SINGULARITIES SYMPOSIUM — ŁOJASIEWICZ 70
BANACH CENTER PUBLICATIONS, VOLUME 43
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

SOME OPTIMAL CONTROL APPLICATIONS OF REAL-ANALYTIC STRATIFICATIONS AND DESINGULARIZATION

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The name of Stanisław Łojasiewicz has been associated in my mind with mathematical research since early on in my career, because it was through his work that I experienced for the first time in my life the thrill of studying in depth a major piece of new mathematics and following the thoughts of a creative mind.

This happened thirty years ago, in 1966, when I reached the last semester of my undergraduate studies in mathematics in Buenos Aires, Argentina. The final requirement was a "trabajo de seminario superior", for which the student was expected to write a detailed expository monograph based on an important recent research article. My topic was S. Łojasiewicz's theorem on the division of distributions by real-analytic functions.

Subsequently, my path diverged from his until 1977, when I learned from the work of Pavol Brunovsky —[1, 2]— *about the possibility of using subanalytic sets in control theory. This was new to me but, thanks to my own previous exposure to semianalytic sets and stratifications through Łojasiewicz's work, it was something I was ready for.*

Much later, the name "Łojasiewicz" entered my life again in a different way, when I collaborated with Prof. Łojasiewicz's son S. Łojasiewicz Jr. in 1983–84 in [4]. *More recently, a brilliant idea of S. Łojasiewicz Jr. has played a decisive role in my own work* [11, 12, 13, 14] *on the maximum principle of optimal control.*

So the celebration of Professor Łojasiewicz's 70th birthday has a special meaning to me, and this paper is dedicated to him with deep admiration and gratitude.

1. Introduction. Real analyticity has important consequences in control theory, and real-analytic control systems have much nicer properties than smooth systems. In this paper we describe some of these properties, classifying them into two kinds: in Section 4 we focus on elementary consequences of real analyticity, i.e. results that follow by using no more than the analytic continuation property for functions of one real variable. Although

Received by the editors: September 27, 1996. Revised: December 16, 1997.
Research supported in part by NSF Grant DMS95-00798 and AFOSR Grant 0923.
1991 *Mathematics Subject Classification*: Primary 32B20; Secondary 93C10.
The paper is in final form and no version of it will be published elsewhere.

at this point nothing deep about real analyticity is used, we choose to include these elementary results for two reasons, namely,

(a) that they already show that real-analytic systems are strikingly different from smooth ones,

(b) that they are needed for our subsequent discussion of more sophisticated uses of real analyticity.

In Section 5 we look at a result on trajectory structure whose proof uses real-analytic desingularization. Since the proof of this result has only been presented in a very rough outline in a proceedings paper of an engineering conference, we give a fairly detailed proof. We also present, in Section 6, some curious applications of the result to observability theory, related to ideas of J.-P. Gauthier and I. Kupka. As a preliminary, in Section 2 we present some general definitions and introduce notations, and in Section 3 we define smooth control systems and present some of their basic properties.

In addition to the applications presented here, there are many others that we will not discuss for lack of space. For example, there are results on subanalyticity of reachable sets and piecewise analyticity of value functions, and there is also a theory of universal inputs. Some of these other applications are discussed in [10].

2. Basic definitions and notations.

2.1. *Smoothness, manifolds, tangent and cotangent spaces and bundles, vector fields, flows.* In this paper, "smooth" means "of class C^∞", and "manifold" means "smooth, finite-dimensional, paracompact manifold without boundary, not necessarily of pure dimension". (Paracompactness is of course equivalent to the property that every connected component is σ-compact, and also to the existence of a smooth Riemannian metric on M.) In particular, a manifold can have components of arbitrarily high dimension. (The fact that manifolds are not assumed to be of pure dimension will play a crucial role in Section 5 below.)

We use T_xM, T_x^*M, TM, T^*M, $T^\# M$ to denote, respectively, the tangent and cotangent spaces of a manifold M at a point $x \in M$, the tangent and cotangent bundles of M, and the cotangent bundle of M with the zero section removed. We use $C^\infty(M)$ and $V^\infty(M)$ to denote, respectively, the set of all smooth functions and that of all smooth vector fields on M, so $V^\infty(M)$ is a Lie algebra, if the Lie bracket $[f,g]$ of $f,g \in V^\infty(M)$ is defined in the usual way, by regarding f and g as first-order differential operators on $C^\infty(M)$ and letting $[f,g] = fg - gf$. Given any smooth vector bundle E, we use π_E to denote the projection map from E onto its base space. If M is a real-analytic manifold, then we use $C^\omega(M)$, $V^\omega(M)$ to denote the subsets of $C^\infty(M)$, $V^\infty(M)$ whose members are real-analytic. Then $V^\omega(M)$ is a Lie subalgebra of $V^\infty(M)$.

If $f \in V^\infty(M)$, we use exponential notation for the flow of f, so $t \mapsto x\,e^{tf}$ is the integral curve of f that goes through x at time $t = 0$. (Having the map e^{tf} act on the *right* rather than on the left is convenient in many applications. For example, if f, g are smooth vector fields, then the first t-derivative of $xe^{tf}e^{tg}e^{-tf}e^{-tg}$ at $t = 0$ vanishes, and the second derivative is $2x(fg - gf)$, i.e. $[f,g](x)$, so $xe^{tf}e^{tg}e^{-tf}e^{-tg} = x + t^2[f,g](x) + o(t^2)$, which is the correct Campbell-Hausdorff formula. Had we insisted on having the maps act on the left, we would have found the wrong formula.) Each map $x \mapsto x\,e^{tf}$, $t \in \mathbb{R}$, is then a smooth diffeomorphism, defined on a —possibly empty— open subset $\mathrm{Dom}(e^{tf})$ of M, and mapping $\mathrm{Dom}(e^{tf})$ onto $\mathrm{Dom}(e^{-tf})$. The set $\Delta(f) = \{(x,t) : t \in \mathbb{R},\ x \in \mathrm{Dom}(e^{tf})\}$ is then open in $M \times \mathbb{R}$, and $M \times \{0\} \subseteq \Delta(f)$.

2.2. *Submanifolds and leaves.* A *submanifold* of a manifold M is a subset S of M endowed with a manifold structure such that the inclusion map from S to M is a smooth immersion. (If M is paracompact then S is automatically paracompact as well.) A submanifold of M is *embedded* if it is a topological subspace of M.

An important class of submanifolds, intermediate between that of all submanifolds and that of embedded submanifolds, is that of "leaves", defined as follows. First, recall that a topological subspace of a topological space X is a subset S of X endowed with a topology such that, if μ is an arbitrary map from a topological space Y to X such that $\mu(Y) \subseteq S$, then μ is continuous as a map into X iff it is continuous as map into S. By analogy with this, we define —following P. Stefan, cf. [6], Section 1, Part I, p. 2— a *leaf* in a smooth manifold M to be a submanifold S of M such that, if $\mu : N \mapsto M$ is an arbitrary map from a manifold N to M such that $\mu(N) \subseteq S$, then μ is smooth as a map into M iff it is smooth as a map into S. It is easy to see that if a subset S admits a manifold structure σ with respect to which it is a leaf, then this structure is unique. (If σ_1, σ_2 are two such structures, then the inclusion $\iota : S \mapsto M$ is smooth from (S, σ_1) to M, and $\iota(S) \subseteq S$, so ι is smooth from (S, σ_1) to (S, σ_2). Similarly, ι is also smooth from (S, σ_2) to (S, σ_1). So the identity map from (S, σ_1) to (S, σ_2) is a diffeomorphism, and then $\sigma_1 = \sigma_2$.) So we can talk without ambiguity about a *subset* —rather than a submanifold— of M being a leaf.

It is clear that an embedded submanifold is a leaf, but there are examples of leaves that are not embedded submanifolds. (For example, if f is a smooth vector field, then every maximal integral curve of f is a leaf. Clearly, such a curve need not be embedded, since M could be a torus, and f a vector field on M whose orbits are dense.) On the other hand, not every submanifold is a leaf. (For example, a figure eight in the plane is not.) Actually, a sufficient condition for a subset S of M to be a leaf is the following:

(L) *For every $s \in S$ there exists a smooth diffeomorphism Φ from a neighborhood V of s in M to a product $C_1 \times C_2$ of open cubes in Euclidean spaces, such that*

(a) *for each $c_2 \in C_2$ the set $\Phi^{-1}(C_1 \times \{c_2\})$ is a subset either of S or of $M \backslash S$,*

(b) *the set $\{c_2 \in C_2 : \Phi^{-1}(C_1 \times \{c_2\}) \subseteq S\}$ does not contain a nonconstant smooth curve.*

Condition (L) is obviously verified when S is an embedded submanifold, and also when S is an orbit of a vector field, in which case (L) is trivial if S is a single point, and follows from the flow-box theorem if S is not a point. The fact that orbits of vector fields have property (L) can be generalized to orbits of *sets* of vector fields, as we now explain.

2.3. *Orbits.* If F is a set of smooth vector fields on a manifold M, an *F-invariant set* is a subset S of M having the property that, whenever $f \in F$, $t \in \mathbb{R}$, and x belongs to $S \cap \mathrm{Dom}(e^{tf})$, it follows that $x\, e^{tf} \in S$. (If S only satisfies the weaker conclusion that $x\, e^{tf} S$ if $t \geq 0$, then S will be called *forward F-invariant.* There is an obvious analogous definition of "backward invariance".) A nonempty minimal F-invariant set is said to be an *F-orbit.* Then two points x, x' lie in the same orbit iff for some m there exist $f_1, \ldots, f_m \in F$ and $t_1, \ldots, t_m \in \mathbb{R}$ such that $x' = x\, e^{t_1 f_1} e^{t_2 f_2} \ldots e^{t_m f_m}$. So "$x$ and x' are in the same F-orbit" is an equivalence relation. It follows that the set of all F-orbits is a partition of M.

We will repeatedly use the following "orbit theorem" (Sussmann [7]):

THEOREM 1. *Let M be a smooth manifold and let F be a set of smooth vector fields on M. Let S be an F-orbit. Then S satisfies condition* (L)*, so in particular S is a leaf in M. Endowed with its unique leaf structure, S is connected and has the property that the tangent space T_xS at a point $x \in S$ is the linear span of the set of all vectors*

$$v = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} x' \, e^{t_1 f_1} e^{t_2 f_2} \ldots e^{(t_i+\varepsilon) f_i} \ldots e^{t_m f_m}, \tag{1}$$

ranging over all possible choices of the positive integer m, the vector fields $f_1, \ldots, f_m \in F$, the numbers $t_1, \ldots, t_m \in \mathbb{R}$, the point $x' \in S$, and the index $i \in \{1, \ldots, m\}$, such that $x' \, e^{t_1 f_1} e^{t_2 f_2} \ldots e^{t_m f_m} = x$.

It is clear that every $f \in F$ is tangent to all the F-orbits. So, if $L(F)$ is the Lie algebra of vector fields generated by F, it follows that every $g \in L(F)$ is tangent to all the F-orbits.

2.4. *Hamiltonian vector fields and momentum functions.* If M is a smooth manifold, then every smooth function $H : T^*M \mapsto \mathbb{R}$ gives rise in a well known way —using the canonical symplectic structure of T^*M— to a *Hamilton vector field* $\overrightarrow{H} \in V^\infty(T^*M)$. The space $C^\infty(T^*M)$, endowed with the Poisson bracket $(H, K) \mapsto \{H, K\}$, is a Lie algebra, and the map $H \mapsto \overrightarrow{H}$ is a Lie algebra homomorphism from $C^\infty(T^*M)$ to $V^\infty(T^*M)$. If $f \in V^\infty(M)$, then the real-valued function $T^*M \ni (x, z) \mapsto h_f(x, z) \stackrel{\text{def}}{=} \langle z, f(x) \rangle$ is the *momentum function* —or *switching function*— corresponding to f.

The map $V^\infty(M) \ni f \mapsto h_f \in C^\infty(T^*M)$ is a Lie algebra homomorphism. Therefore the map $V^\infty(M) \ni f \mapsto \overrightarrow{h_f} \in V^\infty(T^*M)$ is a Lie algebra homomorphism as well. We will write $f^* \stackrel{\text{def}}{=} \overrightarrow{h_f}$, and call f^* the *Hamiltonian lift* of f. It is clear that the projection $\xi = \pi_{T^*M} \circ \Xi$ of an integral curve Ξ of f^* is an integral curve of f. Conversely, if I is an interval, $\xi : I \mapsto M$ is an integral curve of f, $t \in I$, and $z \in T^*_{\xi(t)}M$, then there exists a unique integral curve $\Xi : I \mapsto T^*M$ such that $\Xi(t) = (\xi(t), z)$ and $\xi = \pi_{T^*M} \circ \Xi$. The curve Ξ is entirely contained in $T^\# M$ iff $z \neq 0$.

3. Smooth control systems. A *control system* is a triple $\Sigma = (M, U, f)$ such that M is a smooth manifold, U is a compact metric space, and $f : M \times U \mapsto TM$ is a continuous map such that each partial map $M \ni x \mapsto f(x, u) \in TM$ is a vector field on M. Often, we will use the notation $f_u(x)$ as an alternative for $f(x, u)$, to emphasize the fact that f_u is a vector field. Also, we will use the expression "the control system $\dot{x} = f(x, u)$, $x \in M$, $u \in U$" as an alternative name for the system (M, U, f). We use $F(\Sigma)$ to denote the set $\{f_u : u \in U\}$, so $F(\Sigma)$ is a set of continuous vector fields on M.

We will be interested in control systems having extra regularity properties. Suppose that $k, \ell \in \{0, 1, \ldots\} \cup \{+\infty\}$ and $\ell \leq k$. We say that a control system $\Sigma = (M, U, f)$ is *of type* $C^{k,\ell}$ if

(a) every vector field f_u is of class C^k,

(b) all the partial derivatives of order $\leq \ell$ of $f(x, u)$ with respect to x are jointly continuous with respect to x and u.

(The meaning of the last condition is obvious in terms of local coordinates. Alternatively, it can be stated invariantly as follows: for every $\varphi \in C^\infty(M)$ and every ℓ-tuple

$(X_1, \ldots, X_\ell)$ in $V^\infty(M)$, the function $M \times U \ni (x, u) \mapsto X_1 X_2 \ldots X_\ell f_u \varphi(x) \in \mathbb{R}$ is continuous.)

Of special interest to us will be the systems Σ of type $C^{\infty,0}$, because for such systems $F(\Sigma) \subseteq V^\infty(M)$, and then the iterated Lie brackets of all orders of the members of $F(\Sigma)$ are defined. We will call such systems *smooth*.

Remark 1. The hypotheses made here are much more restrictive than is customary in control theory. For example, one often has to consider systems where f is time-dependent, and the set U is a more general metric space, not necessarily compact. Actually, the best general setting is one where U is a general abstract set with no additional structure, in which case continuity with respect to u is not a meaningful requirement, but one has to be more careful when defining "admissible control". We have chosen the above definitions, and we will be paying special attention to the class of smooth systems, because the purpose of this paper is to explore the consequences of *real analyticity*. In order to do that, we want to talk about real-analytic systems, defined below, and compare them with smooth systems that need not be real-analytic, to see which new features occur because of real analyticity. In view of this limited objective, we choose to use a narrowly defined class of "smooth systems" for comparison purposes, so we can focus on the "C^ω as opposed to C^∞" distinction.

3.1. *Controls, trajectories, reachability.* A *control* for a control system $\Sigma = (M, U, f)$ is a measurable U-valued function η defined on an interval I. If $\eta : I \mapsto U$ is a control, a *trajectory* for η (or "generated by η") is a locally absolutely continuous map $\xi : I \mapsto M$ such that $\dot{\xi}(t) = f(\xi(t), \eta(t))$ for almost all $t \in I$. A *trajectory-control pair* of a control system Σ is a pair (ξ, η) such that η is a control and ξ is a trajectory generated by η.

Given a control $\eta : I \mapsto U$, a $t \in I$, and an $x \in M$, then a trajectory $\xi : I \mapsto M$ of η such that $\xi(t) = x$ always exists locally. (More precisely: there always exists an $\varepsilon > 0$ such that there is a $\xi : I \cap]t - \varepsilon, t + \varepsilon[\mapsto M$ which is a trajectory of the restriction of η to $I \cap]t - \varepsilon, t + \varepsilon[$ and satisfies $\xi(t) = x$.) If the system is of class $C^{1,1}$, then the trajectory $\xi : I \mapsto M$ generated by a control η and satisfying an initial condition $\xi(t) = x$, if it exists, is necessarily unique. This uniqueness property does not necessarily hold for $C^{1,0}$ systems, or even for $C^{\infty,0}$ systems. (Consider, for example, the system $\dot{x} = 3u \sin \frac{x}{u^2}$, $-1 \leq u \leq 1$, the control $\eta(t) = t$, and the initial condition $\xi(0) = 0$. Then $\xi(t) \equiv 0$ is a solution. To find another solution, write $x = t^3 y$, so $\dot{x} = 3t^2 y + t^3 \dot{y}$, and then $3t^2 y + t^3 \dot{y} = 3t \sin ty$. Write $\sin z = z + z^3 h(z)$, where h is a smooth function. Then $3t \sin ty = 3t^2 y + 3t^4 y^3 h(ty)$, so $\dot{y} = 3ty^3 h(ty)$, which has solutions with any nonzero initial condition $y(0)$.) However, the uniqueness property holds for some $C^{k,0}$ systems, so we give them a name: we say that system is of class $C^{k,u}$ if it is of class $C^{k,0}$ and the uniqueness property of trajectories holds. (For example, the system $\dot{x} = u$, $\dot{y} = u \sin \frac{x}{u^2}$ is of class $C^{\infty,u}$ but not of class $C^{\infty,1}$.)

A curve $\xi : I \mapsto M$ is a *trajectory* of a control system $\Sigma = (M, U, f)$ if there is a control $\eta : I \to U$ such that ξ is a trajectory for η.

Remark 2. If $\xi : I \mapsto M$ is a locally absolutely continuous curve for which there is a function $I \ni t \mapsto \eta(t) \in U$ such that $\dot{\xi}(t) = f(\xi(t), \eta(t))$ for almost every t, then ξ is a trajectory of Σ. In other words, if ξ is "almost a trajectory", in the sense that it satisfies all the conditions for being a trajectory, except for the fact that the "control" η is not necessarily measurable, then one can always choose a measurable control η' such that

$f(\xi(t), \eta'(t)) = f(\xi(t), \eta(t))$ for almost every t, so ξ is a true trajectory. This fact follows from standard measurable selection theorems.

We say that a point x' is Σ-*reachable from* x *in time* t if there exists a trajectory-control pair (ξ, η) of Σ such that $\xi(0) = x$ and $\xi(t) = x'$. (In that case, we also say that ξ *goes from* x *to* x' *in time* t, or that η *steers* x to x'.) We use $R_{\Sigma,t}(x)$ to denote the set of all x' that are reachable from x in time t, and write $R_{\Sigma,I}(x)$, if I is an arbitrary subset of $[0, \infty[$, to denote the set $\bigcup_{t \in I} R_{\Sigma,t}(x)$. When $I = [0, \infty[$, we just write $R_\Sigma(x)$, and refer to this set as the *reachable set from* x.

We can also consider reachable sets using restricted classes of controls. For example, $R^{pc}_{\Sigma,t}(x)$, $R^{pc}_{\Sigma,I}(x)$, $R^{pc}_{\Sigma}(x)$ are defined exactly like the sets without the superscript, except that only piecewise constant controls are allowed.

3.2. *The orbits of a smooth control system.* If we are given a smooth control system $\Sigma = (M, U, f)$, then the $F(\Sigma)$-orbits will be referred to as the Σ-*orbits*, or the *orbits of* Σ. It is clear that if S is a Σ-orbit then each vector field f_u has a well defined and smooth restriction $f_u \lceil S$, and the map from $S \times U$ to TS that sends (x, u) to $f(x, u)$ is continuous. Therefore Σ has a well defined restriction $\Sigma \lceil S$ to each Σ-orbit S, which is also a smooth control system. Clearly, if Σ is of class $C^{\infty,\ell}$ for $\ell > 0$, or of class $C^{\infty,u}$, then the same is true of $\Sigma \lceil S$.

Every trajectory of $\Sigma \lceil S$ is a trajectory of Σ. Moreover,

LEMMA 1. *Every trajectory of a* $C^{\infty,u}$ *system* Σ *is entirely contained in an orbit.*

Proof. To see this, we let $\xi : I \mapsto M$ be a trajectory corresponding to a control η. For each $t \in I$, let $S(t)$ be the orbit that contains $\xi(t)$. We show that the map $t \mapsto S(t)$ is locally constant. If $t \in I$ and $S = S(t)$, then there is a well defined trajectory ζ of $\Sigma \lceil S$ on some interval $]t - \varepsilon, t + \varepsilon[\cap I$, corresponding to the restriction $\eta_{t,\varepsilon}$ of η to $]t - \varepsilon, t + \varepsilon[\cap I$, and such that $\zeta(t) = \xi(t)$. Then ζ is a trajectory of Σ for $\eta_{t,\varepsilon}$. By the uniqueness property, $\zeta = \xi$ on $]t - \varepsilon, t + \varepsilon[\cap I$. This shows that $\xi(t') \in S$ for $t' \in I$, t' near t.

So $t \mapsto S(t)$ is locally constant. Since I is connected, $S(t)$ is independent of t, and our conclusion is proved. ∎

Remark 3. The above result need not be true for $C^{\infty,0}$ systems. For example, for the system $\dot{x} = u \sin \frac{x}{u^2}$, $-1 \leq u \leq 1$ there are three orbits, namely, $]-\infty, 0[$, $\{0\}$, and $]0, +\infty[$, but we have already seen how to construct a trajectory that starts at 0 but does not stay in the set $\{0\}$.

3.3. *The accessibility Lie algebra.* Let Σ be a smooth control system. We write $L(\Sigma)$ for $L(F(\Sigma))$, so $L(\Sigma)$ is the Lie algebra of vector fields generated by $F(\Sigma)$. We refer to $L(\Sigma)$ as the *accessibility Lie algebra* of Σ.

3.4. *Algebraic accessibility and Lie bracket relations.* Given an arbitrary set U, we use $\Lambda(U)$ to denote the free Lie algebra over $\mathbb{R}$ with generators F_u, $u \in U$. For a smooth control system $\Sigma = (M, U, f)$, there is a unique Lie algebra homomorphism P_Σ from $\Lambda(U)$ to $L(\Sigma)$ —"plugging in the f_u for the indeterminates F_u"— that maps each generator F_u to the corresponding vector field f_u. Clearly, P_Σ is onto. Given a point $x \in M$, we can consider the *evaluation map* $E_{x,\Sigma} : L(\Sigma) \mapsto T_x M$ given by $E_{x,\Sigma}(g) = g(x)$, for $g \in L(\Sigma)$.

We say that the system Σ has the *algebraic accessibility property* at the point x if $E_{x,\Sigma}(L(\Sigma)) = T_x M$.

We write $\mathrm{REL}(x,\Sigma)$ to denote the kernel of the map $P_\Sigma \circ E_{x,\Sigma}$. Then $\mathrm{REL}(x,\Sigma)$ is a Lie subalgebra of $\Lambda(U)$. We refer to $\mathrm{REL}(x,\Sigma)$ as the *set of Lie bracket relations at* x for Σ. For example, if $u,v,w\in U$, then the expression $F_u+[F_v,F_w]-3[F_u,[F_v,F_u]]$ is a Lie bracket relation at x for Σ if and only if $f_u(x)+[f_v,f_w](x)-3[f_u,[f_v,f_u]](x)=0$.

3.5. *The positive form of Chow's theorem.* Let $\Sigma=(M,U,f)$ be a smooth control system. We say that Σ has the *accessibility property* (resp. the *piecewise constant accessibility property*) from a point $x\in M$ if the set $R_{\Sigma,[a,b]}(x)$ (resp. $R^{pc}_{\Sigma,[a,b]}(x)$) has nonempty interior in M whenever $0\le a<b$.

THEOREM 2. *If a smooth control system $\Sigma=(M,U,f)$ has the algebraic accessibility property at a point x, then it has the piecewise constant accessibility property from x.*

Proof. To prove this, we use an argument essentially due to A. Krener. Notice first that we can assume that U is finite and the algebraic accessibility condition holds at every $x'\in M$.

Fix a,b such that $0\le a<b$. For each m, use $A_m(a,b)$ to denote the set of those $\tau=(\tau_1,\ldots,\tau_m)\in\,]0,+\infty[^m$ such that $a<\tau_1+\ldots+\tau_m<b$. Then $A_m(a,b)$ is an open subset of $]0,\infty[^m$.

Let Q be the set of all possible triples $(m,\tau,\mathbf{u})$ such that m is a nonnegative integer, $\tau=(\tau_1,\ldots,\tau_m)\in A_m(a,b)$, and $\mathbf{u}=(u_1,\ldots,u_m)\in U^m$. For $(m,\tau,\mathbf{u})\in Q$ as before, let

$$Z(m,\tau,\mathbf{u})=xe^{\tau_1 f_{u_1}}e^{\tau_2 f_{u_2}}\ldots e^{\tau_m f_{u_m}}. \tag{2}$$

Let Q_0 be the set of all $(m,\tau,\mathbf{u})\in Q$ such that $Z(m,\tau,\mathbf{u})$ exists. For $(m,\tau,\mathbf{u})\in Q_0$, let $\rho(m,\tau,\mathbf{u})$ be the rank of the differential at τ of the map $\tau'\mapsto Z(m,\tau',\mathbf{u})$. Choose $(m,\tau,\mathbf{u})\in Q_0$ such that $\rho(m,\tau,\mathbf{u})$ has the largest possible value, and let ν be this value. Then the map $B=Z(m,\cdot,\mathbf{u})$ has constant rank near τ, so by the implicit function theorem there exists a neighborhood W of τ in $A_m(a,b)$ such that the set $B(W)$ is an embedded ν-dimensional submanifold of M. We show that every f_u is tangent to S. If this were not so, there would exist $y\in S$ and $u\in U$ such that $f_u(y)\notin T_yS$. Let $y=B(\tilde\tau)$, $\tilde\tau\in W$. Then the map $\psi:(\tau',\theta)\mapsto B(\tau')e^{\theta f_u}$ has rank $\nu+1$ at $(\tilde\tau,0)$. So ψ has rank $m+1$ at $(\tilde\tau,\theta)$ if $\theta>0$ is small enough. Clearly, $(\tilde\tau,\theta)\in A_{m+1}(a,b)$ if $\theta>0$ is small enough. Then, if $\theta>0$ is small, we have $(m+1,(\tilde\tau,\theta),\mathbf{u}^*)\in Q_0$ and $\rho(m+1,(\tilde\tau,\theta),\mathbf{u}^*)=\nu+1$, if $\mathbf{u}^*=(\mathbf{u},u)$. This contradicts the maximality of ν.

Since every f_u is tangent to S, it follows that every $g\in L(\Sigma)$ is tangent to S. The algebraic accessibility condition then implies that S is open in M, and this proves our conclusion, since $S\subseteq R^{pc}_{\Sigma,]a,b[}(x)$. ∎

Theorem 2 has two important consequences, for systems $\Sigma=(M,U,f)$ of class $C^{\infty,u}$ that have the algebraic accessibility property at every point.

First, let $x\in M$, $0\le a<b$, $x'\in R_{\Sigma,[a,b]}(x)$. Let W be an open neighborhood of x'. We claim that $R^{pc}_{\Sigma,[a,b[}(x)\cap W\ne\emptyset$. To see this, suppose $x'\in R_{\Sigma,t}(x)$, $t\in[a,b]$. Let $\eta:[0,t]\mapsto U$ be a control that steers x to x', and let $\xi:[0,t]\mapsto M$ be the corresponding trajectory. Find a sequence of piecewise constant controls $\eta_j:[0,t]\to U$ such that $\eta_j(t')\to\eta(t')$ for almost all $t'\in[0,t]$. (The existence of such a sequence is easily proved.) A simple application of Ascoli's theorem shows that for large enough j there are trajectories $\xi_j:[0,t]\mapsto M$ generated by η_j such that $\xi_j(0)=x$ and $\xi_j\to\xi$ uniformly on $[0,t]$. If $t<b$, then $\xi_j(t)\in R^{pc}_{\Sigma,[a,b[}(x)\cap W\ne\emptyset$ for large enough t. If $t=b$ then $\xi_j(s)\in R^{pc}_{\Sigma,[a,b[}(x)\cap W\ne\emptyset$ if j is large enough, and $s<b$ is sufficiently close to b.

Pick $x'' \in R^{pc}_{\Sigma,\hat{t}}(x) \cap W$, $\hat{t} \in [a,b[$. Applying Theorem 2 to x'' and the time θ, where $\theta > 0$ is chosen so that $R_{\Sigma,[0,\theta]}(x'') \subseteq W$ and $\theta < b - \hat{t}$, we find that the interior of $R^{pc}_{\Sigma,[0,\theta]}(x'')$ is a nonempty subset of W. Since it is clear that $R^{pc}_{\Sigma,[0,\theta]}(x'') \subseteq R^{pc}_{\Sigma,[a,b]}(x)$, we conclude that W contains an interior point of $R^{pc}_{\Sigma,[a,b]}(x)$. Since W is an arbitrary open neighborhood of x', which is an arbitrary point of $R_{\Sigma,[a,b]}(x)$, we have shown that

$$R_{\Sigma,[a,b]}(x) \subseteq \mathrm{Clos}\Big(\mathrm{Int}\Big(R^{pc}_{\Sigma,[a,b]}(x)\Big)\Big) \text{ whenever } x \in M,\ 0 \leq a < b. \tag{3}$$

Next, suppose that $x' \in \mathrm{Int}\Big(R_{\Sigma,[a,b]}(x)\Big)$, where $0 \leq a \leq b$. Then we can pick an open neighborhood W of x' such that $W \subseteq R_{\Sigma,[a,b]}(x)$. We can then apply Theorem 2 to the system $\Sigma^r = (M, U, -f)$ —i.e. "Σ ran in reverse"— and a sufficiently small time $\theta > 0$, and conclude that $R^{pc}_{\Sigma^r,[0,\theta]}(x') \cap W$ contains a nonempty open set Ω. Then $\Omega \subseteq R_{\Sigma,[a,b]}(x)$. Pick $y \in \Omega$. Then y can be reached from x, at some time $\hat{t} \in [a,b]$, by means of a control $\eta : [0,\hat{t}] \mapsto U$ and a corresponding trajectory ξ. Let $\{\eta_j\}_{j=1}^{\infty}$ be a sequence of piecewise constant controls defined on $[0,\hat{t}]$ such that $\eta_j(t') \to \eta(t')$ for almost all $t' \in [0,\hat{t}]$. Once again, it follows from standard properties of ordinary differential equations that, if our system is of class $C^{1,u}$, then the trajectories $\xi_j : [0,\hat{t}] \mapsto M$ for η_j such that $\xi_j(0) = x$ exist for large j, and converge uniformly to ξ. Let $y_j = \xi_j(\hat{t})$. Then there is a j such that $y_j \in \Omega$. Since $y_j \in R^{pc}_{\Sigma,[a,b]}(x)$ and $x' \in R^{pc}_{\Sigma,[0,\theta]}(y_j)$ —because $y_j \in \Omega \subseteq R_{\Sigma^r,[0,\theta]}(x')$— we see that $x' \in R_{\Sigma,[a,b+\theta]}(x)$. Since θ is arbitrarily small, we have shown that

$$\mathrm{Int}\Big(R_{\Sigma,[a,b]}(x)\Big) \subseteq \bigcap_{\theta>0} R^{pc}_{\Sigma,[a,b+\theta]}(x) \text{ whenever } x \in M,\ 0 \leq a \leq b. \tag{4}$$

In other words, every point in the interior of the reachable set from x in some time belonging to $[a,b]$ can be reached from x by means of a piecewise constant control in time belonging to $[a, b+\theta]$, for any $\theta > 0$.

So we have proved:

THEOREM 3. *If $\Sigma = (M, U, f)$ is a control system of class $C^{\infty,u}$ that has the algebraic accessibility property at every point, then* (3) *and* (4) *hold.*

Remark 4. It is easy to give examples where the number θ of (4) cannot be taken to be equal to 0. For example, consider the system $\Sigma = (M, U, f)$ whose dynamics is given by $\dot{\rho} = u(1 - \frac{1}{4}(v - \cos\rho)^2)$, where $\rho \in S^1 = M$ and the controls u, v satisfy $|u| \leq 1$, $|v| \leq 1$, so $U = [-1,1] \times [-1,1]$. Let the initial condition be $\rho(0) = 0$. Then $R_{\Sigma,[0,\pi]}(0) = R_{\Sigma,\pi}(0) = M$, so π is an interior point of $R_{\Sigma,[0,\pi]}(0)$. However, the only controls that steer 0 to π in time $\leq \pi$ are the ones given by $u(t) = \alpha$, $v = \cos t$ —where $\alpha = 1$ or $\alpha = -1$— which are not piecewise constant.

3.6. *The maximum principle.* If $\Sigma = (M, U, f)$ is a control system of class $C^{k,\ell}$, with $k \geq \ell \geq 1$, then each vector field f_u, $u \in U$, has a Hamiltonian lift $f^*_u \in V^{\infty}(T^*M)$, which is a vector field of class C^{k-1}. If we define $f^*(z,u) = f^*_u(z)$ for $z \in T^*M$, then $\Sigma^* = (T^*M, U, f^*)$ is a control system of class $C^{k-1,\ell-1}$. Moreover, when $\ell = 1$, Σ^* is of class $C^{k-1,u}$. (In coordinates, if Σ is a system $\dot{x} = f(x,u)$, then Σ^* is the system $\dot{x} = f(x,u)$, $\dot{z} = -z \cdot \frac{\partial f}{\partial x}(x,u)$. Given a control η and initial conditions $x(0) = \bar{x}$, $z(0) = \bar{z}$, the equation $\dot{x} = f(x, \eta(t))$ has a unique solution ξ because Σ is of class $C^{1,1}$, and then

$\dot{z} = -z \cdot \frac{\partial f}{\partial x}(\xi(t), \eta(t))$ also has a unique solution because it is of the form $\dot{z} = z.A(t)$ with $A(\cdot)$ bounded and measurable.)

The control system Σ^* defined above is the *Hamiltonian lift* of Σ. The projection $\xi = \pi_{T^*M} \circ \Xi$ of a trajectory Ξ of Σ^* generated by a control η is a trajectory of Σ generated by η. Conversely, if I is an interval, $\xi : I \mapsto M$ is a trajectory of Σ generated by η, $t \in I$, and $z \in T^*_{\xi(t)}M$, then there exists a unique trajectory $\Xi : I \mapsto T^*M$ of Σ^* generated by η such that $\Xi(t) = (\xi(t), z)$ and $\xi = \pi_{T^*M} \circ \Xi$. The curve Ξ is entirely contained in $T^\# M$ iff $z \neq 0$. If $\gamma = (\xi, \eta)$ is a trajectory-control pair of Σ, and Ξ is a trajectory of Σ^* generated by η and such that $\pi_{T^*M} \circ \Xi = \xi$, then the trajectory-control pair $\Gamma = (\Xi, \eta)$ of Σ^* is called a *Hamiltonian lift* of γ. If in addition Ξ is contained in $T^\# M$, then Γ is a *nontrivial* Hamiltonian lift of γ.

The function $H_\Sigma : T^*M \times U \mapsto \mathbb{R}$ given by $H_\Sigma(x, z, u) = z \cdot f(x, u)$ is called the *Hamiltonian* of Σ.

A trajectory-control pair $\Gamma = (\Xi, \eta)$ of Σ^* is said to be *Hamiltonian minimizing* if

$$H_\Sigma(\Xi(t), \eta(t)) = \min\{H_\Sigma(\Xi(t), u) : u \in U\} \text{ for almost every } t. \tag{5}$$

If in addition $H_\Sigma(\Xi(t), \eta(t)) = 0$ for almost every t, then Γ is *null-minimizing*.

An *extremal* of Σ is a trajectory-control pair that admits a nontrivial null-minimizing Hamiltonian lift.

The following result is one version of the maximum principle of optimal control theory.

THEOREM 4. *Let Σ be a control system of class $C^{1,1}$. Let $\gamma = (\xi, \eta)$ be a trajectory-control pair of Σ such that the domain of η is an interval $[0, T]$. Then either*

$$\xi(T) \in \bigcap_{\varepsilon > 0} \mathrm{Int}\Big(R_{\Sigma,]T-\varepsilon, T+\varepsilon[}(\xi(0))\Big), \tag{6}$$

or γ is an extremal.

The maximum principle says, roughly, that if a point $\hat{x} = \xi(T)$ is reachable in time T from another point $\bar{x} = \xi(0)$ by means of a trajectory $\xi : [0, T] \mapsto M$ and corresponding control η, then a necessary condition for $\hat{x}$ to belong to the boundary of the reachable set from $\bar{x}$ is that the pair $\gamma = (\xi, \eta)$ be an extremal. The statement we have presented gives more precise information, since it says that, if γ is not an extremal, then not only is $\hat{x}$ an interior point of $R_\Sigma(\bar{x})$, but in fact it is an interior point of $R_{\Sigma,]T-\varepsilon, T+\varepsilon[}(\bar{x})$ for every $\varepsilon > 0$. (In other words, given any $\varepsilon > 0$ one can fill up some neighborhood W_ε of $\hat{x}$ with points that are reachable from $\bar{x}$ in times between $T - \varepsilon$ and $T + \varepsilon$.)

4. Real-analytic control systems: elementary properties. A *real-analytic control system* is a smooth control system $\Sigma = (M, U, f)$ such that M is a real-analytic manifold, U is a compact subanalytic subset of some other real-analytic manifold N, and $f : M \times U \mapsto TM$ is jointly real-analytic. (For simplicity, the reader can choose to interpret the last condition in the most obvious way, namely, that f is the restriction to $M \times U$ of some real-analytic map on an open subset Ω of $M \times N$ such that $M \times U \subseteq \Omega$. What we actually need for all the results of this paper is the weaker assumption that, for some compact real-analytic manifold V and surjective real-analytic map $\Phi : V \mapsto U$, the composite map $M \times V \ni (x, v) \mapsto f(x, \Phi(v))$ is real-analytic. So, for example, the control system $\dot{x} = \sqrt{u}$, $u \in U = [0, 1]$, is real-analytic in our sense, since we can desingularize U by taking, e.g., $V = S^1$, $\Phi(\theta) = \sin^4 \theta$.)

It is clear that a real-analytic control system is of class $C^{\infty,\infty}$.

4.1. *Integral manifolds.* In the real-analytic case, Theorem 1 can be strengthened by giving a much more precise characterization of the tangent spaces to the orbits. Precisely:

THEOREM 5. *If M is a real-analytic manifold and $F \subseteq V^\omega(M)$, then the tangent space T_xS of an F-orbit S at a point $x \in S$ is*

$$T_xS = L(F)(x) \overset{\text{def}}{=} \{g(x) : g \in L(F)\}. \tag{7}$$

Proof. We present the complete proof, because it is very short —modulo the orbit theorem— and shows exactly where and how analyticity is used. As will become clear shortly, the only fact about analyticity used in the proof is the analytic continuation theorem for functions of one real variable.

We have to show that every vector v of the form (1) is in $L(F)(x)$. For this purpose, it suffices to show that, if $x \in M$, $w \in L(F)(x)$ and $f \in F$, then $we^{tf} \in L(F)(xe^{tf})$. (Here we are using we^{tf} to denote the object usually called $(e^{tf})_*w$, i.e. the tangent vector $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\xi(\varepsilon)e^{tf}$, if ξ is a C^1 curve such that $\xi(0) = x$ and $\dot\xi(0) = w$.) To prove this, pick $W \in L(F)$ such that $W(x) = w$. Then $we^{tf} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} xe^{\varepsilon W}e^{tf}$. Letting $\xi(t) = xe^{tf}$, we have $we^{tf} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\xi(t)e^{-tf}e^{\varepsilon W}e^{tf}$. Let $z(s) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\xi(t)e^{-sf}e^{\varepsilon W}e^{sf}$. Then $z(s)$ is a tangent vector at $\xi(t)$ for all s such that $\xi(t)e^{-sf}$ is defined, and in particular for all s between 0 and t. Clearly, $z(s) = \xi(t)e^{-sf}We^{sf}$. Then

$$\frac{dz}{ds}(s) = \xi(t)e^{-sf}[W, f]e^{sf}, \tag{8}$$

and, more generally,

$$z^{(k)}(s) = \xi(t)e^{-sf}W^k e^{sf}, \tag{9}$$

where $W^0 = W$ and $W^{k+1} = [W^k, f]$. It follows that $z^{(k)}(0) \in L(F)(\xi(t))$ for every k. Since z is analytic, we conclude that $z(s) \in L(F)(\xi(t))$ for all s between 0 and t. (*This is the only place where analyticity is used.*) In particular, $we^{tf} = z(t) \in L(F)(\xi(t))$, and the proof is complete. ■

So, if M is real-analytic and $F \subseteq V^\omega(M)$, then M is partitioned into leaves S —the F-orbits— such that each S is an *integral manifold of* $L(F)$, i.e. a connected C^ω leaf in M such that $T_xS = L(F)(x)$ for all $x \in S$. These manifolds are obviously *maximal*, in the sense that if S' is an integral manifold of $L(F)$, then S' is an open submanifold of one of the sets S. When F is itself a Lie algebra of vector fields, this yields the *Hermann-Nagano theorem* (cf. Hermann [3], Nagano [5]):

THEOREM 6. *If M is a real-analytic manifold and L is a Lie subalgebra of $V^\omega(M)$, then M is partitioned into maximal integral submanifolds of L.*

4.2. *The equivalence theorem.* An important corollary of the Hermann-Nagano theorem is the following *equivalence theorem*:

THEOREM 7. *Let $\Sigma^i = (M^i, U, f^i)$, $i = 1, 2$, be smooth control systems such that M_1 and M_2 are real-analytic manifolds and the vector fields f^1_u, f^2_u are real-analytic for every u*(1)*. Let $x^1 \in M^1$, $x^2 \in M^2$ be such that the algebraic accessibility condition holds*

(1) In particular, the systems could be real-analytic in the sense of our definition, but for Theorem 7 there is no need to require analyticity or even smoothness of the dependence with respect to u.

for Σ^i *at* x^i *for* $i = 1, 2$. *Then the following two conditions are equivalent*:

(i) $\mathrm{REL}(x^1, \Sigma^1) = \mathrm{REL}(x^2, \Sigma^2)$,

(ii) *there exists a real-analytic diffeomorphism* Φ *from a neighborhood* U^1 *of* x^1 *in* M^1 *onto a neighborhood* U^2 *of* x^2 *in* M^2 *such that* $D\Phi(x).f^1(x, u) = f^2(\Phi(x), u)$ *for every* $x \in U^1$.

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. The fact that (i) $\Rightarrow$ (ii) is proved by an elementary application of the well known graph method of E. Cartan: we let $M = M^1 \times M^2$, and construct the graph G of the map Φ by letting G be the maximal integral manifold of $L(\Sigma)$, where Σ is the system (M, U, f), and $f(y^1, y^2, u) = (f^1(y^1, u), f^2(y^2, u))$, using the obvious identification $T_{(y^1,y^2)}M \sim T_{y^1}M^1 \times T_{y^2}M^2$. Let $x = (x^1, x^2)$. Then the accessibility property for each system Σ^i implies that the corresponding projection $\pi^i : G \ni (y^1, y^2) \mapsto y^i \in M^i$ is a submersion near x. Then (i) implies that $\ker D\pi^1(x) = \ker D\pi^2(x) = \{0\}$, because

$$P_{\Sigma^i}(V)(x^i) = 0 \Leftrightarrow V \in \mathrm{REL}(x^i, \Sigma^i), \tag{10}$$

so

$$D\pi^1(x)(v^1, v^2) = 0 \Leftrightarrow v^1 = 0 \Leftrightarrow v = 0 \Leftrightarrow v^2 = 0 \Leftrightarrow D\pi^2(x)(v^1, v^2) = 0 \tag{11}$$

if $v = (v^1, v^2) \in T_xG$, since v is of the form $(P_{\Sigma^1}(V)(x^1), P_{\Sigma^2}(V)(x^2))$ for some $V \in \Lambda(U)$.

So the maps π^i are in fact local diffeomorphisms near x. We then define

$$\Phi = \pi^2 \circ (\pi^1)^{-1},$$

and it is easy to verify that Φ has the desired properties. ∎

Theorem 7 is important because it says that, for a real-analytic control system $\Sigma = (M, U, f)$ having the algebraic accessibility property at a point x, all the C^ω-diffeomorphism-invariant properties of the system near the point x are determined, in principle, by the Lie bracket relations at x. (Examples of local properties that are *not* determined by the Lie bracket relations: (1) whether or not Σ has the algebraic accessibility property at x, (2) whether or not the reachable sets $R_{\Sigma,t}(x)$ are convex for small t.)

In other words, *for systems having the algebraic accessibility property, the Lie bracket relations of* Σ *at* x *form a* **complete set of invariants** *under the pseudogroup of local real-analytic diffeomorphisms*. This is one of several reasons why Lie brackets play such an significant role in control theory.

4.3. *An elementary property of reachable sets.* Theorems 3 and 5, together with Lemma 1, imply that reachable sets for real-analytic systems have some special properties that are not true in general for smooth systems.

THEOREM 8. *Suppose that* $\Sigma = (M, U, f)$ *is a real-analytic control system,* $x \in M$, *and* $0 \le a < b$. *Let* $\mathcal{R} = R_{\Sigma,[a,b]}(x)$. *Then there exists a real-analytic leaf* S *in* M *such that* $\mathcal{R} \subseteq S$ *and* $\mathcal{R}$ *is in fact contained in the closure, relative to* S, *of its interior relative to* S. *In particular,* $\mathcal{R}$ *has integer Hausdorff dimension.*

Proof. Let S be the Σ-orbit through x. By Lemma 1, $\mathcal{R} \subseteq S$. By Theorem 5, S is an integral manifold of $L(\Sigma)$. Then the restriction $\Sigma\lceil S$ of Σ to S is an analytic system having the algebraic accessibility property at every point. Clearly, $\mathcal{R} = R_{\Sigma\lceil S,[a,b]}(x)$. By Theorem 3, $\mathcal{R}$, regarded as a subset of S, is contained in the closure of its interior. ∎

Remark 5. The property of Theorem 8 can definitely fail for smooth systems. For example, let Σ be the system $\dot{x} = f(x) + ug(x)$ in $\mathbb{R}^2$, where $u \in U = [-1, 1]$, $f = \frac{\partial}{\partial x_1}$,

$g = \varphi(x_1)\frac{\partial}{\partial x_2}$, and $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is a smooth function such that $\varphi(s) = 0$ for $s \leq 1$ and $\varphi(s) > 0$ for $s > 1$. Let $\bar{x} = (0,0)$. Then for any time $T > 1$, the reachable set $R_{\Sigma,[0,T]}(\bar{x})$ has nonempty interior but is not contained in the closure of its interior.

4.4. *An open problem.* Is it true for general smooth systems $\Sigma = (M, U, f)$ that every reachable set $R_{\Sigma,[0,T]}(\bar{x})$ has integer Hausdorff dimension? The answer is not known.

5. A weak regularity theorem, using desingularization. We now present an example of a much more delicate property of real-analytic systems, whose proof uses the desingularization theorem in a crucial way. A simpler version of this result, for the case when the control space U is the interval $[-1, 1]$ (in which case desingularization is not needed), has appeared in Sussmann [9].

THEOREM 9. *Let $\Sigma = (M, U, f)$ be a real-analytic control system, and let $\bar{x} \in M$, $\hat{x} \in M$, $T > 0$ be such that $\hat{x}$ can be reached from $\bar{x}$ in time T. Then there exists a control $\eta : [0,T] \mapsto U$ that steers $\bar{x}$ to $\hat{x}$ and is analytic on an open dense subset of the interval $[0,T]$.*

Proof. We will first prove a weaker result:

(WR) *Under our hypotheses, if $\varepsilon > 0$ is arbitrary, then there exist a time $T' \in [T-\varepsilon, T+\varepsilon]$ and a control $\eta : [0,T'] \mapsto U$ that steers $\bar{x}$ to $\hat{x}$ and is analytic on an open dense subset of the interval $[0,T']$.*

We prove that (WR) implies the desired conclusion. To show this, we apply (WR) to the system $\check{\Sigma}$ given by $\dot{x} = f(x,u)$, $\dot{y} = 1$, on the manifold $M \times \mathbb{R}$. It is clear that x' is Σ-reachable from x in time t iff (x',t) is $\check{\Sigma}$-reachable from $(x,0)$. Under our hypothesis, $(\hat{x},T)$ is $\hat{\Sigma}$-reachable from $(\bar{x},0)$. It then follows from (WR) that $(\hat{x},T)$ is $\hat{\Sigma}$-reachable from $(\bar{x},0)$ in some time T' by means of a control which is analytic on an open dense subset of $[0,T']$. If $[0,T'] \ni t \mapsto (\xi(t),\tau(t))$ is the corresponding trajectory, then $\tau(0) = 0$, $\tau(T') = T$, and $\dot{\tau} \equiv 1$. So $T' = T$. Therefore $(\hat{x},T)$ is $\hat{\Sigma}$-reachable from $(\bar{x},0)$ in time T by means of a control which is analytic on an open dense subset of $[0,T]$. So $\hat{x}$ is Σ-reachable from $\bar{x}$ in time T by means of a control which is analytic on an open dense subset of $[0,T]$.

So it suffices to prove (WR). Moreover, the desingularization theorem implies that it suffices to assume that U is a real-analytic manifold which is a finite union of tori.

So from now on we assume that U is a finite union of tori, and try to prove (WR). The proof will be "by induction on U", so our first task will be to assign to each possible U an "index" $\nu(U)$ belonging to some well-ordered set, so as to be able to do induction.

We let $\nu_k(U)$ be the number of k-dimensional connected components of U, and define $\nu(U)$ to be the sequence $(\nu_0(U), \nu_1(U), \ldots)$. Then $\nu(U) \in \mathcal{N}$, where $\mathcal{N}$ is the set of all sequences $(n_0, n_1, n_2, \ldots)$ of nonnegative integers such that $n_k = 0$ for all but finitely many k's.

The set $\mathcal{N}$ is well ordered by the binary relation $\preceq$ where, by definition,

$$(n_0, n_1, n_2, \ldots) \preceq (m_0, m_1, m_2, \ldots)$$

iff there does not exist a $k \in \{0, 1, \ldots\}$ with the property that $n_k > m_k$ and $n_j = m_j$ for $j > k$. If $\mathbf{n} = (n_0, n_1, n_2, \ldots)$ and $\mathbf{m} = (m_0, m_1, m_2, \ldots)$, we write $\mathbf{n} \prec \mathbf{m}$ if $\mathbf{n} \preceq \mathbf{m}$ and $\mathbf{n} \neq \mathbf{m}$.

We will prove our conclusion by induction on $\nu(U)$. That is, we will assume from now on that (WR) is true for all systems $\Sigma' = (M', U', f')$ with $\nu(U') \prec \nu(U)$, and will prove (WR) for $\Sigma = (M, f, U)$.

We now fix $\varepsilon > 0$, pick two points $\bar{x}$, $\hat{x}$ in M, and choose a trajectory $\xi : [0, T] \mapsto M$ of Σ, corresponding to a control $\eta : [0, T] \mapsto U$, and such that $\xi(0) = \bar{x}$, $\xi(T) = \hat{x}$. We let $\kappa = 1 + \frac{\varepsilon}{T}$, and observe that $\frac{T}{\kappa} > T - \varepsilon$. The trajectory ξ is then entirely contained in a maximal integral submanifold $\tilde{M}$ of $L(\Sigma)$, so we may as well assume that $\tilde{M} = M$, i.e. that Σ has the algebraic accessibility property at each point and M is connected.

It is clear that our desired conclusion is local, in the sense that, if it holds on all members of an open covering $\mathcal{M}$ of M, then it holds on M. (This is trivial: pick $t_0, \ldots, t_m \in [0, T]$ such that $0 = t_0 < t_1 < \ldots < t_m = T$, with the property that $\xi([t_{i-1}, t_i])$ is contained in a member of $\mathcal{M}$ for each i. We can then find a curve $\xi_i' : [a_i, b_i]$, defined on an interval of length $L_i = b_i - a_i$, such that $\frac{t_i - t_{i-1}}{\kappa} \leq L_i \leq \kappa(t_i - t_{i-1})$, $\xi_i'(a_i) = \xi(t_{i-1})$, $\xi_i'(b_i) = \xi(t_i)$, and ξ_i' is generated by a control η_i' which is analytic on an open dense subset of $[a_i, b_i]$. After suitably translating the time intervals $[a_i, b_i]$, and concatenating the curves ξ_i' and the controls η_i', we get a trajectory $\xi' : [0, T'] \mapsto M$ generated by a control $\eta' : [0, T] \mapsto U$ such that η' is analytic on an open dense subset of $[0, T']$, $\xi'(0) = \xi(0)$, $\xi'(T') = \xi(T)$, and $\frac{T}{\kappa} \leq T' \leq \kappa T$. So $T - \varepsilon \leq T' \leq T + \varepsilon$, and then (WR) holds.)

It follows from the above that we can assume that M is an open subset of $\mathbb{R}^n$ for some n, so that in particular T^*M is naturally identified with $M \times \mathbb{R}^n$, and $T^\# M$ with $M \times (\mathbb{R}^n \backslash \{0\})$. We make this assumption from now on.

We let $\mathcal{R}$ denote the Σ-reachable set $R_{\Sigma,]T-\frac{\varepsilon}{2}, T+\frac{\varepsilon}{2}[}(\bar{x})$. We will treat separately the cases $\hat{x} \in \mathrm{Int}(\mathcal{R})$, $\hat{x} \notin \mathrm{Int}(\mathcal{R})$. The former case is easily disposed of. Indeed, Theorem 3 tells us that $\hat{x}$ is reachable from $\bar{x}$ in time t for some $t \in [T - \varepsilon, T + \varepsilon]$ by means of a piecewise constant control, so our conclusion follows.

We now look at the case $\hat{x} \notin \mathrm{Int}(\mathcal{R})$. Let $\gamma = (\xi, \eta)$. Then the maximum principle tells us that γ is an extremal, so it has a null-minimizing nontrivial Hamiltonian lift $\Gamma = (\Xi, \eta)$. This means that Ξ is a trajectory of the Hamiltonian lift $\Sigma^* = (T^*M, U, f^*)$, generated by η, contained in $T^\# M$, projecting down to ξ, and such that

$$0 = H_\Sigma(\Xi(t), \eta(t)) = \min\{H_\Sigma(\Xi(t), u) : u \in U\} \text{ for almost all } t. \tag{12}$$

(The minimization part of (12) is not going to be used in what follows.) We then let

$$C = \{(z, u) \in T^\# M : H_\Sigma(z, u) = 0\}. \tag{13}$$

Then C is a real-analytic subset of $T^\# M \times U$.

If S is a submanifold of $T^\# M$, we use C_S^{tan} to denote the set of all pairs $(s, u) \in C$ such that $s \in S$ and $f^*(s, u)$ is tangent to S at s. It is clear that if S is subanalytic then C_S^{tan} is a subanalytic subset of $T^\# M \times U$ and a real-analytic subset of $S \times U$.

We now construct a stratification $\mathcal{S}$ of $T^\# M$ such that the following is true for each stratum S of $\mathcal{S}$:

(I) *S is a relatively compact subanalytic subset and a real-analytic embedded submanifold of $T^\# M$, and there exist a compact real-analytic manifold D_S and a real-analytic map $\Phi_S : S \times D_S \mapsto T^\# M \times U$, such that D_S is a finite union of tori, and*

(I.a) *$\Phi_S(S \times D_S) = C_S^{\mathrm{tan}}$ and $\Phi_S(s, y) \in \{s\} \times U$ for every $(s, y) \in S \times D_S$,*

(I.b) *$\nu(D_S) \prec \nu(U)$.*

We will first prove that if $\mathcal{S}$ exists then our conclusion follows, and then we will show how to construct $\mathcal{S}$. Let us call a subset S of $T^{\#}M$ "good" if it satisfies (I), and let us call a stratification $\mathcal{S}$ of $T^{\#}M$ "good" if every stratum of $\mathcal{S}$ is good. Let $\mathcal{S}$ be a good stratification of $T^{\#}M$.

Let $S(t)$ denote, for $t \in [0,T]$, the stratum that contains $\Xi(t)$. Define

$$\mu(t) = \dim(S(t)).$$

Let I be set of $t \in [0,T]$ such that μ is constant on some neighborhood of t in $[0,T]$. Then I is relatively open and dense in $[0,T]$. (Openness is trivial. Density follows from the frontier and dimension axioms: if $t \in [0,T]$ and $\varepsilon > 0$, pick $t' \in [0,T] \cap]t-\varepsilon, t+\varepsilon[$ that maximizes $\mu(t')$. If μ was not constant near t', then there would exist t_j such that $t_j \to t'$ and $\mu(t_j) < \mu(t')$. But then we can assume, by passing to a subsequence, that all the $\Xi(t_j)$ belong to the same stratum S, whose dimension is smaller than that of $S(t')$. Since $\Xi(t_j) \to \Xi(t')$, this says that $\Xi(t')$ is in the closure of S, which is a contradiction, since $\Xi(t) \in S(t')$ and $\dim(S(t')) > \dim(S)$.)

Let $\mathcal{J}$ be the set of connected components of I, so $\mathcal{J}$ is finite or countably infinite, the members of $\mathcal{J}$ are pairwise disjoint relatively open subintervals of $[0,T]$, and $I = \bigcup \mathcal{J}$. If $J \in \mathcal{J}$, then the restriction Ξ_J of Ξ to J is entirely contained in a stratum S_J of $\mathcal{S}$. (This follows from the connectedness of J and the fact that the map $t \mapsto S(t)$ has to be locally constant on J. To prove this last fact, pick $t \in J$ and assume that there are t_j converging to t and such that $S(t_j) \neq S(t)$. Then we can pass to a subsequence and assume that all the $S(t_j)$ are equal to a fixed stratum S', and from the dimension and frontier axioms we get $\mu(t) = \dim(S(t)) < \dim(S') = \mu(t_j)$, contradicting the fact that $t \in I$.)

Let $g_J(s,v) = f^*(s, \Phi_{S_J}(v))$, for $s \in S_J$, $v \in D_{S_J}$. It is then clear that the vector $\dot{\Xi}_J(t) = f^*(\Xi_J(t), \eta(t))$ is tangent to S_J for almost all $t \in J$. We know that $\eta(t)$ belongs to $\{u : (\Xi(t), u) \in C\}$ for almost every $t \in J$. So $\eta(t) \in \{u : (\Xi(t), u) \in C_{S_J}^{\tan}\}$ for almost every $t \in J$. This means that for almost every $t \in J$ there exists a $v(t)$ belonging to D_{S_J} such that $\eta(t) = \Phi_{S_J}(v(t))$. Then $\dot{\Xi}_J(t) = g_J(\Xi_J(t), v(t))$. As explained in Remark 2, the function v can be chosen to be measurable. It then follows that Ξ_J is a trajectory of the control system $\Sigma_J = (S_J, D_{S_J}, g_J)$, which is exactly like our original system Σ, but with a control space D_{S_J} such that $\nu(D_{S_J}) \prec \nu(U)$. So the inductive hypothesis can be applied to Σ_J and the restriction Ξ_K of Ξ to any nontrivial compact subinterval $K = [a_K, b_K]$ of J, yielding the conclusion that there exist a trajectory-control pair (Ξ'_K, η'_K) of Σ_J, defined on a compact interval $K' = [a'_K, b'_K]$, such that $\Xi'_K(a'_K) = \Xi_K(a_K)$, $\Xi'_K(b'_K) = \Xi(b_K)$, and η'_K is analytic on an open dense subset of K'. Moreover, the length $L_{K'} = b'_K - a'_K$ of K' can be chosen so that $|L_{K'} - L_K| < \theta$, where $L_K = b_K - a_K$, and $\theta > 0$ is arbitrary.

We choose $\theta = \theta_K$ for each K so that $\frac{1}{\kappa} L_K < L_{K'} < \kappa L_K$. We also choose $a'_K = a_K$, as we certainly can given that our system is autonomous. Then Ξ'_K is defined on the interval $[a_K, a_K + L_{K'}]$. We let $\Xi^*_K : [a_K, b_K] \mapsto S_J$, $\eta^*_K : [a_K, b_K] \mapsto D_{S_J}$ be the maps given by

$$\Xi^*_K(t) = \Xi'_K\Big(a_K + \frac{L_{K'}}{L_K}(t - a_K)\Big), \ \eta^*_K(t) = \eta'_K\Big(a_K + \frac{L_{K'}}{L_K}(t - a_K)\Big). \tag{14}$$

Then $\Xi^*_K(a_K) = \Xi(a_K)$, $\Xi^*_K(b_K) = \Xi'_K(a_K + L_{K'}) = \Xi(b_K)$, and Ξ^*_K satisfies the differential equation $\dot{\Xi}^*_K(t) = \omega_K f^*_J(\Xi^*_K(t), \eta'_K(t))$ on K, where the constant ω_K lies between $\frac{1}{\kappa}$ and κ.

We now pick, for each $J \in \mathcal{J}$, a discrete subset E_J of J whose closure contains both endpoints of J. We then let $\mathcal{K}(J)$ be the set of all compact subintervals $K = [a_K, b_K]$ such that $a_K < b_K$, $a_K \in E_J$, $b_K \in E_J$, and $E_J \cap]a_K, b_K[= \emptyset$. We construct $\Xi_K^* : K \mapsto S_J$ and $\eta_K^* : K \mapsto D_{S_J}$ as above for each $K \in \mathcal{K}(J)$, and do this for all $J \in \mathcal{J}$. We then define a curve $\Xi' : [0,T] \mapsto T^\# M$ by first letting $\Xi'(t) = \Xi(t)$ if $t \in [0,T]\backslash I$ and then, if $t \in I$, picking the unique $J \in \mathcal{J}$ such that $t \in J$, and a $K \in \mathcal{K}(J)$ such that $t \in K$, and letting $\Xi'(t) = \Xi_K^*(t)$. The set K is unique unless $t \in E_J$, in which case there are two possible choices of K, both of which give $\Xi'(t) = \Xi(t)$. So Ξ' is well defined. Similarly, we define $\eta'(t) = \Phi_{S(t)}(\eta_K^*(t))$ for $t \in K \in \mathcal{K}(J)$, $J \in \mathcal{J}$, and $\eta'(t) = \eta(t)$ for $t \notin I$. We also define $\omega(t) = \omega_K(t)$ for $t \in K \in \mathcal{K}(J)$, $J \in \mathcal{J}$, and $\omega(t) = 1$ for $t \notin I$.

We now show that the curve Ξ' is Lipschitz, and satisfies the differential equation

$$\dot{\Xi}'(t) = \omega(t) f^*(\Xi'(t), \eta'(t)) \text{ for almost all } t \in [0,T]. \tag{15}$$

To see this, we first let $S(\Xi)$ be the set of strata $S \in \mathcal{S}$ such that $\Xi([0,T])$ intersects S. Then $S(\Xi)$ is finite, because the set $\Xi([0,T])$ is compact. Let $Q = \bigcup\{S : S \in S(\Xi)\}$. Then $||f^*(x,z,u)||$ is bounded by a constant $c > 0$ as long as $(x,z) \in Q$. So Ξ is Lipschitz with constant c. Also, Ξ' is contained in Q by construction, and each curve Ξ_K^*, for $K \in \mathcal{K}(J)$, $J \in \mathcal{J}$, is a solution of the equation $\dot{\Xi}_K^*(t) = \omega_K(t) f^*(\Xi_K^*(t), \eta'(t))$ on K. So each Ξ_K^* is Lipschitz with constant κc. If $0 \le t_1 < t_2 \le T$, we will show that

$$||\Xi'(t_2) - \Xi'(t_1)|| \le \kappa c (t_2 - t_1). \tag{16}$$

Suppose first that $\Xi'(t_1) = \Xi(t_1)$ and $\Xi'(t_2) = \Xi(t_2)$. Then the inequality

$$||\Xi'(t_2) - \Xi'(t_1)|| \le c(t_2 - t_1)$$

holds, because $||\Xi(t_2) - \Xi(t_1)|| \le c(t_2 - t_1)$. So (16) holds.

Next suppose that $\Xi'(t_1) = \Xi(t_1)$ but $\Xi'(t_2) \ne \Xi(t_2)$. Then $t_2 \in I$, so $t_2 \in J$ for some $J \in \mathcal{J}$. Therefore $t_2 \in K = [a_K, b_K]$ for some $K \in \mathcal{K}(J)$. If $t_1 \ge a_K$, then both t_1 and t_2 are in K, so $\Xi'(t_1) = \Xi_K^*(t_1)$ and $\Xi'(t_2) = \Xi_K^*(t_2)$, and then $||\Xi'(t_2) - \Xi'(t_1)|| \le \kappa c(t_2 - t_1)$, because Ξ_K^* is Lipschitz with constant κc. If $t_1 < a_K$, then

$$\begin{aligned} ||\Xi'(t_2) - \Xi'(t_1)|| &\le ||\Xi'(t_2) - \Xi'(a_K)|| + ||\Xi'(a_K) - \Xi'(t_1)|| \\ &\le \kappa c(t_2 - a_K) + ||\Xi(a_K) - \Xi(t_1)|| \le \kappa c(t_2 - t_1), \end{aligned}$$

so (16) holds.

A similar argument shows that (16) is true if $\Xi'(t_1) \ne \Xi(t_1)$ and $\Xi'(t_2) = \Xi(t_2)$. Finally, if $\Xi'(t_1) \ne \Xi(t_1)$ and $\Xi'(t_2) \ne \Xi(t_2)$, then both t_1 and t_2 are in I. If both belong to the same interval $K \in \mathcal{K}(J)$, for a $J \in \mathcal{J}$, then (16) follows because Ξ_K^* is Lipschitz with constant κc. If they do not, then there is a t such that $t_1 < t < t_2$ and $\Xi'(t) = \Xi(t)$, in which case the results for the previous cases apply, and

$$\begin{aligned} ||\Xi'(t_2) - \Xi'(t_1)|| &\le ||\Xi'(t_2) - \Xi'(t)|| + ||\Xi'(t) - \Xi'(t_1)|| \\ &= ||\Xi'(t_2) - \Xi(t)|| + ||\Xi(t) - \Xi'(t_1)|| \le \kappa c(t_2 - t_1), \end{aligned}$$

so (16) holds as well. So Ξ' is Lipschitz with constant κc.

To prove that equation (15) holds, we first observe that the derivative $\dot{\Xi}'(t)$ is clearly equal to $\omega(t) f^*(\Xi'(t), \eta'(t))$ for almost every $t \in J$, if $J \in \mathcal{J}$. So it suffices to prove that $\dot{\Xi}'(t) = f^*(\Xi'(t), \eta'(t))$ for almost all $t \in [0,T]\backslash I$. Now, if $t \in [0,T]\backslash I$, then $\Xi'(t) = \Xi(t)$ and $\eta'(t) = \eta(t)$. So we have to prove that $\dot{\Xi}'(t) = f^*(\Xi(t), \eta(t))$ for almost all $t \in [0,T]\backslash I$. We know that almost every point of $[0,T]\backslash I$ is a point of density of $[0,T]\backslash I$, and $\dot{\Xi}(t)$ exists and is equal to $f^*(\Xi(t), \eta(t))$ for almost all $t \in [0,T]$. We also know that $\dot{\Xi}'(t)$ exists for almost all $t \in [0,T]$, because Ξ is Lipschitz. So there is a subset B of $[0,T]\backslash I$ of measure zero such that, if $t \in [0,T]\backslash I$ but $t \notin B$, then t is a point of density of $[0,T]\backslash I$,

$\dot{\Xi}(t) = f^*(\Xi(t), \eta(t))$, and $\dot{\Xi}'(t)$ exists. Any such t is an accumulation point of $[0,T]\backslash I$, so the limit $\dot{\Xi}'(t) = \lim_{h\to 0, h\neq 0} \frac{\Xi'(t+h)-\Xi'(t)}{h}$ can be computed by just letting $t+h$ vary in $[0,T]\backslash I$. Since $\Xi' \equiv \Xi$ on $[0,T]\backslash I$, we conclude that $\dot{\Xi}'(t) = \dot{\Xi}(t) = f^*(\Xi(t), \eta(t))$, as desired.

So we have shown that the curve Ξ' satisfies (15). We now eliminate the multiplicative factor $\omega(t)$ by reparametrizing the time interval. To do this, we define a reparametrization map $\tau : [0,T] \mapsto [0,T']$ by letting $\tau(r) = \int_0^r \omega(s)\,ds$. (Here we let $T' \stackrel{\text{def}}{=} \tau(T) = \int_0^T \omega(s)\,ds$.) Then $\frac{1}{\kappa} \leq \frac{d\tau}{dt} \leq \kappa$, so τ is a homeomorphism.

We then define $\Xi''(t) = \Xi'(r)$ and $\eta''(t) = \eta'(r)$, if $t = \tau(r)$. Then

$$\frac{d\Xi''}{dt}(t) = f^*(\Xi''(t), \eta''(t)) \text{ for almost all } t \in [0,T']. \tag{17}$$

Then Ξ'' is a trajectory of Σ^* generated by η''. Clearly, $\Xi''(T') = \Xi(T)$ and $\Xi''(0) = \Xi(0)$. So, if we define $\xi'' = \pi_{T^*M} \circ \Xi''$, it follows that ξ'' is a trajectory of Σ such that $\xi''(T') = \hat{x}$ and $\xi''(0) = \bar{x}$, and ξ'' is also generated by η''. Moreover, η'' is analytic on an open dense subset of $[0,T']$. Finally, T' satisfies $\frac{T}{\kappa} \leq T' \leq \kappa T$, so $T - \varepsilon \leq T' \leq T + \varepsilon$. Therefore the proof of (WR) is complete, modulo the assumption that $\mathcal{S}$ exists.

We now prove the existence of a good stratification $\mathcal{S}$. Recall that M is assumed to be an open subset of $\mathbb{R}^n$. For $k \in \{0, \ldots, 2n+1\}$, call a stratification $\mathcal{S}$ "k-good" if all the strata of $\mathcal{S}$ of dimension $\geq k$ are good. We need to prove the existence of a 0-good stratification. We do this by proving the existence, for every $k \in \{0, \ldots, 2n+1\}$, of a k-good stratification $\mathcal{S}_k$ with real-analytic subanalytic relatively compact strata. This is done by induction with respect to $k' = 2n+1-k$. When $k' = 0$, i.e. $k = 2n+1$, we let $\mathcal{S}_{2n+1}$ be any stratification of $T^{\#}M$ (i.e. of $M \times (\mathbb{R}^n\backslash\{0\})$) with real-analytic relatively compact subanalytic strata.

We now assume that $k \in \{1, 2, \ldots, 2n+1\}$ and a k-good stratification $\mathcal{S}_k$ with real-analytic subanalytic relatively compact strata exists. We want to construct a $(k-1)$-good stratification $\mathcal{S}_{k-1}$, also with real-analytic subanalytic relatively compact strata.

Let $\mathcal{A}$ be the set of all strata of $\mathcal{S}_k$ of dimension $k-1$. Let $\mathcal{U}$ be the set of connected components of U. For each $S \in A$, and each $U' \in \mathcal{U}$, we let $C(S, U')$ be the set of all $s \in S$ such that $\{s\} \times U' \subseteq C_S^{\tan}$. It is easy to see that each set $C(S, U')$ is subanalytic. Since $\mathcal{U}$ is finite, the family of sets $C(S, U')$, for all $S \in \mathcal{A}$, $U' \in \mathcal{U}$, is locally finite in $T^{\#}M$. So by standard stratification theorems there exists a stratification $\tilde{\mathcal{S}}$ with real-analytic subanalytic strata which is a refinement of $\mathcal{S}_k$ and is compatible with every set $C(S, U')$, $S \in \mathcal{A}$, $U' \in \mathcal{U}$. (We say that a set $\mathcal{H}$ of sets is *compatible* with a set L if every member of $\mathcal{H}$ either disjoint from or a subset of L.)

Let $\tilde{\mathcal{A}}$ be the set of all $(k-1)$-dimensional strata of $\tilde{\mathcal{S}}$ that are contained in an $S \in \mathcal{A}$. If $S \in \tilde{\mathcal{A}}$ and $U' \in \mathcal{U}$, then either $\{s\} \times U' \subseteq C_S^{\tan}$ for all $s \in S$, or $\{s\} \times U' \not\subseteq C_S^{\tan}$ for all $s \in S$. (Notice that if $s \in S$, $S \in \tilde{\mathcal{A}}$, $S \subseteq S' \in \mathcal{A}$, then $(s,u) \in C_S^{\tan} \Leftrightarrow (s,u) \in C_{S'}^{\tan}$.)

For a stratum $S \in \tilde{\mathcal{A}}$, we let $\mathcal{U}(S)$ be the set of those $U' \in \mathcal{U}$ having the property that $\{s\} \times U' \subseteq C_S^{\tan}$ for all $s \in S$.

We now make use of the assumption that Σ has the algebraic accessibility property at every point, to prove the following crucial fact:

(*) *If $S \in \tilde{\mathcal{A}}$, then $\mathcal{U}(S) \neq \mathcal{U}$.*

To prove (*), we assume that $S \in \tilde{\mathcal{A}}$ is such that $\mathcal{U}(S) = \mathcal{U}$, and try to derive a contradiction. Our assumption says that the vector $f^*(s,u)$ is tangent to S for every

$s \in S$, $u \in U$, and in addition $H_\Sigma(s,u) = 0$ for $s \in S$, $u \in U$. Let $h^u : T^\# M \mapsto \mathbb{R}$ be the function $(x,z) \mapsto H_\Sigma(x,z,u)$. Then h^u is the momentum function corresponding to the vector field f_u, and f_u^* is the Hamiltonian vector field $\overrightarrow{h^u}$. Let L be the subset of $L(\Sigma)$ consisting of all vector fields $X \in L(\Sigma)$ such that the momentum function h_X vanishes identically on S. Then L is obviously a linear space. Moreover, $f_u \in L$ for all $u \in U$, since h^u vanishes on S. Suppose that, for some ℓ, L contains all the brackets of the form

$$Y = [f_{u_1}, [f_{u_2}, [\ldots, [f_{u_{\ell-1}}, f_{u_\ell}] \ldots]]]. \tag{18}$$

Let $X = [f_u, Y]$, where Y is of the form (18). Then $h_X = h_{[f_u,Y]} = \{h^u, h_Y\}$. Moreover, the Poisson bracket $\{h^u, h_Y\}$ is the directional derivative $\overrightarrow{h}^u h_Y$, i.e. $f_u^* h_Y$. Since $h_Y \equiv 0$ on S, and f_u^* is tangent to S, we conclude that $h_X \equiv 0$ on S, so $X \in L$. This proves that L contains all the brackets of the form (18), for all ℓ. Therefore $L = L(\Sigma)$. Given $s = (x,z) \in S$, the accessibility condition tells us that $T_x M = L(\Sigma)(x) = L(x)$. So every vector $v \in T_x M$ is of the form $X(x)$ for some $X \in L$. Therefore $h_X \equiv 0$ on S, so $h_X(x,z) = 0$. But this says that $z(v) = 0$. So z annihilates $T_x M$. Therefore $z = 0$, contradicting the fact that $(x,z) \in T^\# M$. This completes the proof of (*).

Now let $S \in \tilde{\mathcal{A}}$ and pick $U' \in \mathcal{U}$ such that $U' \notin \mathcal{U}(S)$. For $s \in S$, let $Z_{S,U'}(s)$ be the set of those $u \in U'$ such that $(s,u) \in C_S^{\rm tan}$. We then know that $Z_{S,U'}(s)$ is a proper subset of U'. Moreover, $u \in Z_{S,U'}(s)$ iff $f^*(s,u)$ is tangent to S and $H_\Sigma(s,u) = 0$. This clearly implies that $Z_{S,U'}(s)$ is an analytic subset of U'. Since U' is connected, and $Z_{S,U'}(s) \neq U'$, we conclude that $Z_{S,U'}$ is an analytic subset of U' of positive codimension.

With S, U' as before, we now let $Z_{S,U'} = \{(s,u) : u \in Z_{S,U'}(s)\}$. Then $Z_{S,U'}$ is an analytic subset of $S \times U'$ and a subanalytic relatively compact subset of $T^\# M \times U'$. We let $\bar{Z}_{S,U'}$ be the closure of $Z_{S,U'}$ in $T^\# M \times U'$. Then $\bar{Z}_{S,U'}$ is subanalytic and compact. Since the dimension of each fiber $Z_{S,U'}(s)$ is $< \dim U'$, we can conclude that, if we write $\Delta_{S,U'} = \dim \bar{Z}_{S,U'}$, then $\Delta_{S,U'} < \dim S + \dim U'$.

We now use the desingularization theorem to find, for each $S \in \tilde{\mathcal{A}}$ and each $U' \in \mathcal{U}$ such that $U' \notin \mathcal{U}(S)$, a compact analytic manifold $N_{S,U'}$ of dimension $\Delta_{S,U'}$ and a real-analytic map $\Psi_{S,U'}$ from $N_{S,U'}$ onto $\bar{Z}_{S,U'}$. Let $\psi_{S,U'}$ be the composite of $\Psi_{S,U'}$ with the projection $(x,z,u) \mapsto (x,z)$ from $T^\# M \times U$ onto $T^\# M$. Then $\psi_{S,U'}$ is a real-analytic map such that $\psi_{S,U'}(N_{S,U'})$ is the closure $\bar{S}$ of S.

Let $\hat{N}_{S,U'} = \psi_{S,U'}^{-1}(S)$. Then $\hat{N}_{S,U'}$ is open in $N_{S,U'}$. Let $\hat{\Psi}_{S,U'}$, $\hat{\psi}_{S,U'}$ be the restrictions of $\Psi_{S,U'}$, $\psi_{S,U'}$ to $\hat{N}_{S,U'}$. Then $\hat{\psi}_{S,U'}$, regarded as a map into S, is proper and surjective. Moreover, the graph of $\hat{\psi}_{S,U'}$ is a subanalytic subset of $T^\# M \times N_{S,U'}$.

Let $S_{{\rm crit},U'}$ be the set of critical values of $\hat{\psi}_{S,U'}$. Then $S_{{\rm crit},U'}$ is a subanalytic subset of $T^\# M$, which is relatively closed as a subset of S, and has dimension less than $\dim(S)$.

Let $\mathcal{S}^*$ be a real-analytic subanalytic stratification of $T^\# M$ which is a refinement of $\tilde{\mathcal{S}}$, is compatible with all the sets $S_{{\rm crit},U'}$ for all $S \in \tilde{\mathcal{A}}$ and all $U' \in \mathcal{U} \backslash \mathcal{U}(S)$, and consists of strata that are analytically diffeomorphic to balls.

We then let $\mathcal{S}_{k-1}$ be the union of

(a) the set of all strata of $\mathcal{S}_k$ of dimension $\geq k$,

(b) the set of all strata of $\mathcal{S}^*$ that are contained in a stratum of $\mathcal{S}_k$ of dimension $< k$.

Then $\mathcal{S}_{k-1}$ is a stratification.

We now prove that $\mathcal{S}_{k-1}$ is $(k-1)$-good. It is clear that every stratum of $\mathcal{S}_{k-1}$ is relatively compact, subanalytic, and a real-analytic submanifold of $T^\# M$. The strata of dimension $\geq k$ are in $\mathcal{S}_k$, so they are good.

Let $S \in \mathcal{S}_{k-1}$ be such that $\dim S = k-1$. Then $S \subseteq S'$ for some $S' \in \tilde{\mathcal{S}}$ such that $\dim S' = k-1$. Therefore $S' \subseteq S''$ for some $S'' \in \mathcal{S}_k$. Clearly, S'' must be $(k-1)$-dimensional, so $S'' \in \mathcal{A}$, and then $S' \in \tilde{\mathcal{A}}$. From the fact that $\tilde{\mathcal{S}}$ is compatible with the sets $C(S'', U')$ for all $U' \in \mathcal{U}$ it follows that for each $U' \in \mathcal{U}$ either

(i) $(s,u) \in C^{\tan}_{S''}$ for all $s \in S'$, $u \in U'$,

or

(ii) for every $s \in S'$ the set $\{u \in U' : (s,u) \in C^{\tan}_{S''}\}$ is a proper real-analytic subset of U' of positive codimension.

The set of components $u' \in \mathcal{U}$ for which (i) holds is precisely what we called $\mathcal{U}(S')$. For $U' \notin \mathcal{U}(S')$, we have constructed a manifold $\hat{N}_{S',U'}$ and a real-analytic map $\hat{\Psi}_{S',U'}$: $N_{S',U'} \mapsto S'$ such that $\hat{\Psi}_{S',U'}(\hat{N}_{S',U'}) = Z_{S',U'}$, and $\dim \hat{N}_{S',U'} = \dim Z_{S',U'} < \dim U' + k - 1$. The composite $\hat{\psi}_{S',U'}$ of $\hat{\Psi}_{S',U'}$ with the projection $(s,u) \mapsto s$ is a proper real-analytic map from $\hat{N}_{S',U'}$ onto S'. Since $S \in \mathcal{S}^*$, the set $S'_{\mathrm{crit},U'}$ either contains S or is disjoint from S. The possibility that $S \subseteq S'_{\mathrm{crit},U'}$ is excluded because $\dim S = \dim S' = k-1$. So $S \cap S'_{\mathrm{crit},U'} = \emptyset$.

Let $N^*_{S,U'} = \hat{\psi}^{-1}_{S',U'}(S)$, for $U' \in \mathcal{U}(S')$. Let $\Psi^*_{S,U'}$, $\psi^*_{S,U'}$ be the restrictions of $\hat{\Psi}_{S',U'}$, $\hat{\psi}_{S',U'}$ to $N^*_{S,U'}$. Then $\psi^*_{S,U'}$ is a proper real-analytic submersion from $N^*_{S,U'}$ onto S. Since S is analytically diffeomorphic to a ball, $\psi^*_{S,U'}$ is globally analytically trivial, in the sense that there exist a real-analytic manifold $D^*_{S,U'}$ and a C^ω diffeomorphism $Y_{S,U'}$ from $S \times D^*_{S,U'}$ onto $N^*_{S,U'}$ such that $\psi^*_{S,U'} \circ Y_{S,U'}(s,r) = s$ for all $s \in S$, $r \in D^*_{S,U'}$. (To prove this, it suffices to assume that S is a ball. Put a real-analytic Riemannian metric on $N^*_{S,U'}$ (2). Using this metric, the vector fields $X_i = \frac{\partial}{\partial x_i}$ can be lifted to vector fields $\tilde{X}_i$ on $N^*_{S,U'}$ that are orthogonal to the fibers. If $\bar{s} = (0,\ldots,0)$ is the center of S, and we let $D_{S,U'} = (\psi^*_{S,U'})^{-1}(\bar{s})$, then we can define

$$Y_{S,U'}((s_1,\ldots,s_{k-1}),r) = r\, e^{s_1\tilde{X}_1}\ldots e^{s_{k-1}\tilde{X}_{k-1}}. \tag{19}$$

It is clear that this choice of $D^*_{S,U'}$ and $Y_{S,U'}$ satisfies our requirements.)

We now apply the desingularization theorem again and find, for each component $U' \in \mathcal{U}\backslash\mathcal{U}(S')$, a pair $(D^{**}_{S,U'}, W_{S,U'})$, such that

(i) $D^{**}_{S,U'}$ is a compact manifold of the same dimension as $D^*_{S,U'}$,

(ii) $D^{**}_{S,U'}$ is a finite union of tori,

(iii) $W_{S,U'}$ is a real-analytic map from $D^{**}_{S,U'}$ onto $D^*_{S,U'}$.

Finally, we define D_S to be the disjoint union of the connected components of U that belong to $\mathcal{U}(S')$, and the manifolds $D^{**}_{S,U'}$, for $U' \in \mathcal{U}\backslash\mathcal{U}(S')$. We then define maps $\Phi_S : S \times D_S \mapsto T^{\#}M \times U$ by letting $\Phi_S(s,r) = (s,r)$ if $r \in U' \in \mathcal{U}(S')$, and

$$\Phi_S(s,r) = \Psi^*_{S,U'}\Big(Y_{S,U'}\Big((s, W_{S,U'}(r))\Big)\Big) \tag{20}$$

if $r \in U' \in \mathcal{U}\backslash\mathcal{U}(S')$.

Then condition (I.a) holds. To conclude our proof, we have to show that (I.b) holds, i.e. that $\nu(D_S) \prec \nu(U)$.

To see that $\nu(D_S) \prec \nu(U)$, observe that D_S was constructed by taking the union of certain components of U, and replacing each remaining component U' by a manifold

(2) Such a metric exists on an arbitrary real-analytic manifold by Grauert's theorem, but here the existence is trivial because our manifold is an open subset of a finite union of tori.

of strictly smaller dimension(3). We know from (*) that at least one component of U is replaced by a manifold of smaller dimension. Let ℓ be the largest of the dimensions of the components of U that are so replaced. Then $\nu_j(D_S) = \nu_j(U)$ for $j > \ell$, and $\nu_\ell(D_S) < \nu_\ell(U)$. So $\nu(D_S) \prec \nu(U)$, and our proof is complete. ∎

Remark 6. The analogue of Theorem 9 for smooth systems is false. For simplicity, we just explain the situation for the case when $U = [-1,1]$. It is easy to show that *given any $T > 0$ and a completely arbitrary measurable function $\eta : [0,T] \mapsto [-1,1]$, there exist a control system in $\mathbb{R}^3$ of the special form $\dot{x} = f(x) + ug(x)$, $u \in [-1,1]$, with vector fields f, g of class C^∞, and a pair $(\bar{x}, \hat{x})$ of points in $\mathbb{R}^3$, such that*

(a) *η steers $\bar{x}$ to $\hat{x}$,*

(b) *no other control does.*

The construction of f, g, $\bar{x}$ and $\hat{x}$ is rather simple, so we review it. Let $h(t) = \int_0^t \eta(s)\,ds$, for $t \in [0,T]$. Let K be the graph of h, i.e. $K = \{(x,y) \in \mathbb{R}^2 : x \in [0,T],\ y = h(x)\}$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^∞ such that $\psi(x,y) \geq 0$ for all (x,y), and $\psi(x,y) = 0$ iff $(x,y) \in K$. We then take our system to be the one given by $\dot{x} = 1$, $\dot{y} = u$, $\dot{z} = \psi(x,y)$, and we let $\bar{x} = (0,0,0)$, $\hat{x} = (T, h(T), 0)$. The proof that this works is trivial.

Remark 7. Theorem 9, together with Remark 6, show that the regularity properties of trajectories for real-analytic systems are fundamentally different from those that hold in the smooth case. To be precise, let us define, for a given control set U, a *sufficient class of controls*, for a given class $\mathbf{\Sigma}$ of systems, to be a subset $\mathbf{U}$ of the class $M(U)$ of all measurable U-valued controls defined on intervals of the form $[0,T]$, having the property that, whenever $\Sigma \in \mathbf{\Sigma}$ and a control η steers a state $\bar{x}$ to a state $\hat{x}$, then there is a control η' belonging to $\mathbf{U}$ that steers $\bar{x}$ to $\hat{x}$.

What we have in mind is classes $\mathbf{U}$ characterized by "regularity properties". For example, if it was true —as is often implicitly assumed in many nonrigorous books and papers on optimal control— that whenever two points can be connected by a trajectory arising from a general control, then the points can also be joined by a trajectory generated by a piecewise continuous control, then the class of piecewise continuous controls would be sufficient. Such a simple statement is not true, however, for the class of smooth systems. In fact, Remark 6 says that *for the class of smooth systems there is no sufficient class smaller than the class of all controls*, i.e. that every conceivable pathology does occur. Theorem 9 says that for the class of all real-analytic systems there is a proper subset of $M(U)$ which is sufficient, namely, the class of all controls η such that η is real-analytic on $\mathrm{Dom}(\eta)$. This is still a very large class, but Theorem 9 at least shows that *something special happens for real-analytic systems that has no counterpart for smooth systems.*

5.1. *Open problems.* It is not known if any of the following stronger versions of Theorem 9 are true:

(I) same statement as in Theorem 9, except that η is analytic on an open dense subset of the interval $[0,T]$ whose complement has measure zero.

(II) same statement as in Theorem 9, except that η is analytic on an open dense subset of the interval $[0,T]$ whose complement is countable.

(3) The components of U are of course of pure dimension, but the manifolds that replace them need not be. The *dimension* of a manifold that is not of pure dimension is the maximum of the dimensions of its components. The dimension of a compact manifold is finite.

The answers to these questions are unknown even for control-affine systems in $\mathbb{R}^3$ of the form $\dot{x} = f(x) + ug(x)$, $u \in [-1, 1]$, with f, g real-analytic vector fields. It is known that the even stronger version in which the set of points of nonanalyticity is finite can fail, as shown by a famous example due to Fuller. (There is an entire book [15] devoted to variations on the Fuller problem.)

Remark 8. The regularity problem is often stated as a question about the **regularity of optimal trajectories**. Suppose a trajectory-control pair (ξ, η) minimizes an integral $\int_0^T L(\xi(t), \eta(t))\,dt$ among all trajectory-control pairs that go from $\bar{x}$ to $\hat{x}$. Can one infer from this that η must have some extra regularity property? To get a good problem, one must exclude degenerate cases where pathological optimal controls exist for trivial reasons, e.g. because every control steering $\bar{x}$ to $\hat{x}$ is optimal. (For example, take any control system, and let $L \equiv 0$.) The best way to do this is to reformulate the problem as a question about "regularity of sufficient classes": can one identify, for a given class $\mathcal{P}$ of optimal control problems (i.e. of 4-tuples $(\Sigma, L, \bar{x}, \hat{x})$), a class $\mathbf{U}$ of controls with the property that, whenever a problem in $\mathcal{P}$ has a solution, then it has a solution in $\mathbf{U}$? (Notice that in the special case when a problem in $\mathcal{P}$ has a unique solution it will follow that this solution is in $\mathbf{U}$, so for these problems we get a "regularity property of solutions" in the ordinary sense.) Theorem 9 implies that, for the class of real-analytic problems (i.e. problems where Σ and L are real-analytic), one can take $\mathbf{U}$ to be the class of controls that are analytic on an open dense subset of their domain. (The proof is trivial: apply Theorem 9 to the "augmented system" $\dot{x} = f(x, u)$, $\dot{y} = L(x, u)$.) Remark 6 implies that no such class exists for smooth problems. (The control η constructed in the remark is optimal —for any choice of L— because it is the only control that steers $\bar{x}$ to $\hat{x}$.)

The open questions stated above for the reachability problem are also open for optimal control.

6. Visibility and observability. We now consider real-analytic control systems $\Sigma = (M, U, f)$ with a real-analytic "output map" $h : M \times U \mapsto \mathbb{R}^m$. Given a control $\eta : [0, T] \mapsto U$ and a point $x \in M$, we can define the *output* $\rho_{x,\eta}$ corresponding to x and η to be the function $t \mapsto h(\xi(t), \eta(t))$, where ξ is the trajectory generated by η such that $\xi(0) = x$. (If ξ is not defined on the whole interval $[0, T]$ then $\rho_{x,\eta}(t)$ will not be defined for all t.)

Let us say that a control η *sees*(4) a state x if the output $\rho_{x,\eta}$ is not almost everywhere zero wherever it is defined.

THEOREM 10. *For a real-analytic system, if every real-analytic control sees every state, then every measurable control sees every state.*

Proof. Let Σ' be the new system $\dot{x} = f(x, u)$, $\dot{y} = \|h(x, u)\|^2$, on $M \times \mathbb{R}$. Suppose $\eta : [0, T] \mapsto U$ is a measurable control that fails to see a state x. This means that —after restricting η to a smaller interval if necessary— there exists a trajectory $\xi : [0, T] \mapsto M$ generated by η, such that $\xi(0) = x$ and $h(\xi(t), \eta(t)) = 0$ for almost every $t \in [0, T]$. Using the same control η for Σ', with the initial condition $(x, 0)$, we see that $(\xi(T), 0)$ is reachable from $(x, 0)$. Therefore there exists a control $\eta' : [0, T'] \mapsto U$ which is analytic on an open dense subset of $[0, T']$ and steers $(x, 0)$ to $(\xi(T), 0)$. Let ξ' be the corresponding trajectory, and write $\xi'(t) = (\xi_0'(t), y(t))$. Then $y(0) = y(T') = 0$,

(4) A better word would be "detects", but the term "detectability" already has a different, established meaning in control theory.

and $y(T') = \int_0^{T'} \|h(\xi'(t), \eta'(t))\|^2\,dt$. So $h(\xi'(t), \eta'(t)) = 0$ for almost every t. Now pick a nontrivial subinterval $J = [a, b]$ of $[0, T']$ such that η' is real-analytic on J. Let $x^* = \xi'(a)$. Then the control $[0, b-a] \ni t \mapsto \eta'(t+a) \in U$ is real-analytic and does not see the state x^*. ∎

J.-P. Gauthier and I. Kupka call a control system $\Sigma = (M, U, f)$ with an output h as before *strongly observable* with respect to a class $\mathcal{C}$ of controls if for every pair of distinct initial states x_1, x_2 and every control in $\mathcal{C}$ the corresponding outputs are not identical. They proved the following:

THEOREM 11. *If a real-analytic system is strongly observable with respect to the class of all real-analytic controls, then it is strongly observable with respect to the class of all measurable controls.*

Let us show that the Gauthier-Kupka result follows from Theorem 10.

Consider the system $\dot{x}_1 = f(x_1, u)$, $\dot{x}_2 = f(x_2, u)$, on the manifold

$$\tilde{M} = (M \times M) \backslash \Delta, \tag{21}$$

where $\Delta = \{(x, x) : x \in M\}$, and take the output to be $h(x_1, u) - h(x_2, u)$. Then the hypothesis of Theorem 11 says that every analytic control sees every state of the new system, and the desired conclusion says that every measurable control sees every state. Theorem 10 then applies and yields Theorem 11 as a corollary. ∎

R e m a r k 9. It is easy to construct counterexamples showing that the obvious smooth analogues of Theorems 10 and 11 are false. For a counterexample to Theorem 10 we pick a function $\varphi : [0, 1] \mapsto \mathbb{R}$ which is continuous, nowhere differentiable, and such that $|\varphi(t)| \leq 1$ for all $t \in [0, 1]$. We let K be the graph of the indefinite integral of φ, i.e. the set of all $(x, y) \in \mathbb{R}^2$ such that $0 \leq x \leq 1$ and $y = \int_0^x \varphi(t)\,dt$. Then K is a compact subset of $\mathbb{R}^2$, so there is a function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ whose set of zeros is exactly K. We then consider the system $\dot{x} = 1$, $\dot{y} = u$, $u \in [-1, 1]$, with output $\psi(x, y)$. Then a control η fails to see a state $(\bar{x}, \bar{y})$ iff the trajectory for η that starts at $(\bar{x}, \bar{y})$ is entirely contained in K. So every control of class C^1 sees every state, but the continuous control φ does not see the state $(0, 0)$.

A similar, though slightly more complicated, construction gives a counterexample to the smooth analogue of Theorem 11.

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