathcal{R}_{\beta,a,t}\} \tag{7}$$

is subanalytic in $\mathbb{R}^{n+1}$. (Proof: define $\tilde{F} : I \to \mathbb{R}^{(n+1)\times m}$ by letting $\tilde{F}(t) : \mathbb{R}^m \to \mathbb{R}^{n+1}$ be the linear map that sends $v \in \mathbb{R}^m$ to $(F(t)v, 1)$. Let $\tilde{\mathcal{R}}_{\beta,a,t}$ be the set of all points of the form $(0,a) + \int_a^t \tilde{F}(s)w(s)$, for some $w \in BB(I,V)$. Then the sets $Q_t = \cup_{a \le \tau \le t} \tilde{\mathcal{R}}_{\beta,a,t}$ are subanalytic. Clearly, $\hat{G} = \cup_{a \le t} Q_t = \cup_{k=1}^{\infty} \hat{H}_k$, where $H_1 = Q_{a+1}$, and $H_k = Q_{a+k} \backslash Q_{a+k-1}$ for $k \ge 2$. The sets H_k are obviously subanalytic, and the family $\{H_k\}_{k=1}^{\infty}$ is locally finite, so $\hat{G}$

is subanalytic as well.)

From this one concludes that *the graph G of the function τ is subanalytic.* Indeed,

$$G = \left\{(\gamma, t) \in \hat{G} : (\forall t' \in \mathbb{R})(t' < t \Rightarrow (\gamma, t') \notin \hat{G})\right\}, \tag{8}$$

and the subanalyticity of G now follows from general properties of subanalytic sets (cf. [38]). Finally, once we know that the graph of a function τ is subanalytic, this implies many useful properties of the function that can be stated without mentioning subanalyticity. For example, the function τ is "piecewise analytic," in the following precise sense: *there exists a τ-locally finite partition of the domain $D(\tau)$ of τ into connected embedded real-analytic submanifolds on each of which τ is real-analytic.* (By definition, $D(\tau) = \{\gamma : \tau(\gamma) < +\infty\}$. A set $\mathcal{S}$ of subsets of $D(\tau)$ is *τ-locally finite* if $\{S \in \mathcal{S} : S \cap C \neq \emptyset\}$ is finite for every compact subset C of $D(\tau)$ on which τ is bounded. Notice that in general τ is lower semicontinuous but not continuous on $D(\tau)$, so there can be compact subsets of $D(\tau)$ on which τ is unbounded.)

The idea of using the theory of subanalytic sets in optimal control theory to prove properties of the minimum-time function is due to P. Brunovský, cf. [19, 22]. In his 1978 paper [19], Brunovský used the bound on the number of switchings to prove the existence of a *regular synthesis* for linear control systems of the form (2) with constant matrices A, B, in a sense similar (but not identical) to the one introduced by V. G. Boltyanski in 1966 in [10]. (Brunovský's precise definition is given in [19], cf. also [26, 29]. A regular synthesis is essentially a piecewise analytic optimal feedback control, and the problem of the existence of such a synthesis is closely related to that of the piecewise analyticity of the minimum-time function.) Brunovský used the bang-bang theorem of [5] rather than Levinson's version, and for that reason had to make the unnecessary assumption that the system was *normal*, i.e. satisfied the general position requirement of [5].

The technique of proving bounds on the number of switchings, and using them to establish subanalyticity of attainable sets and minimum-cost functions, has been used for many other classes of *nonlinear* control systems (cf. Brunovský [22], Sussmann [21, 20, 23, 29, 24, 25, 32, 33, 34, 37, 40], Bressan [31], Schättler [35, 36]). This has led to the discovery that these questions are related to many other interesting mathematical issues in analysis, dynamical systems and differential geometry. We now outline some of these issues, to show how *Levinson's result on the number of switchings, though extremely special if narrowly interpreted, is in fact very fruitful as a source of new generalizations if given a broader interpretation.*

To begin with, it is not surprising that as soon as one goes from linear to nonlinear systems the bang-bang property in general no longer holds. For example, consider the nonlinear control system

$$\dot{x} = 1 - y^2, \qquad \dot{y} = u, \tag{9}$$

in the xy plane, where the scalar control u takes values in the interval $[-1, 1]$. Suppose we are interested in a control $t \to u(t)$ that transfers the initial state $(0,0)$ to the terminal state $(1,0)$ in minimum time. It is clear that, if we define $u(t) \equiv 0$ for $t \in [0,1]$, then the corresponding solution of (9) with initial condition $(0,0)$ at time 0 is the curve $t \to (t, 0)$. So this control transfers $(0,0)$ to $(1,0)$ in time 1. On the other hand, if $[a,b] \ni t \to (x(t), y(t)) \in \mathbb{R}^2$ is any trajectory of (9) such that $(x(a), y(a)) = (0,0)$ and $(x(b), y(b)) = (1,0)$, then

$$1 = x(b) - x(a) = \int_a^b (1 - y(t)^2)dt \leq b - a,$$

with equality holding if and only if $y(t) \equiv 0$, in which case $u(t) \equiv 0$. This shows that the minimum time for the transfer from $(0,0)$ to $(1,0)$ is exactly equal to 1, and the only control that achieves this transfer is given by $u(t) \equiv 0$, which is not bang-bang!

The preceding example shows that the bang-bang theorem fails to hold as soon as control systems with nonlinearities are considered. This, however, is true if one insists on the strict interpretation that the crucial point of Levinson's result is its *bang-bang* aspect. If, on the other hand, one focuses on the *bound on the number of switchings* as the critical issue, one discovers that there are large classes of nonlinear systems for which such bounds hold, *except that now one has to consider piecewise constant controls with values in subsets $\tilde{V}$ of the polyhedron P other than the set V of its vertices.* For example, for the system (9) it turns out that if one takes $\tilde{V} = \{-1, 0, 1\}$, then the conclusion of Levinson's result still holds. Another example of this situation had in fact been studied by L. Dubins (cf. [3]), in his analysis of a problem originally posed by A. A. Markov in 1887, cf. [1], on the shortest paths in the plane with a prescribed bound on the curvature. In this problem one has a control polyhedron P equal to the interval $[-1, 1]$, and the bound on the number of switchings holds with $\tilde{V} = \{-1, 0, 1\}$. A closely related problem was studied by Reeds and Shepp in 1990 (cf. [41], and also [42] for a control-theoretic analysis of the Markov-Dubins and Reeds-Shepp problems). In this case, the control polyhedron is the square $P = [-1, 1] \times [-1, 1]$ in $\mathbb{R}^2$, and it turns out that the conclusion of Levinson's theorem holds if one takes $\tilde{V}$ to be a set of *six* points of P, namely, the four vertices plus the points $(1, 0)$ and $(-1, 0)$.

If one wants to extend Levinson's theorem —properly reinterpreted— to general nonlinear systems, it is not enough to consider general piecewise constant controls instead of bang-bang ones. For example, consider the system

$$\dot{x} = 1 - \Big(y - x(1-x)\Big)^2, \qquad \dot{y} = u, \tag{10}$$

in $\mathbb{R}^2$, with scalar control u taking values in the interval $[-1, 1]$ as before. Suppose we want to transfer the initial state $(0, 0)$ to the terminal state $(1, 0)$ in minimum time. The control $t \to 1 - 2t$ effects the desired transfer in time 1 (with $y(t) = t(1-t)$, $x(t) = t$), and an argument similar to the one given in our analysis of (9) shows that 1 is the minimum time, and $t \to 1 - 2t$ is the only minimum-time control. Clearly, this minimum-time control is not piecewise constant.

What is happening here is that there is a singular arc S in the xy plane, given by the equation $y = x - x^2$. In order to construct a family of trajectories which is "sufficient for time-optimality" (i.e., paraphrasing Halkin, a family $\mathcal{T}$ such that "anything which can be done with an arbitrary admissible trajectory can also be done with a trajectory belonging to $\mathcal{T}$"), one has to consider finite concatenations of basic pieces selected from a larger class: in addition to the "bang trajectories" (i.e. trajectories arising from constant controls equal to $+1$ or to -1) one also has to use the singular arcs, i.e. the trajectories that are contained in S. (Notice that we have shifted from talking about "controls" to talking about "trajectories". This change of point of view is absolutely necessary. For example, for the problem of (10) the control $t \to 1 - 2t$ is "singular," and must be included as one of the basic constituents of the more general optimal controls, only when it gives rise to a trajectory contained in S.)

The above examples suggest a general research program in the theory of nonlinear control systems. Briefly, the program involves the identification of large classes of problems for which one can prove analogues of Levinson's bang-bang theorem with bounds on the number of switchings in which the basic constituent pieces are allowed to be more general than constant

V-valued controls. Then, one still seeks to establish that, for a reasonable class $\mathcal{B}$ of such basic pieces, there is a sufficient family of trajectories consisting of finite concatenations of members of $\mathcal{B}$, with some *a priori* bound on the number of pieces. To pursue this program beyond the level of particular examples, one needs mathematical tools and a language for talking about general properties of nonlinear systems. This language turns out to be that of *Lie brackets* and *Lie algebras of vector fields.*

First of all, notice that, if we consider control systems of the form (2), then the conclusion of Levinson's theorem (except for the convexity of the reachable sets) is *invariant under arbitrary nonlinear changes of coordinates.* That is, if one takes a system of the form (2) —with a polyhedral control set P, as before— and performs a smooth invertible nonlinear transformation $y = \Phi(x)$ of the state space (where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one, onto, and of class C^1 with an inverse also of class C^1), then the transformed system

$$\dot{y} = \frac{\partial \Phi}{\partial x}\Big(\Phi^{-1}(y)\Big)\Big[A(t)\Phi^{-1}(y) + B(t)w\Big] \tag{11}$$

will in general be highly nonlinear, but will still have Levinson's bang-bang property with bounds on the number of switchings. However, it will no longer be true in general for (11) that the reachable sets are convex, since nonlinear transformations do not preserve convexity. Thus the bang-bang property is not really a property of *linear systems*, but of *systems that are equivalent to a linear system under a possibly nonlinear diffeomorphism.* From a nonlinear perspective, one should then be able to characterize those systems that are equivalent to linear ones under a diffeomorphism.

For control systems of the form

$$\dot{x} = f_0(x) + u_1 f_1(x) + \ldots + u_m f_m(x)\,, \tag{12}$$

one can introduce invariants, expressed in terms on the Lie brackets of the vector fields f_i, whose vanishing is equivalent to the property of "linearizability under a change of coordinates." So the bang-bang theorem can be restated as a theorem about nonlinear systems for which those invariants vanish. But then, quite remarkably, it turns out that, for the bang-bang theorem with bounds on the number of switchings to hold, one does not need the invariants to vanish, and it suffices for a much weaker set of relations to hold. This is carried out in detail in [20] for systems $\dot{x} = f(x) + ug(x)$, with a scalar control belonging to $[-1, 1]$. (The case of vector-valued controls has not been completely settled as yet.) It turns out that linearizability under a nonlinear change of coordinates is equivalent to the vanishing of all iterated Lie brackets of f and g that involve at least two g's. For the bang-bang property to hold, it suffices that some of these brackets be expressible as linear combinations of some other brackets. For example, it suffices that every bracket with two g's be expressible as a linear combination of brackets with only one g and of smaller degree. Notice that for the system (9) we have

$$f = (1 - y^2)\frac{\partial}{\partial x}\,,\;\; g = \frac{\partial}{\partial y}\,,\;\; [f, g] = 2y\frac{\partial}{\partial x}\,,\;\; [g, [f, g]] = 2\frac{\partial}{\partial x}\,. \tag{13}$$

Then $[g, [f, g]]$ never vanishes, and therefore (9) cannot be linearized by a local change of coordinates near any point of the plane. On the other hand, the real obstruction to the validity of the bang-bang theorem in a neighborhood of a point p is the impossibility of expressing $[g, [f, g]]$ as a linear combination of g and $[f, g]$ near p. As long as $[g, [f, g]]$ admits such an expression, the bang-bang property holds near p. It is clear from (13) that $[g,[f,g]]$ is a linear combination of g and $[f,g]$ near p for every p, except when p lies on the x-axis. This is the true "differential geometric" explanation of the occurrence for (9) of non bang-bang optimal controls, and of the fact that they occur precisely on the x axis and nowhere else.

A second line of research naturally motivated by Levinson's result proceeds by dropping the "bang-bang" requirement and concentrating on the more general property of the existence of *some* sufficient family —possibly involving singular controls— that can be used to prove subanalyticity of reachable sets or piecewise analyticity of the minimum-time function. A general theorem in this direction has been proved by Brunovský in [22], and a complete analysis of real-analytic problems in the plane was given in [32, 33, 34]. The general question of the piecewise regularity of optimal trajectories for real-analytic nonlinear systems is still open. There exist examples of problems where "infinite chattering" occurs: the only optimal control is bang-bang but has infinitely many switchings. The simplest such example is *Fuller's problem*: for the system $\dot{x} = y$, $\dot{y} = u$, $\dot{z} = x^2$, with control constraint $|u| \leq 1$, it can be shown that there is infinite chattering for suitably chosen initial and terminal conditions in $\mathbb{R}^3$, cf. [6, 14, 39], and the book [43], which is entirely devoted to the Fuller phenomenon and its generalizations. One of the major open problems in the theory is to decide whether, for real-analytic control systems of the form (12) with a polyhedral control set P, it is always true that, when there exists a measurable P-valued time-optimal control $[a,b] \to u(t)$ steering a point x_- to a point x_+, it follows that there exists a time-optimal control $[a,b] \to \tilde{u}(t)$ that also steers x_- to x_+ and is real-analytic on the complement of a closed countable subset S of $[a,b]$. (Fuller's example shows that the answer is "no" if S is required to be finite rather than countable. In [30] it is shown that the answer is "yes" if S is only required to be closed and nowhere dense, but it is not known —even for systems in $\mathbb{R}^3$ with a scalar control— if "closed and nowhere dense" can be strengthened to "closed and of measure zero" or "closed and countable." It is also shown in [30] that, if the vector fields f_i are only required to be of class C^∞ rather than real-analytic, then even the weaker conclusion with S closed and nowhere dense can fail.)

A third natural question is that of the subanalyticity of reachable sets for linear systems (2) with a control set P which is compact, convex and semialgebraic but not necessarily a polyhedron. In [28] an example is given of a linear system in $\mathbb{R}^3$, with constant matrices A, B, and a rather simple two-dimensional control set $P = \{(u_1, u_2) : |u_2| \leq 1 - u_1^2\}$, for which the reachable sets and the minimum-time function fail to be subanalytic. In [27] it is proved that for linear systems $\dot{x} = Ax + Bw$, $w \in P$, with P compact convex subanalytic, the minimum-time function is real-analytic on an open dense subset of its domain whose complement is a countable union of real-analytic submanifolds of positive codimension. It is still an open question whether, for such systems, the reachable sets are always stratifiable and the value functions are piecewise analytic.

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