MATHEMATICS 503, FALL 2005

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The midterm exam. There will be only one midterm exam, on **Thursday November 3**.

Postings. I have posted the following items in my Math 503 Web site (https://www.math.rutgers.edu/~sussmann/math503page.html): (1) an elementary proof of the fundamental theorem of algebra, without using analytic function theory, basically along the lines of what we did in class, (2) a purely combinatorial, formal power series proof (without using differentiation of series), sent to me by Doron Zeilberger, that $\log(\exp(X)) = X$.

The final take-home exam. In lieu of a final exam in class, I am assigning a set of homework problems. They are due on **Thursday, December 1, 2005.**

Here are the problems:

1. Book, page 93, problem 3.
2. Book, page 132, problems 1, 3.
3. Book, page 149, problem 3.
4. Book, page 158, problems 2, 3, 4, 5, 7, 11.
5. Book, pages 164-165, problems 3, 8, 12.
6. Book, pages 170-171-172, problems 1, 4, 8, 12.
7. Book, pages 204-205-206-207, problems 3, 9, 12, 14, 24.
8. Book, page 238, problem 12.
9. Book, page 307, problems 1, 2.
10. Book, page 311, problem 6.
11. Prove: if Ω is open in $\mathbb{C}$, $\gamma : [a, b] \mapsto \Omega$ is continuous, and $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge uniformly on every compact subset of Ω to a function $f : \Omega \mapsto \mathbb{C}$, then $\int_\gamma f_n(z)\, dz \to \int_\gamma f(z)\, dz$.

*In Problems 12, 13 and 14: A **curve in** Ω is a continuous map $\gamma : [a, b] \mapsto \Omega$ (for some $a, b \in \mathbb{R}$ such that $a < b$). A curve $\gamma : [a, b] \mapsto \Omega$ is **closed** if $\gamma(b) = \gamma(a)$. A **loop** is a closed curve. A loop $\gamma : [a, b] \mapsto \Omega$ is **simple** if γ is one-to-one on the interval $\{t : a \leq t < b\}$. A **homotopy in** Ω is a continuous map $H : [a, b] \times [c, d] \mapsto \Omega$ (for some $a, b \in \mathbb{R}$ such that $a < b$ and $c < d$). If $H : [a, b] \times [c, d] \mapsto \Omega$ is a homotopy, then we define H_s, for each $s \in [c, d]$, to be the curve given by $H_s(t) = H(t, s)$ for $(t, s) \in [a, b] \times [c, d]$. A homotopy $H : [a, b] \times [c, d] \mapsto \Omega$ is a **loop-homotopy** if all the curves H_s are loops, i.e., if $H(b, s) = H(a, s)$ for every $s \in [c, d]$. A homotopy $H : [a, b] \times [c, d] \mapsto \Omega$ is* *a **fixed-point homotopy** if all the curves H_s have the same endpoints, i.e., if $H(a,s) = H(a,c)$ and $H(b,s) = H(b,c)$ for every $s \in [c,d]$. Given two curves $\gamma, \delta : [a,b] \mapsto \Omega$, a **homotopy from** γ **to** δ is a homotopy $H : [a,b] \times [c,d] \mapsto \Omega$ such that $H_c = \gamma$ and $H_d = \delta$. The curves $\gamma, \delta : [a,b] \mapsto \Omega$ are **homotopic in** Ω if there exists a homotopy from γ to δ. The curves $\gamma, \delta : [a,b] \mapsto \Omega$ are **fixed-point homotopic in** Ω if there exists a fixed-point homotopy from γ to δ. The curves $\gamma, \delta : [a,b] \mapsto \Omega$ are **loop-homotopic in** Ω if there exists a loop-homotopy from γ to δ. (Naturally, γ and δ cannot be fixed-point homotopic unless they have the same endpoints. Also, γ and δ cannot be loop-homotopic unless they are both loops.)*

12. Prove that, if Ω is an open connected subset of $\mathbb{C}$, then any two curves $\gamma, \delta : [0,1] \mapsto \Omega$ are homotopic.

    Remark: This says that the notion of "homotopy," without any extra requirement such as "fixed point" or "loop," is completely uninteresting.

13. Let $\gamma : [a,b] \mapsto \mathbb{C}$ be a simple loop. Suppose that γ contains a nontrivial straight-line segment. (This means: "there exist α, β such that $a \leq \alpha < \beta < b$ having the property that $\gamma(t) = \gamma(\alpha) + \frac{t-\alpha}{\beta-\alpha}(\gamma(\beta) - \gamma(\alpha))$ for $\alpha \leq t \leq \beta$.") Let $|\gamma|$ be the set of all points $\gamma(t)$, $t \in [a,b]$. **Prove:** $\mathbb{C}\backslash|\gamma|$ is not connected. (Hint: prove that the winding number $W(\gamma, z)$ changes when z crosses the segment. First you will have to prove that for some point p lying on the segment and some small disc D centered at p the set $D \cap |\gamma|$ contains no points of $|\gamma|$ other than those that belong to the intersection of D with the segment.)

    Remark: The result of Problem 12 is a very simple special case of a theorem that seems completely obvious but is rather hard to prove, known as the *Jordan curve theorem*. The theorem says that if γ is a simple loop then $\mathbb{C}\backslash|\gamma|$ has exactly two connected components (that is, "$|\gamma|$ divides $\mathbb{C}$ into two pieces"). An important part of the proof is to show that $\mathbb{C}\backslash|\gamma|$ cannot be connected. In this problem you are asked to prove that conclusion when the loop γ contains a segment. It turns out that, with just a little bit more work, using the implicit function theorem, one can also prove rather easily that $\mathbb{C}\backslash|\gamma|$ is not connected if γ contains an embedded arc of class C^1. (This means: "there exist α, β such that $a \leq \alpha < \beta < b$ having the property that γ is continuously differentiable on $[\alpha, \beta]$ and $\gamma'(t) \neq 0$ for all $t \in [\alpha, \beta]$.") The truly hard things are: (a) to prove that $\mathbb{C}\backslash|\gamma|$ is not connected if γ is just a continuous simple loop, without assuming that γ contains a smooth embedded arc, and (b) to show that there cannot be moro than two connected components of $\mathbb{C}\backslash|\gamma|$. But, of course, I am **not** asking you to prove that! All I am asking you to do is to prove that $\mathbb{C}\backslash|\gamma|$ is not connected if γ contains a nontrivial segment.

14. An open subset Ω of $\mathbb{C}$ is *simply connected* if every closed curve $\gamma : [0,1] \mapsto \Omega$ is loop-homotopic in Ω to a constant curve. Prove that Ω is simply connected if and only if, whenever $\gamma : [0,1] \mapsto \Omega$ and $\delta : [0,1] \mapsto \Omega$ are curves in Ω that have the same endpoints (i.e., are such that $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$), it follows that γ and δ are fixed-endpoint homotopic in Ω.